DISCONTINUITY OF THE HAUSDORFF DIMENSION OF HYPERBOLIC SETS

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Abstract. We prove that the Hausdorff dimension of a hyperbolic basic set may vary discontinuously with the dynamics if the dimension of the ambient manifold is bigger than two. This loss of continuity is associated to the occurrence of intersections between the stable (resp. unstable) manifold and the strong unstable (resp. strong stable) manifold of some periodic point.

1. Introduction

We begin by recalling a few basic facts concerning hyperbolicity and Hausdorff dimension, see e.g. [HP] and [F] for details.

Given a $\mathcal{C}^r$-diffeomorphism $f\colon M \to M$, $1 \le r \le \infty$, we say that a compact f-invariant set $\Lambda_f \subset M$ is *hyperbolic* if there is a splitting $E^s_{\Lambda_f} \oplus E^u_{\Lambda_f}$ of the tangent bundle $T_{\Lambda_f}M$ and there are constants $C > 0$, $\lambda < 1$, such that

$$||D_x f^n(v)|| \le C\lambda^n ||v|| \quad \text{and} \quad ||D_x f^{-n}(w)|| \le C\lambda^n ||w||$$

for all $v \in E^s_x$, $w \in E^u_x$, $x \in \Lambda_f$ and $n \ge 1$. We call Λ_f a *basic set* if it is *transitive* (i.e. there is a dense orbit of f in Λ_f) and *isolated* (i.e. $\Lambda_f = \cap_{i\in\mathbb{Z}} f^i(U)$ for some neighbourhood U of Λ_f) and contains a dense subset of periodic points. An important feature of basic sets is their persistence under perturbation of the dynamics: there is a neighbourhood $\mathcal{V}^r$ of f in $\mathrm{Diff}^r(M)$ such that for all $g \in \mathcal{V}^r$

$$\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(U) \quad (\text{the } \textit{continuation} \text{ of } \Lambda_f)$$

is a basic set of g and, moreover, $g|\Lambda_g$ is *conjugate* to $f|\Lambda_f$: there is a homeomorphism $h\colon \Lambda_f \to \Lambda_g$ such that $(g|\Lambda_g) \circ h = h \circ (f|\Lambda_f)$.

Given $\alpha > 0$, the *Hausdorff α-measure* of a compact metric space X is

$$m_\alpha(X) = \lim_{\varepsilon\to 0^+} \inf \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^\alpha,$$

where the infimum is taken over all finite coverings $\mathcal{U}$ of X by sets with diameter less than ε. Then there is a unique $d \in [0,\infty]$ such that $m_\alpha(X) = \infty$ if $\alpha < d$ and $m_\alpha(X) = 0$ if $\alpha > d$. One calls d the *Hausdorff dimension* of X and writes

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$d = HD(X)$. Here we make use of the (easy) fact that the Hausdorff dimension is non-increasing under Lipschitz maps.

The Hausdorff dimension of basic sets of surface diffeomorphisms depends in a quite regular way on the diffeomorphism: the function

$$\mathcal{V}^r \ni g \mapsto HD(\Lambda_g)$$

is continuous, McCluskey-Manning [MM] (see also [PV1]), and even of class $\mathcal{C}^{r-1}$, Mañé [M].

The purpose of this article is to prove that such a regularity of the Hausdorff dimension breaks down in higher dimensional manifolds:

Theorem. *Suppose M is an m-manifold, $m \geq 3$. Then, for any $1 \leq r \leq \infty$, the function*

$$\mathcal{V}^r \ni g \mapsto HD(\Lambda_g)$$

introduced above is, in general, not continuous.

Let us give a brief sketch of the proof of this result, details being provided in the next section. Clearly, it is no restriction to consider $M = \mathbb{R}^{n+1}$, $n \geq 2$, and we do so from now on. We begin by taking a $\mathcal{C}^r$-diffeomorphism F of $\mathbb{R}^n$ with a basic set Σ (a *horseshoe*) such that $F|\Sigma$ is conjugate to the full shift on two symbols. We assume that $HD(\Sigma) < 1$. Next, we let $1 < \lambda < 2$ and consider the diffeomorphism

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad (X, x) \mapsto (F(X), \lambda x).$$

Note that f also has a horseshoe $\Lambda_f = \Sigma \times \{0\}$ and $HD(\Lambda_f) = HD(\Sigma) < 1$. Let P be some fixed point of F in Σ. Then $(P, 0)$ is a hyperbolic fixed point of f and

$$W^s((P,0), f) = W^s(P, F) \times \{0\} \text{ and } W^u((P,0), f) = W^u(P, F) \times \mathbb{R}.$$

For simplicity, we assume that every expanding eigenvalue of $DF(P)$ is larger than 2 and then the strong unstable manifold of $(P, 0)$ is

$$W^{uu}((P,0), f) = W^u(P, F) \times \{0\}.$$

Since $W^s(P, F)$ and $W^u(P, F)$ meet transversely at some $Q \in \mathbb{R}^n$, the manifolds $W^s((P,0), F)$ and $W^{uu}((P,0), F)$ have a quasi-transverse intersection at $(Q, 0)$. Now we consider arcs of $\mathcal{C}^r$-diffeomorphisms $\{f_t\}_{t \in [-1,1]}$, with $f_0 = f$, unfolding generically this intersection and we prove that the continuation Λ_t of $\Lambda_0 = \Lambda_f$ satisfies

$$HD(\Lambda_t) \geq 1 \text{ for every small } t \neq 0$$

(actually, the strict inequality holds). Clearly this implies the Theorem.

