Centre de Physique Théorique*- CNRS - Luminy, Case 907
F–13288 Marseille Cedex 9 - France

ON THE PURITY OF THE LIMITING GIBBS STATE FOR THE ISING MODEL ON THE BETHE LATTICE

P.M. BLEHER [1], J. RUIZ [2] and V.A. ZAGREBNOV[3]

Abstract

We give a proof that for the Ising model on the Bethe lattice, the limiting Gibbs state with zero effective field (disordered state) persists to be pure for temperature below ferromagnetic critical temperature T_c^F till the critical temperature T_c^{SG} of the corresponding spin-glass model. This new proof revises the one proposed in [1].

Key-Words : Pure Gibbs state, Ising-Bethe model, spin-glass.

September 1994

CPT-94/P.3074

anonymous ftp or gopher : cpt.univ-mrs.fr

---

* Unité Propre de Recherche 7061

1 Department of Mathematical Sciences, Indiana University, Purdue University at Indianapolis, 402 N. Blackford Street, Indianapolis, IN 46202-3216 USA
E-mail address: bleher@math.iupui.edu

2 E-mail address: ruiz@cpt.univ-mrs.fr

3 and Département de Physique, Université d'Aix-Marseille II
E-mail address: zagrebnov@cpt.univ-mrs.fr

1. The question about the possibility of the disordered limiting Gibbs state (i.e., with zero effective field) of the ferromagnetic Ising model on the Bethe lattice, to be pure below the ferromagnetic critical temperature T_c has been formulated as an open problem in [2,3] (some non–homogeneous pure Gibbs states were constructed in [4,5]). In [1] it was proved that the disordered phase is pure up to the spin–glass critical temperature T_c^{SG}, but as was later pointed out to the author by Hans-Otto Georgii, the proof contained a gap. A corrected version of the proof [6] was based on an inequality valid for the Bethe lattice τ^k of degree $k \leq 6$. In the present note we give a new proof valid for all k.

2. Let τ^k be the Bethe lattice of degree $k \geq 2$ such that exactly $(k+1)$ edges come out of any of it vertex. Then the Ising model is defined by the Hamiltonian

$$H(\sigma) = -\sum_{x,y} J_{xy}\, \sigma(x)\sigma(y) \tag{2.1}$$

where the sum is over pairs of nearest neighbors x, y and $\sigma(x) = \pm 1$ The ferromagnetic model corresponds to $J_{xy} = J > 0$, while the spin-glass model corresponds to $J_{xy} = \pm J$ $(J > 0)$, where $\{J_{xy}\}$ are independent random variables with $\Pr\{J_{xy} = +J\} = \Pr\{J_{xy} = -J\} = 1/2$ for any pair xy. Denote

$$\theta = \tanh \beta, \qquad \beta = J/k_B T, \tag{2.2}$$

where T is the temperature. Then for the ferromagnetic model the critical value $\theta_c^F = 1/k$ (see e.g. [3]), while for the spin-glass model $\theta_c^{SG} = 1/\sqrt{k}$, see e.g. [7]. The main result of this note is the following theorem

Theorem. *The limiting Gibbs state with zero effective field (disordered phase) is pure for* $0 < \theta \leq \theta_c^{SG}$.

Remark. For $0 < \theta \leq \theta_c^F$ the limiting Gibbs state is unique, and so trivially pure. On the other hand, for $\theta > 1/\sqrt{k}$ the disordered phase is not pure [2,4]. Thus our main result concerns the interval $[1/k, 1/\sqrt{k}]$.

