Splitting equivariant dynamics

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Summary. We prove that any dynamical system on a G-manifold M which is equivariant under the G action, can be decomposed into the semidirect product of an autonomous dynamics in the G-orbit space $\Omega = M/G$, and a dynamics (depending on the G-orbit) on G. This result is actually a corollary of Michel theorem [1] on the geometry of symmetry breaking, and uses the same ingredients for the proof. It permits to unify a number of known and useful results in the literature, as discussed here.

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1. Introduction

Let M be a smooth ($\mathcal{C}^\infty$) manifold, on which the smooth action of a compact Lie group G is defined; i.e., M is a smooth G-manifold. For ease of notation, we will consider M to be a smooth submanifold of $\mathbf{R}^N$, but our construction and results will not depend on this assumption.

Let us consider the algbera $\mathcal{V}$ of G-invariant real functions on M, i.e.

$$\mathcal{V} = \{V(x) \ : \ V : M \to \mathbf{R} \ ; \ V(gx) = V(x) \ \forall g \in G\} \tag{1}$$

and the algebra $\mathcal{M}$ of smooth G-equivariant vector fields on M, i.e.

$$\mathcal{M} = \{\varphi : M \to TM \ : \ \varphi = f(x)\nabla \ ; \ f(gx) = (D_g f)(x) \ \forall g \in G\} \tag{2}$$

Here we have denoted the transformed of x by $g \in G$ (through the action defined on M) simply by gx, as we consider only one G action; and the action of g on TM by D_g.

The set $\mathcal{M}$ is a Lie algebra once equipped with usual commutator of vector fields $[\varphi, \psi] = \varphi\psi - \psi\varphi$; we could equivalently consider the algebra

$$\mathcal{E} = \{f : M \to \mathbf{R}^N \ ; \ f(gx) = (D_g f)(x) \ \forall g \in G\} \tag{2'}$$

in which case the Lie algebra operation is the Lie-Poisson bracket $\{f, g\} = (f \cdot \nabla)g - (g \cdot \nabla)f$.

We want to consider G-equivariant dynamical systems on M,

$$\dot{x} = f(x) \qquad f \in \mathcal{E} \tag{3}$$

A subclass of these, of special interest, is given by gradient flows, i.e. by the case

$$\dot{x} = f(x) = \nabla V(x) \qquad V \in \mathcal{V} \tag{4}$$

In this case, we would be mainly interested in fixed points of the flow, i.e. in critical points of V; indeed, these represent asymptotic solutions to the dynamical system.

In order to study (3), (4) we will now recall a construction due to L. Michel [1] (see also [2-5]), based on the geometry of G-action in M, and developed in the frame of variational problems; we will then show that this gives as a corollary a nice and useful splitting lemma for the general case of equivariant dynamics (3), which represents the main result of this note. The remaining parts of this paper will be devoted to discuss applications of the splitting lemma, and to show how it permits to unify a number of known (and, again, useful) results in equivariant dynamics and in particular in equivariant bifurcation theory.

2. The orbit space

Let $\omega[x]$ be the G-orbit through x in M, i.e.

$$\omega[x] = \{y \in M \ : \ y = gx \ , \ g \in G\} = \{Gx\} \tag{5}$$

Clearly, belonging to the same G-orbit is an equivalence relation in M, and we can consider the space of such equivalence classes or, the *orbit space*

$$\Omega = M/G \tag{6}$$

Results on the geometry of orbit space are collected e.g. in [6,7] (see also [2,4,8]); here we will just recall the results which are useful for the present discussion.

Lemma I. Let G be a compact Lie group and M a smooth G-manifold; then $\Omega = M/G$ is a semialgebraic manifold.

We recall that an *integrity basis* for the G-action on M is a set $\{\theta_1, ..., \theta_k\}$ of functions $\theta_i \in \mathcal{V}$ such that any smooth (algebraic) function $V(x) \in \mathcal{V}$ can be written as a smooth (algebraic) function $\widetilde{V}$ of the θ's, $V(x) = \widetilde{V}(\theta_1(x), ..., \theta_k(x))$. The integrity basis is called *minimal* if no one of the θ's can be dropped still having an integrity bases.

