AGREEMENT PERCOLATION AND PHASE COEXISTENCE IN SOME GIBBS SYSTEMS

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Abstract. We extend some relations between percolation and the dependence of Gibbs states on boundary conditions known for Ising ferromagnets to other systems and investigate their general validity: percolation is defined in terms of the agreement of a configuration with one of the ground states of the system. This extension is studied via examples and counterexamples, including the antiferromagnetic Ising and hard core models on bipartite lattices, Potts, many layered Ising and continuum Widom–Rowlinson models. In particular our results on the hard square lattice model make rigorous observations made in Hu and Mak (1989) and (1990) on the basis of computer simulations. Moreover we observe that the (naturally defined) clusters of the Widom–Rowlinson model play (for the WR–model itself) the same role that the clusters of the Fortuin–Kasteleyn measure play for the ferromagnetic Potts models. The phase transition and percolation in this system can be mapped into the corresponding liquid–vapor transition of a one component fluid.

1. Introduction

The low temperature phases of matter can generally be thought of as small thermal perturbations of corresponding ground states. This is particularly simple for the case of a classical lattice system whose configuration is specified by $\{\sigma(x)\}$ with x on some regular lattice $\mathcal{L}$, and $\sigma(x)$ a spin variable taking one of a finite number of values at each site. Given a local interaction having a finite number of periodic ground state configurations (PGSC), we can then take these PGSC as boundary conditions for the Gibbs measures in Λ, i.e. consider the Gibbs distribution at

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inverse temperature β with boundary conditions given by a PGSC on Λ^c, the complement of Λ, of some box Λ (see Section 2 for precise definitions). For fixed Λ and $\beta > 0$ the boundary condition has a definite influence on the probability distribution of the spins in the bulk of Λ. In particular, in the limit $\beta \nearrow \infty$, the Gibbs measure in Λ becomes concentrated on the extension inside Λ of the PGSC imposed outside Λ. We are interested in knowing whether this influence persists for β large but finite when we take the volume Λ to be macroscopic. In other words, is there an ordered state in the infinite volume ($\Lambda \nearrow \mathcal{L}$) system dependent upon the PGSC imposed as boundary conditions on the finite box Λ?

If there is such a *memory* of the state with respect to boundary conditions *at infinity*, one sometimes calls the corresponding ground state thermally stable. The well known Peierls argument provides such a result for the ferromagnetic Ising model and the Pirogov–Sinai theory, Pirogov and Sinai (1976), and its extensions study the genericity of this scenario. Under certain conditions, e.g. when there are a finite number of thermally stable periodic ground states, it allows one to construct the low temperature phase diagram of the system (for a review see Sinai (1982), (1986) Zahradník (1984), Bricmont and Slawny (1985), Slawny (1986)).

In this note, we investigate the geometric or percolation picture of this memory effect, considered by many authors (Russo (1979), Fortuin and Kasteleyn (1972), Georgii (1988)): to make precise the intuition that the influence of the boundary conditions must propagate via "interacting sites" from infinity to the center of the system if the corresponding non–zero temperature state is to be stable. We are thus led to the question: in what sense is the finite temperature state corresponding to a ground state accompanied by the presence of an infinite connected cluster on which the PGSC is realized. This question is fully answered at sufficiently low temperatures where the proof of the existence of different phases, determined by different GSC boundary conditions, via the Peierls argument or Pirogov–Sinai theory actually proves the existence in each such phase of just such a cluster and no other. Our interest here is therefore primarily the extension of well known low temperature results to higher temperatures where the Peierls and Pirogov-Sinai arguments fail. We are particularly interested in the question of whether, for two dimensional systems with particular symmetries, the existence of such an ordered state is equivalent to the percolation of that and only that ground state configuration. This is known for example in the case of the standard ferromagnetic Ising model without external magnetic field. As we show below, this picture extends to other models such as the antiferromagnetic Ising model, the hard square model and the Widom–Rowlinson model. A weaker statement is proven for the Potts model. However, we will also give examples (see Section 3(c) below) in which the existence of an ordered state does not imply percolation of the corresponding ground state.

We emphasize that our setup is different from that in the Fortuin–Kasteleyn representation. Here the percolation clusters are defined directly in terms of the Gibbs state configuration. Therefore in general we do not expect to have direct relations between e.g. correlation functions in the Gibbs state and corresponding percolation probabilities. Still we will see that for the model in Section 4 such a relation can in fact be established.

In the next Section we present the general framework. This is implemented by the examples of Section 3 in the case of lattice systems. Section 4 is devoted to the continuum Widom–Rowlinson model.

2. General framework

We present the notation here in case of lattice systems. The continuum model is contained in Section 4.

The lattice. We restrict our attention to the d–dimensional lattice $\mathbb{Z}^d$, $d \geq 2$. This restriction is made for notational convenience and possible generalizations will be noted later on. By $x \sim y$ we mean that x and $y \in \mathbb{Z}^d$ are nearest neighbors. Given $\Lambda \subset \mathbb{Z}^d$, $\partial\Lambda$ will denote the outer boundary of Λ, i.e. $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : y \sim x \text{ for some } y \in \Lambda\}$. In the sequel Λ is always a finite subset of $\mathbb{Z}^d$.

The configuration space. The single site state space is denoted by S and it contains a finite number of elements ($|S| = q$). An infinite volume configuration $\sigma = \{\sigma(x)\}_{x \in \mathbb{Z}^d}$ is an element of $\Omega = S^{\mathbb{Z}^d}$. A configuration σ is periodic if there is a $(k_1, \ldots, k_d) \in \mathbb{Z}^d$ such that if $y = x + (n_1 k_1, \ldots, n_d k_d)$ for some $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, then $\sigma(x) = \sigma(y)$.

The Hamiltonian and its ground states. The translation invariant Hamiltonian H consists (for definiteness) of the nearest neighbor interaction $U : S \times S \to \mathbb{R}$ and the self–energy $V : S \to \mathbb{R}$

$$H(\sigma) = H(\sigma; J, h) = -J \sum_{x \sim y} U(\sigma(x), \sigma(y)) - h \sum_{x} V(\sigma(x)) \tag{2.1}$$

in which the infinite sums are formal. The real parameters $J \geq 0$ and h in (2.1) play the roles of a coupling coefficient and magnetic field (or chemical potential).