3. Let $\tau^k = (V, L, i)$ be the Caley tree of order k with a root vertex $x_* \in V$. Here V is the set of vertices, L is the set of edges and i is the incidence function which corresponds to each edge $l \in L$ its end-points $x_1, x_2 \in V$. There is a distance $d(x, y)$ on V which is the length of the minimal path from x to y, assuming that the length of any edge is 1. Denote by

$$W_n = \{x \in V : \ d(x_*, x) = n\}$$

the sphere of radius n on V, and by

$$V_n = \{x \in V : \ d(x_*, x) \leq n\}$$

the ball of radius n, so that

$$V_n = \cup_{m=0}^n W_m.$$

For any $x \in W_n,\ n = 0, 1, 2, \ldots,$ denote by

$$S(x) = \{y \in W_{n+1} :\ d(x, y) = 1\}.$$

Let $\{h(x),\ x \in V\}$ be a set of real numbers satisfying for each $x \in V$ the recursive relation

$$h(x) = \sum_{y \in S(x)} f_\theta(h(y)), \quad f_\theta(h) = \operatorname{artanh}(\theta \tanh(h)). \tag{3.1}$$

Define the Gibbs probability distribution on the configurational space

$$\Sigma(V_n) = \{\sigma_n = \{\sigma(x) = \pm 1,\ x \in V_n\}\}$$

by the formula

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left(\beta \sum_{x,y \in V_n :\, d(x,y)=1} \sigma(x)\sigma(y) + \sum_{x \in W_n} h(x)\sigma(x)\right). \tag{3.2}$$

Then (3.1) implies the consistency of μ_n for different n so that there exists a Gibbs measure μ on the infinite configurational space

$$\Sigma(V) = \{\sigma = \{\sigma(x) = \pm 1,\ x \in V\}\}$$

whose finite–dimensional distributions are μ_n. The function $h(x),\ x \in V$, which satisfies (3.1), is called an effective field. Remark, that $h(x) \equiv 0$ satisfies (3.1), so that the finite–dimensional distributions

$$\mu_n^{\#}(\sigma_n) = Z_n^{-1} \exp\left(\beta \sum_{x,y \in V_n :\, d(x,y)=1} \sigma(x)\sigma(y)\right) \tag{3.3}$$

are consistent and generate a Gibbs measure $\mu^{\#}$, the disordered phase.

To prove that $\mu^{\#}$ is pure we will prove that the spins $\{\sigma(x),\ x \in V_n\}$ and $\{\sigma(x),\ x \in W_N\}$ are asymptotically independent with respect to $\mu^{\#}$, when $N \to \infty$ and n is fixed. To that end we first fix $N > 0$ and define recursively for every $x \in V_{N-1}$ a random variable h_x such that

$$h_x = \sum_{y \in S(x)} f_\theta(h_y), \qquad \forall\, x \in V_{N-2}, \tag{3.4}$$

with the initial data

$$h_x = \beta \sum_{y \in S(x)} \sigma(y), \qquad \forall\, x \in W_{N-1}. \tag{3.5}$$

Eqs. (3.4), (3.5) define h_x as a function of $\sigma(y),\ y \in W_N \cap V_x$, where V_x is the subtree growing from x. It is to be noted that in general the random variables $\{h_x,\ x \in V_{N-1}\}$ depend on N.

Lemma 3.1. *For every $n \le N-1$, the joint distribution of $\sigma_n = \{\sigma(x),\ x \in V_n\}$ and $h^{(n)} = \{h_x,\ x \in W_n\}$ with respect to $\mu^\#$ is given by*

$$\mu^\#(\sigma_n, h^{(n)}) = Z^{-1} \exp\left(-H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\right) \prod_{x \in W_n} \nu_x(h_x), \tag{3.6}$$

where the probability distributions $\nu_x(h)$ are defined by the recursive equations

$$\nu_x(h) = \frac{\displaystyle\sum_{h_y,\ldots,h_z:\, f(h_y)+\ldots+f(h_z)=h} \left(\prod_{v \in S(x)} a(h_v)\,\nu_v(h_v)\right)}{\displaystyle\sum_{h_y,\ldots,h_z} \left(\prod_{v \in S(x)} a(h_v)\,\nu_v(h_v)\right)}, \qquad \forall\, x \in V_{N-2}, \tag{3.7}$$

where

$$S(x) = (y, \ldots, z), \qquad f(h) = f_\theta(h), \qquad a(h) = [1 + (1-\theta^2)\sinh^2 h]^{1/2}, \tag{3.8}$$

and by the initial condition

$$\nu_x(h) = Z^{-1} \sum_{\beta\,[\sigma(y)+\ldots+\sigma(z)]=h} 1\,, \qquad \forall\, x \in W_{N-1}. \tag{3.9}$$