It is interesting to contrast this construction with some of the results in [PV2], where geometric invariants of hyperbolic basic sets in any dimension were considered, in a context of bifurcations of diffeomorphisms. Indeed, by Section 4 in that paper, invariants such as the Hausdorff dimension, the limit capacity or the thickness of basic sets, do vary continuously with the dynamics, *if one avoids homoclinic trajectories in strong stable or strong unstable manifolds* (such as we are making use of here). On the other hand, our present arguments, see also the construction of *cs-blenders* in [BD, Section 2], suggest that explosion of the Hausdorff dimension is a fairly general phenomenon in situations involving such *strong* homoclinic trajectories. In this direction we state

Conjecture. *Given any m-dimensional manifold M, $m \geq 3$, and $1 \leq r \leq \infty$, there is a codimension 1 submanifold $\mathcal{W}^r$ of $\mathrm{Diff}^r(M)$ such that every $f \in \mathcal{W}^r$ has a hyperbolic basic set Λ_f containing some strong homoclinic intersection and f is a point of upper semi-discontinuity of the Hausdorff dimension of (the continuation of) Λ_f.*

We close this section by posing the following natural question:

Question. *Is the Hausdorff dimension of basic sets always a lower semi-continuous function of the dynamics?*

2. Proof of the Theorem

Now we fill-in the details of our argument. In order to keep the exposition as transparent as possible we deal with a fairly simple example even if, clearly, the present construction has a rather more general scope. As we already said, we start with a diffeomorphism F of $\mathbb{R}^n$, $n \geq 2$, exhibiting a horseshoe Σ

$$\Sigma = \bigcap_{i \in \mathbb{Z}} F^i(R), \qquad \text{with } R = [-1,1]^n \text{ say,}$$

and $F^{-1}(R) \cap R$ consisting of two connected components $\hat{D}_1$, $\hat{D}_2$. We take F to be affine on each of these components: there are s, $u \geq 1$, with $s + u = n$, and linear maps $S_i \colon \mathbb{R}^s \to \mathbb{R}^s$, $U_i \colon \mathbb{R}^u \to \mathbb{R}^u$, such that

$$DF|\hat{D}_i = \begin{pmatrix} S_i & 0 \\ 0 & U_i \end{pmatrix} \text{ and } ||S_i||, ||U_i^{-1}|| < 1 \text{ for } i = 1, 2.$$

In particular, $\hat{D}_i = [-1,1]^s \times D_i$, $D_i \subset [-1,1]^u$, for $i = 1, 2$. In what follows (x^s, x^u) are the usual coordinates in $R = [-1,1]^s \times [-1,1]^u$ and we suppose the fixed point P of F to be located at $(0^s, 0^u) \in \hat{D}_1$.

Define the smooth arc $\{f_t\}_{t \in [-1,1]}$ of diffeomorphisms of $\mathbb{R}^{n+1}$ by

$$f_t(x^s, x^u, x) = \begin{cases} (F(x^s, x^u), \lambda x) \text{ if } x^u \in D_1, \\ (F(x^s, x^u), \lambda x - t) \text{ if } x^u \in D_2. \end{cases}$$

We let $1 < \lambda < 2$ and $||U_1^{-1}|| < 1/2$ so that the fixed point $\mathcal{O} = (0^s, 0^u, 0)$ of f_t has

$$W^{uu}_{\mathrm{loc}}(\mathcal{O}, f_t) = \{0^s\} \times [-1,1]^u \times \{0\} \text{ and } W^s_{\mathrm{loc}}(\mathcal{O}, f_t) = [-1,1]^s \times \{0^u\} \times \{0\}.$$

On the other hand, given any $x^s \in [-1,1]^s$ there are x^s_1, $x^s_2 \in [-1,1]^s$ such that

$$\begin{aligned} &f_t(\{x^s\} \times [-1,1]^u \times \{x\}) \supset \\ &\qquad \supset (\{x^s_1\} \times [-1,1]^u \times \{\lambda x\}) \cup (\{x^s_2\} \times [-1,1]^u \times \{\lambda x - t\}) \end{aligned} \tag{1}$$

for every $x \in \mathbb{R}$. Hence,

$$f_t(W^{uu}_{\mathrm{loc}}(\mathcal{O}, f_t)) \supset \{0^s_2\} \times [-1,1]^u \times \{-t\}$$

(note that $0^s_1 = 0^s$) and so the arc $\{f_t\}_{t \in [-1,1]}$ unfolds generically the quasi-transverse intersection of $W^s(\mathcal{O}, f_0)$ and $W^{uu}(\mathcal{O}, f_0)$ at $(0^s_2, 0^u, 0)$.