Lemma II. If G is a compact Lie group acting smoothly in M, the algebra of smooth (algebraic) invariant functions on M admits a finite dimensional minimal integrity basis.

Lemma III. Let $\{\theta_1, ..., \theta_k\}$ be a minimal integrity basis for the G-action in M. Then, each G-orbit belongs to one common level set of $\{\theta_1, ..., \theta_k\}$, and conversely for any common level set of $\{\theta_1, ..., \theta_k\}$ in M there is at least a G-orbit whose closure coincides with the common level set.

These results were proved by Hilbert [9] for the algebraic case, and extended by Schwarz [10] to the smooth case (see also [11]). The statement of lemma III is sometimes summarized by saying that "invariants do separate orbits".

We will denote by $\theta : M \to \mathbf{R}^k$ the mapping which associates to a point $x \in M$ the values $\theta_i(x)$ of the functions of a minimal integrity basis. Let us denote by $\widetilde{M}$ the image of M under θ, i.e. $\theta(M) = \widetilde{M} \subseteq \mathbf{R}^k$. Clearly, Ω is then diffeomorphic to $\widetilde{M}$, and can therefore be thought as a semialgebraic submanifold of R^k.

Lemma IV. If G is a compact Lie group acting smoothly in M, the orbit space $\Omega = M/G$ is diffeomorphic to a semialgebraic manifold in $\mathbf{R}^k$, where k is the dimension of a minimal integrity basis $\{\theta_1, ..., \theta_k\}$ for the G action; this semialgebraic manifold is $\widetilde{M} = \theta(M)$.

Let us now go back to the dynamical system (3); this induce a dynamical system in Ω. Indeed, let us consider the evolution of the invariant functions $\theta_i(x)$ when $x(t)$ evolves according to (3), i.e. the evolution of θ's along solutions to (3). In this case,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = (f(x) \cdot \nabla)\theta(x) \tag{7}$$

Now, θ is G-invariant, and the operator $(f \cdot \nabla)$ is also G-invariant, as it is the contraction of a contravariant and a covariant object, so that the r.h.s. of (7) is an invariant function, and therefore, according to Lemma III, can be written in terms of the θ's. Equivalently, the time derivative of the θ's is clearly also G-invariant, and therefore can be written in terms of the θ's themselves. Therefore, we can rewrite (7) as

$$\dot{\theta} = \widetilde{f}(\theta) \tag{8}$$

We have thus proved that the dynamical system (3) on M projects to a dynamical system (8) in $\Omega \simeq \widetilde{M}$

We can therefore split the dynamics in two parts: an autonomous part on $\widetilde{M} \simeq \Omega$, and an orbit-dependent dynamics on group variables (which does *not* mean "tangent to group orbits"). It should be mentioned that a great deal of this construction survives when we consider non-compact and/or infinite dimensional Lie groups G and smooth (Banach) manifolds M (we need Banach manifolds to define an equivariant retraction [1,2,4,8]); we will not discuss this point, and the limitations of such an extension, here; the interested reader is referred e.g. to [4,6,7].

We will now discuss how the present construction is related to the geometry of G-action in M; this discussion will also make clear the reason why we employed the term "group variables".

3. Stratification

To any point $x \in M$ we associate its isotropy group $G_x \subseteq G$ by

$$G_x = \{ g \in G \ : \ gx = x \} \tag{9}$$

Let us consider the equivalence relation between subgroups of G given by conjugacy in G; i.e., if $H \subseteq G$,

$$C[H] = \{ K \subseteq G \ : \ K = gHg^{-1} \ , \ g \in G \} \tag{10}$$

This equivalence relation induces an equivalence relation in M, known as *isotropy stratification*; the equivalence classes - or *strata* (latin for layers) - being defined as