Consider $\sigma, \eta \in \Omega$ such that $\sigma(x) = \eta(x)$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, for some finite $\Lambda \subset \mathbb{Z}^d$. Then σ and η are said to be *equal at infinity* or one is an *excitation* of the other. Their finite relative Hamiltonian $H(\sigma|\eta)$ is then

$$
\begin{aligned}
H(\sigma|\eta) &\equiv H(\sigma) - H(\eta) \\
&\equiv -J \sum_{x \sim y} [U(\sigma(x), \sigma(y)) - U(\eta(x), \eta(y))] - h \sum_{x} [V(\sigma(x)) - V(\eta(x))]
\end{aligned}
\tag{2.2}
$$

in which the sum is now only over a finite number of terms. A configuration $\eta \in \Omega$ is a *ground state* if $H(\sigma|\eta) \geq 0$ for any excitation σ of η. The set of periodic ground state configurations (PGSC) is denoted by $g(H)$.

The Gibbs measure. Take $\eta \in g(H)$. The finite volume (Λ) Gibbs state with η boundary conditions is the probability measure on the product space S^Λ defined by

$$\mu^\eta_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_\Lambda(\beta, \eta)} \exp[-\beta H(\sigma^\eta_\Lambda | \eta)] \tag{2.3}$$

for $\sigma_\Lambda \in S^\Lambda$. In (2.3) $\beta > 0$ plays the role of the inverse temperature, $Z_\Lambda(\beta, \eta)$ is the normalization constant and

$$\sigma^\eta_\Lambda(x) = \begin{cases} \sigma_\Lambda(x) & \text{if } x \in \Lambda \\ \eta(x) & \text{if } x \in \Lambda^c = \mathbb{Z}^d \setminus \Lambda \end{cases} \tag{2.4}$$

Observe that σ^η_Λ is an excitation of η. The infinite volume Gibbs measure μ^η_β is obtained taking the limit as $\Lambda \nearrow \mathbb{Z}^d$ along suitable subsequences: μ^η_β is a measure on Ω endowed with the product sigma algebra (for details see Georgii (1988), Simon (1993)). By phase coexistence at β we mean that $\mu^\eta_\beta \neq \mu^\sigma_\beta$ for some $\eta, \sigma \in g(H)$.

Agreement with the PGSC. To understand the geometrical structure of such phases fix $\eta \in g(H)$ and the map $s_\eta : \Omega \to \{0,1\}^{\mathbb{Z}^d}$ expressing agreement or disagreement with the ground state η

$$(s_\eta(\sigma))(x) = \begin{cases} 1 & \text{if } \sigma(x) = \eta(x) \\ 0 & \text{if } \sigma(x) \neq \eta(x) \end{cases} \tag{2.5}$$

We denote by $\nu^\eta_{\beta,\Lambda} = \mu^\eta_{\beta,\Lambda} s_\eta^{-1}$ the measure on $\{0,1\}^\Lambda$ induced by the map s_η (with a slight abuse of notation). As before we can take the limit $\Lambda \nearrow \mathbb{Z}^d$; write it as ν^η_β.

Percolation of the ground states. Percolation is defined in terms of regular site percolation for the measure ν^η_β: for $s \in \{0,1\}^{\mathbb{Z}^d}$ define the clusters

$$C(s,\alpha) = \{x \in \mathbb{Z}^d : s(x) = \alpha\} \tag{2.6}$$

in which $\alpha = 0, 1$. Call $\overline{C}(\sigma,\alpha)$ the maximal connected component containing the origin of $\mathbb{Z}^d$ ($\overline{C}(s,\alpha) = \emptyset$ if $s(0) \neq \alpha$). The probability of $\eta-$percolation (respectively non $\eta-$percolation) in the phase with boundary condition η is defined as

$$\theta^\eta_\beta(\alpha) = \nu^\eta_\beta(\{s : |\overline{C}(s,\alpha)| = \infty\}) \tag{2.7}$$

with $\alpha = 1$ (respectively $\alpha = 0$). We have $\eta-$percolation, or agreement percolation, (respectively non $\eta-$percolation) if $\theta^\eta_\beta(1) > 0$ (respectively $\theta^\eta_\beta(0) > 0$). Observe that not having $\eta-$percolation in general does not imply having non $\eta-$percolation.

Attractive measures. If S is ordered ($\geq$), Ω is ordered ($\succ$) by defining $\sigma' \succ \sigma$ if $\sigma'(x) \geq \sigma(x)$ for all $x \in \mathbb{Z}^d$. The order chosen for S may depend on the site x. A measurable event A is said to be increasing if $\sigma \in A$ and $\sigma' \succ \sigma$ implies $\sigma' \in A$. A measure μ on Ω is attractive (FKG) if $\mu(A \cap B) \geq \mu(A)\mu(B)$ for all A, B increasing events, see Fortuin *et al.* (1971).

We are interested in the verification of three main statements:

a) we are asking whether phase coexistence implies percolation, i.e. given $\eta, \eta' \in g(H)$ is it true that

$$\mu_\beta^\eta \neq \mu_\beta^{\eta'} \implies \theta_\beta^\eta(1) > 0 \tag{2.8a}$$

b) whether the reverse implication is true when $d = 2$

$$\theta_\beta^\eta(1) > 0 \iff \mu_\beta^\eta \neq \mu_\beta^{\eta'} \tag{2.8b}$$

c) still for $d = 2$, can we rule out the possibility of disagreement percolation

$$\theta_\beta^\eta(0) = 0 \tag{2.8c}$$

For the models we will be dealing with, if $d > 2$ the statement (2.8b) fails or it is expected to fail (see Aizenman *et al.* (1987) and references therein).

3. Lattice models

As already mentioned in the introduction, the case in which $S = \{-1, +1\}$, $U(a, b) = ab$ and $h = 0$ (ferromagnetic Ising model) is well understood. In this case $g(H) = \{\eta_+, \eta_-\}$ ($\eta_+(x) = +1, x \in \mathbb{Z}^d$ and $\eta_- = -\eta_+$) and for $\beta > \beta_c$ ($1/\beta_c$ is the critical temperature) there are two distinct translation invariant extremal states obtained by taking η_+ and η_- as boundary conditions and the proof of the statements (2.8a), (2.8b) and (2.8c) can be found in Coniglio *et al* (1976).

a) Ising antiferromagnet.