Denote by Λ_t the continuation for f_t, small t, of the basic set $\Lambda_0 = \Sigma \times \{0\}$ of f_0. Observe that Λ_t coincides with the closure of the set $H(\mathcal{O}, f_t) = W^s(\mathcal{O}, f_t) \cap W^u(\mathcal{O}, f_t)$ of all (transverse) homoclinic points of $\mathcal{O}$.

Lemma. *Let $t > 0$ (resp. $t < 0$) be close to zero and $J \subset [0,t]$ (resp. $J \subset [t,0]$) be an open interval. Then there are $x^s \in [-1,1]^s$ and $j \geq 0$ such that*

$$f_t^j(\{0^s\} \times [-1,1]^u \times J) \supset \{x^s\} \times [-1,1]^u \times \{\lambda^{-1}t\}.$$

In particular, $(\{0^s\} \times [-1,1]^u \times J) \cap H(\mathcal{O}, f_t) \neq \emptyset$.

As a consequence of this Lemma,

$$\pi\left(\overline{H(0,f_t)}\right) \supset [0,t], \quad \text{where } \pi\colon [-1,1]^s \times [-1,1]^u \times \mathbb{R} \to \mathbb{R}, \ \pi(x^s, x^u, x) = x.$$

Since π is a Lipschitz map, it follows that

$$HD(\Lambda_t) \geq HD(\pi(\Lambda_t)) \geq 1,$$

which proves the Theorem. We are left to give the

Proof of the Lemma. We suppose $t > 0$, the case $t < 0$ being completely analogous. Consider the affine functions

$$\pi_{1,t}\colon [0, \lambda^{-1}t] \to [0,t], \ \pi_{1,t}(y) = \lambda y,$$
$$\pi_{2,t}\colon [\lambda^{-1}t, t] \to [0,t], \ \pi_{2,t}(y) = \lambda y - t.$$

Note that $\pi_{2,t}$ is well defined since $1 < \lambda < 2$. By (1)

$$\begin{aligned} f_t(\{x^s\} \times [-1,1]^u \times \{x\}) &\supset \{x_1^s\} \times [-1,1]^u \times \{\pi_{1,t}(x)\} \quad \text{if } x \in [0, \lambda^{-1}t], \\ f_t(\{x^s\} \times [-1,1]^u \times \{x\}) &\supset \{x_2^s\} \times [-1,1]^u \times \{\pi_{2,t}(x)\} \quad \text{if } x \in [\lambda^{-1}t, t]. \end{aligned} \tag{2}$$

Write $I_0 = J$ and $z_0 = 0^s$. If I_0 contains $\lambda^{-1}t$ then there is nothing to prove. Hence, we may assume that either $I_0 \subset (0, \lambda^{-1}t)$ or $I_0 \subset (\lambda^{-1}t, t)$. We let $i = 1$ in the first case and $i = 2$ in the second one and we write $I_1 = \pi_{i,t}(I_0)$ and $z_1 = 0_i^s$. Then, by (2),

$$f_t(\{0^s\} \times [-1,1]^u \times I_0) \supset \{z_1\} \times [-1,1]^u \times I_1$$

As above, if I_1 contains $\lambda^{-1}t$ then we are done. Otherwise we apply the previous procedure inductively: for each $j \geq 1$, if $\lambda^{-1}t \notin I_{j-1}$ then we construct an open interval $I_j \subset [0,t]$ and a point $z_j \in [-1,1]^s$ so that

$$I_j = \pi_{1,t}(I_{j-1}) \text{ if } I_{j-1} \subset [0, \lambda^{-1}t] \quad \text{and} \quad I_j = \pi_{2,t}(I_{j-1}) \text{ if } I_{j-1} \subset [\lambda^{-1}t, t]$$

and

$$f_t(\{z_{j-1}\} \times [-1,1]^u \times I_{j-1}) \supset (\{z_j\} \times [-1,1]^u \times I_j).$$

Since $\text{length}\,(I_j) = \lambda \cdot \text{length}\,(I_{j-1})$ and $\lambda > 1$ there must be a first j such that $\lambda^{-1}t \in I_j$. This ends the proof of the first part of the Lemma.

As for the second one, it is now a direct consequence. Observe that

$$f_t(\{x^s\} \times D_2 \times \{\lambda^{-1}t\}) \cap W^s_{\text{loc}}(\mathcal{O}, f_t) \neq \emptyset$$

and so $\{x^s\} \times [-1,1]^u \times \{\lambda^{-1}t\}$ intersects $W^s(\mathcal{O}, f_t)$ at some point Q_t. Then, taking $j \geq 0$ as above,

$$f_t^{-j}(Q_t) \in (\{0^s\} \times [-1,1]^u \times J) \subset W^u(\mathcal{O}, f_t),$$

which also means that $f_t^{-j}(Q_t) \in H(\mathcal{O}, f_t)$. This completes our argument. □

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