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COMONOTONE APPROXIMATION POINTWISE ESTIMATES FOR TWICE DIFFERENTIABLE FUNCTIONS

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Abstract

An estimate for polynomial approximation of a twice differentiable comonotone (i.e. piecewise-monotone) function is obtained in terms of $k-th$ modulus of continuity of the function in question.

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Introduction.

1^0.We denote by C - the space of continuous functions $f:[-1,1]\to R$ endowed by uniform norm

$$\| f \|:=\max_{x\in[-1,1]}|f(x)|;\quad C^r:=\left\{f:f^{(r)}\in C\right\},\ r\in\mathbf{N};\ C^0:=C; I:=[-1,1].$$

Let $s\in\mathbf{N}, Y=Y_s$ is a collection of $s+1$ distinct points $y_i\in I$. For the fix collection Y we denote by $\Delta(Y)-$ the set of functions $f\in C$ such that f is nondecreasing on $[y_{i+1},y_i]$ when $i-$ even; f is nonincreasing, when $i-$ odd (that is $\Delta(Y)-$ the set of piecewise-monotone functions). Functions $f\in\Delta(Y)$ are called comonotone each other.

For the case $s=1$, that is for the set of monotone functions the direct estimates of approximation by monotone polynomials are investigated in the works of G.G.Lorentz, K.L.Zeller, R.A.De Vore, A.S.Shvedov, R.K.Beatson, X.M.Yu, D.Leviatan, I.A.Shevchuk almost so full as in unconstrained approximation. Therefore everywhere below $s>1, s\in\mathbf{N}$. Let us denote by $\mathcal{P}_n-$ the spase of algebraic polynomials of degree $\le n,\ n\in\mathbf{N},\ \ \mathcal{P}_n(Y):=\mathcal{P}_n\cap\Delta(Y)$,

$$E_n^*(f):=\inf_{\mathcal{P}\in\mathcal{P}_n(Y)}\| f-\mathcal{P} \|$$

$-$ the value of the best uniform approximation of a function $f\in\Delta(Y)$ by the polynomials $\mathcal{P}\in\mathcal{P}_n(Y)$.

D.J.Newman, E.Passow and L.Raymon proved an estimate (see, for example [5])

$$E_n^*(f)\le B_Y\omega_1(f;n^{-1}),\quad n\in\mathbf{N}, \tag{1.1}$$

where $\omega_1(f;t)-$ the modulus of continuity of $f\in C$, and the constant B_Y depends only of Y. G.L.Iliev [6] established that the constant B_Y in (1.1) may be changed by the constant B_s, depending only of s. A.S.Shvedov [7], (see also K.M.Yu [8]) stregthened the estimate (1.1), replasing the first modulus of continuity $\omega_1(f;t)$ by the second modulus of continuity $\omega_2(f;t)$; namely the estimate

$$E_n^*(f)\le B_Y\omega_2(f;n^{-1}),\quad n\in\mathbf{N}, \tag{1.2}$$

was proved.

Besides, it turned out, that the constant B_Y in (1.2) can't be replaced by the constant B_s (see [7]). Estimate (1.2) yields

$$E_n^*(f)\le \frac{B_Y}{n}\omega_1(f';n^{-1}),\quad n\in\mathbf{N}, \tag{1.3}$$

where $f\in C^1\cap\Delta(Y)$. Similary to (1.1) constant B_Y in (1.3) can be replaced by the constant B_s, see R.K.Beatson and D.Leviatan [9]. For the smoothness more than two the following estimates of E.Passow, L.Raymon and J.A.Roulier [3] are known:
if $f\in C^{(j+s)}(I)\bigcap\Delta(Y)$, then

$$E_n^*(f)\le B_j 2^s\frac{\| f^{(j+s)} \|}{n^j},\quad n>2(s-1+j),$$

$$E_n^*(f) \le B_{Y,j} n^2 \frac{\| f^{(j+s)} \|}{n^{j+s}}, \quad n > 4(s+1+j).$$

Recall that the $k-$th modulus of continuity of a function $f = f(x)$ continuous on $[a,b]$ is the function

$$\omega_k(t; f; [a,b]) = \sup_{h\in[0,t]} \sup_{x\in[a,b-kh]} |\sigma_h^k(f;x)|, \quad t \in [0,(b-a)/k],$$

where

$$\sigma_h^k(f;x) := \sum_{i=0}^{k} (-1)^{k-i} \left(\begin{array}{c} k \\ i \end{array} \right) f(x+ih)$$

is the $k-$ th order finite difference of f at x with step h.

Put

$$\rho := \rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}, \quad x \in I, \ n \in \mathbf{N}.$$

In this paper the theorem 1 is proved, which provides a coapproximation estimate the same as unconstrained estimate established by S.M.Nikol'skii [13], A.F.Timan [14], V.K.Dzyadyk [15], G.Freud[16], Yu.A.Brudnyi [17].

Theorem 1. *Let $k \in \mathbf{N}$. If $f \in C^2 \cap \Delta(Y)$ then for every integer $n > N_Y$ there exists an algebraic polynomial $\mathcal{P}_n \in \mathcal{P}_n(Y)$ such that*

$$|f(x) - \mathcal{P}_n(x)| \le B_{s,k} \rho_n^2(x) \omega_k(f''; \rho_n(x); I)$$

for all $x \in I$, where the integer N_Y depends only of Y, the constant,$B_{s,k}$ depends only of s and k.

The following theorem 2 is a corollary from theorem 1.

Theorem 2. *Let $k \in \mathbf{N}$. If $f \in C^2 \cap \Delta(Y)$ then for every integer $n \ge k+1$ there exists an algebraic polynomial $\mathcal{P}_n \in \mathcal{P}_n(Y)$ such that*

$$|f(x) - \mathcal{P}_n(x)| \le B_{Y,k} \rho_n^2(x) \omega_k(f''; \rho_n(x); I)$$

for oll $x \in I$, where the costant $B_{Y,k}$ depends only of Y and k.

Theorem 1 and a well-known Dzjadyk's inverse theorem (see, for example [10], p. 263, see also A.T.Timan [11], §6.2.3) provide the theorem 3 – constructive characteristic of $Lip^*\alpha \cap \Delta(Y)$ classes for $\alpha > 2$.

Theorem 3. *Let $\alpha > 2$. The function $f \in Lip^*\alpha \cap \Delta(Y)$ iff there exists a sequence of polynomials $\mathcal{P}_n \in \mathcal{P}_n(Y)$ such that*

$$\left\| \frac{f - \mathcal{P}_n}{\rho^\alpha} \right\| = O(1), \quad n \to \infty.$$

Let us also formulate the theorems 1 and 2 corollary for the class W^r, $r \in \mathbf{N}$, of functions which have the $(r-1)$-th absolutely continuous derivative on I and $|f^{(r)}(x)| \le 1$ a.e. on I.