$\mathbb{Z}^d$ is bipartite, that is it can be split into two sublattices (in this case the points with even sum of coordinates $\mathbb{Z}^d_e$ and the ones with odd sum $\mathbb{Z}^d_o$) such that if $x, y \in \mathbb{Z}^d_e$ (or $x, y \in \mathbb{Z}^d_o$) then $x \not\sim y$. Fix $S = \{-1, +1\}$ and take

$$U(a,b) = -ab \qquad V(a) = a \tag{3.1}$$

If $|h| < 2dJ$, then $g(H) = \{\eta_e, \eta_o\}$ in which $\eta_e(x) = +1$ if $x \in \mathbb{Z}^d_e$, $\eta_e(x) = 0$ otherwise and $\eta_o = -\eta_e$ (see for example Dobrushin and Shlosman (1985)). The phase transition in this model has been studied in Dobrushin (1968) and Heilmann (1974). Because of the bipartite structure, flipping the spins on the even (or odd) sites makes the model into a ferromagnetic Ising model with a staggered magnetic field. In particular for magnetic field $h = 0$, there is the usual Curie point T_c (the critical temperature for the ferromagnetic model above).

Proposition 3.1:. *With the choices of (3.1) in (2.1), for any h and any $J \geq 0$, statements (2.8a), (2.8b) and (2.8c) hold.*

Proof. Take $\eta = \eta_e$. For any Λ containing the origin, we have that if $\mu^{\eta_e}_\beta \neq \mu^{\eta_o}_\beta$

$$\begin{aligned}
&\nu^{\eta_e}_\beta(s(0) = 1 | s(x) = 0 \text{ for all } x \in \partial\Lambda) = \\
&\mu^{\eta_e}_\beta(\sigma(0) = +1 | \sigma(x) = \eta_o(x), \text{ for all } x \in \partial\Lambda) = \\
&\mu^{\eta_o}_\beta(\sigma(0) = +1 | \sigma(x) = \eta_o(x), \text{ for all } x \in \partial\Lambda) \leq \\
&\mu^{\eta_o}_\beta(\sigma(0) = +1) < \nu^{\eta_e}_\beta(s(0) = +1)
\end{aligned} \tag{3.2}$$

The first equality is a change of notation and the second one is the Markov property. The first inequality follows from the attractivity of the measure $\mu^{\eta_o}_\beta$ with respect to the order relation $\sigma \succ \sigma'$ iff $\sigma(x)\eta_e(x) \geq \sigma'(x)\eta_e(x)$ for all $x \in \mathbb{Z}^d$. By Theorem 2 of Bricmont *et al.* (1987) we have (2.8a).

If $\mu_\beta^{\eta_e} = \mu_\beta^{\eta_o}$ then, by the fact that $\mu_\beta^{\eta_e} T_i^{-1} = \mu_\beta^{\eta_o}$ ($T_i\sigma(x) = \sigma(x - e_i)$, e_i is the unit vector in the direction $i \in \{1, \ldots, d\}$) we have that $\nu_\beta^{\eta_e} = \nu_\beta^{\eta_o} \equiv \nu_\beta$. This implies that ν_β, besides being attractive (in the usual order), is reflection invariant and ergodic under translations. Hence if $d = 2$ the main Theorem of Gandolfi *et al.* (1988) applies and (2.8b) is proven. If $\mu_\beta^{\eta_e} \neq \mu_\beta^{\eta_o}$, μ_β^{η}'s are not T_i-invariant, but only T_i^2-invariant. Hence we have to use an extension (Klein and Yang (1993)) of the result in Gandolfi *et al.* (1988) to get (2.8c). $\square$

The results of Proposition 3.1 can be extended to more general bipartite lattices.

b) Hard core lattices.

The hard core (or hard squares) lattice model with activity λ is defined by the infinite volume limit ($\Lambda \nearrow \mathbb{Z}^d$) of the measure $\mu_{\lambda,\Lambda}$ on the product space $\{0,1\}^\Lambda$

$$\mu_{\lambda,\Lambda}^{\gamma}(\sigma_\Lambda) = \frac{\chi(\sigma_\Lambda^{\eta_\gamma})\lambda^{N_\Lambda}}{Z_\gamma(\lambda,\Lambda)} \tag{3.3}$$

In (3.3) $\gamma \in \{e, o\}$ and $\eta_e(x) = 1$ if x is even, $\eta_e(x) = 0$ if x is odd ($\eta_o(x) = 1 - \eta_e(x)$ for all x). Further, $N_\Lambda = \sum_{x\in\Lambda} \sigma_\Lambda^{\eta_\gamma}(x)$ with σ_Λ^η as defined in (2.4) ; $Z_\gamma(\lambda,\Lambda)$ is a normalization and for $\sigma \in \{0,1\}^{\mathbb{Z}^d}$

$$\chi(\sigma) = \begin{cases} 1 & \text{if } \sigma(x)\sigma(y) = 0 \text{ for all } x \sim y \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Hence the model can be seen as a gas of hard (i.e. non overlapping) squares (see Fig.1) or diamonds with fugacity λ. Call $C(\eta) = \cup_{x:\eta(x)=1} Q(x)$ ($Q(x) = \{y \in \mathbb{R}^d : \sum_{i=1}^d |y_i - x_i| = 1\}$) and denote by $C_0(\eta)$ the connected component of $C(\eta)$ that contains the origin (two squares touching at a corner are connected, see Fig.1). There are clearly two different type of connected components of C : the ones for which the squares are centered on even sites and the ones for which they are centered on odd sites. We will call them type e and type o clusters. For $\gamma, \delta \in \{e, o\}$ define

$$\theta_\lambda^\gamma(\delta) = \mu_\lambda^\gamma(\{\eta : C_0(\eta) \text{ is of type } \delta \text{ and unbounded}\}) \tag{3.5}$$

(3.5) is clearly the analogue of (2.7).

The hard core model can be seen as a limit of the antiferromagnetic Ising model for $\beta \to \infty$ and $h \to 2dJ$ along $\beta(h - 2dJ) = \text{cotan}\theta$ ($\theta \in (0, \pi)$). The phase diagram point $(2dJ, 0)$ is highly degenerate, since there are infinitely many (in general nonperiodic) ground states. In these limits map $\sigma \in \{-1, +1\}^{\mathbb{Z}^d} \to \eta \in \{0, 1\}^{\mathbb{Z}^d}$ by setting $\eta(x) = 1$ if $\sigma(x) = -1$ and $\eta(x) = 0$ if $\sigma(x) = +1$. We get a hard square model with activity $\lambda = \exp(-2\text{cotan}\theta)$ (see Dobrushin *et al.* (1985) for details). This picture suggests that the critical fugacity (if it exists) should correspond to θ_c (see Fig.2). We rephrase (2.8a), (2.8b), (2.8c) into

Proposition 3.2. *For the hard square model*

$$\mu_\lambda^e \neq \mu_\lambda^o \Longrightarrow \theta_\lambda^e(e) > 0 \tag{3.6a}$$