Remark. Lemma 3.1 allows the following extension. Let μ be a Gibbs measure generated, according to (3.2), by some set of real numbers $\{h(x),\ x \in V\}$ which satisfy the consistency equations (3.1), and let $\{h_x,\ x \in W_{N-1}\}$ be random variables defined through (3.4) and (3.5). Then the joint distribution $\mu(\sigma_n, h^{(n)})$ has the form (3.6) with probability distributions $\nu_x(h)$ defined by the recursive equations (3.6) and by the initial condition

$$\nu_x(h) = Z^{-1} \sum_{\beta\,[\sigma(y)+\ldots+\sigma(z)]=h} \exp[h(y)\sigma(y) + \ldots + h(z)\sigma(z)]\,, \qquad \forall\, x \in W_{N-1}. \tag{3.10}$$

When $h(x) \equiv 0$, (3.10) reduces obviously to (3.9). If, in addition, the coupling constant J_{xy} depend on xy then again the formulas (3.2)–(3.8), (3.10) are valid with

$$\beta = J_{xy}/k_B T, \qquad \theta = \tanh(J_{xy}/k_B T),$$

and some natural modifications in these formulas. The proof of this extension of Lemma 3.1 is similar to the proof of Lemma 3.1.

Proof of Lemma 3.1. The proof of (3.7) is based on the following identity:

$$\sum_{\sigma(y)=\pm 1} \exp[\beta\sigma(x)\sigma(y) + h_y\sigma(y)] = Z\,a(h_y)\exp\left[f_\theta(h_y)\,\sigma(x)\right], \quad \text{if} \quad \sigma(x) = \pm 1, \tag{3.11}$$

where $Z = \cosh\beta$. To prove this indentity notice that according to $\sigma(x) = \pm 1$, it reduces to two identities:

$$\exp(\beta + h_y) + \exp(-\beta - h_y) = Z\, a(h_y) \exp f_\theta(h_y),$$
$$\exp(-\beta + h_y) + \exp(\beta - h_y) = Z\, a(h_y) \exp[-f_\theta(h_y)].$$

If we divide the first identity by the second one then we obtain

$$\frac{\exp(\beta + h_y) + \exp(-\beta - h_y)}{\exp(-\beta + h_y) + \exp(\beta - h_y)} = \exp[2 f_\theta(h_y)],$$

which follows easily from the formula $\exp[2 \operatorname{artanh} z] = (1+z)/(1-z)$. If we multiply the first identity by the second one then we obtain

$$\cosh^2 \beta + \sinh^2 h_y = Z^2 a^2(h_y),$$

which follows easily from the formula $a^2(h) = 1 + \sinh^2 h / \cosh^2 \beta$. Thus (3.11) is proved. Multiplying (3.11) over $y \in S(x)$ we obtain (3.7) by induction. (3.9) follows directly from (3.3). Lemma 3.1 is proved. $\diamondsuit$

4. By (3.9) and (3.7) the measures $\nu_x(h)$ coincide for all $x \in W_n$ so we will denote them by $\nu_n(h)$. By (3.9) and (3.7) $\nu_n(h)$ is symmetric, hence

$$\sum_h \sinh h\, \nu_n(h) = 0.$$

Let

$$D_n = \sum_h \sinh^2 h\, \nu_n(h).$$

Lemma 4.1. *If $k\theta^2 \le 1$ then*

$$D_{n-1} \le k\theta^2 D_n. \tag{4.1}$$

Proof. It is convenient to rewrite (3.4) and (3.7) in the form

$$h = \sum_{j=1}^{k} f(h_j), \qquad f(h) = f_\theta(h),$$

and

$$\nu_{n-1}(h) = \frac{\displaystyle\sum_{h_1,\ldots,h_k : f(h_1)+\ldots+f(h_k)=h} \left(\prod_{j=1}^{k} a(h_j)\nu_n(h_j)\right)}{\displaystyle\sum_{h_1,\ldots,h_k} \left(\prod_{j=1}^{k} a(h_j)\nu_n(h_j)\right)}. \tag{4.2}$$