Theorem 4. *If $f \in W^r \cap \Delta(Y)$, $r \geq 2$, then for every integer $n \geq r-1$ there exists a polynomial $\mathcal{P}_n \in \mathcal{P}_n(Y)$ such that*

$$\left\| \frac{f-\mathcal{P}_n}{\rho^r} \right\| \leq B_{Y,r}.$$

Remark. Theorem 3 is true also for $0<\alpha<2$ [12] and $\alpha=2$, theorem 4 is true for $r=1,2$. Respective papers are to be pablished.

For the methodic purpose we shall prove theorem $1'$ equivalent to the theorem 1.

Everywhere below k is integer, $k>1$; $\omega-(k-1)$-majorant, that is $\omega=\omega(t)$, $t\geq 0$, is a continuos and nondecreasing function with $\omega(0)=0$ and $t^{-(k-1)}\omega(t)$ nonincreasing. We write $\omega \in \Phi^{k-1}$ iff ω is $(k-1)-$ majorant.

Set

$$W^2H^{\omega}_{k-1} := \{f : f \in C^2 \quad \text{and} \quad \omega_{k-1}(t;f'';I) \leq \omega(t), \quad \text{where} \quad \omega \in \Phi^{k-1}\},$$

$$W^1H^{\varphi}_{k} := \{f : f \in C^1 \quad \text{and} \quad \omega_{k}(t;f';I) \leq \varphi(t), \quad \text{where} \quad \varphi \in \Phi^{k}\}.$$

It is well known an embedding

$$W^2H^{\omega}_{k-1} \subset W^1H^{\varphi}_{k},$$

if $\varphi(t)=t\omega(t)$.

Denote by A_i, B_i, c_i, R_i different positive numbers (constants) which may depend only of k and s.

Put

$$\Pi := \Pi(x) := \Pi(x;Y) := \prod_{i=1}^{s-1}(x-y_i).$$

Theorem 1'. *If $f \in W^2H^{\omega}_{k-1}$ and $f'(x)\Pi(x) \geq 0$, $x \in I$, then there exists a number $N=N(Y,k,s)$ and a constant c_1 such that for every $n>N$ there exists an algebraic polynomial $\mathcal{P}_n=\mathcal{P}_n(x)$ of degree $\leq n$ for which the inequalities*

$$\mathcal{P}_n'(x)\Pi(x) \geq 0 \tag{1.4}$$

and

$$|f(x)-\mathcal{P}_n(x)| \leq c_1\rho^2\omega(\rho). \tag{1.5}$$

are valid.

Auxiliary assertions.

2^0. Everywhere below $x \in I$, $\beta := \arccos x$, $y \in I$, $\alpha := \arccos y$, $n \in N$, $n \neq 1$, $r := 24ks+3k+s+2$. We denote by

$$D_{2r+1,n,r}(y,x) := \frac{1}{(2r)!}\frac{\partial^{2r+1}}{\partial x^{2r+1}}(x-y)^{2r}\int\limits_{\beta-\alpha}^{\beta+\alpha} J_{n,r}(t)dt$$

– the Dzjadyk-type polynomial kernel (see [1], §15), where

$$J_{n,r}(t)=\frac{1}{\gamma_{n,r}}\left[\frac{\sin(nt/2)}{sin(t/2)}\right]^{2(r+1)},\quad \gamma_{n,r}=\int\limits_{-\pi}^{\pi}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2(r+1)}dt,$$

is the Jackson-type kernel.

Let the function $g=g(x)$ be continuons on I. By $L_k(x,g)$ we denote the Lagrange polynomial of degree $\le k$ which interpolates g at the points $-1+2i/k,\ i=\overline{0,k}$.
Lemma 1 ([1,p.135]). *If $g\in W^2H^{\omega}_{k-1}$, then the polynomial*

$$D_n^{(0)}(x;g):=D_n(x;g):=\int\limits_{-1}^{1}\left(g(y)-L_k(y;g)\right)D_{2r+1,n,r}(y,x)dy+L_k(x;g)\tag{2.1}$$

of degree $<(r+1)(n-1)$ and its derivatives $D_n^{(1)}(x;g),\ D_n^{(2)}(x;g)$ for all $\delta>0$ and $x\in I$ satisfy the inequalities

$$|g^{(p)}(x)-D_n^{(p)}(x;g)|\le\rho^{2-p}\Bigg(R_1\omega_{k-1}\left(\rho,f'',[x-\delta,x+\delta]\cap I\right)$$

$$+R_2\left(\frac{\rho}{\delta}\right)^{r-2k-2}\omega(\rho)\Bigg),\qquad p=0\vee 1\vee 2,\tag{2.2}$$

in particular

$$|g^{(p)}(x)-D_n^{(p)}(x;g)|\le R_1\rho^{2-p}\omega(\rho).\tag{2.3}$$

For a fix n and every $j=\overline{1,n}$ put

$$\beta_j:=j\pi/n,\quad \overline{\beta_j}:=(j-1/2)\pi/n,$$

$$\beta_j^0:=\begin{cases}(j-1/4)\pi/n, & \text{if}\quad j<n/2,\\ (j-3/4)\pi/n, & \text{if}\quad j\ge n/2,\end{cases}$$

$$x_j:=\cos\beta_j,\ \overline{x}_j:=\cos\overline{\beta}_j,\ x_j^0:=\cos\beta_j^0,$$

$$t_{j,n}(x):=\left(x-x_j^0\right)^{-2}\cos^2 2n\arccos x+\left(x-\overline{x}_j\right)^{-2}\sin^2 2n\arccos x$$

– an algebraic polynomial of degree $4n-2$. Set $x_{-2}=x_{-1}=x_0=1,\ x_{n+1}=x_{n+2}=-1$.

The following inequalities hold (see [1],p.142,120).

$$\rho<h_j<5\rho,\ x\in I_j;\tag{2.4}$$

$$h_{j\pm1}<3h_j;\tag{2.5}$$

$$\min\left\{\left(x-x_j^0\right)^{-2},\ \left(x-\overline{x}_j\right)^{-2}\right\}\le t_{j,n}(x)$$

$$\leq \max\left\{\left(x-x_j^0\right)^{-2},\ (x-\overline{x}_j)^{-2}\right\},\ x\in I; \tag{2.6}$$

$$t_{j,n}\leq 10^3 h_j^{-2},\ x\in I_j; \tag{2.7}$$

$$\rho^2<4h_j\left(|x-x_j|+h_j\right),\ x\in I; \tag{2.8}$$

$$h_j^2<64\rho\left(|x-x_j|+\rho\right),\ x\in I; \tag{2.9}$$

$$2\left(|x-x_j|+h_j\right)>|x-x_j|+\rho>\left(|x-x_j|+h_j\right)/8,\qquad x\in I. \tag{2.10}$$

Let us take $N_Y\in\mathbf{N}$ such, that every segment $[y_{i+1},y_i]$, $i=\overline{0,s-1}$, containes at least three different segments I_j, and everywhere below $n\geq N_Y$.

