Approximate symmetries in dynamical systems

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Summary.

The knowledge of exact symmetries of a differential problem allows to reduce it, and sometimes to completely solve it; it also allows to obtain exact solutions. If we are looking for approximate rather than exact solutions, then approximate symmetries are as good as exact ones; moreover, they can be determined perturbatively. In this paper we introduce and study approximate symmetries, together with some applications to the determination of approximate solutions of dynamical systems.

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1. Symmetries and approximate symmetries for dynamical systems

Let us consider an autonomous dynamical system (DS) of the form

$$\dot{x} = f(x) \tag{1}$$

with $x \in M \subseteq \mathbf{R}^n$, $f : M \to TM$. Under a coordinate transformation generated by $s : M \to TM$, i.e.

$$x \to \widetilde{x} = x + \varepsilon s(x) \tag{2}$$

this is transformed, in accordance with prolongation formula [1-3], to a new DS $\dot{\widetilde{x}} = \widetilde{f}(\widetilde{x})$ where

$$\widetilde{f} = f + \varepsilon\{f, s\} \tag{3}$$

and where we have introduced the Lie-Poisson bracket

$$\{\varphi, \psi\} := (\varphi \cdot \nabla)\psi - (\psi \cdot \nabla)\varphi \ . \tag{4}$$

The Lie-point time-independent (LPTI) symmetries of (1) are therefore given by vector fields (VF) of the form

$$\sigma := s(x) \cdot \nabla \tag{5}$$

with s solution of

$$\{f, s\} = 0 \ . \tag{6}$$

If $\varphi := f(x) \cdot \nabla$ is the VF corresponding to (1), i.e. the generator of the dynamical flow, then (6) is just the commutation condition

$$[\varphi, \sigma] = 0 \ . \tag{7}$$

In order to determine explicitely σ, we should solve (6), i.e. a set of linear non-homogeneous PDEs for $s(x)$ (the *determining equations*),

$$f^j(x)\frac{\partial s^i}{\partial x^j} = \frac{\partial f^i}{\partial x^j} s^j \tag{8}$$

Such equations can be studied by the method of characteristics [4,5], or by means of other procedures (see e.g. [2,6,7]); suppose anyway we want to solve them perturbatively. In particular, assume that we have an isolated fixed point x_0 (we can assume $x_0 = 0$) for (1), and expand f and s around this (notice that necessarily $s(0) = 0$ as well); i.e. let us write

$$f(x) = \sum_{m=0}^{\infty} f_m(x) \tag{9}$$

$$s(x) = \sum_{m=0}^{\infty} s_m(x) \tag{10}$$

where f_m, s_m are homogeneous polynomials of degree $m+1$ (issued by Taylor expansion in our case). Writing (6) in terms of these, we get a series of equations

$$\sum_{j=0}^{k} \{f_j, s_{k-j}\} = 0 \qquad k = 0, 1, 2, ... \tag{11}$$

Notice that if we have solved the equations in this sequence for $k \le k_0 - 1$, i.e. have determined $s_0, ..., s_{k_0 - 1}$, the k_0 equation reads

$$\{f_0, s_{k_0}\} = - \sum_{j=1}^{k_0} \{f_j, s_{k_0 - j}\} := F_{k_0}(x) \tag{12}$$

where the r.h.s. is a known function (cf. [8]).

It is maybe worth remarking explicitely that $\{f_0, .\}$ is known in Normal Form (NF) theory as the *homological operator* associated to (the linear part of) f [5,9,10]. We will write, for $f_0(x) = Ax$,

$$\{f_0, .\} := \mathcal{L}_A(.) \tag{13}$$

so that (12) is an equation of the form

$$\mathcal{L}_A(s_k) = F_k(x) \tag{14}$$

which can be solved if and only if $F_k(x) \in \mathrm{Ran}(\mathcal{L}_A)$; moreover s_k is determined up to a function in $\mathrm{Ker}(\mathcal{L}_A)$. For the relation of this approach to the Poincaré-Dulac procedure for transforming a DS (equivalently, a VF) into its NF, see [6, 9-14]; this relation will also be discussed in the following.

A little thinking, or some attempts to consider concrete cases, shows that if we choose a $s_0 \in \mathrm{Ker}(\mathcal{L}_A)$ (notice the first of (11) is just $\mathcal{L}_A(s_0) = 0$ or, with $s_0(x) = Sx$, simply $[A, S] = 0$), in general we can solve the equations (11), (14) only up to some order k; i.e., at some $k + 1$ the F_{k+1} will not belong to $\mathrm{Ran}(\mathcal{L}_A)$ and (14) will not admit a solution. In this case, we have an **approximate symmetry** of order k.

Let us write, for given f and *general* s,

$$\{f, s\} := r \quad ; \quad r(x) = \sum_{m=0}^{\infty} r_m(x) \tag{15}$$

With this notation, we have the following

Definition. *If in (15) we have $r_m = 0 \ \forall m \le k$, then $\sigma = s(x) \cdot \nabla$ is an approximate symmetry of order k for (1).*

The purpose of this note is indeed to discuss approximate symmetries, and their use in the study of DS.

Approximate symmetries were introduced and studied in [15] (see also [16]), but here we will proceed in an indipendent way and with different goals. Our work will be quite related to previous work on NF theory [8,14,17], and we have indeed tried to keep to a notation similar to the one used there.

2. The set of approximate symmetries of a dynamical system

Let f be given, and let $\sigma_1 = s_1(x) \cdot \nabla$ and $\sigma_2 = s_2(x) \cdot \nabla$ be approximate symmetries of the same order k for $\varphi := f(x) \cdot \nabla$. We consider

$$[\sigma_1, \sigma_2] \equiv \sigma := s(x) \cdot \nabla \quad ; \quad s(x) = \sum_{m=0}^{\infty} s_m(x) \tag{16}$$

and would like to know if σ is also an approximate symmetry. The commutator

$$[\varphi,\sigma] \equiv \rho = r(x)\cdot\nabla \quad ; \quad r(x) = \sum_{m=0}^{\infty} r_m(x) \tag{17}$$

is also written, using Jacobi identity,

$$\rho = [\sigma_1,[\varphi,\sigma_2]] - [\sigma_2,[\varphi,\sigma_1]] \equiv [\sigma_1,\chi_2] - [\sigma_2,\chi_1] \tag{18}$$

Now, both χ_1 and χ_2 have only terms of order greater than k, so that $r_m(x) = 0\ \forall m \leq k$. We have therefore proved the following lemma I (lemma II is obvious from the definition)

Lemma I. The set $\mathcal{G}_\varphi^{(k)}$ of approximate symmetries of order k for a VF φ has the structure of a Lie algebra.

Lemma II. The chain of Lie algebras $\mathcal{G}_\varphi^{(k)}$ obeys $\mathcal{G}_\varphi^{(k+1)} \subseteq \mathcal{G}_\varphi^{(k)}$.

Let I_φ be the algebra of invariant functions under φ, i.e. of functions ζ such that $\varphi\cdot\zeta = 0$, or

$$f^j(x)\frac{\partial\zeta}{\partial x^j} = 0 \tag{19}$$

¿From the point of view of the DS (1), I_φ is just the algebra of constants of motion.