Moreover if $d = 2$

$$\theta_\lambda^e(e) > 0 \Longleftrightarrow \mu_\lambda^e \neq \mu_\lambda^o \tag{3.6b}$$

and

$$\theta_\lambda^e(o) = 0 \tag{3.6c}$$

Proof. Define the partial order $\eta' \succ \eta$ if $\eta'(x) \geq \eta(x)$ for $x \in \mathbb{Z}_e^d$ and $\eta'(x) \leq \eta(x)$ if $x \in \mathbb{Z}_o^d$. It is easy to check that $\mu_{\lambda,\Lambda}$ and its infinite volume limit are attractive. The proof is then the same of the one of Proposition 3.1. $\square$

Remark. The motivation for Proposition 3.2 was provided by the work of C.K. Hu and K.S. Mak (Hu and Mak (1989) and (1990)) in which a similar result is conjectured on the basis of computer simulations. In Hu and Mak (1989) and (1990) they discuss also the case of hard core particles on a triangular lattice, the hard hexagon model. Our result extends easily to other bipartite lattices, such as the hexagonal one. The triangular lattice with nearest neighbor bonds is not bipartite, so our proof does not work. In view of Hu and Mak (1989) and (1990), one expects that Proposition 3.2 still holds for this model, but it is unclear to us, especially in view of the results of the Section 3.d on *many layer* Ising models, whether Proposition 3.1 ($d = 2$) holds for the whole domain of coexisting phases of the antiferromagnetic Ising model on the triangular lattice (in this case, for $h \in (0, 6J)$, $g(H)$ contains three configuration).

c) ***Many layer*** **models.**

Given a model with configuration space $\{-1,+1\}^{\mathbb{Z}^d}$ and Hamiltonian H_1 we can define a family of new models indexed by integers $Q \geq 2$. Take $S = \{-1,+1\}^Q$ so that the infinite volume configuration is of the form $\xi = (\sigma_1, \ldots, \sigma_Q) \in S^{\mathbb{Z}^d}$ (to be identified with $\{-1,+1\}^{\mathbb{Z}^d} \times \ldots \times \{-1,+1\}^{\mathbb{Z}^d}$ (Q copies)). Define the formal Hamiltonian as

$$H(\xi) = \sum_{i=1}^{Q} H_1(\sigma_i) \tag{3.7}$$

so that for boundary conditions $\omega = (\eta_1, \dots, \eta_Q)$, $\mu^{\omega}_{\beta,\Lambda} = \mu^{\eta_1}_{1,\beta,\Lambda} \times \dots \times \mu^{\eta_Q}_{1,\beta,\Lambda}$ (where the subscript 1 refers to the system with Hamiltonian H_1) is the finite volume Gibbs state with respect to H. Observe that $g(H) = (g(H_1))^Q$ (with the previous identification).

Duplicated Ising model.

Take $Q = 2$ and H_1 as in (2.1), characterized by $U(a,b) = ab$ and $h = 0$. As observed before $g(H_1) = \{\eta_+, \eta_-\}$ and so $g(H) = \{\omega_{++}, \omega_{+-}, \omega_{-+}, \omega_{--}\}$, where $\omega_{++} = (\eta_+, \eta_+)$ and so on.

Proposition 3.3. *For the duplicated Ising model statements (2.8a), (2.8b) and (2.8c) hold.*

Proof. We want to estimate the expectation value of the sum of the spins at the origin given that $\xi(y) = (\sigma_1(y), \sigma_2(y)) \neq (+1,+1)$ for $y \in \partial\Lambda$. In that case $\sigma_1(y) + \sigma_2(y) \leq 0$ for $y \in \partial\Lambda$, so that by attractivity

$$
\begin{aligned}
&\mu^{\omega_{++}}_{\beta}(\sigma_1(0) + \sigma_2(0) | s_{\omega_{++}}(\xi) = 0 \text{ on } \partial\Lambda) \leq \\
&\mu^{\omega_{++}}_{\beta}(\sigma_1(0) + \sigma_2(0) | (\sigma_1, \sigma_2) : \sigma_2(y) = -\sigma_1(y) \text{ for all } y \in \partial\Lambda) = \\
&\sum_{\sigma \in \{-1,+1\}^{\partial\Lambda}} \Big\{ \Big[\mu^{\sigma}_{1,\beta}(\sigma_1(0)) + \mu^{-\sigma}_{1,\beta}(\sigma_2(0)) \Big] \cdot \\
&\mu^{\omega_{++}}_{\beta}(\xi(x) = (\sigma_1(x), -\sigma_1(x)), x \in \partial\Lambda | s_{\omega_{++}}(\xi) = 0 \text{ on } \partial\Lambda) \Big\} = 0
\end{aligned}
\tag{3.8}
$$

On the other hand if $\mu^{\eta_+}_{1,\beta} \neq \mu^{\eta_-}_{1,\beta}$

$$
\mu^{\omega_{++}}_{\beta}(\sigma_1(0) + \sigma_2(0)) \equiv 2m^\star(\beta) > 0 \tag{3.9}
$$

Apply Theorem 2 of Bricmont *et al.* (1987) to get (2.8*a*). For what concerns (2.8*b*) and (2.8*c*), it is straightforward to see that $\nu^{\omega_{++}}_{\beta} = \mu^{\omega_{++}}_{\beta} s^{-1}_{\omega_{++}}$ is reflection invariant, ergodic under translation and attractive. Hence we can apply Gandolfi *et al.* (1988) to get (2.8b) and (2.8c). $\square$

Remark. Note that as the phase transition is second order ($\mu_{1,\beta}^{\eta+}(\sigma(0))^+ \equiv m^\star(\beta)$ is continuous at $\beta = \beta_c$), the density of $(+1,+1)$ just below the critical temperature is only slightly above $1/4$ and still the $(+,+)$ spins percolate in the $(+,+)-$ state[1]. In the same way, the density of sites $x \in \mathbb{Z}^2$ where $(\sigma_1(x), \sigma_2(x)) \neq (+,+)$ is there only slightly below $3/4$ and still, in the $(+,+)-$ state, they do not percolate.

The question therefore arises whether one can go arbitrarily far and construct examples where there is percolation for arbitrarily low densities or where there is no percolation no matter how large one makes the density. Such examples in fact exist (see for example Molchanov and Stepanov (1983)) but they are rather singular. It may well be that a minimum density for having percolation actually exists for *good* Markov fields.