We set

$$O_i:=\begin{cases}(x_{j+1},x_{j-1}), & \text{if}\quad x_j=y_i,\\ (x_{j+1},\ x_{j-2}), & \text{if}\quad y_i\in(x_j,x_{j-1}),\end{cases}$$

$$O:=\bigcup_{i=1}^{s-1}O_i.$$

Let us write $j\in W$ if $I_j\cap O=\emptyset,\ j=\overline{1,n}$.

The following simple estimates we shall need

$$\left|\frac{\Pi(x)}{\Pi(y)}\right|\leq\left(\frac{|x-y|}{\rho_n(y)}+1\right)^{s-1},\quad x\in I,\ y\in I\setminus O, \tag{2.11}$$

$$\left|\frac{\Pi(x)}{\Pi(x_j)}\right|\leq 3\left(\frac{|x-x_j|}{h_j}+1\right)^{s-1},\quad x\in I,\ j\in W. \tag{2.12}$$

Denote $b_1:=6ks,\ b_2:=r-2k-1+\left[\frac{s+k-1}{2}\right],\ b:=b_1\vee b_2$;

$$d_{j,n}:=d_{j,n}(b;Y):=\int_{-1}^{1}t_{j,n}^b(y)\Pi(y)dy;$$

$$T_{j,n}(x):=T_{j,n}(x;b;Y):=\frac{1}{d_{j,n}(b;Y)}\int_{-1}^{x}t_{j,n}^b(y)\Pi(y)dy;$$

$$\bar{d}_{j,n}:=\bar{d}_{j,n}(b;Y):=\int_{-1}^{1}(y-x_j)\left(x_{j-1}-y\right)t_{j,n}^{b+1}(y)\Pi(y)dy;$$

$$\tilde{T}_{j,n}(x):=\tilde{T}_{j,n}(x;b;Y):=\frac{1}{\bar{d}_{j,n}(b,Y)}\int_{-1}^{x}(y-x_j)\left(x_{j-1}-y\right)t_{j,n}^{b+1}(y)\Pi(y)dy;$$

$$\bar{T}_{j,n}(x):=\bar{T}_{j,n}(x;b;Y):=\left(T_{j,n}(x;b;Y)-\tilde{T}_{j,n}(x;b;Y)\right)\operatorname{sign}\ d_{j,n}(b;Y);$$

Define

$$\chi_j(x):=0,\quad\text{if } x\leq x_j,\quad \chi_j(x):=1,\quad\text{if } x>x_j;$$

$$\Gamma := \Gamma(x; j; b) := \left(\frac{h_j}{|x - x_j| + h_j}\right)^{2b} \left|\frac{\Pi(x)}{\Pi(x_j)}\right|.$$

Lemma 2. *If $j \in W$, then*

$$\text{sign } d_{j,n} = \text{sign } \bar{d}_{j,n} = \text{sign } \Pi(x_j); \tag{2.13}$$

$$B_1 h_j^{1-2b} |\Pi(x_j)| \le |d_{j,n}| \le B_2 h_j^{1-2b} |\Pi(x_j)|; \tag{2.14}$$

$$B_3 h_j^{1-2b} |\Pi(x_j)| \le |\bar{d}_{j,n}| \le B_4 h_j^{1-2b} |\Pi(x_j)|; \tag{2.15}$$

$$T'_{j,n}(x)\Pi(x)\text{sign } d_{j,n} \ge 0, \quad x \in I; \tag{2.16}$$

$$\bar{T}'_{j,n}(x)\Pi(x) \ge 0, \quad x \in I \setminus I_j; \tag{2.17}$$

$$\left|T'_{j,n}(x)\right| \le B_5 \frac{1}{h_j}\Gamma, \quad x \in I; \tag{2.18}$$

$$\left|\bar{T}'_{j,n}(x)\right| \le B_6 \frac{1}{h_j}\Gamma, \quad x \in I; \tag{2.19}$$

$$\left|\bar{T}'_{j,n}(x)\right| \ge \left|T'_{j,n}(x)\right| \ge B_7 \frac{1}{h_j}\Gamma, \quad x \in I \setminus I_j; \tag{2.20}$$

$$|\chi_j(x) - T_{j,n}(x)| \le B_8 \left(\frac{hj}{|x - x_j| + h_j}\right)^{2b-s}, \ x \in I; \tag{2.21}$$

$$|\bar{T}_{j,n}(x)| \le B_9 \left(\frac{h_j}{|x - x_j| + h_j}\right)^{2b-s}, \ x \in I. \tag{2.22}$$

Proof. For convenience assume $j < n/2$, that is in particular $x_j^0 - x_j > (\bar{x}_j - x_j)/2 > h_j/4$. Since by (2.5) $x_{j-1} - \bar{x}_j > h_j/4$, so $\bar{x}_j - x_j^0 \le 3h_j/8$.

Let us represent $d_{j,n}, \bar{d}_{j,n}$ as a sums

$$d_{j,n} = d_{j,n,-1} + d_{j,n,0} + d_{j,n,1} := \int_{-1}^{x_j} + \int_{x_j}^{x_{j-1}} + \int_{x_{j-1}}^{1} t_{j,n}^b(y)\Pi(y)dy.$$

$$\bar{d}_{j,n} = \bar{d}_{j,n,-1} + \bar{d}_{j,n,0} + \bar{d}_{j,n,1} := \int_{-1}^{x_j} + \int_{x_j}^{x_{j-1}} + \int_{x_{j-1}}^{1} (y - x_j)(x_{j-1} - y)\, t_{j,n}^{b+1}(y)\Pi(y)dy.$$

Applying the estimates

$$|x - x_j| + h_j < 8 \min\left\{|x - x_j^0|, |x - \bar{x}_j|\right\}, \quad x \in I \setminus I_j;$$

$$(x - x_j)(x - x_{j-1}) < (4/3) \min\left\{\left(x - x_j^0\right)^2, (x - \bar{x}_j)^2\right\}, \ x \in I \setminus I_j,$$

(2.6) and (2.12) we obtain

$$\left|\bar{d}_{j,n,-1}\right| < 8^s|\Pi(x_j)|h_j^{1-s}\int\limits_{-\infty}^{x_j}\left(x-x_j^0\right)^{s-1-2b}dx < \frac{2^s4^{2b}}{2b-s}|\Pi(x_j)|h_j^{1-2b}$$

$$:= B_{10}\,|\Pi(x_j)|\,h_j^{1-2b} \tag{2.23}$$

Similary

$$\left|\bar{d}_{j,n,1}\right|, |d_{j,n,\pm 1}| < B_{10}\Pi(x_j)|h_j^{1-2b}. \tag{2.24}$$