$$\sigma[x] = \{ y \in M \ : \ G_y \in C[G_x] \} \tag{11}$$

Since $y = gx$ implies $G_y = gG_xg^{-1}$, we have that points on the same G-orbit belong to the same stratum, or

$$\omega[x] \subseteq \sigma[x] \tag{12}$$

The isotropy stratification of M does therefore induce an isotropy stratification on Ω; indeed, let us denote

$$C[\omega] = C[G_x] \ , !x \in \omega \tag{13}$$

the *isotropy type* of the orbit ω; the strata are then defined by

$$\Sigma[\omega] = \{ \nu \in \Omega \ : \ C[\nu] = C[\omega] \} \tag{14}$$

Let us now consider, for any subgroup $H \subseteq G$, the *fixed space*

$$M^H = \{ x \in M \ : \ gx = x \ \forall g \in H \} \tag{15}$$

In particolar, if $H = G_x$, we will just write M^x for M^{G_x}. In this case, (15) reads also

$$M^x = \{ y \in M \ : \ G_y \subseteq G_x \} \tag{16}$$

Notice also that we have $M^{gx} = g\left(M^x\right)$.

It is clear that for any point $x \in M$, $\omega[x]$ is transversal to M^x (unless $\omega[x] = x$, i.e. $G_x = G$). More precisely, the tangent space T_xM admits the following decomposition:

$$T_xM = T_x\omega[x] \oplus N_x\omega[x] \tag{17}$$

$$N_x\omega[x] = T_xM^x \oplus N_x^{(1)}\omega[x] \equiv N_x^{(0)}\omega[x] \oplus N_x^{(1)}\omega[x] \tag{18}$$

where $N_x\omega[x] \subseteq T_xM$ is the normal subspace to $\omega[x]$ in x, and $N_x^{(0)}, N_x^{(1)}$ denote the invariant and non-invariant (under G_x) parts of $N_x\omega[x]$. Notice that (17),(18) are G_x-invariant decompositions, by definition, and are also equivariants: i.e.,

$$\begin{aligned} T_{gx}\omega[gx] =& D_g\left(T_x\omega[x]\right) \\ T_{gx}M^{gx} =& D_g\left(T_xM^x\right) \\ T_{gx}N_{gx}^{(1)}\omega[gx] =& D_g\left(N_x^{(1)}\omega[x]\right) \end{aligned} \tag{19}$$

Indeed, the first two of these follow immediately from the properties of $\omega[x]$ and M^x; the third one follows from these and (17),(18).

One can easily prove [1,2,4,6,7] that

Lemma V.
$$T_x\sigma[x] = T_x\omega[x] \oplus T_x M^x \equiv T_x\omega[x] \oplus N_x^{(0)}\omega[x] \tag{20}$$

Moreover, it can be proved [1,2,4,12] that

Lemma VI. For every G-equivariant vector field $\varphi \in \mathcal{M}$, $\varphi = f(x)\cdot\nabla$, $\varphi : x \to T_x\sigma[x]$

Actually, by the same construction which yields lemma VI, one can also prove that:

Lemma VII. For every G-equivariant vector field $\varphi \in \mathcal{M}$, $\varphi : x \to T_x M^x$

Notice that Lemma VII implies Lemma VI. A simplest proof of lemma VII is obtained by remarking that if $\widetilde{x} \simeq x + \varepsilon f(x)$, then by equivariance of f, $g\widetilde{x} \simeq gx + \varepsilon f(gx)$, so that $gx = x \;\; => \;\; g\widetilde{x} = \widetilde{x}$.

Finally, we should recall some results concerning the geometry of $\widetilde{M} \approx \Omega$ and the geometry of group action.

Since Ω is a semialgebraic manifold, we can define on it a (Whitney) stratification in the sense of algebraic geometry [13]; call E_i^α the corresponding strata, and

$$\widehat{E}^\alpha = \cup_i E_i^\alpha \tag{21}$$

so that

$$\widehat{E}^{\alpha+1} \subseteq \partial\widehat{E}^\alpha \tag{22}$$

and the generic stratum will be $\widehat{E}^0$. It is remarkable that this stratification is compatible with the isotropy stratification Σ, i.e. that each isotropy stratum is the union of algebraic strata. This means in particular that nearly all points in M have the same isotropy type.