Conjecture. *Given μ on the product space $S^{\mathbb{Z}^d}$ ($|S| = q < \infty$), a translation invariant pure Gibbs state for some isotropic nearest neighbor interaction, there is a constant $c(d) > 0$ (independent of q) such that, if $\mu(\sigma(0) = a) < c(d)$ then $\{x : \sigma(x) = a\}$ does not percolate ($a \in S$).*

In particular, we know of no such Gibbs–state in two dimensions where one has percolation of a spin value with density less than $1/4$. We believe for example that there is no $(+,+,+)-$percolation in the triplicated $+1$ state $\mu_{1,\beta}^+ \times \mu_{1,\beta}^+ \times \mu_{1,\beta}^+$ at temperatures close to the critical one. Obviously, in the conjecture, $c(d) \to 0$ as

[1]The threshold for Bernoulli site percolation on $\mathbb{Z}^2$ is about 0.59

$d \to \infty$ (as does $p_c(d)$, the critical percolation probability for Bernoulli percolation).

A result in the direction of the conjecture is the following

Proposition 3.4. *Take H_1 of the ferromagnetic Ising model ($U(a,b) = ab$) with $h = 0$. Take any β, including $\beta > \beta_c$ for which there is phase coexistence. There is $Q(\beta, d)$ such that for all $Q \geq Q(\beta, d)$*

$$\theta^{\omega}_{\beta}(1) = 0 \tag{3.10}$$

Proof. By direct computation

$$\max_{\xi'} \mu^{\omega}_{\beta}(\xi(x) = (1, ...1)|\xi'(y), y \sim x) = \left(\frac{1}{1 + \exp(-4d\beta J)}\right)^{Q} \tag{3.11}$$

we can now take Q sufficiently large so that

$$\left(\frac{1}{1 + \exp(-4d\beta J)}\right)^{Q} < p_c(d) \tag{3.12}$$

A standard domination argument concludes the proof (if $d = 2$, by using $p_c(2) > 1/2$ and $\beta_c = (1/2J)\log(1 + (\sqrt{5}/2))$ we get $Q(\beta_c, 2) = 25$). $\square$

Hence we have an example in which the measure is attractive, but nevertheless phase coexistence does not imply percolation. Of course Proposition 3.4 also holds for the *many layer* version of other Markov fields.

Remark. If in $\mu^{\omega +-}_{\beta}$ there is with probability one some circuit $\partial\Lambda$ around the origin on which the two coordinates agree, i.e. $\sigma_1(y) = \sigma_2(y), y \in \partial\Lambda$, then $\mu^{\eta +}_{1,\beta} = \mu^{\eta -}_{1,\beta}$ (the effect of the boundary will be clearly cancelled by conditioning on the circuit $\partial\Lambda$). In other words if $\mu^{\eta +}_{\beta} \neq \mu^{\eta -}_{\beta}$ (respectively $\mu^{\eta_1}_{1,\beta} \neq \mu^{\eta_2}_{1,\beta}$ for $\eta_1, \eta_2 \in \Omega$), there must be *disagreement percolation* in $\mu_\beta^{\omega++}$ (respectively in $\mu_{1,\beta}^{\eta_1} \times \mu_{1,\beta}^{\eta_2}$), as it is done in van den Berg (1993) (see also van den Berg and Maes (1994) for another coupling). *Disagreement percolation* means that there is an infinite cluster on which $\sigma_1(x) \neq \sigma_2(x)$.

This is applied in van den Berg and Steif (1994) for the hard core model of above. They take independently two realizations (σ_1, σ_2) according to the product coupling $\mu_\lambda^e \times \mu_\lambda^o$. A site $x \in \mathbb{Z}^2$ is a site of disagreement if $\sigma_1(x) \neq \sigma_2(x)$. They prove that $\mu_\lambda^e = \mu_\lambda^o$ if and only if $\mu_\lambda^e \times \mu_\lambda^o(\{(\sigma_1, \sigma_2)$ has an infinite path of disagreement$\}) = 0$. Using our general formulation we can strengthen this result somewhat : if $\mu_\lambda^e \neq \mu_\lambda^o$ not only will we get disagreement percolation in the above sense but in the state $\mu_\lambda^e \times \mu_\lambda^o$ this percolation will be over sites x where

$$\begin{aligned}(\sigma_1(x), \sigma_2(x)) &= (\eta_e(x), \eta_o(x)) \\ &= (1,0) \text{ for } x \text{ even} \\ &= (0,1) \text{ for } x \text{ odd}\end{aligned} \tag{3.13}$$

The reason is that $\mu_\lambda^e \neq \mu_\lambda^o$ is equivalent to $\mu^e \times \mu^o \neq \mu^o \times \mu^e$ implying the stability in $\mu_\lambda^e \times \mu_\lambda^o$ of the configuration (η_e, η_o) as given in (3.13).

d) The q–state Potts model.

In this case $S = \{1, \dots, q\}$. The Hamiltonian (2.1) is specified by taking $J > 0, h = 0$ and $U(a,b) = 0$ if $a = b$, $U(a,b) = -1$ if $a \neq b$.

It is straightforward to see that $g(H) = \{\eta_a : a \in S\}$ where $\eta_a(x) = a$ for all $x \in \mathbb{Z}^d$.

A very useful way to analyze the Potts model is to take the FK–representation of Fortuin and Kasteleyn (1972). For that we let $\ell(b) = 0, 1$ be a bond configuration. A bond $b =< xy >$ is connecting nearest neighbors $x \sim y \in \mathbb{Z}^d$ and it can be *open*

($\ell(b)=1$) or *closed* ($\ell(b)=0$). In $\Lambda\subset\mathbb{Z}^d$ we fix a bond configuration ℓ by assigning to all bonds $b=<xy>$ (connecting nearest neighbors $x\sim y$ at least one of which is inside Λ) the value 1 or 0. For bond percolation see the definitions in Grimmett (1994). We define the following expectation for local functions $f(\sigma_\Lambda)$ of the Potts model variables σ_Λ in the volume Λ with boundary conditions ξ :

$$\langle f\rangle_\Lambda^\xi(\ell)=\frac{1}{q^{n_\Lambda(\ell)}}\sum_{\sigma_\Lambda}f(\sigma_\Lambda)\prod_{b=<xy>\cap\Lambda\neq\emptyset:\ell_b=1}\delta(\sigma_\Lambda^\xi(x),\sigma_\Lambda^\xi(y)) \tag{3.14}$$