By the aid of (2.7), (2.12), (2.23) and (2.24) easy calculations yield right estimates in (2.14) and (2.15). To estimate $|d_{j,n}|$ and $|\bar{d}_{j,n}|$ from below we notice, that

$$|d_{j,n,0}| > \int\limits_{x_j^0}^{\bar{x}_j} t_{j,n}^{2b}(x)|\Pi(x)|dx,$$

$$\left|\bar{d}_{j,n,0}\right| > \int\limits_{x_j^0}^{\bar{x}_j} t_{j,n}^{2b}(x)|\Pi(x)|dx.$$

Since

$$(x-x_j)(x_{j-1}-x) > \frac{3}{4}\max\left\{\left(x-x_j^0\right)^2,\ (x-\bar{x}_j)^2\right\}, \quad x\in\left[x_j^0,\ \bar{x}_j\right],$$

$$|\Pi(x)| > 2^{1-s}|\Pi(x_j)|,$$

so

$$|d_{j,n,0}|,\ \left|\bar{d}_{j,n,0}\right| > |\Pi(x_j)|3\cdot 2^{-s}\int\limits_{x_j^0}^{\frac{\bar{x}_j+x_j^0}{2}}(x-\bar{x}_j)^{-2b}dx$$

$$= |\Pi(x_j)|3\cdot 2^{-s}\frac{1}{2b-1}\left(\bar{x}_j-x_j^0\right)^{1-2b}\left(2^{2b-1}-1\right)$$

$$> \left(\frac{8}{3}\right)^{2b-1}\left(2^{2b-1}-1\right)3\cdot 2^{-s}\frac{1}{2b-1}h_j^{1-2b}\,|\Pi(x_j)| =: 2B_3h_j^{1-2b}\,|\Pi(x_j)|\,.$$

Obviously,

$$B_3 > 2B_{10}, \tag{2.25}$$

which implies the validity of the left estimates in (2.14) and (2.15).

Besides (2.13), (2.16), (2.17) are follow from (2.25) .

The estimates (2.18), (2.19) and (2.20) are the corollaries from inequalities (2.15), (2.14), (2.6) and (2.7).

The estimate (2.21) follows from (2.18), (2.12) and representations

$$T_{j,n}(x) - \chi_j(x) = \int\limits_{-1}^{x} T'_{j,n}(y)dy, \quad \text{when} \quad x \le x_j;$$

$$\chi_j(x) - T_{j,n}(x) = \int\limits_{x}^{1} T'_{j,n}(y)dy, \quad \text{when} \quad x > x_j.$$

The same estimate is valid for the polynomial $\tilde{T}_{j,n}(x)$, which together with (2.21) give (2.22). Lemma 2 is proved.

Everywhere below $\varphi \in \Phi^k$.

Corollary from Lemma 2. *If $j \in W$ then*

$$h_j\varphi(h_j)\left|\bar{T}'_{j,n}(x)\right| \le B_{11}\varphi(\rho)\left(\frac{\rho}{|x-x_j|+\rho}\right)^{\frac{2b-s-k+1}{2}}, \ x \in I; \tag{2.26}$$

$$h_j\varphi(h_j)\left|\bar{T}'_{j,n}(x)\right| \ge B_{12}\varphi(\rho)\left(\frac{\rho}{|x-x_j|+\rho}\right)^{4b+k}\left|\frac{\Pi(x)}{\Pi(x_j)}\right|, \ x \in I \setminus I_j; \tag{2.27}$$

$$\varphi(h_j)|\chi_j(x) - T_{j,n}(x)| \le B_{13}\varphi(\rho)\left(\frac{\rho}{|x-x_j|+\rho}\right)^{\frac{2b-s-k}{2}}, x \in I; \tag{2.28}$$

$$\varphi(h_j)\left|\bar{T}_{j,n}(x)\right| \le B_{14}\varphi(\rho)\left(\frac{\rho}{|x-x_j|+\rho}\right)^{\frac{2b-s-k}{2}}, \ x \in I; \tag{2.29}$$

Indeed, since $h_j^2 < 64\rho(|x-x_j|+\rho)$ so $\varphi(h_j) \le \varphi(8\sqrt{\rho(|x-x_j|+\rho)}) \le 8^k\rho^{-\frac{k}{2}}(|x-x_j|+\rho)^{\frac{k}{2}}, \ \varphi(\rho)$, similarly $\varphi(\rho) = 2^k h_j^{-\frac{k}{2}}(|x-x_j|+h_j)^{\frac{k}{2}}\varphi(h_j)$, therefore (2.26)– (2.29) follow from (2.11), (2.12) and respectively (2.19), (2.20), (2.21) and (2.22).

3^0. Starting from the collection $\{x_j\}$ and $\{y_i\}$ we construct the collection Z of points $\{z_p\}$. To the collection Z we include all points $y_i, i = \overline{0,s}$ and every point x_j with $j \in W$. Renumber points of the colection Z in descending order. Put $z_0 = 1$. If $z_p = x_j, j \in W$, then let the number $j(p) = j$. If $z_p \in Y$, then we set $j(p) = \max\{j : j \in W \text{ and } x_j > z_p\}$. The collection Z contains $n - s^*$ points, $s < s^* + 1 < 2s$. Put $b_1 := 6ks$.

Lemma 3. *Let a function $g \in \Delta(Y)$ and $|g'(x)| \le \varphi(\rho), \quad x \in I$. Then the polynomial*

$$G_n(x) := G_n(x; g) := g(-1) + \sum_{q=1}^{n-s^*}(g(z_{q-1}) - g(z_q))\,T_{j(q),n}(x; b_1; Y)$$

of degree $< 4b_1 n$ is comonotone with g(x) on I, that is $G'_n(x)\Pi(x) \ge 0$ and

$$|g(x) - G_n(x; g)| \le c_2\rho\varphi(\rho), x \in I.$$

Proof. Let $x \in (z_p, z_{p-1}]$, then

$$g(x) - G_n(x;g) = g(x) - g(z_{p-1}) + \sum_{q=1}^{n-s^*} (g(z_{q-1}) - g(z_q)) \left(\chi_{j(q)}(x) - T_{j(q),n}(x;b_1;Y)\right)$$

(when $z_p \in Y$ the point z_{p-1} in the last formula must be replaced by z_p).

For every $q = \overline{1, n-s^*}$ the following estimates

$$z_{q-1} - z_q \le 13h_{j(q)},$$

$$\frac{1}{9}\rho_n(\theta) \le h_{j(q)} \le 45\rho_n(\theta), \qquad \theta \in [z_q, z_{q-1}] \tag{3.1}$$

are valid. Indeed, $[z_q, z_{q-1}] \subset \bigcup_{v=-2}^{2} I_{j(q)+v}$.