Denote

$$s = \sinh h, \qquad s_j = \sinh h_j, \qquad t_j = \tanh h_j.$$

We have

$$s^2 = \sinh^2 h = \sinh^2(f(h_1) + \ldots + f(h_k)) = \frac{\tanh^2(f(h_1) + \ldots + f(h_k))}{1 - \tanh^2(f(h_1) + \ldots + f(h_k))},$$

and

$$\tanh(x_1 + \ldots + x_k) = \frac{\sum_{\text{odd } p} \sum_{j_1 < j_2 < \ldots < j_p} y_{j_1} \ldots y_{j_p}}{\sum_{\text{even } p} \sum_{j_1 < j_2 < \ldots < j_p} y_{j_1} \ldots y_{j_p}}, \qquad y_j = \tanh x_j,$$

so

$$\tanh(f(h_1) + \ldots + f(h_k)) = \frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}},$$

and

$$\begin{aligned} s^2 = \sinh^2 h &= \frac{\left(\frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}}\right)^2}{1 - \left(\frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}}\right)^2} \\ &= \frac{\left(\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1} \ldots t_{j_p}\right)^2}{\prod_{j=1}^{k} (1 - \theta^2 t_j^2)}. \end{aligned}$$

Using the symmetry of $\nu_n(h)$ we obtain that

$$\begin{aligned} D_{n-1} &= \sum_h \sinh^2 h \, \nu_{n-1}(h) \\ &= \frac{\sum_{h_1, \ldots, h_k} \left(\sum_{\text{odd } p} \theta^{2p} \sum_{j_1 < j_2 < \ldots < j_p} t_{j_1}^2 \ldots t_{j_p}^2\right) \prod_{j=1}^{k} [(1 - \theta^2 t_j^2)^{-1} a(h_j) \nu_n(h_j)]}{\sum_{h_1, \ldots, h_k} \left(\prod_{j=1}^{k} a(h_j) \nu_n(h_j)\right)}. \end{aligned}$$

Now we substitute

$$t_j^2 = \frac{s_j^2}{1 + s_j^2}$$

and obtain

$$D_{n-1} = \frac{\sum_{h_1, \ldots, h_k} \left(\sum_{\text{odd } p} \theta^{2p} \sum_{j_1 < j_2 < \ldots < j_p} s_{j_1}^2 \ldots s_{j_p}^2 \prod_{j \neq j_1, \ldots, j_p} (1 + s_j^2)\right) \prod_{j=1}^{k} a^{-1}(h_j) \nu_n(h_j)}{\sum_{h_1, \ldots, h_k} \left(\prod_{j=1}^{k} a(h_j) \nu_n(h_j)\right)}. \qquad (4.3)$$

To estimate the RHS of (4.3) we use FKG inequality.

Lemma 4.2 (FKG inequality). *Assume that $b(x)$, $c(x)$ and $d(x)$ are functions on a finite set X such that*

$$\begin{aligned} &d(x) \geq 0 \qquad \forall\, x \in X; \\ &[b(x_1) - b(x_2)]\,[c(x_1) - c(x_2)] \geq 0 \qquad \forall\, x_1, x_2 \in X; \\ &\sum_x d(x) > 0, \qquad \sum_x c(x)d(x) > 0. \end{aligned}$$

Then

$$\frac{\sum_x b(x)c(x)d(x)}{\sum_x c(x)d(x)} \geq \frac{\sum_x b(x)d(x)}{\sum_x d(x)}. \tag{4.4}$$