Let $\mathcal{G}_\varphi$ be the algebra of exact symmetries for φ; it is well known that [18]:

Lemma III. The set $\mathcal{G}_\varphi$ has, beyond the structure of Lie algebra, the structure of a *module* over the algebra I_φ.

We can define **approximate constants of motion** by considering functions $\zeta : M \to \mathbf{R}$ and their expansion around $x = 0$ (up to an additional constant $\zeta(0)$)

$$\zeta(x) = \sum_{m=0}^{\infty} z_m(x) \tag{20}$$

The derivative of ζ along the flow of φ is given by

$$(f\cdot\nabla)\zeta = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(f_k\cdot\nabla)z_{m-k}(x) \equiv \sum_{m=0}^{\infty} w_m(x) \tag{21}$$

Let us now consider $\sigma = (s(x)\cdot\nabla)$, and a function $\zeta : M \to \mathbf{R}$; consider $\widetilde{\sigma} = (\zeta(x)s(x)\cdot\nabla)$. We have

$$\{f,\zeta s\} = [(f\cdot\nabla)\zeta]\, s + \zeta(f\cdot\nabla)s - \zeta(s\cdot\nabla)f = [(f\cdot\nabla)\zeta]\, s + \zeta\{f,s\} \tag{22}$$

This suggests the following definition:

Definition. *If in (21) we have $w_m(x) \equiv 0\ \forall m \leq k$, we say that ζ is an approximate constant of motion of order k for (1).*

Let us denote by $I_\varphi^{(k)}$ the set of approximate constants of motion of order k for f, $\varphi := (f\cdot\nabla)$. Clearly, we have that the product of ζ_1,ζ_2 in $I_\varphi^{(k)}$ is still in $I_\varphi^{(k)}$, i.e. we have

Lemma IV. The set $I_\varphi^{(k)}$ is an (abelian) algebra, and the chain of algebras $I_\varphi^{(k)}$ obeys $I_\varphi^{(k+1)} \subseteq I_\varphi^{(k)}$.

The following lemma is an immediate consequence of (21):

Lemma V. The set $\mathcal{G}_\varphi^{(k)}$ has, beyond the structure of Lie algebra, the structure of a *module* over the algebra $I_\varphi^{(k)}$.

Let us also remark that if a VF σ has $s_m = 0 \ \forall m \le k$, then it is automatically a (trivial) approximate symmetry of order k for any DS. Similarly, any function ζ such that $\zeta(0)$ =const (possibly zero) and $z_m = 0$ for $m \le k$ will automatically be a (trivial) constant of motion of order k. Thus, approximate symmetries (constant of motions) are only determined up to such trivial ones. In the following we will only consider nontrivial approximate symmetries and constants of motions.

3. Approximate symmetries and normal forms

As recalled in sect.1, the construction of approximate symmetries bears some resemblance with the construction of normal forms (NF); in this section we will discuss the relations between the two procedures (for details on NF theory, we refer to [5,9,10], see also Sect. 6).

In order to fix notation, let us sketchily recall the Poincaré- Dulac procedure to transform the DS (1) into NF. For the sake of simplicity, we will make the assumption - common in NF theory - that

$$[A, A^+] = 0 \tag{23}$$

where A is the matrix defined by $(Df)(0) \equiv A$, as in sect.1 (i.e., $f_0(x) = Ax$). Most of our results hold even without (23), but would then require an heavier notation; see [17] for details.

Let us explicitly remark that, as usual in NF theory, the series of transformations we are dealing with are in general purely *formal* series, i.e. no assumption is made on their convergence. A complete discussion and a list of results concerning the convergence and analyticity properties of these series can be found in [10]. When the NF transformation is not convergent, we have to operate a partial NF transformation, i.e. transform the DS into NF only up to some finite order[(1)] k; it should not be surprising to remark that approximate symmetries of order k are in this case essentially equivalent to exact symmetries, as far as the NF is concerned; see also the remark at the end of the present section.

It should be stressed that the relation between symmetry and convergence of the NF transformation has attracted attention in recent times [19,20]; in particular, Bruno and Walcher [20] have proved for an ample class of DS in R^2 the remarkable result that the system admits a convergent transformation into NF if and only if it admits a local one-parameter group of symmetries (the results can presumably be generalized to higher dimensions).

Writing $f(x)$ in the form (9), we look for a series of near-identity coordinate transformations

$$x \to \widetilde{x} := x + h_k(x) \tag{24}$$

(1) More generally, in any concrete computation we can implement the Poincaré- Dulac algorithm only up to finite orders.

with h_k homogeneous of order $(k+1)$; transformations (24) should be performed successively for $k = 2, 3, ...$ (we do not discuss linear changes of coordinates).

Under (24), f is transformed to a new $\widetilde{f} = \sum \widetilde{f}_m$, and we have in particular

$$\begin{aligned} \widetilde{f}_m =& f_m \qquad \text{for } m < k \\ \widetilde{f}_k =& f_k + \mathcal{L}_A(h_k) \end{aligned} \tag{25}$$

while the $\widetilde{f}_m$ with $m > k$ are changed in a complicate way. Assumption (23) guarantees $[\mathrm{Ran}(\mathcal{L}_A)]^c = \mathrm{Ker}(\mathcal{L}_A)$, so that by repeatedly applying (24) we can transform (1) into a DS

$$\dot{x} = g(x) = \sum_{m=0}^{\infty} g_m(x) \tag{26}$$

in such a way that $g_0(x) = Ax$ and all the g_m satisfy

$$\mathcal{L}_A(g_m) = 0 \quad \text{or equivalently} \quad \{Ax, g_m(x)\} = 0 \tag{27}$$

Obviously, the same as (27) also holds for g *tout court*,

$$\mathcal{L}_A(g) = 0 \quad \text{or equivalently} \quad \{Ax, g(x)\} = 0 \tag{28}$$

If A admits *no* resonant relations among its eigenvalues [5,9,10] (see also [8,17]), then $\mathrm{Ker}(\mathcal{L}_A)$ is trivial, and (26) is a *linear system*, i.e. $g(x) \equiv Ax$.

Notice that if $\varphi = (f \cdot \nabla)$ admits some symmetry $\sigma = (s \cdot \nabla)$, the geometric relation $[\varphi, \sigma] = 0$ is obviously invariant under changes of coordinates, and therefore under (24), but the expression of s in coordinates will change, similarly to what happens for f. Indeed, under (24), we have $\widetilde{s}_m = s_m$ for $m < k$, and

$$\widetilde{s}_k = s_k + \mathcal{L}_S(h_k) \tag{29}$$

where $S = (Ds)(0)$, or $s_0(x) = Sx$, as already defined in Sect.1, and $\mathcal{L}_S$ is the homological operator associated to S: i.e. $\mathcal{L}_S(.) = \{s_0, .\}$ (cf. (13)). Nevertheless we stress that, if we write $\psi = (g \cdot \nabla)$, then the algebras $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$ of symmetries of (1) and of (26) are just the same algebra.