Taking in to account (2.4) and (2.5) we find

$$\rho_n(\theta) \le \max\left\{h_{j(q)}; h_{j(q)\pm 1}; h_{j(q)} \pm 2\right\} \le 9h_{j(q)}, \quad h_{j(q)} \le 45\rho_n(\theta).$$

Since $x \in (z_p, z_{p-1}]$, the condition of lemma , (3.1) and (2.8) yield

$$|g(z_{p-1}) - g(x)| \le |g(z_{p-1}) - g(z_p)| \le (z_{p-1} - z_p)\,\varphi\left(9h_{j(p)}\right)$$

$$\le 13 \cdot 9^k h_{j(p)} \varphi\left(h_{j(p)}\right) \le c_3 \rho\varphi(\rho),$$

$$|g(z_{q-1}) - g(z_q)| \le 13 \cdot 9^k h_{j(q)} \varphi\left(h_{j(q)}\right).$$

This and (2.28) provide

$$|g(x) - G_n(x;g)| \le c_3\rho\varphi(\rho) + c_4\rho^2\varphi(\rho) \sum_{q=1}^{n-s^*} \frac{h_{j(q)}}{\left(\left|x - x_{j(q)}\right| + \rho\right)^2}$$

$$\le c_3\rho\varphi(\rho) + c_4\rho^2\varphi(\rho) \sum_{j=1}^{n} \frac{h_j}{(|x - x_j| + \rho)^2} \le$$

$$\le c_6\rho\varphi(\rho).$$

It follows from the choise of the numbers $j(q)$ and from the construction of $T_{j,n}(x;b_1;Y)$ (see (2.13)) that $G_n \in \Delta(Y)$. Lemma 3 is proved.

4^0. For every $i = \overline{1, s-1}$ we denote by j_i the number, for which $y_i \in I_{j_i}$ (if there are two such numbers then we choose the greater of them). Put

$$y_i^* := x_{j_i}, \quad \text{if} \quad x_{j_i-1} - y_i \le y_i - x_{j_i}; \qquad y_i^* := x_{j_i-1} \quad \text{in the opposit case;}$$

$$Y_i := (Y \setminus \{y_i\}) \cup \{y_i^*\};$$

$$\breve{T}_{i,n}(x) := \bar{T}_{j_i-2,n}(x; b_2; Y_i);$$

$$K^*(x) := \min_{i=\overline{1,s-1}} |x - y_i|; \qquad K_n(x) := \min\left\{1; \frac{K^*(x)}{\rho}\right\}.$$

Lemma 4. *If a function $g \in W^2 H^\omega_{k-1}$ and $g'(y_i) = 0$ for every $i = \overline{1, s-1}$, then the polynomial*

$$Q_n(x; g) := D_n(x; g) - \sum_{i=1}^{s-1} \frac{D_n'(y_i; g)}{\breve{T}_{i,n}'(y_i)} \breve{T}_{i,n}(x)$$

of degree $< 5b_2 n$ for any $\delta > 0$ and $x \in I$ satisfies the inequalities

$$|g(x) - Q_n(x; g)| \le c_7 \rho^2 \omega(\rho), \tag{4.1}$$

$$\left|g'(x) - Q_n'(x; g)\right| \le \rho\Bigg(c_8 \omega_{k-1}\left(g''; \rho; [x-\delta, x+\delta] \cap I\right)$$
$$+ c_9 \left(\frac{\rho}{\delta}\right)^{r-2k-2} \omega(\rho)\Bigg) K_n(x), \tag{4.2}$$

in particular

$$\left|g'(x) - Q_n'(x; g)\right| \le c_8 \rho \omega(\rho) K_n(x), \quad x \in I. \tag{4.3}$$

Proof. Let us make use of approximate properties of the polynomial $D_n(x, g)$, given in the Lemma 1 and show that for the polynomial $Q_n(x, g)$ the inequality (4.2) holds. Obviously, it is sufficient to prove (4.2) for $\delta > 10\rho$. Let us fix $i = \overline{1, s-1}$. Inequalities (2.19) and (2.20) yield

$$\alpha_i(x) := \left|\frac{\breve{T}_{i,n}'(x)}{\breve{T}_{i,n}'(y_i)}\right| \le B_{15} \left(\frac{h_{j_i-2}}{|x - x_{j_i-2}| + h_{j_i-2}}\right)^{2b_2} \left|\frac{\Pi(x; Y_i)}{\Pi(y_i; Y_i)}\right|$$
$$\le B_{16} \left(\frac{h_{j_i-2}}{|x - x_{j_i-2}| + h_{j_i-2}}\right)^{2b_2 - s + 1} K_n(x)$$
$$\le B_{17} \left(\frac{\rho}{|x - x_{j_i-2}| + h_{j_i-2}}\right)^{b_2 - \frac{s-1}{2}} K_n(x), \quad x \in I \setminus O_i. \tag{4.4}$$

Let $\rho < \delta < 24\left(|x - x_{j_i-2}| + h_{j_i-2}\right)$. Then for $x \in I \setminus O_i$, taking into account (2.3) and (2.4) one can write the inequality

$$\left|D_n'(y_i; g)\right| \alpha_i(x) \le R_1 \rho_n(y_i) \omega\left(\rho_n(y_i)\right) B_{17} \left(\frac{\rho}{|x - x_{j_i-2}| + h_{j_i-2}}\right)^{b_2 - \frac{s-1}{2}} K_n(x)$$
$$\le c_{10} \rho \omega(\rho) \left(\frac{\rho}{|x - x_{j_i-2}| + h_{j_i-2}}\right)^{b_2 - \frac{s-1}{2} - \frac{k}{2}} K_n(x)$$
$$\le c_{11} \rho \omega(\rho) \left(\frac{\rho}{\delta}\right)^{b_2 - \frac{s+k-1}{2}} K_n(x). \tag{4.5}$$

If $\delta \geq 24\left(|x-x_{j_i-2}|+h_{j_i-2}\right) > 2\left(|x-y_i|+h_{j_i-2}\right)$, then we shall make use from (2.2) and (4.4), noting that $\left[y_i-\frac{\delta}{2}, y_i+\frac{\delta}{2}\right] \subset [x-\delta, x+\delta]$:

$$\begin{aligned}
|D_n'(y_i;g)|\,\alpha_i(x) &\leq \rho_n(y_i)\Bigg(R_1\omega_{k-1}\left(\rho_n(y_i);g'';[x-\delta,x+\delta]\cap I\right) \\
&\quad + R_2\left(\frac{\rho_n(y_i)}{\delta/2}\right)^{r-2k-2}\omega(\rho_n(y_i))\Bigg)\,\alpha_i(x) \\
&\leq c_{12}\rho\Bigg(\omega_{k-1}\left(\rho;g'';[x-\delta,x+\delta]\cap I\right)\left(\frac{\rho}{|x-x_{j_i-2}|+h_{j_i-2}}\right)^{b_2-\frac{s+k-1}{2}} \\
&\quad + \left(\frac{|x-y_i|+\rho}{\rho}\right)^{\frac{r-2k-2}{2}}\left(\frac{\rho}{\delta}\right)^{r-2k-2}\omega(\rho) \\
&\quad \times\left(\frac{|x-y_i|+\rho}{\rho}\right)^{\frac{k}{2}}\left(\frac{\rho}{|x-x_{j_i-2}|+h_{j_i-2}}\right)^{b_2-\frac{s-1}{2}}\Bigg)K_n(x) \\
&\leq \rho\Bigg(c_{12}\omega_{k-1}(\rho;g'';[x-\delta,x+\delta]\cap I)+c_{13}\left(\frac{\rho}{\delta}\right)^{r-2k-2}\omega(\rho) \\
&\quad \times\left(\frac{\rho}{|x-y_i|+\rho}\right)^{b_2+1+\frac{k-r-s+1}{2}}\Bigg)K_n(x), \quad x\in I\setminus O_i, \qquad (4.6)
\end{aligned}$$

in this case. Taking into account that $b_2=r-2k-1+\left[\frac{s+k-1}{2}\right]$ we find that for $x\in I\setminus O_i$ the inequality (4.2) follows from (2.2), (4.5) and (4.6).

