A splitting lemma for equivariant dynamics

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Summary. It is shown that a number of interesting and useful results in equivariant dynamics can be seen as a consequence of a general "splitting principle". Both the results and the principle are actually embodied by the Michel theory of symmetry breaking.

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1. Introduction

The purpose of this short note is to state (and prove) explicitely a "splitting lemma" for equivariant dynamics; this allows to separate the equivariant dynamics on a G-manifold M into the "semidirect product" of a dynamics in the G-orbit space $\Omega = M/G$ and a dynamics on G.

Some readers will notice that the result given below, as well as the construction employed and the (idea of the) proof, just follow from the statement and the proof of Michel theorem [1] on the geometry of symmetry breaking (see also [2,3] for discussion and applications of this, and [4,5] for extensions). Thus, it could seem not so useful to give an explicit statement and proof of the present lemma.

On the other side, several results have appeared after [1] was published, which could be considered as corollaries - or special cases - of Michel theorem, and which were obtained with no reference to [1]. Among these, we mention the "Equivariant Branching Lemma" [6] and the "Reduction Lemma" [7], which play a key role in the developement (and applications) of Equivariant Bifurcation Theory [8,9], as discussed in [10]. Also, some more recent results can be seen as natural in the light of the splitting lemma; among these we mention the beautiful work by Krupa [11] on bifurcation from a relative equilibrium, and the very recent contribution by Chossat and Koenig [12].

Thus, with the present note we aim on the one side at pointing out how these different results are manifestations of the same basic facts; and on the other at pointing out how the Michel theory encompasses and unifies these different aspects.

Michel theory, developed in the context of elementary particle theory, has often been overlooked in the context of equivariant dynamics (as a significant exception to this, we mention the far-reaching works by Field and Richardson [13]); this fact is presumably due to lack of communication among different fields, and among Physics and Mathematics communities in general; it is hoped that the present note will call the attention of the readers to this unifying and beautiful theory.

We give here an elementary and self-contained derivation of the splitting lemma (SL in the following); a more complete discussion, considering also the relation of the SL to the results mentioned above and to Michel theory, will be given elsewhere [14]; we also postpone the discussion of more general cases (e.g. non-compact Lie groups) to there.

2. Group action

Let M be a smooth (i.e. $\mathcal{C}^\infty$) real manifold, which we will think as embedded in R^n for ease of notation. Let G be a connected compact Lie group acting smoothly in M. We denote by $\Omega = M/G$ the G-orbit space (or orbit space *tout court*) for the action of G on M, i.e. the space of equivalence classes of points of M for the (equivalence) relation of "lying on the same G-orbit"; elements od Ω will be denoted as ω, while we write $\omega[x] \subseteq M$ for the set of points of M on the same G-orbit as x:

$$\omega[x] := \{ y \in M \ : \ y = gx, \ g \in G \} \equiv \{Gx\} \tag{1}$$

At any point $x \in M$, let us consider the invariant splitting of the tangent space $T_x M$ given by

$$T_x M = T_x \omega[x] \oplus N_x \omega[x] \tag{2}$$

where $T\omega[x]$ (respectively $N\omega[x]$) is the tangent bundle (respectively the normal bundle) to $\omega[x]$.

Let us consider the Lie algebra $\mathcal{G}$ of G; this will have generators $\{\eta_1, \ldots, \eta_r\}$ which we will write in a concrete realization for the G-action on M as

$$\eta_j = \sum_{i=1}^n \varphi^i_{(j)}(x) \cdot \frac{\partial}{\partial x^i} \tag{3}$$

so that $T_x\omega[x]$ is the span of $\{\varphi_{(1)}(x), \ldots, \varphi_{(r)}(x)\}$ (notice that these are not necessarily linearly independent at all points $x \in M$).

To any point $x \in M$ we can associate an *isotropy subgroup* G_x and a *conjugated fixed space* M^x by

$$G_x := \{ g \in G \ : \ gx = x \} \tag{4}$$

$$M^x := \{ y \in M \ : \ gy = y \ \forall g \in G_x \} \equiv \{ y \in M \ : \ G_x \subseteq G_y \} \tag{5}$$

Notice that for points on the same G-orbit we have

$$G_{gx} = g G_x g^{-1} \tag{6}$$

$$M^{gx} = g(M^x) \tag{7}$$

Subgroups of G which are conjugated in G should be seen as equivalent; based on this observation one is naturally led to Michel theory, i.e. to consider the *isotropy stratification* of M and Ω [1]; For this we refer to [1-3]; we will also discuss this in [14], while here we choose to limit ourselves to simpler considerations.