If (26) reduces to the linear form (independently of any assumption on the presence of resonances for the linear part A), i.e. it is just

$$\dot{x} = Ax \tag{30}$$

then the algebra $N(A)$ of matrices commuting with A is a symmetry algebra for (30): indeed any matrix $S \in N(A)$ defines a linear VF $\sigma = (Sx \cdot \nabla)$ which is a symmetry for the DS. We can easily extract from this an abelian algebra of n independent symmetries: e.g. the algebra of matrices $A^m, m = 1, ..., n$ if they are linearly independent, so that the $A^m x$ span a basis of $\mathbf{R}^n$ for any $x \neq 0$ [(2)]. Another (isomorphic) $n-$dimensional abelian algebra of symmetries is provided, if the eigenvalues of A are real, by the dilations

[(2)] Notice that, if the $A^m, m = 1, ..., n$ are not independent, the system (30) is reducible, thanks to (23), i.e. splits into independent DS of smaller dimension.

$\rho_j = y_j(\partial/\partial y_j)$ (no sum over j) along each eigenvector $y_j,\ j = 1, \ldots, n$. We can partially reverse this result, precisely we get (see also [17]):

Proposition 1. *If a DS can be linearized, then it admits n independent commuting symmetries, which can be simultaneously taken into linear form by a coordinate transformation. Conversely, if there is a coordinate system in which the DS admits n linearly independent linear commuting symmetries $\sigma_j = (S_j x \cdot \nabla)$ such that all the matrices S_j are diagonalizable, then the DS can be linearized.*

Proof. In the coordinates in which the system is linear, it is easy to construct n linear independent commuting symmetries. Conversely, given n commuting symmetries $\sigma_j = (S_j x \cdot \nabla)$, the n commuting matrices S_j can be simultaneously diagonalized: $S_j \to \mathrm{diag}(\beta_1^{(j)}, \ldots, \beta_n^{(j)})$; now, with respect to the basis spanned by the n (independent) vectors $\beta^{(j)} \equiv (\beta_1^{(j)}, \ldots, \beta_n^{(j)})$, the symmetries $\sigma_j = (S_j x \cdot \nabla)$ become just the n independent dilations along the directions $\beta^{(j)}$. A DS admitting these n symmetries is necessarily linear.

If (the linear part A of) the DS (1) is resonant, we cannot transform, in general, (1) to the linear NF (30), but only to the general NF (26),(28). However, if the lowest order of resonances in A is $(k+1)$, then (26) will be written as

$$\dot{x} = Ax + \sum_{m=k+1}^{\infty} g_m(x) \tag{31}$$

so that the NF is linear up to order k. In this case, any matrix $S \in N(A)$ will define a VF $\sigma = (Sx \cdot \nabla)$ which is an *approximate* linear symmetry of order k for (31). This corresponds to the fact that (31) is linear, and therefore integrable, up to order k.

We have therefore proved the:

Proposition 2. *If there is a coordinate system in which (1) admits n linearly independent linear abelian approximate symmetries of order k, $\sigma_j = (S_j x \cdot \nabla)$ such that the matrices S_j are diagonalizable, then the NF of the DS is linear up to order k. In particular, if A^m, $m = 1, ..., n$ are linearly independent, then the NF for (1) is linear up to order k if and only if there is a coordinate system in which the $(A^m x \cdot \nabla)$ are approximate symmetries of order k for (1).*

Notice that in this statement, no mention is made of the orders at which A admits resonances. Indeed, if r is the lowest order at which A has resonances, the above proposition is trivial for $k \leq r$, but it does also hold for $k > r$. Actually, a specific DS will have a specific NF, and some of the resonance terms allowed by (26)-(28) could be absent. In general, we could say that while exact symmetries determine the structure (26)-(28) of general NF expansion, approximate symmetries give a more precise determination of lowest order terms in this expansion. In particular, once we have determined the exact symmetries of (1) and therefore the general structure of the NF admitted by such symmetries - which is in general more restrictive than (26)-(28), see [8,17] - we can further restrict the *a priori* NF (i.e. without actually performing the NF reduction) by considering approximate symmetries.

The following proposition is a trivial extension of a result in [8] and can be proved along the lines of the original result. Similar extensions could be obtained for all the results contained in [8] and [12,17], again with proofs equivalent to the ones for exact symmetries.

Proposition 3. *If the DS $\dot{x} = f(x)$ admits the approximate symmetry $\sigma = (s \cdot \nabla)$ of order k, and $f(0) = s(0) = 0$; $(Df)(0) = A$, $(Ds)(0) = S$, and S is assumed to satisfy the condition $[S, S^+] = 0$* [(3)]*, then $\varphi = (f \cdot \nabla)$ and σ can be put in* **Joint Normal Form** *up to order k. That is, there is a series of Poincaré-Dulac changes of coordinates such that in the final coordinates $\varphi = (g(x) \cdot \nabla)$, $\sigma = (v(x) \cdot \nabla)$, and for $m \leq k$*

$$
\begin{aligned}
g_m \in & \mathrm{Ker}(\mathcal{L}_A) \cap \mathrm{Ker}(\mathcal{L}_S) \\
v_m \in & \mathrm{Ker}(\mathcal{L}_A) \cap \mathrm{Ker}(\mathcal{L}_S)
\end{aligned}
$$

This result - and extensions of other results to approximate symmetries - tell something quite intuitive: *if we want to determine the truncation of the NF to order k, then approximate symmetries of order k are as good and useful as exact symmetries.*

In this respect, it should be observed that in actual computations one does always consider truncations of the NF: indeed, even if (1) presents only finitely many terms, in general its NF (26) is an infinite series, so that we have to truncate it if we want to perform computations for practical purposes. Beside this, (26) is in general only a formal series; one is then particularly interested in its asymptoticity [5,9,10], which also leads to consider truncations of (26) rather than the full (infinite) formal series. As mentioned in the beginning of the present section, in these cases we have to consider only a partial NF transform, so that the constraints posed on the NF by an exact symmetry and by an approximate symmetry of sufficiently high order (at least of the same order as the truncation of the NF) are exactly equivalent. Approximate symmetries of lower order will also impose extra constraint on the NF unfolding, and this *a priori* constraint on the explicit form of the DS once transformed into NF can simplify the implementation of the Poincaré- Dulac algorithm. *This shows that approximate symmetries should be given (at least) the same relevance as exact symmetries in NF theory.*

4. Approximate symmetries and approximate solutions to dynamical systems

The knowledge of symmetries of a DS allows to obtain new solutions from known ones, by just applying symmetry transformations; this fact can also be taken as a definition of symmetry (i.e., a symmetry is a transformation which takes solutions into solutions).

We are considering in this paper LPTI symmetries, i.e. symmetries of the form (5), which act on the x alone and do not transform - neither depend on - the time coordinate.

Let the solution to (1) with initial datum $x(t_0) = x_0$ be given by $x(t) = \Phi(t; x_0)$, and let $T_\varphi^{(\tau)}$ be the operator of advance in time along (1), i.e.

$$
T_\varphi^{(\tau)} \Phi(t; x_0) = \Phi(t + \tau; x_0) \tag{32}
$$

(3) As for the analogous condition (23) for A, we introduce here this condition just for simplicity. All the results hold, suitably modified, even for generic A and S; see [17].