Let now $x\in O_i$. We need to estimate an expression

$$g'(x)-D_n'(x;g)+\frac{D_n'(y_i;g)}{\breve{T}_{i,n}'(y_i)}\breve{T}_{i,n}'(x):=\beta(x)$$

For this purpose we involve Dzjadyk's inequality for the algebraic polinomial derivative modulus (see [10], p.257 or [1],§22) and using the estimate

$$|\alpha_i(\theta)|\leq B_{16}\left(\frac{h_{j_i-2}}{|\theta-x_{j_i-2}|+h_{j_i-2}}\right)^{2b_2-s+1},\theta\in I$$

obtain

$$|\alpha_i'(\theta)|\leq B_{18}\frac{1}{\rho_n(\theta)}\left(\frac{h_{j_i-2}}{|\theta-x_{j_i-2}|+h_{j_i-2}}\right)^{2b_2-s+1},\theta\in I.$$

Similarly to (4.5) and (4.6) we find

$$|D_n'(y_i;g)\alpha_i'(\theta)|\leq c_{14}\left(\omega_{k-1}(g'';\rho;[x-\delta,x+\delta]\cap I)+\left(\frac{\rho}{\delta}\right)^{r-2k-2}\omega(\rho)\right)$$

$$=: c_{14}\Omega, \qquad \theta \in O_i.$$

It follows from (2.2) that

$$\left|g''(\theta) - D_n''(\theta; g)\right| \le c_{15}\Omega, \qquad \theta \in O_i.$$

Recalling the equality $g'(y_i) = 0$ we find

$$|\beta(x)| = \left|\int_{y_i}^{x} \beta'(u)du\right| \le |x - y_i|(c_{14} + c_{15})\Omega \le c_{16}\rho\Omega K_n(x). \tag{4.7}$$

Now for $x \in O$ the inequality (4.2) follows from (4.5)$-$(4.7).

The inequality (4.1) follows from (2.3), (2.20), (2.29) and inequalities

$$\left|\sum_{i=1}^{s-1} \frac{D_n'(y_i; g)}{\breve{T}_{i,n}'(y_i)} \breve{T}_{i,n}(x)\right| \le \sum_{i=1}^{s-1} c_{17}\rho_n(y_i)\omega(\rho_n(y_i))$$

$$\times \left(\frac{\rho}{|x - x_{j_i - 2}| + \rho}\right)^{b_2 - \frac{s}{2}} h_{j_i - 2} \left|\frac{\Pi\left(x_{j_i - 2}; Y_i\right)}{\Pi(y_i; Y_i)}\right| \le c_{18}\rho^2\omega(\rho).$$

Lemma 4 is proved.

Lemma 5. *For an arbitrary set E, consisting of the segments I_j, $j \in W$, the polynomial*

$$U_n(x; E) := \sum_{j: I_j \subset E} \varphi(h_j) h_j \bar{T}_{j,n}(x; b_1; Y)$$

of degree $\le 5b_1 n$ satisfies the following inequelities

$$|U_n(x; E)| \le B_{19}\rho\varphi(\rho), \qquad x \in I; \tag{4.8}$$

$$U_n'(x; E)\Pi(x) \ge 0, \qquad x \in I \setminus E; \tag{4.9}$$

$$\left|U_n'(x; E)\right| \le B_{20}\varphi(\rho), \qquad x \in E; \tag{4.10}$$

$$\left|U_n'(x; E)\right| \ge B_{21}\varphi(\rho)\left(\frac{\rho}{dist(x, E) + \rho}\right)^{4b_1 + k + s - 1} K_n(x), \qquad x \in I \setminus E. \tag{4.11}$$

Proof. Let us prove (4.11). For every $x \in I \setminus E$, among all numbers $j : I_j \subset E$ we choose the number j^* such that $|x - x_{j^*}| = \min\limits_{j: I_j \subset E} |x - x_j|$ (if there are two such numbers, we take the largest of them). Applying (2.27) and (2.17) we write

$$\left|U_n'(x; E)\right| \ge B_{12}\varphi(\rho)\left(\frac{\rho}{|x - x_{j^*}| + \rho}\right)^{4b_1 + k} \left|\frac{\Pi(x)}{\Pi(x_{j^*})}\right|, \quad x \in I \setminus E. \tag{4.12}$$

Fix $i = \overline{1, s-1}$ and notice that for $x \in O_i$ the inequality

$$\left|\frac{x - y_i}{x_j - y_i}\right| \ge c_{19} K_n(x) \frac{\rho}{|x - x_j| + \rho}, \qquad j \in W, \tag{4.13}$$

holds. Now we collect together (4.12), (2.11) and (4.13), apply the estimate

$$|x - x_{j^*}| + \rho < 6(\mathrm{dist}(x, E) + \rho), \qquad x \in I \setminus E$$

and find (4.11).

The estimate (4.10) follows from (2.26) and the choice of b_1 ($b_1 = 6ks$), namely

$$|U_n'(x; E)| \le c_{20}\rho\varphi(\rho) \sum_{j: I_j \subset E} \frac{h_j}{(|x - x_j| + \rho)^2} \left(\frac{\rho}{|x - x_j| + \rho} \right)^{\frac{2b_1 - s - k + 1}{2} - 3}$$

$$\le c_{20}\rho\varphi(\rho) \sum_{j: I_j \subset E} \frac{h_j}{(|x - x_j| + \rho)^2} \le 2c_{20}\varphi(\rho), \quad x \in I. \tag{4.14}$$

Similarly to (4.14) the estimate (4.8) follows from (2.29). The estimate (4.9) follows from (2.27). Lemma 5 is proved.

5^0. **Lemma 6.** *Let a function $g \in H_k^\varphi$, an integer $j = \overline{1, n - 20k}$ and the set*

$$J_j := \bigcup_{\nu=0}^{20k} I_{j+\nu}.$$

are given. If among the segments $I_{j+\nu}$ there exist $2k - 1$ seqments $I_{j+\nu_p}, 0 \le \nu_1 < \nu_2 < \ldots < \nu_{2k-1} \le 20k$, such that for each $p = \overline{0, 2k - 1}$ a point $\tilde{x}_{j+\nu_p} \in I_{j+\nu_p}$ can be found for which

$$|g(\tilde{x}_{j+\nu_p})| \le \varphi(\rho_n(\tilde{x}_{j+\nu_p})),$$

then for any $x \in J_j$ we have

$$|g(x)| \le c_{21}\varphi(\rho).$$

Lemma 6 can be proved using the Whitney's inequality [18], [19].