Let us now focus on the orbit space Ω; by a theorem of Hilbert [15] (for polynomials) and Schwartz [16] (for $\mathcal{C}^\infty$ functions), there is a *minimal integrity basis* (MIB) - in general, non unique - of G-invariant $\mathcal{C}^\infty$ functions (respectively, polynomials) $\{I_1(x), ..., I_h(x)\}$, such that any G-invariant $\mathcal{C}^\infty$ function (respectively, polynomial) $F(gx) = F(x)$ can be written as a $\mathcal{C}^\infty$ function (respectively, a polynomial function) of the I_j's, i.e. $F(x) = \widetilde{F}(I_1(x), ..., I_h(x))$ [1-3,15,16]. The orbit space Ω is then a semialgebraic manifold, diffeomorphic to a semialgebraic set $S_\Omega \subseteq R^h$.

An invariant function $F : M \to R$, $F(gx) = F(x)$, can also be seen as a function on the G-orbit space; we denote the corresponding function as $F_\omega : \Omega \to R$; we also write $\omega : M \to \Omega$ the function which associates to $x \in M$ the corresponding orbit as a point in Ω, so that $F \equiv F_\omega \cdot \omega$. If we denote by $I : M \to R^h$ the map associating to $x \in M$ the values $\{I_1(x), ..., I_h(x)\}$ of functions in the chosen MIB, and by $\omega : M \to \Omega$ the map associating to a point $x \in M$ the corresponding element of Ω, we have the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{I} & R^n \\ \Big\downarrow{\scriptstyle \omega} & & \Big\downarrow{\scriptstyle \widetilde{F}} \\ \Omega & \xrightarrow{F_\omega} & R \end{array}$$

In practice, invariant functions $I(x)$ are determined by asking $\eta \cdot I = 0 \ \forall \eta \in \mathcal{G}$; using (3), this leads to a system of linear PDEs of the form

$$\sum_{i=1}^{n} \varphi^i_{(j)}(x) \frac{\partial I}{\partial x^i} = 0 \qquad j = 1, ..., h \tag{8}$$

or equivalently to the characteristic equations [17,18]

$$\frac{dx^1}{\varphi^1_{(j)}} = ... = \frac{dx^n}{\varphi^n_{(j)}} \qquad j = 1, ..., h \tag{9}$$

3. Equivariant dynamics

Let us now consider a (smooth) dynamical system defined on M; we write this in coordinate form as

$$\dot{x} = f(x) \qquad f : M \to TM \tag{10}$$

or, more intrinsically, we consider a vector field ψ on M, whose coordinate expression is $\psi = \sum_i f^i(x)\partial/\partial x^i$. We consider G-equivariant vector fields: this means

$$[\psi, \eta] = 0 \qquad \forall \eta \in \mathcal{G} \tag{11}$$

or, in the coordinate form (10),

$$f(gx) = (D_g f)(x) \qquad \forall g \in G \tag{12}$$

where D_g denotes the action of g on TM.

Let us consider the time evolution of a G-invariant quantity $I(x)$ under the flow of ψ:

$$\dot{I} \equiv \frac{dI}{dt} = (\psi \cdot \nabla) I = f^i(x) \frac{\partial I}{\partial x^i} \tag{13}$$

In particular we can consider I to be one of the I_j in the chosen MIB; then the time evolution of any G-invariant function F is given - in the notation introduced above - by

$$\dot{F}(x) = \sum_{j=1}^{h} \frac{\partial \widetilde{F}}{\partial I_j} \dot{I}_j \tag{14}$$

It is just a simple remark that the $\dot{I}_j$ are themselves G-invariant quantities: thus, there exist functions $\alpha_j : R^h \to R$ such that

$$\dot{I}_j = f^i(x) \frac{\partial I_j}{\partial x^i} := \alpha_j(I_1, ..., I_h) \tag{15}$$

In other words, ψ induces an *autonomous* dynamical system - i.e. a vector field $A = \alpha_j \partial / \partial I_j$ - on S, and thus on Ω. This is also *smooth* since all the mappings involved are smooth [1-3,15,16], and since singular points of S turn out to correspond necessarily to invariant sets and zeroes of the vector field [1-3,14].