Here δ is the Kronecker delta, $n_\Lambda(\ell)$ is the number of connected ℓ-clusters in the volume Λ so that the expectation (3.7) is normalized. The configuration σ_Λ^ξ is defined as in (2.4). The reason for introducing (3.7) is that the Potts model expectations in volume Λ with boundary conditions ξ can be written as

$$\mu_{\beta,\Lambda}^\xi(f)=\frac{1}{Z_\Lambda(\beta,\xi)}\sum_\ell\prod_{b\cap\Lambda\neq\emptyset}p^{\ell_b}(1-p)^{1-\ell_b}q^{n_\Lambda(\ell)}<f>_\Lambda^\xi(\ell)\equiv\nu_{q,\beta,\Lambda}^{FK}(<f>_\Lambda^\xi(\cdot)) \tag{3.15}$$

when we put $p=1-e^{-\beta J}$. Here $\nu_{q,\beta,\Lambda}^{FK}$ denotes the finite volume Fortuin–Kasteleyn measure (or random cluster measure) on the bond configurations whose weights are defined by (3.15). The infinite volume measure will be denoted by $\nu_{q,\beta}^{FK}$ (see Grimmett (1994) and (1994')).

Proposition 3.5. *For the Potts model (2.8a) holds. Moreover if $d=2$ and $\mu_\beta^\eta\neq\mu_\beta^{\eta'}$ for $\eta,\eta'\in g(H)$ then*

$$\theta_\beta^\eta(0)=0 \tag{3.16}$$

Proof. (2.8a) is easily proven by observing that

$$\mu^{\eta_a}_{\beta,\Lambda}(\sigma(0)=a)=$$
$$\frac{1}{q}+\left(\frac{q-1}{q}\right)\nu^{FK}_{q,\beta,\Lambda}(\{\ell : 0 \text{ is connected to } \partial\Lambda \text{ by a chain of open bonds}\}) \quad (3.17)$$

and that given the coupling between σ and ℓ implicit in (3.14) and (3.15), $\sigma(x)=1$ if x belongs to one of the bonds in the infinite cluster of open bonds. By (3.17) the latter exists a.s. in the coexistence region (see Fortuin–Kasteleyn (1972) and Aizenman *et al.* (1988)).

To prove (3.16), let us consider the FK measure $\nu^{FK}_{q,\beta}$ associated to the extremal Potts measure $\mu^{\eta_a}_{\beta}$. ¿From its construction (see Grimmett (1994) and (1994')), besides being translation and rotation invariant, it is ergodic under translations. Moreover it is known that this measure is also attractive (Fortuin and Kasteleyn (1972), Aizenmann *et al.* (1988), Grimmett (1994) and (1994')). A straightforward adaptation of Gandolfi *et al.* (1988) allows to conclude that if there is percolation of open bonds, outside any box containing the origin there is a circuit of open bonds surrounding the origin (a.s.). Again by the coupling between σ and ℓ, we conclude that if $\theta^{\eta_a}(1)>0$, then $\mu^{\eta_a}_{\beta}$–a.s. every point is surrounded by a circuit on which $\sigma(x)=a$. By (2.8a) we conclude. □

Remark. Note that in $d=2$ for $q>4$ the "magnetization" $\mu^a[\sigma(0)=a]$ is believed to be discontinuous at $\beta=\beta_c$, see Baxter (1982). Therefore, in the two–dimensional Potts model the lowest density of a ground state configuration for which we know there is percolation is $1/4+\epsilon$ (for arbitrary $\epsilon>0$) and is obtained for $q=4$ in the corresponding Gibbs state just below the critical point[2].

e) Widom–Rowlinson lattice model.

The statements (2.8a), (2.8b) and (2.8c) hold also for a class of models first introduced in Wheeler and Widom (1970). They are "spin 1" models with single site state space $S = \{-1, 0, +1\}$ and Hamiltonian (2.1) determined by $U(a, b) = ab(1-ab), V(a) = a^2, 0 \le J < \infty$ and $h > 0$. This model was shown to have a phase transition by Lebowitz and Gallavotti (1971) and to be attractive by Lebowitz and Monroe (1972).

The detailed analysis of these models follows along standard lines.

4. Continuum model

The continuum Widom–Rowlinson (WR-) model (Widom and Rowlinson (1970) consists of particles of type A and type B having positions in $\mathbb{R}^d$ and fugacities z_A and z_B whose interaction consists of the hard core constraint that the centers of any two particles of different type must be at least distance R from each other. In Cassandro *et al.* (1973) it is shown how this model can be obtained from a lattice model of the type described in Section 3(e) above.

More precisely, we take $\Lambda \subset \mathbb{R}^d$ a finite Borel set and let $x = (x_1, \dots, x_{N_A})$ (respectively $y = (y_1, \dots, y_{N_B})$ denote the position of particles A (respectively B), $x_i, y_i \in \Lambda$ for $i = 1, \dots, N_A$ and $j = 1, \dots, N_B$. Call X the space of $\sigma-$finite integer valued measures over Λ (and its Borel sets); our probability space will be $\Omega = X \times Y$ ($X = Y$ will have the topology of weak convergence, that character-

[2] This is similar to the case of duplicated Ising variables, see example 3(c)

izes its Borel sets). By separability, any element of X can be written as $\sum_{i\in I}\delta_{x_i}$ ($I\subset\mathbb{Z}, |I|<\infty$) and so we will use the notation $N_A(\omega)$, $N_B(\omega)$, $x(\omega)$ and $y(\omega)$ for $\omega\in\Omega$ with obvious meaning. The constraint that is imposed is determined by the hard core length R:

$$\min_{i,j}|x_i-y_j|>R \tag{4.1}$$

Letting $I[x,y]$ denote the indicator function corresponding to (4.1), we put

$$Z^{\Lambda}_{N_A,N_B}=\int_{\Lambda^{N_A}\times\Lambda^{N_B}}\mathrm{d}\lambda_{N_A}(x)\mathrm{d}\lambda_{N_B}(y)I[x,y] \tag{4.2}$$

for $\mathrm{d}\lambda_N(x)=\mathrm{d}^dx_1\ldots\mathrm{d}^dx_N$ the $N-$product Lebesgue measure. Fixing the fugacities $z_A,z_B>0$, the grand-canonical partition function of the WR-model is then

$$\Xi=\Xi(\Lambda,z_A,z_B)=\sum_{N_A,N_B}\frac{z_A^{N_A}z_B^{N_B}}{N_A!N_B!}Z^{\Lambda}_{N_A,N_B} \tag{4.3}$$

We will be mostly interested in the case $z_A=z_B=z$, for which we adopt the notation $\Xi(\Lambda,z)$.