The following inequality will be quoted.

$$\mathrm{mes} J_j \le c_{22}\rho, \qquad x \in J_j. \tag{5.1}$$

Put

$$M = \max \left\{ \frac{2c_8(c_{22} + 1)^{4b_1 + k + s - 1}}{B_{21}}, \ \frac{14^{4b_1 + k + s - 1} c_8}{B_{21}}, \ 1 \right\}$$

and

$$A = \max \{2c_8 + MB_{20}, \ 1\}.$$

Everywhere below $\varphi(t) := t\omega(t)$.

Definition 1. We shall write $j \in V_1$, if

$$|f'(x)| \le Ac_{21}\varphi(\rho)$$

for each $x \in I_j$; $\qquad j \in V_2$ if $j \notin V_1, O \cap \bigcup\limits_{\nu=-3}^{3} I_{j+\nu} = \emptyset$ and

$$|f'(x)| \geq A\varphi(\rho)$$

for each $x \in I_j$; $\qquad j \in V_3$, if $j \notin V_1 \cup V_2$. Put

$$E_1 := \bigcup_{j \in V_1} I_j, \quad E_2 := \bigcup_{j \in V_2} I_j, \quad E_3 := \bigcup_{j \in V_3} I_j.$$

The set E_3 (if it is not empty) consists of nonintersecting segment $[a_\nu, b_\nu] \subset I$.
Lemma 7. *Every from nonintersecting segments $[a_\nu, b_\nu]$, constituting the set E_3, consists of no more then $20k$ segments I_j. Other words, if $j \in V_3$, then among the numbers $j, j+1, \ldots, j+20k$ there are at least one number j^0 such that $j^0 \notin V_3$.*
Lemma 7 follows from lemma 6.

Henceforth we shall assume that $E_2 \neq \emptyset$, else theorem $1'$ wold follow from the lemmas 7 and 3.

Let $j \in V_3$. We denote by $\left[a_{\nu(j)}, b_{\nu(j)}\right]$ – that of nonintersecting segments $[a_\nu, b_\nu]$, constituting the set E_3, which contains the segment I_j.
Definition $1'$. We shall write $j \in V_{3,1}$, if $j \in V_3$ and $E_1 \cap [a_{\nu(j)}, b_{\nu(j)}] \neq \emptyset$. Denote

$$V_{3,2} := V_3 \setminus V_{3,1}, \quad V_4 := V_{3,1} \cup V_1, \quad V_5 := V_{3,2} \cup V_2;$$

$$E_{3,1} := \bigcup_{j \in V_{3,1}} I_j, \quad E_{3,2} := \bigcup_{j \in V_{3,2}} I_j; \quad E_4 := E_{3,1} \cup E_1, \quad E_5 := E_{3,2} \cup E_2.$$

We shall write $j \in V_6$, if $j \in V_4$ and $I_j \cap E_5 \neq \emptyset$. Put

$$E_6 = \bigcup_{j \in V_6} I_j.$$

We shall write $j \in V_7$, if $j \in V_4 \setminus V_6$ and $I_j \cap E_6 \neq \emptyset$. Put

$$E_7 = \bigcup_{j \in V_7} I_j.$$

Finally, we shall write $j \in V_8$, if $j \in V_4 \setminus (V_6 \cup V_7)$ and $I_j \cap E_7 \neq \emptyset$. Put

$$E_8 = \bigcup_{j \in V_8} I_j.$$

For every $j = \overline{1,n}$ we define the function

$$S_j(x) := \int\limits_{x_j}^{x} (y - x_j)^k (x_{j-1} - y)^k dy \left(\int\limits_{x_j}^{x_{j-1}} (y - x_j)^k (x_{j-1} - y)^k dy \right)^{-1}. \tag{5.2}$$

The derivative $f'(x)$ we represent as a sum of a "litle" function $g_1 = g_1(x)$ and a "large" function $g_2 = g_2(x)$ as follows.

Definition 2. We put $g_1(x) = 0$ for $x \in E_5 \cup E_6$; $g_1 = f'(x)$, for $x \in E_4 \setminus (E_6 \cup E_7)$. For $x \in I_j$ with $j \in V_7$ we put $g_1(x) := 0$, when $(j+1) \in V_6$ and $(j-1) \in V_6$; $g_1(x) := (1 - S_j(x))\, f'(x)$, when $(j+1) \notin V_6$; $g_1(x) = S_j(x) f'(x)$, when $(j-1) \notin V_6$.

Set $g_2(x) := f'(x) - g_1(x)$,

$$f_1(x) := f(-1) + \int_{-1}^{x} g_1(y)dy, \quad f_2(x) := \int_{-1}^{x} g_2(y)dy,$$

(that is $f(x) = f_1(x) + f_2(x)$.)

Lemma 8. *The following inequalities are valid*

$$|g_1(x)| \le A_1 \varphi(\rho), \qquad x \in I; \tag{5.3}$$

$$\omega_{k-1}(g_2'; t; I) \le A_2 t^{-1} \varphi(t), \qquad t \ge 0. \tag{5.4}$$

The first inequality of Lemma 8 can be proved using Whitney's inequality [18], [19] and Lemma 6. The second can be shown as in [20].

Proof of the theorem 1'.

6^0. We choose the number n_1 such , that conditions

$$A_2 c_9 \left(\frac{\rho_{n_1}(x)}{\rho} \right) \le c_8, \tag{6.1}$$

$$65^{4b_1+k+s-1} c_8 A_2 \left(\frac{\rho_{n_1}(x)}{\rho} \right) \le M B_{21}, \qquad x \in I \tag{6.2}$$

are fulfiled. For this purpose it is sufficient to take $n = A_3 n$, where

$$A_3 = \max \left\{ \left[\frac{A_2 c_9}{c_8} + 1 \right], \left[\frac{65^{4b_1+k+s-1} c_8 A_2}{M B_{21}} + 1 \right] \right\}.$$

We denote by

$$\bar{Q}_{n_1}(x; f_2) := Q_{n_1}(x; f_2) + M U_n(x; E_2)$$

the sum of polinomials, defined in lemmas 4 and 5. The inequality

$$\left| f_2(x) - \bar{Q}_{n_1}(x; f_2) \right| \le c_{23} \rho^2 \omega(\rho), \qquad x \in I \tag{6.3}$$

follows from (4.1) and (4.8).