Lemma I. *A smooth G-equivariant dynamical system on the G-manifold M, induces a smooth dynamical system on the orbit space $\Omega = M/G$.*

Given a manifold M, we can think to choose a subset $M_0 \subseteq M$ such that each G-orbit in M intersects M_0 precisely once (so that Ω is isomorphic to M_0); then any point $x \in M$ can be described in terms of two coordinates $y \in M_0$ and $g \in G$ just as

$$x \simeq (y, g) \Leftrightarrow x = gy \qquad y \in M_0 \tag{16}$$

(notice g in (16) is only defined up to G_y).

Let us decompose the flow of $x(t)$ into the direction tangent to the G-orbit $\omega[x(t)]$ and into that orthogonal to the same G-orbit, see (2). Corresponding to this, the flow of ψ will be decomposed into a flow A in the orbit space Ω, and a flow along G-orbits or, equivalently, a flow on G. Notice that the latter is only defined up to elements of $G_{x(t)}$ (which is a kind of gauge invariance).

The fact we define in this way a *smooth flow* on G can be seen as a consequence of general theorems in Bredon [17] (chapt.VI), and depends crucially on our smoothness assumptions.

Again, it is important in this respect that orbits with greater isotropy subgroups are invariant sets for the flows [1-3,13], which leads us again to rely on Michel theory. The smoothness of the flow on G also amounts to the possibility of choosing $M_0 \subseteq M$ above, as a smooth (embedded) submanifold of M.

Notice that the flow on G will, in general, depend on the orbit we are on. In other words, let us choose coordinates $\{\theta_1, ..., \theta_k\}$ on G; then the flow on G, i.e. the time evolution of the θ_j's, will depend on the I_j's, namely will be described by equations of the form

$$\dot{\theta}_j = \beta_j(\theta_1, ..., \theta_k; I_1, ..., I_h) \tag{17}$$

We will summarize our discussion as follows:

Splitting Lemma. (Invariant formulation) *Let M be a smooth G-manifold, with G a smooth compact connected Lie group acting smoothly on M; let ψ be a smooth G-equivariant vector field on M. Let $\Omega = M/G$ be the G-orbit space on M. Then, ψ induces an autonomous vector field A on Ω, and a Ω-dependent flow on G.*

Splitting Lemma. (Coordinate formulation) *Let $M \subseteq R^n$ be a smooth manifold, G a smooth compact connected Lie group acting smoothly on M. Choose coordinates $\{\theta_1, ..., \theta_k\}$ in G, and coordinates $\{I_1, ..., I_h\}$ in $\Omega = M/G$ (i.e. a MIB for G). Consider a smooth G-equivariant dynamical system in M,*

$$\dot{x} = f(x) \qquad f : M \to TM$$

$$f(gx) = (D_g f)(x) \qquad \forall g \in G$$

Then, the evolution of the $\{\theta; I\}$ is governed by a "triangular system"

$$\begin{aligned} \dot{I} =& \alpha(I) \\ \dot{\theta} =& \beta(I, \theta) \end{aligned}$$

4. Remarks and extensions

Remark 1. Symmetry considerations do also grant the invariance of specific submanifolds of M under G-equivariant flows: e.g., the manifolds M^x introduced above are invariant, so that one can restrict to such invariant manifolds, simplifying considerably the study of the dynamics. In particular, in this case only the subgroup $N_G(G_x) \subseteq G$ is accessible to the dynamics (here $N_G(H)$ is the normalizer of $H \subseteq G$ in G), and only $D_G(G_x) := N_G(G_x)/G_x$ acts effectively in M^x. Therefore, for a given initial datum $X(0) = x_0$, we restrict to dynamics in Ω and flow on $D_G(G_{x_0})$.

Remark 2. In some cases - e.g. for H a maximal isotropy subgroup of G - the group $D_G(H)$ is particularly simple [17], and we can provide stronger results [10].