If $\sigma = (s(x) \cdot \nabla)$ is a symmetry of (1), $\sigma \in \mathcal{G}_\varphi$, and $\Lambda^{(a)}$ the transformation corresponding to

$$\Lambda^{(a)} x = e^{a\sigma} x \qquad a \in \mathbf{R} \tag{33}$$

then, on solutions, this means that if $x(t) = \Phi(t; x_0)$ is a solution to (1), also $y(t) = \Lambda x(t)$ is a solution (we write Λ for $\Lambda^{(a)}$, for ease of notation), i.e.:

$$y(t) := \Lambda \Phi(t; x_0) = \Phi(t; \Lambda x_0) \equiv \Phi(t; y_0) \tag{34}$$

where

$$y_0 = \Lambda x_0 \ . \tag{35}$$

Let us now consider $\sigma \in \mathcal{G}_\varphi^{(k)}$, i.e. an *approximate* symmetry of order k for (1). Let again consider a solution $x(t) = \Phi(t; x_0)$, and let $\Lambda = e^{a\sigma}$ denote a transformation on M. We denote by Ψ the Λ transform of Φ, i.e.

$$y(t) = \Lambda x(t) = \Lambda \Phi(t; x_0) = \Psi(t; y_0) \tag{36}$$

Since σ is *not* an exact symmetry of φ, we have in general

$$\Psi(t; y_0) \neq \Phi(t; y_0) \tag{37}$$

On the other side, since σ is an approximate symmetry, we expect that $\Psi(t; y_0)$ is an approximation of $\Phi(t; y_0)$. If this is the case, knowledge of the solution issuing from x_0 and of an approximate symmetry, gives an approximation to other solutions. Let us now write, for ease of notation, $\Phi_t(.)$ for $\Phi(t, .)$, and similarly $\Psi_t(.)$ for $\Psi(t, .)$. We consider, for an initial point $y_0 = \Lambda x_0$, the exact solution $y(t)$, and the curve $\widetilde{y}(t)$ obtained by applying the transformation Λ (which will be thought as an approximate symmetry) to the exact solution $x(t) = \Phi_t(x_0)$; these curves are

$$y(t) = \Phi_t(y_0) \quad , \quad \widetilde{y}(t) = \Psi_t(y_0) \equiv \Lambda \Phi_t \Lambda^{-1}(y_0) \tag{38}$$

We are interested in the error we have if considering $\widetilde{y}(t)$ instead than $y(t)$, i.e. we want to know in which sense $\widetilde{y}(t)$ is an approximate solution. We have therefore to consider the operator

$$\Theta_t \equiv \Phi_t - \Psi_t = \Phi_t - \Lambda \Phi_t \Lambda^{-1} \equiv [\Phi_t, \Lambda] \ \Lambda^{-1} \tag{39}$$

The commutator appearing here can be written as

$$[\Phi_t, \Lambda] = \left[e^{a\sigma}, e^{t\varphi}\right] = \sum_{m,n} \frac{t^m a^n}{m!n!} \left[\varphi^m, \sigma^n\right] \tag{40}$$

and if $[\varphi, \sigma] \simeq O(x^k)$, each of these commutators contributes with $(m \cdot n)$ terms $O(x^k)$, so we get as a rough estimate

$$[\Phi_t, \Lambda] \simeq \sum_{m,n} \frac{t^m a^n}{m!n!} mn O(x^k) \simeq e^{at} O(atx^k) \tag{41}$$

Thus, for small t, the error $w(t) = |\widetilde{y}(t) - y(t)|$ grows with a speed of order $|\Lambda x_0|^k = |y_0|^k$.

This could be better illustrated by some explicit computation on a concrete example. Let us consider the very simple system

$$\begin{aligned} \dot{x}_1 &= -\, x_1 + x_2^k \\ \dot{x}_2 &= -\, x_2 \end{aligned} \tag{42}$$

this admits rotations as an approximate symmetry of order $(k-1)$: we compute e.g. a solution $x(t)$ with initial datum $x(0) = (1,0)$; we then consider an initial condition $y(0)$ obtained by rotating $x(0)$ through a (small) angle θ; we then compute the solution $y(t)$ with initial datum $y(0)$, and the curve $\widetilde{y}(t)$ obtained by rotating through the same angle θ the curve $x(t)$. It is easily seen that in this case the error $w(t)$ is (uniformly, with respect to t) of order $O(|\theta|^k)$. It can be directly verified (possibly by means of numerical computations) that, in many cases, the solution $\widetilde{y}(t)$ obtained in this way provides in fact a quite good approximation of the exact solution $y(t)$: this is particularly true especially if the DS admits some (approximate) constants of motion and/or it has a Hamiltonian structure. In this case, actually, due to the presence of the conserved quantity and the oscillating nature of solutions, we should rather apply averaging and Nekhoroshev-like theorems [5,9,21] to provide estimates, but this is beyond the scope of the present paper.

5. Approximate symmetries and perturbation of dynamical systems

Considerations similar to the ones presented above also apply if we have a different kind of "perturbation" problem: suppose that (1) is a known "integrable" system, and $\widetilde{f}$ is another problem, which we could consider as a perturbation (not necessarily "small") of (1), i.e.

$$\dot{x} = \widetilde{f}(x) = f(x) + a r(x) \tag{43}$$

If we are able to solve the equation for s

$$\{f(x), s(x)\} = r(x) \tag{44}$$

then (approximate) solutions of (43) can be obtained by applying the appropriate symmetry operator $\Lambda = e^{a\sigma}$, $\sigma = (s \cdot \nabla)$, to solutions of (1). In the notation of the previous section, $\Phi(t; x_0)$ will now be the solution to $\dot{x} = f(x)$, i.e. to (43) for $a = 0$, and $\Psi(t; x_0)$ will be solution to $\dot{x} = \widetilde{f}(x)$.

The idea is then that if we have an approximate symmetry of the system $\dot{x} = f(x)$ which satisfies (44), once applied to $\Phi(t; x_0)$ this will generate a curve $\Psi(t; y_0)$ which is an approximation of the curve $\Phi(t; y_0)$, but which provides a solution to (43). Notice that if s satisfies (44) exactly, we have in this way an exact solution to (43), while if (44) is satisfied only up to some finite order (as it will generally be the case) we have in this way an approximate solution to (43).

In other words, operating with a transformation Λ^{-1}, studying time evolution, and then applying again Λ, is equivalent to apply a coordinate transformation Λ^{-1}, study time evolution in the new coordinates, and then passing back to the old ones by the transformation Λ: this is precisely the approach used in NF theory: in fact, as we have already stressed, approximate symmetry theory (and computations) bears a strong resemblance with NF theory (and computations); see also Sect. 6 for further clarifications on this point. Indeed, (44) is just the homological equation we should solve to eliminate the terms $r(x)$.

Clearly, a special but important case is obtained when (1) is a linear system, see (30), corresponding to the linear part of (43); this corresponds to the discussion in Sect. 3. Indeed, such a σ would be an approximate symmetry for $\widetilde{f}$, allowing for elimination of low-order nonlinear terms.