The points on strata other than the generic one will have greater isotropy groups, and correspondingly smaller G-orbits. Indeed, if we consider the generic stratum $\widetilde{M}_0$ in $\widetilde{M}$, and correspondingly $M_0 = \theta^{-1}\left(\widetilde{M}_0\right) \subseteq M$, M_0 is a G-fiber bundle, $M_0 = \widetilde{M}_0 \times G$. In general, M is foliated but not fibered by G, as the leaves (the G-orbits) through $x \in M\backslash M_0$ are not isomorphic to leaves through $x_0 \in M_0$; the non-generic strata are therefore also called *singular*.

These considerations also suggest the following construction: let $B \subset M$ be a subset of M such that $\theta(B) = \widetilde{M}$, i.e. $P \simeq \Omega$, or B meets any G-orbit in M in one and only one point.

Lemma VIII. If G is a compact Lie group and M a smooth G-manifold, then there exists a B_0 with the above properties and such that B_0 is a smooth submanifold (in general, with boundary) of M.

Once we have find a B, clearly $M = G(B)$, and any point $X \in M$ can be identified by a pair $x = (b, g)$ of coordinates $b \in B$, $g \in G$, such that

$$x = gb \tag{23}$$

We stress that the rep[resentation (23) is *not* unique, in general, for a given B: e.g., in (23) we could also choose $g' = gh$, with $h \in G_b$.

If the B is chosen as the B_0 of lemma VIII, we have that

Lemma IX. Let $M = G(B_0)$, with B_0 as in lemma VIII; consider a smooth curve $x(s) \in M$; then, we can write $x(s) = g(s)\cdot b(s)$, with $b(s)$ a smooth curve in B_0, and $g(s)$ a smooth curve in G.

Therefore, in general, a smooth dynamical system (3) on the G-manifold M can be also written as

$$\begin{aligned} \dot{g}(t) =& F_1(g,b) \\ \dot{b}(t) =& F_2(g,b) \end{aligned} \tag{24}$$

As we have remarked before, however, the evolution of the θ's depends only on the θ's themselves, so that (24) is actually in "triangular" form

$$\begin{aligned} \dot{g}(t) =& F_1(g,b) \\ \dot{b}(t) =& F_2(b) \end{aligned} \tag{25}$$

We will summarize our discussion as follows:

Splitting lemma. *Let M be a smooth G-manifold, with G a compact Lie group. Let B be a smooth submanifold of M, with $\theta(B) = \widetilde{M} \approx \Omega$, where $\Omega = M/G$ is the G-orbit space, and $\theta : M \to \mathbf{R}^k$ is the map associated to an integrity basis. Any point $x \in M$ is then written (in a non-unique way) as $x = gb$, where $g \in G$, $b \in B$. Let a smooth G-equivariant dynamical system $\dot{x} = f(x)$ be defined on M; this induces a G-invariant dynamical system $\dot{\theta} = \widetilde{f}(\theta)$ on Ω. Also, with the above representation of x, the dynamical system is written in the triangular form (25), and is therefore equivalent to (the semidirect product of) an autonomous dynamics in Ω (equivalently, in B) and a ω-dependent dynamics on G.*

It should be stressed that if G is not compact, the splitting lemma does not hold in general, but we are however still guaranteed that the splitting is possible locally.

Results going in the same direction as the splitting lemma can be found in the literature; see e.g. [14] and [15], to quote two recent works, and also see next section to see existing results which can be unified by the above splitting lemma. It seems, however, that the result has not been stated - and presumably is not known - in general terms, as the *ad hoc* proofs of these works do also confirm.

4. Discussion I - General features

Before considering some examples in full detail, we would like to discuss some special case and applications of the splitting lemma. We start by general remarks.