So far we have not spoken of boundary conditions. Obviously we can fix the position of some particles introducing extra constraints (beyond (4.1)). For example, we speak of boundary conditions of type A if we replace $I[x,y]$ in (4.1) by $I_A[x,y]=I[x,y]I_A[y]$ where $I_A[y]$ is the indicator function corresponding to

$$\inf_{j,x\in\Lambda^c}|y_j-x|>R \tag{4.4}$$

Analogous definitions and notations apply for boundary conditions of type B. The grand–canonical partition function is then changed into Ξ_γ, corresponding to the boundary conditions of type $\gamma=A,B$.

The finite volume Gibbs measure μ_Λ^γ for boundary conditions $\gamma = A, B$ gives the probability of finding the particles in certain regions of Λ. If we condition on having N_A type A particles and N_B type B particles in Λ, the random field will have density

$$\frac{\mathrm{d}\mu_\Lambda^\gamma(\cdot)|_{N_A(\omega)=N_A, N_B(\omega)=N_B}}{\mathrm{d}\lambda_{N_A} \times \mathrm{d}\lambda_{N_B}}(\omega) = \frac{1}{Z_{N_A,N_B}^\Lambda} \frac{z_A^{N_A} z_B^{N_B}}{N_A! N_B!} I_\gamma(\omega) \tag{4.5}$$

in which $I_\gamma(\omega) = I_\gamma[x(\omega), y(\omega)]$ $(\omega \in \Omega)$ and analogously for $I(\omega)$, see (4.1). The infinite volume measures are denoted by μ^γ and can be obtained as limit from μ_Λ^γ as $\Lambda \nearrow \mathbb{R}^d$ along suitable subsequences (for examples along spheres). The measure μ^γ does depend on the boundary condition γ if $z_A = z_B = z$ and z is sufficiently large, Ruelle (1971).

Clusters and percolation probability. Define the function $Sp : \Lambda \times \{\omega : I(\omega) = 1\} \to \{A, B, W\}$ as

$$Sp(x) = Sp(x, \omega) = \begin{cases} A & \text{if } \mathrm{dist}(\mathbf{x}(\omega), \{x\}) < R/2 \\ B & \text{if } \mathrm{dist}(\mathbf{y}(\omega), \{x\}) < R/2 \\ W & \text{otherwise} \end{cases} \tag{4.6}$$

We can imagine the function Sp a coloring of Λ in red (A), black (B) or white (W). From now on take $\gamma, \delta \in \{A, B\}$. The γ cluster at the origin $(C_\gamma^0(\omega))$ will then be defined as the connected component of $Sp(\cdot, \omega)^{-1}(\gamma)$ that contains the origin ($C_\gamma^0 = \emptyset$ if $Sp(0) \neq \gamma$). The percolation probability is thus defined as

$$\theta^\gamma(\delta) = \mu^\gamma(\{\omega : \mathrm{diam}(C_\delta^0(\omega)) = \infty\}) \tag{4.7}$$

Proposition 4.1. *Using the notations and definitions above with $z = z_A = z_B$,*

$$\mu^A(Sp(0) = A) - \mu^B(Sp(0) = A) = \theta^A(A) - \theta^A(B) \tag{4.8}$$

implying

$$\theta^A(A) > 0 \Longleftarrow \mu^A \neq \mu^B \tag{4.9a}$$

In $d = 2$*,*

$$\theta^A(B) = 0 \tag{4.9b}$$

and

$$\theta^A(A) > 0 \Longleftrightarrow \mu^A \neq \mu^B \tag{4.9c}$$

Remark 1. (4.8) says that the particle clusters in the WR–model play a similar role as the random clusters in the FK–representation of the Potts model. We believe this to be the first example where such a direct relation between the 'order parameter' and the cluster geometry is found. Note also that by attractivity (Lebowitz and Monroe (1972)) the left hand side of (4.8) is zero if and only if the A-phase is different from the B-phase.

Remark 2. We believe that the results in the Proposition partially extend to the case where a hard core condition is added between alike particles. In that case the measures are not attractive but, as will become clear from the proof, that is not at all crucial here. Since there are extra technicalities and quite a bit of extra notation related to the existence of infinite volume limits if we don't have FKG, we prefer to give the proof only in the case of strict hard cores.

Remark 3. So far we have assumed for simplicity that the fugacities of the two types of particles are equal, $z_A = z_B = z$. Note however that if say $z_A \geq z$, then $\mu^A_{z_A,z}$ stochastically dominates $\mu^A_{z,z}$ where we now explicitly indicate by subscripts the fugacities of A- respectively, B-type particles. This implies that if in $\mu^A_{z,z}$ there is percolation of A-type particles (as in the phase coexistence regime, $z > z_c$), then we get the same result for all A-particle fugacities $z_A \geq z$. Suppose we now integrate out the positions of the B-particles. A simple calculation shows that we get a new measure for the A-particles where now z (previously the fugacity of the B-particles) plays the role of an inverse temperature. In that measure, for $z > z_c$, the A-particles percolate for all values $z_A \geq z$.

Idea of the proof. Consider an A-cluster covering the origin. If it is bounded, then, by the hard core constraint, there is necessarily a white region surrounding it, see Fig. 3. This effectively screens the origin from the external boundary condition. Therefore this contribution to the probability of having an A-particle at the origin is the same in all states. What remains is the probability that the A–cluster extends infinitely far. This idea can be most easily implemented through a discretization of the space.

Proof of Proposition 4.1. Take $\Lambda' \subset \Lambda \subset \mathbb{R}^d$ two spheres. Define the *finite volume* percolation probability as $\theta^\gamma(\delta; \Lambda, \Lambda') = \mu^\gamma_\Lambda(C^0_\delta(\omega) \cap \partial\Lambda' \neq \emptyset)$ and observe that

$$\lim_{\Lambda' \nearrow \mathbb{R}^2} \lim_{\Lambda \nearrow \mathbb{R}^2} \theta^\gamma(\delta; \Lambda, \Lambda') = \theta^\gamma(\delta) \tag{4.10}$$

Take $\epsilon > 0$ small and cover $\mathbb{R}^d$ with a grid of spacing ϵ. Disregarding boundary problems, this naturally defines a partition of $\mathbb{R}^d$ into squares of sidelength ϵ ($\epsilon-$squares).