Proof. We have:

$$\begin{aligned} 0 &\leq \sum_{x_1,x_2} [b(x_1) - b(x_2)]\,[c(x_1) - c(x_2)]d(x_1)d(x_2) \\ &= 2\sum_{x_1} b(x_1)c(x_1)d(x_1)\sum_{x_2} d(x_2) - 2\sum_{x_1} b(x_1)d(x_1)\sum_{x_2} c(x_2)d(x_2), \end{aligned}$$

which implies (4.4). $\diamondsuit$

Applying FKG inequality to the RHS of (4.3), we obtain

$$D_{n-1} \leq \frac{\displaystyle\sum_{h_1,\ldots,h_k} \left(\sum_{\text{odd } p} \theta^{2p} \sum_{j_1<j_2<\ldots<j_p} s_{j_1}^2 \ldots s_{j_p}^2 \prod_{j \neq j_1,\ldots,j_p} (1 + s_j^2) \right) \prod_{j=1}^{k} \nu_n(h_j)}{\displaystyle\sum_{h_1,\ldots,h_k} \left(\prod_{j=1}^{k} a^2(h_j)\nu_n(h_j) \right)},$$

which gives

$$D_{n-1} \leq \frac{\displaystyle\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1 + D_n)^{k-p}}{[1 + (1 - \theta^2)D_n]^k}, \qquad C_k^p = \frac{k!}{p!(k-p)!}. \tag{4.5}$$

Lemma 4.3. *If $k\theta^2 \leq 1$ then*

$$\frac{\displaystyle\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1 + D_n)^{k-p}}{[1 + (1 - \theta^2)D_n]^k} \leq k\theta^2 D_n. \tag{4.6}$$

Proof. Denote for a fixed $D_n > 0$,

$$A(\theta) = \frac{\displaystyle\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1 + D_n)^{k-p}}{k\theta^2 D_n [1 + (1 - \theta^2)D_n]^k} = \frac{\displaystyle\sum_{\text{odd } p} C_k^p \theta^{2(p-1)} D_n^{p-1} (1 + D_n)^{k-p}}{k[1 + (1 - \theta^2)D_n]^k}$$

Then (4.6) is equivalent to

$$A(\theta) \le 1.$$

Observe that $A(\theta)$ is an increasing function of θ, so it is sufficient to prove that

$$A(k^{-1/2}) \le 1. \tag{4.7}$$

By the Newton binomial formula,

$$\begin{aligned} A(k^{-1/2}) &= \frac{\sum\limits_{\text{odd } p} C_k^p k^{-p} D_n^p (1+D_n)^{k-p}}{D_n[1+(1-k^{-1})D_n]^k} \\ &= \frac{(1+D_n+k^{-1}D_n)^k - (1+D_n-k^{-1}D_n)^k}{2D_n[1+(1-k^{-1})D_n]^k} \\ &= \frac{\left(\dfrac{1+(1+k^{-1})D_n}{1+(1-k^{-1})D_n}\right)^k - 1}{2D_n}. \end{aligned} \tag{4.8}$$

Put

$$z = \frac{1+(1+k^{-1})D_n}{1+(1-k^{-1})D_n} - 1 = \frac{2D_n}{k+(k-1)D_n}.$$

Then

$$0 < z < \frac{2}{k-1}, \qquad \text{when} \qquad 0 < D_n < \infty.$$

In addition,

$$D_n = \frac{kz}{2-(k-1)z},$$

and (4.7) is equivalent (due to (4.8)) to

$$(1+z)^k - 1 \le 2D_n = \frac{kz}{1-(k-1)z/2}.$$

Expanding the both sides of the last inequality in the Taylor series in z we reduce it to

$$kz + \frac{k(k-1)}{2}z^2 + \frac{k(k-1)(k-2)}{6}z^3 + \ldots \le kz + \frac{k(k-1)}{2}z^2 + \frac{k(k-1)^2}{4}z^3 + \ldots, \tag{4.9}$$

which is obvious. Thus (4.7) is proved and this finishes the proof of Lemma 4.3. $\diamondsuit$