Remark 1. If $f(x) = (\nabla V)(x)$, i.e. if the dynamical system describes a gradient flow, $f(x)$ is obviously orthogonal to the G-orbit $\omega[x]$ at any $x \in M$. Therefore, the dynamics along the G-orbit is absent, and we just have the dynamics in orbit space.

Remark 2. The situation considered in Remark 1 was the one considered by Michel [1-3]. It should be stressed that, although on the basis of physical considerations he only considered the problem of critical points for an invariant potential, not only his construction is general enough to deal with the general case, but also his original proof does immediately apply to the general case as well. *Our splitting lemma should therefore be seen as a corollary to Michel theorem* [1].

Remark 3. A particularly interesting situation arises when the dynamics in Ω is simple. If $\omega_0 \in \Omega$ is a fixed point for this dynamics, the quantities θ_i can be considered as constants of the motion (for initial data in ω_0), or as asymptotic constants of motion (for initial data in the basis of attraction of ω_0). On the orbit ω_0, we can then apply the momentum map reduction.

Remark 4. Indeed, this corresponds to saying that the whole dynamics takes place on G (i.e. on the G-orbit ω_0).

Remark 5. In the language of Lie-point symmetries, the existence of a fixed point for the dynamics in Ω corresponds to the existence of k simultaneous conditional symmetries [16], or conditional constants of motion[1] [17], see also Remark 3.

Remark 6. If $G_x \in C[H]$, the G-orbit through x is isomorphic to $G/G_x \simeq G/H$.

Remark 7. We have noticed earlier that if $x(0) = x_0$, then $x(t) \in M^{x_0}\ \forall t \geq 0$. We could therefore apply the construction - and the splitting lemma - to M^{x_0} rather than to the whole M. Notice that M^{x_0} could, in general, contain points y such that G_y strictly contains G_{x_0}; this is not the case if G_{x_0} is a maximal isotropy subgroup for the action of G in M. In this case, M^{x_0} is necessarily a fiber bundle; the fibers correspond to the effective action of G on M^{x_0}. The maximal subgroup of G mapping M^{x_0} into itself is $N_G(G_{x_0})$, the normalizer of G_{x_0} in G; obviously G_{x_0} is a normal subgroup of $N_G(G_{x_0})$, and by definition it acts as the identity in M^{x_0}. Therefore, the effective action of G in M^{x_0} is given by $D_{x_0} := N_G(G_{x_0})/G_{x_0}$; if G_{x_0} is a maximal isotropy group in G, then the action of D_{x_0} in M^{x_0} is free. (These remarks are due to Golubitsky and Stewart [12]).

Remark 8. Suppose that we are interested in knowing if our system (3) admits periodic orbits. If the dynamics in Ω drives the system in M to a given G-orbit ω_0, $\omega_0 \simeq G/G_0$, we would then like to know if the system on this orbit admits periodic solutions. If there is a periodic (non stationary) solution $\bar{x}(t) \in \omega_0$, then any $\widetilde{x}(t) = g\bar{x}(t)$ (with g fixed) will also be a periodic solution. In other words, ω_0 must be fibered by S^1 circles (corresponding to the dynamical system flow). This condition can allow us to assert there are no periodic solutions, if ω_0 does not admit an S^1 fibration, e.g. if it has nonvanishing characteristic classes. Similar considerations do also apply for quasiperiodic solutions, requiring a T^k fibration of ω_0.

Remark 9. In some cases, it could be sufficient to have a local S^1 fibration (e.g. if we have local Lie groups rather than proper ones); the analysis of obstructions to a local S^1 fibration of a manifold is considerably more complicate than for global S^1 fibrations; see Gromov *et al.* [18].

Remark 10. Considerations not so different from the above do also apply for periodic or quasiperiodic solutions to the dynamics in orbit space. The number of conditional constants of motion would now be $(k-q)$ for q-periodic solutions in Ω, and the coefficients for the dynamics in G would be q-periodic functions of time.