Applying an approximate symmetry $\sigma = (s \cdot \nabla)$ of order k to a DS (VF) f, we obtain a new DS (VF) $\widetilde{f} = f + ar$ as in (1)-(3). We wonder if σ is also an approximate symmetry for $\widetilde{f}$: precisely, let $r_m = 0$ $\forall m \le k$, so that a $s(x)$ solution to (44) is an approximate symmetry of order k for (1), $\sigma \in \mathcal{G}_\varphi^{(k)}$; consider

$$\{f + ar, s\} \equiv v(x) = \sum_{m=0}^{\infty} v_m(x) \tag{45}$$

recalling that s is a solution to (44), we have

$$v(x) = r(x) + a\{r, s\} \tag{46}$$

and, since $r_m = 0$ $\forall m \le k$, we also have

$$v_m(x) = 0 \quad \forall m \le k \tag{47}$$

In this way we have also proved the following general result:

Lemma VI. Consider a VF φ and the algebra $\mathcal{G}_\varphi^{(k)}$; let $G^{(k)}(\varphi)$ be the set of all the VF obtained by applying $\Lambda = e^{a\sigma}$ to φ, with $a \in \mathbf{R}$, $\sigma \in \mathcal{G}_\varphi^{(k)}$. Then, for any $\psi = e^{a\sigma}\varphi \in G^{(k)}(\varphi)$, we have $\mathcal{G}_\varphi^{(k)} \subseteq \mathcal{G}_\psi^{(k)}$.

The corollary below follows then from applying the lemma to ψ rather than to φ, or from the algebra structure of $\mathcal{G}_\varphi^{(k)}$:

Corollary. In lemma VI we actually have $\mathcal{G}_\varphi^{(k)} = \mathcal{G}_\psi^{(k)}$.

Therefore, if $\widetilde{f}$ is obtained from f by the action of $\sigma = (s \cdot \nabla)$ and the first nonzero nonlinear terms in $\widetilde{f}$ are of order k, then σ is an approximate symmetry of order k for both f and $\widetilde{f}$. Notice that $\widetilde{f}$ could have approximate symmetries of higher order, allowing for linearization up to higher orders according to sect.3.

6. Determination of approximate symmetries and reduction to normal form

We would like, also in view of the considerations presented in the previous section, to come back to the relations existing between approximate symmetries and reduction to normal forms.

We recall that, under the Poincaré-Dulac transformation (24) of order k, terms f_m with $m < k$ are not changed, while f_k is changed according to (25), i.e.

$$f \to \widetilde{f} = f + \mathcal{L}_A(h_k) + O(k+1) \tag{48}$$

On the other side, under an approximate symmetry transformation $x \to e^\sigma x$, $\sigma \in \mathcal{G}_\varphi^{(k-1)}$, we have

$$f \to \widehat{f} = f + r_k + O(k+1) \tag{49}$$

$$r_k = \sum_{j=0}^{k} \{f_j, s_{k-j}\} \tag{50}$$

If we choose $s_m = 0 \ \forall m \le k-1$, which surely gives a (trivial) approximate symmetry of order $k-1$, then

$$r_k = \{f_0, s_k\} = \mathcal{L}_A(s_k) \tag{51}$$

This can also be read as saying that: *solving the homological equation at order k is equivalent to determining an approximate symmetry of order k with trivial $(k-1)$ first terms.*

Let us now consider the case f is in linear NF up to order $(k-1)$; in this case,

$$r_k = \{f_0, s_k\} + \{f_k, s_0\} \equiv \mathcal{L}_A(s_k) - \mathcal{L}_S(f_k) \tag{52}$$

Also, in this case $s(x) = Sx$, with $[A, S] = 0$, gives an approximate symmetry of order at least $(k-1)$, by definition. We can therefore consider $\sigma \in \mathcal{G}_\varphi^{(k-1)}$ with generator

$$s(x) = Sx + s_k(x) + O(k+1) \tag{53}$$

As shown above, we can choose s_k to be precisely the solution of the homological equation; this permits to eliminate the part of f_k lying in $\mathrm{Ran}(\mathcal{L}_A)$ or, with assumption (23), in $[\mathrm{Ker}(\mathcal{L}_A)]^c$. It is natural to wonder if (52) could not be used to eliminate, or reduce, also the part of f lying in $\mathrm{Ker}(\mathcal{L}_A)$. Indeed, due to

$$[\mathcal{L}_A, \mathcal{L}_S] = \mathcal{L}_{[A,S]} \tag{54}$$

(see [17] for a proof), we have that $[A, S] = 0$ implies

$$\mathcal{L}_S : \mathrm{Ker}(\mathcal{L}_A) \to \mathrm{Ker}(\mathcal{L}_A) \tag{55}$$

Let us decompose f_k as

$$f_k = v_k + t_k \quad ; \quad v_k \in \mathrm{Ran}(\mathcal{L}_A) \ , \ t_k \in \mathrm{Ker}(\mathcal{L}_A) \tag{56}$$

Then, by a Poincaré-Dulac transformation, or by a transformation of the form (53) with $S = 0$, we can eliminate v_k. Let us now try to eliminate t_k; we look therefore for a S in $N(A)$ such that

$$\mathcal{L}_S(t_k) = t_k \tag{57}$$

In particular, consider $[S, S^+] = 0$, so that $\mathrm{Ker}(\mathcal{L}_S)$ and $\mathrm{Ran}(\mathcal{L}_S)$ are complementary subspaces. Notice that we can limit ourselves to considering the action of $\mathcal{L}_S$ on homogeneous polynomials of order k, i.e. we deal with finite-dimensional spaces. The t_k is now given, and we should look for a $S \in N(A)$ such that (57) is verified; again, we can choose a basis $\{S_1, \dots, S_r\}$ in $N(A)$, with $r \le n^2$, and consider a finite dimensional operator $\mathcal{A}_f$ acting on matrices and valued in the space of vector functions, defined by

$$[\mathcal{A}_f(S)] = \{f(x), Sx\} + f(x) \tag{58}$$

If $\mathrm{Ker}(\mathcal{A}_{t_k}) \cap N(A)$ is not empty, then by (57) we can eliminate t_k. Otherways, we can simplify it by choosing an appropriate S; this must be such that t_k is not in $\mathrm{Ker}(\mathcal{L}_A) \cap \mathrm{Ker}(\mathcal{L}_S)$, or we would have no simplification. Notice that if f is in joint normal form with all of its symmetries [8], no further simplification is possible.

7. Examples

Example 1.

We discuss first a very simple example, in order to fix the ideas developed in the previous general discussion, by means of a concrete case.

We consider two-dimensional DS $\dot{x} = f(x)$ of the form

$$\begin{aligned} \dot{x}_1 =& \alpha(r^2)x_1 - \beta(r^2)x_2 + h_{[k]}(x_1, x_2) \\ \dot{x}_2 =& \beta(r^2)x_1 + \alpha(r^2)x_2 + g_{[k]}(x_1, x_2) \end{aligned} \tag{59}$$

where $r^2 = x_1^2 + x_2^2$, α and β are polynomials or, more in general, analytic functions of r^2

$$\alpha(r^2) = \sum_{m=0}^{\infty} \alpha_{2m} r^{2m} \ ; \ \beta(r^2) = \sum_{m=0}^{\infty} \beta_{2m} r^{2m} \tag{60}$$

and $h_{[k]}, \ g_{[k]}$ are polynomials (or analytic functions) such that all terms of degree $< k$ are 0 (we take $k \geq 2$ in the following). In this case it is quite easy to determine the approximate symmetries of the DS, which indeed are immediately evident; we will anyway follow our general procedure in order to illustrate it.

