Variational Characterization of the Speed of Propagation of Fronts for the Nonlinear Diffusion Equation

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Abstract

We give a variational characterization for the speed of fronts of the nonlinear diffusion equation $u_t = u_{xx} + f(u)$ with $f > 0$, which permits the calculation of the exact speed for arbitrary f.

1 Introduction

The problem of the asymptotic speed of propagation of the interface between an unstable and stable state has received much attention in connection with different problems of population growth, chemical reactions, pattern formation and others. We refer to [1] for a recent review and references. The best understood of such problems is that of the nonlinear reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1a}$$

with

$$f(0) = f(1) = 0,\ f'(0) > 0 \qquad \text{and} \quad f > 0 \qquad \text{in} \quad (0,1) \tag{1b}$$

for which Aronson and Weinberger [AW][2] have shown that any positive sufficiently localized initial condition $u(x,0)$ evolves into a front that joins the stable state $u = 1$ to $u = 0$. The asymptotic speed at which the front propagates is the minimal speed c^* for which there is a monotonic front joining $u = 1$ to $u = 0$. Moreover they show that the selected speed is bounded above and below by

$$2\sqrt{f'(0)} \leq c^* < 2 \sup \sqrt{\frac{f(u)}{u}} \tag{2}$$

and that the asymptotic selected front approaches the fixed point $u = 0$ exponentially with slope

$$m = -\frac{1}{2}(c^* + \sqrt{c^{*2} - 4f'(0)}). \tag{3}$$

The lower bound $c_L = 2\sqrt{f'(0)}$ is that predicted from a physical argument, the linear marginal stability hypothesis[3]. For concave functions f, the upper and lower bounds coincide and the speed is exactly the linear value. However, the asymptotic speed of propagation can still be the linear value even when the upper and lower bounds do not coincide as explicit examples and a variational characterization[4] which provides improved upper bounds shows. We have recently obtained an improved lower bound[5] on the speed of the front that enables one to decide when the selected speed is greater than the linear value case which is referred to as nonlinear marginal stability selection. There have been several reformulations of Aronson's and Weinberger's rigorous results for the nonlinear diffusion equation aiming to their heuristic extension to other higher order and pattern forming equations[6, 7, 8, 9]. None of these approaches however provide the means to calculate a priori the velocity of the fronts.

The purpose of the present work is to extend our previous result[5] to show a variational characterization of the speed of the fronts of equation (1) which enables its exact calculation for arbitrary f. Our main result is

$$c^* = \begin{cases} \max\{I(g)|g \in \mathcal{D}\} & \text{if } c^* > 2\sqrt{f'(0)} \\ 2\sqrt{f'(0)} & \text{otherwise} \end{cases} \tag{4}$$

where

$$I(g) = 2\frac{\int_0^1 \sqrt{fgh}du}{\int_0^1 gdu} \tag{5}$$

and $\mathcal{D}$ is the space of functions such that

$$g \geq 0, \quad h = -g' > 0 \quad g(1) = 0 \qquad \text{and} \int_0^1 g(u)\, du < \infty \tag{6}$$

In Section 2 we prove this result and in Section 3 an example is given.

2 The Variational Characterization

We are interested in the calculation of the minimal speed for which equation (1) has a monotonic travelling front $u(x,t) = q(z)$ with $z = x - ct$ joining $u = 1$ to $u = 0$. Since the selected speed corresponds to that of a decreasing monotonic front, it is convenient to work in phase space. Calling $p(q) = -dq/dz$, where the minus sign is included so that p is positive, we find that the monotonic fronts are solutions of

$$p(q)\,\frac{dp}{dq} - c^*\,p(q) + f(q) = 0, \tag{7}$$

with

$$p(0) = 0, \qquad p(1) = 0, \qquad p > 0 \quad \text{in} \quad (0,1). \tag{8}$$

In the derivation of the main result we shall make use of the known[2] asymptotic behavior of p at the origin which we briefly recall. Near $q = 0$, since the approach to the fixed point is exponential and given by (3), $p \sim a_1 q$ with $a_1 = -m$. This value of a_1 is the largest root of

$$a_1^2 - ca_1 + f'(0) = 0. \tag{9}$$

We find it convenient to introduce the parameter λ defined by

$$c = \lambda\, a_1$$

then,

$$c = \lambda\sqrt{\frac{f'(0)}{\lambda - 1}} \qquad \text{and} \qquad a_1 = \sqrt{\frac{f'(0)}{\lambda - 1}} \tag{10}$$

It is straightforward to verify that whenever $1 < \lambda < 2$ the above value of a_1 corresponds to the largest root of (9) and therefore to the asymptotic slope at the origin of the selected front[10]. At $\lambda = 2$ the speed attains the linear value c_L.