Remark 11. It is maybe worth mentioning how one could, in practice, write down the dynamics on G. If $\{L_1, ..., L_r\}$ are the generators of $\mathcal{G}$, the Lie algebra of the connected Lie group G, any element $g \in G$ can be written as $g = \exp[\sum_i \lambda_i L_i]$, $\lambda_i \in \mathbf{R}$, so that the dynamics in G can be mapped into a dynamics for the λ_i's.

Remark 12. Notice also that the dynamical system (1) induces a flow which correspond to a one-parameter group Φ (with Lie generator φ); in the case of dynamics on only one G-orbit, this Φ must be a one-parameter subgroup of G. Also, since G transforms solutions into solutions, Φ must be a normal subgroup of G, i.e. φ must be an ideal of $\mathcal{G}$.

Remark 13. The considerations of Remark 12 (and Remarks 8-10 as well) do not apply if ω_0 is not a fixed point for the dynamics in Ω. In this case, to a q-periodic solution $\omega(t)$ correspond a space phase $T^q \otimes_{\rightarrow} G$ for the dynamics in G (here $\otimes_{\rightarrow}$ denotes semidirect product).

[1] It was pointed out to me separately by G. Saccomandi and by G. Marmo that these were already known to Amaldi and Levi Civita.

5. Discussion II - Equivariant bifurcation theory

We would like to present some further remarks, specifically concerned with the application of our splitting lemma, and more generally of Michel theory, to equivariant bifurcation theory.

Remark 14. It is interesting to remark that, nearly at the same time as Michel and Radicati [3], considerations resorting to the same approach based on geometry of symmetry breaking (Lemma VI) were applied to the $SU(3)$ symmetry breaking by Cabibbo and Maiani [5]; their discussion lacks the mathematical generality of Michel theorem [1], but curiously their argument is essentially an anticipation of the "Symmetric Criticality Principle" of Palais [19]. The latter is (or can be seen as) an extension of Michel theorem to the case of variational problems defined by a functional over a space of sections of a fiber bundle (see also [20] for a similar extension in the case of gauge theories and for connections with bifurcation theory).

Remark 15. Two basic tools in equivariant bifurcation theory [21,22,12,23] are the "Equivariant Branching Lemma" (EBL) of Cicogna and Vanderbauwhede [24] (extended to Hopf case by Golubitsky and Stewart [12]) and the "Reduction Lemma" (RL) of Golubitsky and Stewart [12]; these can be seen as corollaries of Michel theorem, although the original proofs were given with little or no reference to Michel's work[2]. Also, these simpler result do not require the introduction of the full Michel theory, but just lemmas V, VI, VII above (about this matter, see also [25,26]). It is interesting that these useful tools, which are at the core of applications of equivariant bifurcation theory, fit so naturally in Michel theory.

Remark 16. In the present setting, the EBL corresponds to bifurcation in the orbit space dynamics (while the dynamics on group variable remains trivial for stationary bifurcations, and goes around a $G/G_x = U(1)$ subgroup of G in the Hopf case). The RL corresponds to Lemma VII above, and the reduced dynamics can be seen as the dynamics on a group (see [23] for concrete examples in the simplest cases; see also the next section).

Remark 17. One of the nicest theoretical accomplishments of equivariant bifurcation theory is the work of Krupa [14] on bifurcation from a relative equilibrium (see also [27]). In the present framework, this corresponds to bifurcation of a fixed point for the dynamics in the orbit space (see remark 3); if this undergoes a Hopf bifurcation, we recover the scenario described by Krupa (see also [26] for an example).

Remark 18. In a recent work, Chossat and Koenig [15] have developed a criterion for the existence of periodic branching solutions, based on the analysis of orbit space. Such a result can be seen as an extension of Michel analysis (originally focused on the existence of critical points for variational problems, or fixed points for the dynamics) to periodic solutions. This is indeed quite similar in spirit to the extension by Golubitsky and Stewart [12] of the EBL [24] to the case of Hopf bifurcations.[3]

Remark 19. The last few remarks suggest that Michel theory of symmetry breaking has been generally overlooked, and that a number of recent interesting results fit naturally into it, although obtained in different ways. It would be not surprising to discover that this theory has yet a number of interesting and useful results (maybe in the form of simple but useful corollaries, see the above remarks) to offer us, such as the splitting lemma presented here. I hope that the present discussion can also serve to induce the readers to familiarize with this beautiful theory and its developements.