Remark 3. Invariant subsets are also granted to exist in Ω, due to symmetry. This can be seen by using Michel's theory of isotropy stratifications [1-3], which leads to a hierarchy of invariant subsets in Ω. It is in this way that one obtains e.g. the Equivariant Branching Lemma [6]

Remark 4. If the dynamical system in Ω admits fixed points (respectively, periodic or multiperiodic orbits), in correspondence to these one has to study autonomous flows (respectively, flows with periodic or multiperiodic coefficients) in G, or more precisely in $D_G(G_{x_0})$. Again, Michel's theory can in some cases grant the existence of fixed points for the flow in Ω on the basis of symmetry considerations alone.

Remark 5. In the situation mentioned in remark 4, one can use a number of powerful tools (momentum mapping, coadjoint orbits theory), as we are reduced to a (simple) dynamics on a Lie group. Notice that for the non-stationary (in Ω) case, we could have troubles if in the presence of a nontrivial topology for Ω. It is conjectured ([17], chapt.I) that this is never the case for $M = R^n$ or $R = D^n$.

Remark 6. Notice that fixed points for the dynamics in Ω correspond to *relative equilibria* for the overall dynamics in M.

5. Simple examples

Example 1. Let $M = R^2$, $G = SO(2)$ acting in the standard way. We pass to polar coordinates (ρ, θ) (with $\rho = (x^2 + y^2)/2$), which are also coordinates for Ω and for G respectively. Indeed, any G-equivariant system is written as

$$\begin{aligned} \dot{x} =& a(\rho)x - b(\rho)y \\ \dot{y} =& b(\rho)x + a(\rho)y \end{aligned}$$

which yields

$$\begin{aligned} \dot{\rho} =& \rho a(\rho) \\ \dot{\theta} =& b(\rho) \end{aligned}$$

Example 2. Let $M = R^4$, $G = SU(2)$ acting by the realification of its defining representation. Then there are three matrices H_1, H_2, H_3 (generating a representation of the $SU(2)$ algebra themselves) [19] such that any G-equivariant system is written as

$$\dot{x} = a(\rho)x + \sum_{i=1}^{3} b_i(\rho) H_i x$$

and we have

$$\dot{\rho} = \rho a(\rho)$$

while the flow along G is given simply by

$$\dot{x} = \sum_{i=1}^{3} b_i(\rho) H_i x$$

Example 3. Let M be the solid torus, on which we take coordinates $\rho \in [0,1]$, $\theta \in S^1$, $\varphi \in S^1$ (say, for pictorial representation, with θ going around the equator of the torus); let $G = SO(2) \times SO(2)$ acting through θ and φ independent rotations, so that ρ parametrizes G-orbits ($\rho = 0$ corresponds then to an exceptional orbit being a circle S^1, all the other G-orbits being tori $T^2 \simeq G$). In these coordinates, G-equivariant systems are written as

$$\dot{\rho} = a(\rho) \quad ; \quad \dot{\theta} = b(\rho) \quad ; \quad \dot{\varphi} = c(\rho)$$

Let us introduce explicitely a control parameter λ, so that $\dot{\rho} = a(\rho, \lambda)$, etc. We make standard bifurcation assumptions, i.e. we assume that $a(\rho, \lambda) < 0$ for ρ large enough and for any λ, and that at $\rho = 0$ we have $\partial a(\rho, \lambda)/\partial \rho < 0$ for $\lambda < 0$, and $\partial a(\rho, \lambda)/\partial \rho > 0$ for $\lambda > 0$ (Michel theorem ensures $a(0, \lambda) = 0 \ \forall \lambda$). Also, we assume that b, c have definite signs for $\lambda \simeq 0$, $\rho \simeq 0$ (this a genericity requirement), e.g. $b(0,0) > 0$, $c(0,0) > 0$.

With these, we have a (symmetry breaking) bifurcation from the relative equilibrium $\rho = 0$ for $\lambda = 0$. In this bifurcation, the symmetry $G_0 \subset G$ corresponding to φ rotations is broken; the splitting lemma guarantees that the bifurcating flow is given by a small deformation of the flow on S^1 times a flow on G_0; this is indeed the kind of results obtained by Krupa [11].

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