For completeness we shall repeat part of the developments given in[5]. Let g be any function in $\mathcal{D}$ defined in (6). Multiplying equation (7) by g/p and integrating with respect to q we find after integrating by parts,

$$c^* = \frac{\int_0^1 \left(h\,p + \frac{f(q)}{p}\,g\right) dq}{\int_0^1 g(q) dq} \tag{11}$$

However since p, h, f, and g are positive, for every fixed q

$$h\,p + \frac{f(q)\,g}{p} \geq 2\sqrt{f\,g\,h} \tag{12}$$

hence

$$c^* \;\geq 2\,\frac{\int_0^1 \sqrt{f\,g\,h}\,dq}{\int_0^1 g\,dq} \tag{13}$$

In order to prove the main result (4), we must show that we can always find a function g for which the equality in (12) and therefore in (13), holds.

Let $v(q)$ be the positive solution of the equation

$$\frac{v'}{v} = \frac{c^*}{p} \tag{14}$$

and choose

$$g = \frac{1}{v'}. \tag{15}$$

First we show that this function belongs to $\mathcal{D}$. Since p is positive there is always a positive solution for v and from (14) we see that v' and therefore g is positive. Also one can always adjust the constant of integration in (14) in such a way to have $g(1) = 0$. We must verify that $h = -g' > 0$. From its definition

$$g' = -\frac{v''}{v'^2} \tag{16a}$$

and taking the derivative of (14) with respect to q we have

$$\frac{v''}{v} - \frac{v'^2}{v^2} = -\frac{c}{p^2}p' \tag{16b}$$

Making use of equations (7) and (14) to eliminate p' and v' in (16b) we find

$$\frac{v''}{v} = \frac{cf}{p^3} \tag{17}$$

Since v, f and p are positive, $v'' > 0$ and $h = -g' > 0$. Finally we must show that the integral of g does not diverge. Since p is continuous and vanishes only at $q = 0$ and $q = 1$, the only divergence of the integral of g can occur at these points. We have chosen $g(1) = 0$ so we must examine the behavior of g at $q = 0$. Near $q = 0$, we know $p \sim a_1 q$ so from (14) we have

$$\frac{v'}{v} \sim \frac{c}{a_1 q} = \frac{\lambda}{q}$$

which implies that $v \sim q^\lambda$ and

$$g \sim \frac{1}{\lambda q^{\lambda - 1}} \tag{18}$$

and therefore

$$\int_0^1 g(q)\, dq < \infty \qquad \text{if} \qquad \lambda < 2 \tag{19}$$

which is precisely the value of λ corresponding to the asymptotic behavior of the selected front.

Having verified that $g \in \mathcal{D}$ we now show that for this choice of g the equality holds in (13). We must verify that

$$hp + \frac{fg}{p} = 2\sqrt{fgh}$$

or equivalently, that

$$hp = \frac{fg}{p}$$

. This follows directly from the definition of g, $hp = -g'p = v''p/v'^2$. Using equations (14), (17) and the definition (15) of g the result follows. Having shown that whenever $\lambda < 2$ there exists a function g for which we obtain the exact speed we have proven (4).

3 Example

In this section we illustrate the results by applying it to the exactly solvable case $f(u) = u(1-u)(1+au)$ for which it is known that

$$c^* = \begin{cases} c = \sqrt{\frac{2}{a}} + \sqrt{\frac{a}{2}} & \text{if } a > 2 \\ 2\sqrt{f'(0)} & \text{if } a < 2 \end{cases}$$

Let

$$g(q) = \frac{(1-q)^{\lambda+1}}{q^{\lambda-1}} \qquad \text{with } \lambda = 1 + \tfrac{2}{a}$$

then

$$h = -g' = \frac{(1-q)^{\lambda}}{q^{\lambda}}(\lambda - 1)(1 + ax)$$

and[11]

$$\int_0^1 g(q)\,dq = \frac{\Gamma(\lambda+2)\Gamma(2-\lambda)}{\Gamma(4)} \qquad \text{if} \quad \lambda < 2$$

We obtain

$$\int_0^1 \sqrt{fgh}\,dq = \sqrt{\lambda - 1}\left(\int_0^1 \frac{(1-q)^{\lambda-1}}{q^{\lambda-1}}dq + a\int_0^1 (1-q)^{\lambda+1}q^{2-\lambda}dq\right)$$

and therefore

$$I(g) = 2\sqrt{\lambda-1}\left(1 + a\frac{\Gamma(3-\lambda)}{\Gamma(2-\lambda)}\frac{\Gamma(4)}{\Gamma(5)}\right)$$

Using the definition of λ and $\Gamma(z+1) = z\Gamma(z)$ we obtain

$$I(g) = \sqrt{\frac{2}{a}} + \sqrt{\frac{a}{2}}$$

which is the exact value for the speed c^*.

4 Conclusion

We have given a variational characterization of the minimal speed for which the nonlinear diffusion equation has monotonic fronts. As [AW] have shown this is the asymptotic speed of propagation of a sufficiently small positive initial condition $u(x, 0)$.

The variational principle we have derived here can also be used to study the dependence of c on the parameters of f. Monotonicity properties can be inmediatly derived. Derivatives of c with respect to parameters of f can be obtained easily using the Feynman–Hellmann formula.

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