[2] It should be stressed that at least some of the authors working at the developement of equivariant bifurcation theory, however, were aware not only of the existence of Michel theorem, but also of its relevance [22].

[3] We remark that, quite curiously, in [15] some results due to Michel are attributed to Field and Richardson [28], although these authors correctly attribute them to Michel in their very interesting and profound works [28].

6. Examples

Finally, we would like to consider in details some concrete examples, showing how the splitting lemma applies to simple dynamical systems.

Example 1. Let $M = \mathbf{R}^2$, $G = SO(2)$. Then, any G-equivariant dynamical system is written as

$$\dot{x} = f(x) \equiv (\alpha(\rho) I + \beta(\rho) J)\, x$$

where $\rho = (x^2 + y^2)/2$, I is the 2×2 identity matrix, and J is the generator of $SO(2)$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The $SO(2)$ orbits are indexed by $\rho \geq 0$, and indeed it is easy to see that

$$\dot{\rho} = (x \cdot \dot{x}) = \alpha(\rho) \cdot \rho \equiv f_\omega(\rho)$$

(notice that $\rho = 0$ is a fixed point for any α, β, as stipulated by Michel theorem).

We can therefore pass to polar coordinates (ρ, φ); in these we have

$$\begin{aligned} \dot{\rho} =& f_\omega(\rho) \equiv \rho \cdot \alpha(\rho) \\ \dot{\varphi} =& f_G(\rho, \varphi) \equiv \beta(\rho) \end{aligned}$$

which realizes the splitting of the dynamics.

Example 2. Let $M = \mathbf{R}^4$, $G = SU(2)$ acting with generators

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad H_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Then, G-equivariant dynamical systems are of the form

$$\dot{x} = f(x) \equiv \left[\alpha(\rho) I + \sum_{i=1}^{3} \beta_i(\rho) K_i \right] x$$

where $\rho = (1/2) \sum_{k=1}^{4} x_k^2$ and

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad K_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The $SU(2)$ orbits are indexed by ρ, and

$$\dot{\rho} = \alpha(\rho) \cdot \rho$$

Again, $x = 0$ is a fixed point for any G-equivariant f.

Notice that for both the present and the previous example, the evolution on the G-orbits which are fixed points for the dynamics in the orbit space (i.e. such that $\alpha(\rho) = 0$) is *linear*; we have therefore a spontaneous linearization for asymptotic dynamics [29]. This fact is related to the fact that we have only *one* invariant (ρ) for the G action on M.

Notice that all orbits - at the exception of $\rho = 0$ - belong to the same stratum, and for $x \neq 0$ we have $G_x = \{e\}$, $M^x = \mathbf{R}^4 \backslash \{0\}$, $\omega[x] \simeq G$.

The group $G = SU(2)$ is isomorphic to the sphere S^3, which admits a fibration by circles S^1, the *Hopf fibration.* Indeed, unless $\beta_1(\rho_0) = \beta_2(\rho_0) = \beta_3(\rho_0) = 0$, the flow by f on the orbits satisfying $\alpha(\rho_0) = 0$ gives raise to periodic solutions.

Example 3. Let $M = \mathbf{R}^3$, $G = SO(3)$. here again the orbits are indexed by $\rho = (1/2)\sum_{i=1}^3 x_i^2$. Now the G-equivariant dynamical systems are of the form

$$\dot{x} = f(x) \equiv \alpha(\rho) x$$

so that the dynamics along G is trivial, and we only have a nontrivial dynamics in Ω,

$$\dot{\rho} = \alpha(\rho) \cdot \rho$$

Thus, G-orbits such that $\alpha(\rho_0) = 0$ correspond to a sphere S^2 of fixed points. This fact is related to the irreducibility of the standard (defining) representation of $SO(3)$, not only on $\mathbf{R}$ but also on $\mathbf{C}$.