From (4.5) and Lemma 4.3 we obtain (4.1). Lemma 4.1 is proved. $\diamondsuit$

Now we observe that (4.9) can be strengthened to

$$kz + \frac{k(k-1)}{2}z^2 + \frac{k(k-1)(k-2)}{6}z^3 + \ldots \le \frac{kz + \dfrac{k(k-1)}{2}z^2 + \dfrac{k(k-1)^2}{4}z^3 + \ldots}{1 + \dfrac{(k-1)^2}{12}z^2},$$

which leads to

$$(1+z)^k - 1 \leq \frac{2D_n}{1 + \dfrac{(k-1)^2}{12}\left(\dfrac{2D_n}{k+(k-1)D_n}\right)^2}$$

and then we get the following

Lemma 4.4.

$$D_{n-1} \leq \frac{k\theta^2 D_n}{1 + \dfrac{(k-1)^2}{3}\left(\dfrac{D_n}{k+(k-1)D_n}\right)^2}. \tag{4.10}$$

This is useful when $k\theta^2 = 1$.

5. *Proof of the Theorem.* The inequality (4.10) says that

$$\lim_{N-n\to\infty} D_n = 0 \tag{5.1}$$

for $0 < \theta \leq 1/\sqrt{k}$. Following the line of reasoning of Section 3 from [1], one can now check that (5.1) implies the extremality of the disordered state $\mu^{\#}$. This means that for any $\varepsilon > 0$, $n > 0$ and any configuration $\sigma_n \subset \Sigma(V_n)$, there exist $N > n$ and a set $\Omega_N \subset \Sigma(W_N)$ such that

(i) $\mu^{\#}(\Omega_N) > 1 - \varepsilon$

(ii) $|\mu^{\#}(\sigma_n \mid \sigma^{(N)}) - \mu^{\#}(\sigma_n)| < \varepsilon$, $\forall \sigma^{(N)} \in \Omega_N$

This gives our main result. $\diamondsuit$

One of the open problems remains about the purity of the limiting Gibbs states in the case of nonzero external field. Another open problem is the characterization of the limiting states for the random field model. The ground states for this model for binary distribution were examined in [8].

Acknowledgements. One of us (P.M.B.) would like to express his sincere gratitude to Hans-Otto Georgii for pointing out a gap in the original proof [1] of the main theorem of the present paper. A corrected version of the proof for $k \leq 6$ was given then in the letter of P.M.B. to H.-O. Georgii [6] and the general case is considered here. P.M.B. thanks the Centre de Physique Théorique–Marseille for hospitality and the Université de Provence for financial support during his visit to CPT–Marseille where this work was done.

References.

1. P.M. Bleher, *Extremity of the disordered phase in the Ising model on the Bethe lattice*, Commun. Math. Phys. **128**, 411-419 (1990).

2. T. Moore, J.L. Snell, *A branching process showing a phase transition*, J. Appl. Prob. **16**, 252-260 (1979).

3. H.-O. Georgii, *Gibbs Measures and Phase Transitions* (De Gruyter Studies in Math., vol. 9) Berlin-New York: De Gruyter 1998.

4. Y. Higuchi, *Remarks on the limiting Gibbs states on a $(d+1)$-tree*, Publ. RIMS Kyoto University **13**, 335-348 (1977).

5. P.M. Bleher and N.N. Ganihodgaev, *On pure phases of the Ising model on the Bethe lattice*, Theory Prob. Appl. **35**, 1-26 (1990).

6. P.M. Bleher, *Letter to H.-O. Georgii from January 21, 1992*.

7. J.M. Carlson, J.T. Chayes, L. Chayes, P. Shetna, D.J. Thouless, *Critical behavior of the Bethe lattice spin-glass*, Europhys. Lett. **5**, 355-360 (1989).

8. R. Bruinsma, *Random field Ising model on a Bethe lattice*, Phys. Rev. B **30**, 289-299 (1984).