To control the errors made by the space discretization we define

$G(\epsilon) = \{\omega : \mathrm{dist}(\mathbf{x}, \mathbf{y}) > R + 6d\epsilon,\ |\mathrm{dist}(\mathbf{x}, \partial\Lambda') - R| > 6d\epsilon,\ |\mathrm{dist}(\mathbf{y}, \partial\Lambda') - R| > 6d\epsilon,$

$$||x_i - x_k| - R| > 6d\epsilon \; i \neq k \in \{1, \dots, N_A(\omega)\}\} \tag{4.11}$$

A simple argument by contradiction yields the existence of $\Delta_\Lambda(\epsilon) \geq 0$, vanishing as $\epsilon \downarrow 0$ and such that

$$\mu_\Lambda^\gamma(G(\epsilon)) > 1 - \Delta_\Lambda(\epsilon) \tag{4.12}$$

with $\gamma \in \{A, B\}$.

The union of a finite number of $\epsilon-$squares is an $\epsilon-$cluster at the origin if the interior of this set is connected and if it contains the origin. Denote by $\mathcal{C}_\epsilon$ the set of $\epsilon-$clusters at the origin ($\emptyset \in \mathcal{C}_\epsilon$).

We are now going to define a subset of Ω characterized by having a certain element of $\mathcal{C}_\epsilon$ as minimal $\epsilon-$covering of the corresponding maximal cluster at the origin. More precisely, given $C \in \mathcal{C}_\epsilon$ define

$$\mathcal{A}_\epsilon(C) = \{\omega : C_A^0(\omega) \subset C \text{ and for all } C' \in \mathcal{C}_\epsilon, C' \subsetneq C, C_A^0(\omega) \not\subset C'\} \tag{4.13}$$

We have then by construction that the probability to find an A-particle at the origin is

$$\mu_\Lambda^A(Sp(0) = A) = \sum_{C \cap \partial\Lambda' = \emptyset} \mu_\Lambda^A(\mathcal{A}_\epsilon(C)) + \sum_{C \cap \partial\Lambda' \neq \emptyset} \mu_\Lambda^A(\mathcal{A}_\epsilon(C)) \tag{4.14}$$

where the sums are over $C \in \mathcal{C}(\epsilon)$. We deal with the two terms in the right hand side of (4.14) separately.

First of all observe that by (4.12)

$$\left| \sum_{C \cap \partial\Lambda' = \emptyset} \mu_\Lambda^A(\mathcal{A}_\epsilon(C)) - \sum_{C \cap \partial\Lambda' = \emptyset} \mu_\Lambda^B(\mathcal{A}_\epsilon(C)) \right|$$

$$\leq 2\Delta_\Lambda(\epsilon) + \left| \sum_C \left(\mu_\Lambda^A(\mathcal{A}_\epsilon(C) \cap G(\epsilon)) - \mu_\Lambda^B(\mathcal{A}_\epsilon(C) \cap G(\epsilon)) \right) \right| \tag{4.15}$$

Define ∂C_W to be that subset of Ω such that $Sp(x,\omega) = W$ for all x contained in an $\epsilon-$square adjacent to the outer boundary of $C \in \mathcal{C}_\epsilon$ (that is the external connected component of the boundary). Since $\mathcal{A}_\epsilon(C) \cap G(\epsilon) \subset \mathcal{A}_\epsilon(C) \cap \partial C_W$ we can continue (4.15) obtaining

$$\left| \sum_{C\cap\partial\Lambda'=\emptyset} \mu_\Lambda^A(\mathcal{A}_\epsilon(C)) - \sum_{C\cap\partial\Lambda'=\emptyset} \mu_\Lambda^B(\mathcal{A}_\epsilon(C)) \right|$$

$$\leq 4\Delta_\Lambda(\epsilon) + \sum_C \left| \mu_\Lambda^A(\mathcal{A}_\epsilon(C) \cap \partial C_W) - \mu_\Lambda^B(\mathcal{A}_\epsilon(C) \cap \partial C_W) \right| \tag{4.16}$$

By the Markov property of the μ_Λ^γ the sum in the right hand side of (4.16) is zero. This is because by conditioning it is straightforward to see that every term in this sum is equal to

$$\frac{\mu_\Lambda(\mathcal{A}_\epsilon(C)|\partial C_W)\mu_\Lambda(I_A|\partial C_W)\mu_\Lambda(\partial C_W)}{\mu_\Lambda(I_A)}$$

$$-\frac{\mu_\Lambda(\mathcal{A}_\epsilon(C)|\partial C_W)\mu_\Lambda(I_B|\partial C_W)\mu_\Lambda(\partial C_W)}{\mu_\Lambda(I_B)} = 0 \tag{4.17}$$

The last equality follows because of the symmetry under exchange $A \leftrightarrow B$. Hence

$$| \sum_{C\cap\partial\Lambda'=\emptyset} \mu_\Lambda^A(\mathcal{A}_\epsilon(C)) - \sum_{C\cap\partial\Lambda')^c=\emptyset} \mu_\Lambda^B(\mathcal{A}_\epsilon(C))| \leq 4\Delta_\Lambda(\epsilon) \tag{4.18}$$

The absolute value of the difference between the second term in the right hand side of (4.14) and $\theta^A(A;\Lambda,\Lambda')$ is by the definition (4.11) of $G(\epsilon)$ smaller than $\Delta_\Lambda(\epsilon)$. Combining this with (4.18) and writing (4.14) also for the measure μ_Λ^B, we get

$$|\mu_\Lambda^A(Sp(0) = A) - \mu_\Lambda^B(Sp(0) = A) + \theta^B(A;\Lambda,\Lambda') - \theta^A(A;\Lambda,\Lambda')| \leq 6\Delta_\Lambda(\epsilon) \tag{4.19}$$

Hence the left hand side of (4.19) is zero. The result follows by taking the limits as indicated in (4.10). In the case of two dimensions ($d = 2$), we use a generalized version of Gandolfi *et al.* (1988) extending their result without difficulty to the continuum. This ensures that $\theta^B(A) = 0$ if $\theta^A(A) > 0$. $\square$

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Caption for fig.1

A portion of a configuration of the hard–square lattice gas. The squares centered on odd sites are painted darker than the ones centered on even sites. In this figure there are three odd clusters and two even clusters.

Caption for fig.2

The expected phase diagram of the antiferromagnetic Ising model for $d = 2$. The shadowed area is the region of phase transition (this is only a qualitative diagram). The hard square limit is obtained taking $J \to \infty$ along the line with slope $\tan(\theta)$ shown on the figure. The phase transition for the hard–square model is expected to happen at $\theta = \theta_c$.

Caption for fig.3

A portion of a configuration of A (darker) and B (lighter) particles. There is an $A-$cluster at the origin and it is separated from the other clusters by a white layer.

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