Notice that examples 1-3 are also examples of the three types of real irreducible representations, according to Schur lemma over $\mathbf{R}$ [30].

Example 4. Let $M = \mathbf{R}^2$ and $G = Z_2^{(1)} \times Z_2^{(2)}$, generated by

$$H_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we have two G-invariants, $\theta_1 = x_1^2$ and $\theta_2 = x_2^2$; any G-equivariant dynamical system is of the form

$$\dot{x}_i = f_i(x_1^2, x_2^2) \cdot x_i$$

The orbit space is the quadrant $\{x_1 \geq 0 \ , \ x_2 \geq 0\}$; and the point $x = 0$ has $G_x = G$, points $(x_1, 0)$ and $(0, x_2)$ (with $x_1 \neq 0$, $x_2 \neq 0$) have, respectively, $G_x = Z_2^{(2)}$ and $G_x = Z_2^{(1)}$; points (x_1, x_2) such that $x_1 x_2 \neq 0$ have $G_x = \{e\}$. Notice that $M^{Z_2^{(1)}} = (0, x_2)$, $M^{Z_2^{(2)}} = (x_1, 0)$; $M^{\{e\}} = M \backslash [\{0\} \cup M^{Z_2^{(1)}} \cup M^{Z_2^{(2)}}]$.

Let us consider the time evolution of θ_1, θ_2:

$$\begin{aligned} \dot{\theta}_1 =& 2 f_1(\theta_1, \theta_2) \cdot \theta_1 \\ \dot{\theta}_2 =& 2 f_2(\theta_1, \theta_2) \cdot \theta_2 \end{aligned}$$

Notice that, in agreement with general results [1-4], the strata are invariant under the flow, so that the axes cannot be crossed, and the motion will remain confined in a quadrant. This also means that the evolution on the (discrete) group G is trivial; this is indeed the only smooth evolution possible on a discrete group.

Example 5. Let $M = \mathbf{R}^2$ and $G = \mathbf{R}$ acting as dilations; notice that here we have a noncompact group. The G-equivariant, i.e. scale-invariant, dynamical systems are of the form

$$\dot{x} = A x$$

with A a numerical matrix; thus, we have linear dynamical systems. The G-orbits correspond to straight lines $y = mx$, and are indexed e.g. by $\theta = \tan(x_2/x_1)$; points along a G-orbit are indexed by $\rho = |x|^2$. Notice that $x = 0$ is a special G-orbit, and also belongs to all the G-orbits. [4]

The difficulty to write down the dynamical systems according to the splitting lemma - i.e., in this case, in polar coordinates - is indeed related to this singularity (although $x = 0$ is a simple fixed point for the dynamics). Notice however that locally, in the neighbourhood of any nonsingular point $x \neq 0$, we can easily write the dynamical system in polar coordinates, and the equation for $\dot{\theta}$ does not depend on ρ. The singularity in $x = 0$ forbids a global description.

Notice that if we take $G = SO(2) \times \mathbf{R}$ (where $\mathbf{R}$ act as before), the problem is not present. Indeed, G-equivariant dynamical systems are now of the form

$$\dot{x} = (aI + bJ)x$$

with J as in example 2; in polar coordinates, this is just

$$\dot{\rho} = a\rho \quad ; \quad \dot{\theta} = b$$

[4] The stratification of the orbit space for noncompact group actions presents some serious problems, and is not always well defined. The case of noncompact groups with a centralizer satisfying a compactness condition is discussed in [2, 22]; a more general discussion of the geometry of non compact group actions is given in [4]. Similar problems occur for infinite groups actions (here the very order relation between subgroups is endangered, as an infinite group can be isomorphic to one of its subgroups); in some case it is possible to obtain partial results, see [20].

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