Let us first consider the linear part of the DS, $\dot{x} = Ax$: the matrix A is given by

$$A = \begin{pmatrix} \alpha_0 & -\beta_0 \\ \beta_0 & \alpha_0 \end{pmatrix} \tag{61}$$

and assume $\beta_0 \neq 0$, so that $N(A)$ is spanned by the two (commuting) matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{62}$$

The corresponding VF $\sigma_0 = (Ix \cdot \nabla)$ and $\sigma_1 = (Jx \cdot \nabla)$ (i.e. respectively the dilation and the rotation symmetries) are therefore approximate symmetries of order (at least) one for (59). Actually, it is easy to see that σ_1 is an approximate symmetry of order k as well, where $k+1$ is the order of $h_{[k]}, \ g_{[k]}$ appearing in (59). Indeed, we have

$$\varphi = f(x) \cdot \nabla \ , \ f(x) = \bar{f}(x) + O(k+1) \ , \ \bar{f}(x) := [\alpha(r^2)I + \beta(r^2)J]x \tag{63}$$

so that $\bar{\varphi} := \bar{f}(x) \cdot \nabla = \alpha(r^2)\sigma_0 + \beta(r^2)\sigma_1$, and for a linear VF $\sigma = Sx \cdot \nabla$ we have

$$\{\bar{f}(x), Sx\} = \alpha[I, S]x + \beta[J, S]x - 2(x, Sx)[\alpha' I + \beta' J]x \tag{64}$$

where $(.,.)$ denotes scalar product. Notice that for $S = J$, the scalar product (x, Jx) vanishes, so that indeed σ_1 is a symmetry of $\bar{f}$, which is in fact rotationally symmetric. For $S = I$, i.e. for σ_0, instead, we have an approximate symmetry up to an order τ if α' and β' vanish up to terms of order $(\tau+1)$. If also $\alpha_0 = 0$, then $\rho = r^2$ is an approximate constant of motion of order τ as well. In the general case, for (59) we have $\tau = 2$, but e.g. if $\alpha_2 = \beta_2 = 0$ and $k \geq 4$, then σ_0 is an approximate symmetry (and ρ an approximate constant of motion) up to order $\tau = 4$.

Notice that if $\alpha_0 = 0$ then (59) is in NF up to terms of order $(k+1)$, and linear up to order τ, i.e. up to the order such that we have two linearly independent approximate symmetries of the given order (if instead $\alpha_0 \neq 0$ the NF is in fact linear).

Let us consider VF of the form $\sigma = \theta(r^2)Sx \cdot \nabla \equiv s(x) \cdot \nabla$; in this case we have

$$\{\bar{f}, s\} = \theta\left(\alpha[I,S] + \beta[J,S]\right)x + 2\theta'\alpha r^2 Sx - 2\theta(x, Sx)\left(\alpha' I + \beta' J\right)x \tag{65}$$

For $S = J$ the possibilities to have approximate symmetries of order k are either that $\theta' = 0$, or $\alpha(r^2) = 0$. In the first case we get the linear rotation symmetry $\sigma = \sigma_1$; in the second one, $\rho = r^2$ is an (approximate) constant of motion of order k, then, according to Lemmas III and V, also any $\theta(r^2)Jx \cdot \nabla$ is a symmetry (which can be actually considered as a trivial one, being simply a multiple of the linear rotation σ_1 on each solution).

For $S = I$, i.e. $\sigma = \theta(r^2)x \cdot \nabla$, if $\beta' = 0$ we also have the symmetry given by $\theta = \text{const} \times \alpha$. The same result can be obtained writing the problem, with the notations of (63), in the form $\bar{f} = \beta_0 Jx + \alpha(r^2)x$, and observing that both VFs $\sigma_1 = Jx \cdot \nabla$ and $\bar{\varphi} = \bar{f} \cdot \nabla$ (which is the generator of the dynamical flow) are symmetries for the DS. This clearly implies that $\alpha(r^2)x \cdot \nabla = \bar{\varphi} - \beta_0\sigma_1$ is a symmetry.

Notice that if $\theta = 1 + O(\tau+1)$ we have however (trivial) approximate symmetries of order τ.

Example 2.

The above discussion is helpful in illustrating a simple example, but surely the introduction of approximate symmetries is not of special help in analyzing the simple system (59). The situation is different if we have a system which is equivalent to (59), but is not given in NF. This case will also illusrate the connection between approximate symmetries and reduction to NF.

Let us consider a system of the general form

$$\begin{aligned} \dot{x} =& f_1(x,y) \\ \dot{y} =& f_2(x,y) \end{aligned} \tag{66}$$

where

$$\begin{aligned} f_1(x,y) =& ax - x^2 + cx^3 - x^4 - bx^4 - y - bxy + 2x^2y - bx^2y + cx^3y + 2bxy^2 + cxy^2 \\ & - 4x^2y^2 - 3bx^2y^2 - by^3 - 4bxy^3 + cxy^3 - 2by^4 \\ f_2(x,y) =& x + bx^3 + by - xy + cx^2y + bx^3y - by^2 + 2xy^2 + bxy^2 + cx^2y^2 \\ & + 2by^3 + cy^3 - 4xy^3 + bxy^3 - 4by^4 + cy^4 \end{aligned} \tag{67}$$

This somewhat arbitrary form is chosen because if we consider systems of the form

$$\begin{aligned} \dot{w} =& aw - z + (w^2+z^2)(cw - bz) - w^4 \\ \dot{z} =& w + bz + (w^2+z^2)(bw + cz) \end{aligned} \tag{68}$$

then by the change of coordinates

$$w = x + xy \ , \ z = y + y^2 \tag{69}$$

we get exactly (truncating at order 4) the above (66-67). Thus, we have a simple way of checking our computations, and we can interpret the results by checking our results on reducibility of systems to simple forms depending on their approximate symmetries. Notice that our computations will be performed for the system in the form (66-67). In this form, it is not at all transparent that the system could be reduced to the simpler form (68); on the other side, the existence of approximate symmetries can ensure us that this can be done, as it is indeed the case. The reader can indeed check, by comparing the results for the symmetries and the form (68) to which the system can be reduced (for the different values of the parameters a, b, c considered in the following) that the statements of propositions given above are indeed satisfied in these concrete examples

Notice also that the linear part of the system (68) is different from the linear part of the systems considered in example 1 above, unless we consider the case $b = a$.

We will look for approximate symmetries of

$$\varphi = f_1(x,y)\partial_x + f_2(x,y)\partial_y \tag{70}$$

in the form

$$\sigma = s_1(x,y)\partial_x + s_2(x,y)\partial_y \tag{71}$$

In looking for approximate symmetries, we will use the decomposition (9),(10) of section 1, and solve equations (11) at successive orders. We will limit to consider approximate symmetries of order up to four.