With the help of lemmas 4, 5, 8 and 7 we shall prove the estimate

$$\bar{Q}_{n_1}'(x; f_2)\Pi(x) = \left(Q_{n_1}'(x; f_2) - f_2'(x) + M U_n'(x; E_2) \right) \Pi(x) + f_2'(x)\Pi(x) \ge 0, \quad x \in I. \tag{6.4}$$

We notice, that the inequality

$$f_2'(x)\Pi(x) \ge 0, x \in I. \tag{6.5}$$

follows directly from the function's f_2 definition.

To prove (6.4) we consider four cases.

1) Let $x \in E_2$. By the construction, the sets E_2 and $E_9 := \cup I_j$, where $j \notin V_2$, $I_j \cap E_2 \neq \emptyset$, doesn't contain any point y_i, $i = \overline{1, s-1}$, therefore here $K_n(x) = K_{n_1}(x) = 1$. Besides, from the definitions 2 and 1 it is seen that the function $f_2'(x)$ is "large" on the set E_2, namely $|f'(x)| \geq A\rho\omega(\rho)$. Hense, for the validity of (6.4) it is sufficient in accordance with (4.2), (4.10) and (5.4) to verify the inequality

$$\rho_{n_1}(x)\left(c_8\omega_{k-1}\left(f_2'';\rho_{n_1}(x);[x-\delta,x+\delta]\cap I\right)+A_2c_9\left(\frac{\rho_{n_1}(x)}{\delta}\right)^{r-2k-2}\omega\left(\rho_{n_1}(x)\right)\right)$$

$$+MB_{20}\rho\omega(\rho) \leq A\rho\omega(\rho),$$

being satisfied for some δ, say equel to ρ. It is not difficult, as the numbers n_1 and A are choosen respectively.

2) Let $x \in E_{3,2} = E_5 \setminus E_2$. Sinse (4.9) is valid, so it is sufficient to prove the inequality

$$\left|Q_{n_1}'(x;f_2) - f_2'(x)\right| \leq M\left|U_n'(x;E_2)\right|. \tag{6.6}$$

Let $\delta = \operatorname{dist}(x, E_7)$. Let us now make use of (4.2), (4.11) and taking into account (5.4) write

$$\left(c_8 + A_2c_9\left(\frac{\rho_{n_1}(x)}{\operatorname{dist}(x,E_7)}\right)^{r-2k-2}\right)\rho_{n_1}(x)\omega(\rho_{n_1}(x))K_{n_1}(x)$$

$$\leq MB_{21}\rho\omega(\rho)\left(\frac{\rho}{\operatorname{dist}(x,E_2)+\rho}\right)^{4b_1+k+s-1}K_n(x). \tag{6.7}$$

From the definition of the set $E_{3,2}$, lemma 7 and (5.1) it is seen , that $\operatorname{dist}(x, E_2) < c_{22}\rho$. Besides, for all $x \in I$ and $n_1 \geq n$ the inequality

$$\rho_{n_1}(x)K_{n_1}(x) \leq \rho K_n(x). \tag{6.8}$$

is true. It follows from this that the choosen n_1 and M supply the validity of (6.7) and hence (6.6).

3) Let $x \in E_4 \setminus (E_6 \cup E_7 \cup E_8)$. Similary to 2) we have to show that the inequality (6.6) is valid. Take $\delta = \operatorname{dist}(x, E_7)$. Taking in to account (4.2), (5.4), (4.11) and equality $f_2'(x) = 0$ we write

$$A_2c_9\left(\frac{\rho_{n_1}(x)}{\operatorname{dist}(x,E_7)}\right)^{r-2k-2}\rho_{n_1}(x)\omega(\rho_{n_1}(x))K_{n_1}(x)$$

$$\leq MB_{21}\rho\omega(\rho)\left(\frac{\rho}{\operatorname{dist}(x,E_2)+\rho}\right)^{4b_1+k+s-1}K_n(x).$$

In accordance with (6.8) and the choice of r and b_1

$$A_2c_9\left(\frac{\rho_{n_1}(x)}{\operatorname{dist}(x,E_7)}\right)^{4b_1+k+s} \leq MB_{21}\left(\frac{\rho}{\operatorname{dist}(x,E_2)+\rho}\right)^{4b_1+k+s-1}. \tag{6.9}$$

Notice, that 14dist$(x, E_7) > \text{dist}(x, E_2) + \rho$. Therefore by virtue of (6.1)

$$14^{4b_1+k+s-1}c_8 \leq MB_{21}.$$

This inequality follows from the choice of the number M, therefore the estimate (6.6) is true.
4) Finally let $x \in E_6 \cup E_7 \cup E_8$. We take $\delta = \infty$. In this case (6.6) follows from (4.2), (4.11), (5.4), (6.2) and equality $K_n(x) = K_{n_1}(x) = 1$.

Thus the estimate (6.4) is valid for all $x \in I$. This with lemma 3 and (6.3) mean that the polynomial

$$\mathcal{P}_{n_1}(x) := \bar{Q}_{n_1}(x; f_2) + G_n(x; f_1)$$

of degree $< \max\{A_3,\ 5\}\, 25ksn$ is desired in theorem $1'$. Theorem $1'$ is proved.

The proof of the theorem 2.

It follows from the theorem $1'$, that it is sufficient to investigate only the case $n = k+1$.

Let us cosider two situations.
1) $s > k$. Sinse $f'(y_i) = 0$ for all $i = \overline{1,k}$, then $L(x; f') := L(x; f'; y_1, \ldots, y_k) \equiv 0$ where $L(x; f'; y_1, \ldots, y_k)$ - the Lagrange polynomial of degree $\leq k-1$, which interpolates the function $f'(x)$ at the points $y_i, i = \overline{1,k}$. We denote $[y_1, \ldots, y_k, x; f']$ $-$ is the divided difference of k $-$th order of the function f' associeted with the points y_i, $\ i = \overline{i,k}$. By [1, p.56] we have

$$|f'(x)| = |[y_1, \ldots, y_k, x; f']| \prod_{i=1}^{k} |x - y_i| \leq c_1(Y)\omega(1)$$

($c_i(Y)-$ constants depending of Y). Therefore the polynomial $\mathcal{P}_k \equiv f(-1)$ is desired in this theorem.
2) Let $s \leq k$. To the collection of points $y_i, i = \overline{1,s}$ let us add $(k-s+1)$ equdistant points $-1 = y_s < y_{s+1} < \ldots < y_{k+1} = y_{s-1}$. Taking in to account that $f \in W^2 H^{\omega}_{k-1}$ by [1, p.56]

$$|f'(x) - L(x; f'; y_1, \ldots, y_k)| \leq c_2(Y)\omega(1) \prod_{i=1}^{k} |x - y_i| \leq c_3(Y)\omega(1) |\prod(x)|.$$

We put

$$\mathcal{P}_k(x) := f(-1) + \int_{-1}^{x} \left(L(u; f'; y_1, \ldots, y_k) + c_3(Y)\omega(1)\Pi(u)\right) du$$

and note, that $\mathcal{P}_k'(x)\Pi(x) \geq 0$. Theorem 2 is proved.

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