At order one we write

$$s_1 = r_1 x + r_2 y \ , \ s_2 = t_1 x + t_2 y$$

and we get simply

$$\begin{aligned} (r_2 + t_1)x - (r_1 + (a-b)r_2 - t_2)y =& 0 \\ [-r_1 + (a-b)t_1 + t_2]x - (r_2 + t_1)y =& 0 \end{aligned}$$

so that at first order

$$\begin{aligned} s_1 =& ((b-a)\alpha_1 + \alpha_2)x + \alpha_1 y \\ s_2 =& -\alpha_1 x + \alpha_2 y \end{aligned}$$

or, in other words, we have two symmetries generated by

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} (b-a)x + y \\ -x \end{pmatrix}$$

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Notice that the second of these correspond to scaling, which is always an approximate symmetry of order one. The general formulas (i.e. the formulas for general a, b, c) become soon quite large; so we will deal with some selected special case. As the complications of the formulas below reveal, the above computations can be easily implemented on an algebraic manipulation program, such as MATHEMATICA or MAPLE.

We consider here the case $a = b = c = 0$ (other cases will be presented in the Appendix); this corresponds for (68) to

$$\dot{w} = -z - w^4 \ ; \ \dot{z} = w \tag{72}$$

i.e. to a rotation plus terms of order four; in the x, y coordinates we get the system

$$\begin{aligned} \dot{x} =& - y - x^2 + 2x^2y - x^4 - 4x^2y^2 \\ \dot{y} =& x - xy + 2xy^2 - 4xy^3 \end{aligned} \tag{73}$$

We will write

$$\begin{aligned} s_1(x,y) =& (r_1x + r_2y) + (r_3x^2 + r_4xy + r_5y^2) + (r_6x^3 + r_7x^2y + r_8xy^2 + r_9y^3) \\ &+ (r_{10}x^4 + r_{11}x^3y + r_{12}x^2y^2 + r_{13}xy^3 + r_{14}y^4) \\ s_2(x,y) =& (t_1x + t_2y) + (t_3x^2 + t_4xy + t_5y^2) + (t_6x^3 + t_7x^2y + t_8xy^2 + t_9y^3) \\ &+ (t_{10}x^4 + t_{11}x^3y + t_{12}x^2y^2 + t_{13}xy^3 + t_{14}y^4) \end{aligned} \tag{74}$$

At order one we get the equations

$$\begin{aligned} r_2x + t_1x - r_1y + t_2y =& 0 \\ -r_1x + t_2x - r_2y - t_1y =& 0 \end{aligned} \tag{75}$$

which gives

$$r_1 = t_2 \ , \ t_1 = -r_2 \tag{76}$$

so that the first order terms of $s(x, y)$ are given by

$$s_1^{(1)}(x,y) = t_2x + r_2y \ , \ s_2^{(1)}(x,y) = -r_2x + t_2y \tag{77}$$

We can now use this form of $s^{(1)}$ and go to the second order equation; in this way we get the equations

$$\begin{aligned} (r_4 + t_2 + t_3)x^2 + (r_2 - 2r_3 + 2r_5 + t_4)xy - (r_4 + t_5)y^2 =& 0 \\ (-r_3 + t_4)x^2 + (-r_4 + t_2 - 2t_3 + 2t_5)xy + (r_2 - r_5 - t_4)y^2 =& 0 \end{aligned} \tag{78}$$

which yield

$$r_3 = r_2 \ , \ r_4 = -t_2 \ , \ r_5 = 0 = t3 \ , \ t_4 = r_2 \ , \ t_5 = -t_2 \tag{79}$$

so that the second order terms in $s(x, y)$ are given by

$$s_1^{(2)}(x,y) = r_2x^2 - t_2xy \ , \ s_2^{(2)}(x,y)r_2xy - t_2y^2 \tag{80}$$

Proceeding in the same way, we get at order three (we write only free constants, and substitute for those expressed in terms of other parameters)

$$\begin{aligned} s_1^{(3)}(x,y) =& (-2t_2 + t_9)x^3 + (-4r_2 - t_8)x^2y + t_9xy^2 + (-2r_2 - t_8)y^3 \\ s_2^{(3)}(x,y) =& (2r_2 + t_8)x^3 + (-2t_2 + t_9)x^2y + t_8xy^2 + t_9y^3 \end{aligned} \tag{81}$$

And finally, at order four

$$\begin{aligned} s_1^{(4)}(x,y) =& - r_2x^4 - t_8x^4 - \frac{22t_2x^3y}{5} + t_9x^3y - 2r_2x^2y^2 - 3t_8x^2y^2 \\ &- \frac{38t_2xy^3}{5} + t_9xy^3 - 4r_2y^4 - 2t_8y^4 \\ s_2^{(4)}(x,y) =& \frac{-3t_2x^4}{5} + 2r_2x^3y + t_8x^3y - \frac{22t_2x^2y^2}{5} + t_9x^2y^2 + 6r_2xy^3 + t_8xy^3 \\ &- \frac{38t_2y^4}{5} + t_9y^4 \end{aligned} \tag{82}$$

Therefore, we have four approximate symmetries of order up to four (counting only nontrivial ones, i.e. up to terms of order higher than four, which we could always add). These are

$$\begin{aligned}
\sigma_\alpha(x,y) =& \Big(x - 2x^3 - xy - \frac{22x^3y}{5} - \frac{38xy^3}{5}\Big)\partial_x \\
&+ \Big[y - 2x^2y - y^2 + \Big(\frac{-3x^4}{5} - \frac{22x^2y^2}{5} - \frac{38y^4}{5}\Big)\Big]\partial_y \\
\sigma_\beta(x,y) =& (x^2 - x^4 + y - 4x^2y - 2x^2y^2 - 2y^3 - 4y^4)\partial_x \\
&+ [-x + 2x^3 + xy + \big(2x^3y + 6xy^3\big)]\partial_y \\
\sigma_\gamma(x,y) =& (-x^4 - x^2y - 3x^2y^2 - y^3 - 2y^4)\partial_x + [x^3 + xy^2 + \big(x^3y + xy^3\big)]\partial_y \\
\sigma_\eta(x,y) =& (x^3 + x^3y + xy^2 + xy^3)\partial_x + [x^2y + y^3 + \big(x^2y^2 + y^4\big)]\partial_y
\end{aligned} \tag{83}$$

Example 3.

Another example, which is of some independent interest due to some peculiarity of its symmetry properties [22], is the following. Let $x \in \mathbf{R}^4$, and consider the "realification" of the standard representation of $SU(2)$ (cf. [22]), acting in $\mathbf{R}^4 \approx \mathbf{C}^2$, generated by real (4×4) matrices H_i, $i = 1,2,3$, such that $[H_i, H_j] = \varepsilon_{ijm} H_m$. These are such that $(x, H_i x) = 0$. There are then real (4×4) matrices K_j, $j = 0, \ldots, 3$, such that $K_0 = I$, $[K_i, K_j] = \varepsilon_{ijm} K_m$ for $i,j,m \neq 0$ (i.e. K_i, $i = 1,2,3$ generate a representation of $SU(2)$ as well), and commuting with the H_j, $[K_j, H_i] = 0$. Thus, a general $SU(2)-$equivariant DS is written as $\big($with $r^2 = (x,x)\big)$

$$\dot{x} = \alpha(r^2)x + \sum_{i=1}^{3} \beta_i(r^2) K_i x \tag{84}$$

Let us assume for definiteness that the linear term $A = \alpha(0)I + \sum_{1=1}^{3} \beta_i(0) K_i$ of this problem is $\neq 0$. Now, if $\alpha(0) \neq 0$, then A does not admit resonances and the DS can be reduced to a linear NF. If $\alpha(0) = 0$, it is not restrictive to assume that the linear part has the form $Ax = \beta_1(0) K_1 x$, $\beta_1(0) \neq 0$ (we can always obtain this form by means of a *linear* transformation of coordinates); then it is easy to verify that all terms multiplying K_2, K_3 are nonresonant with A, and therefore once reduced (or approximately reduced) to NF the problem (84) becomes

$$\dot{x} = \widetilde{\alpha}(r^2)x + \widetilde{\beta}(r^2) K_1 x \tag{85}$$

In this form, it is clear that it admits, in addition to the three symmetries $(H_i x \cdot \nabla)$, the fourth symmetry $(K_1 x \cdot \nabla)$. Therefore, this is an example of a nonlinear problem (in NF) admitting n independent linear (but not commuting) symmetries.

Notice that r^2 is a constant of motion only if α=0, and an approximate constant of motion only up to order τ, where α_τ is the first nonzero coefficient in the expansion of α.

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Appendix

We discuss here some other examples of the kind of those considered in Sect. 7. Precisely, let us choose in Example 2, eqs. (67-68), the case

$$a = c = 0\ ,\ \ b = 1 \tag{A1}$$

so that we have the system

$$\begin{aligned} \dot{w} =& - z - (w^2 + z^2)z - w^4 \\ \dot{z} =& w + z + (w^2 + z^2)w \end{aligned} \tag{A2}$$

or, in the x, y coordinates

$$\begin{aligned} \dot{x} =& - x^2 - 2x^4 - y - xy + x^2y + 2xy^2 - 7x^2y^2 - y^3 - 4xy^3 - 2y^4 \\ \dot{y} =& x + x^3 + y - xy + x^3y - y^2 + 3xy^2 + 2y^3 - 3xy^3 - 4y^4 \end{aligned} \tag{A3}$$

We can proceed in the same way, as with the same notations, as in example 2, sect. 7; in this way we get the result that there are only two symmetries of order up to four (up to terms of higher orders), which are given explicitely by

$$\begin{aligned} \sigma_\alpha(x,y) =& \Big(x - \frac{22x^3}{13} - \frac{5245x^4}{819} - xy - \frac{62x^2y}{13} - \frac{8438x^3y}{819} - \frac{22xy^2}{13} \\ & - \frac{4166x^2y^2}{273} - \frac{30y^3}{13} - \frac{10436xy^3}{819} - \frac{3676y^4}{819} \Big) \frac{\partial}{\partial x} \\ & + \Big(\frac{62x^3}{13} - \frac{73x^4}{63} + y + \frac{66x^2y}{13} + \frac{1618x^3y}{819} - y^2 + \frac{62xy^2}{13} \\ & + \frac{502x^2y^2}{273} + \frac{42y^3}{13} + \frac{2242xy^3}{819} - \frac{2788y^4}{819} \Big) \frac{\partial}{\partial y} \end{aligned} \tag{A4}$$

$$\begin{aligned} \sigma_\beta(x,y) =& \Big(x + x^2 - \frac{22x^3}{13} - \frac{3607x^4}{819} + y - \frac{75x^2y}{13} - \frac{8438x^3y}{819} - \frac{48xy^2}{13} \\ & - \frac{2255x^2y^2}{273} - \frac{17y^3}{13} - \frac{7160xy^3}{819} - \frac{2038y^4}{819} \Big) \frac{\partial}{\partial x} \\ & + \Big(- x + \frac{49x^3}{13} - \frac{73x^4}{63} + xy + \frac{66x^2y}{13} + \frac{799x^3y}{819} + \frac{23xy^2}{13} \\ & + \frac{502x^2y^2}{273} + \frac{16y^3}{13} + \frac{4699xy^3}{819} + \frac{488y^4}{819} \Big) \frac{\partial}{\partial y} \end{aligned} \tag{A5}$$

Presentation of similar complicate formulas for other cases would have little interest; we just mention that e.g. for the case $a = 1$, $b = c = 0$ and for the case $a = c = 1$, $b = -1$, we have again two nontrivial approximate symmetries of order four.

It is maybe interesting to consider a little modification of the case considered above; this amounts to replace the b appearing in the linear part of (68) by an a, so that we have a rotational symmetry; we then take again $a = c = 0, b = 1$. Thus, our equations are now

$$\begin{aligned} \dot{x} =& - x^2 - 2x^4 - y - xy + x^2y + 2xy^2 - 7x^2y^2 - y^3 - 4xy^3 - 2y^4 \\ \dot{y} =& x + x^3 - xy + x^3y - y^2 + 3xy^2 + 2y^3 - 3xy^3 - 4y^4 \end{aligned} \tag{A6}$$

We proceed as usual; up to order two, we get two nontrivial approximate symmetries, given by

$$\sigma_\alpha = (y + xy + x^2)\partial_x + (-x + xy + y^2)\partial_y \tag{A7}$$

$$\sigma_\beta = (x - xy + x^2)\partial_x + (y + xy - y^2)\partial_y \tag{A8}$$

When we try to solve the equation at third order, it turns out that this has solutions for σ_α, but has no solution for σ_β. Thus, σ_β provides an example of an approximate symmetry (of order two) which is *not* an approximate symmetries at higher orders; notice that this corresponds to the discussion of Example 1 above.

As for σ_α, this can be extended to a symmetry of order four, giving

$$\begin{aligned} \sigma_\alpha = & (x^2 + 2x^3 - 3x^4 + y + xy - 4x^2y + 5x^3y - 4x^2y^2 - 2y^3 + 9xy^3 - 4y^4)\partial_x + \\ & (-x + 2x^3 - 6x^4 + xy + 2x^2y + y^2 - 7x^2y^2 + 4xy^3 + 3y^4)\partial_y \end{aligned} \tag{A9}$$

Also, we have symmetries at order four which are trivial at order two (i.e. contain only higher order terms); these are given by

$$\sigma_\gamma = (-x^4 - x^2y + x^3y - 3x^2y^2 - y^3 + xy^3 - 2y^4)\partial_x + (x^3 - 2x^4 + x^3y + xy^2 - 3x^2y^2 + xy^3 - y^4)\partial_y \tag{A10}$$

$$\sigma_\eta = (x^3 - x^4 + x^3y + xy^2 - x^2y^2 + xy^3)\partial_x + (x^2y - x^3y + x^2y^2 + y^3 - xy^3 + y^4)\partial_y \tag{A11}$$

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