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DIFFERENTIAL OPERATOR ALGEBRAS ON COMPACT RIEMANN SURFACES[1]

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Abstract. This talk reviews results on the structure of algebras consisting of meromorphic differential operators which are holomorphic outside a finite set of points on compact Riemann surfaces. For each partition into two disjoint subsets of the set of points where poles are allowed, a grading of the algebra and of the modules of $\lambda-$forms is introduced. With respect to this grading the Lie structure of the algebra and of the modules are almost graded ones. Central extensions and semi-infinite wedge representations are studied. If one considers only differential operators of degree 1 then these algebras are generalizations of the Virasoro algebra in genus zero, resp. of Krichever Novikov algebras in higher genus.

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1. Introduction

It is well-known that the Virasoro algebra together with its representations play a fundamental role in Conformal Field Theory (CFT). Let me recall the realization of the Virasoro algebra which is suitable for the generalization I am considering. The meromorphic vector fields on $\mathbb{P}^1$ (everything is over the complex numbers) which are holomorphic over $\mathbb{P}^1 \setminus \{0, \infty\}$ can be given as $p(z)\frac{d}{dz}$, with $p \in \mathbb{C}[z, z^{-1}]$ a Laurent polynomial. The commutator

$$[\,p(z)\frac{d}{dz}, r(z)\frac{d}{dz}\,] = \big(p(z)\frac{dr}{dz}(z) - r(z)\frac{dp}{dz}(z)\big)\frac{d}{dz} \tag{1}$$

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defines a Lie algebra structure. This Lie algebra is called the Witt algebra. Obviously a basis is given by

$$l_n := z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}\ ,$$

and one calculates immediately $[\, l_n, l_m\,] = (m-n)\, l_{n+m}$. The Witt algebra admits a universal central extension V with one-dimensional center

$$0 \longrightarrow \mathbb{C} \longrightarrow V \longrightarrow W \longrightarrow 0\ .$$

The algebra V is the Virasoro algebra. A basis is given by lifts L_n of l_n and a central element t. The well-known structure equations are

$$[\, L_n, L_m\,] = (m-n)L_{n+m} + \frac{1}{12}(m^3-m)\delta_{m,-n}\, t, \qquad [t, V] = 0\ . \tag{2}$$

It is important to keep in mind that the Virasoro algebra is not only this algebra but this algebra together with the standard grading induced by $\deg(L_n) = n$ and $\deg(t) = 0$. ¿From (2) we see immediately that V is a graded Lie algebra. Such a grading is essential if one wants to construct highest weight representations (e.g. by using semi-infinite wedge forms). In CFT one is exactly looking for such representations. I do not want to go into the applications to CFT here but let me just say that in string theory the Virasoro algebra (or better the Witt algebra) describes the situation of a world sheet of genus zero with one incoming string (at $P = \{z=0\}$) and one outgoing string (at $Q = \{z=\infty\}$) (see Fig. 1).

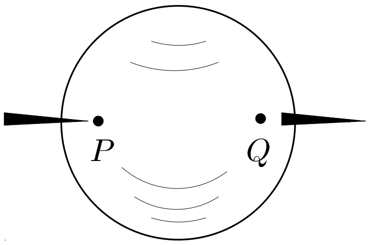

Fig. 1: Virasoro situation ($g = 0,\ N = 2$)

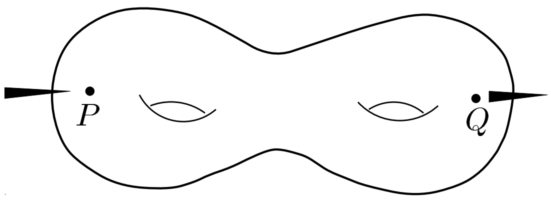

Fig. 2: An example of a Krichever - Novikov situation

¿From this point of view it is quite natural to generalize the situation to Riemann surfaces of arbitrary genus and an arbitrary (but fixed) number of incoming strings and an arbitrary (again fixed) number of outgoing string. For one incoming and one outgoing string this was done by Krichever and Novikov (1987/88) [10] (see Fig. 2) and for arbitrary numbers by myself (1989–) [15],[16] (see Fig. 3).

I will consider the more general situation of differential operators of arbitrary degree. The degree one case corresponds to the vector field case. In this talk I will review some of the results obtained. For more information I have to refer to [16] and [18].

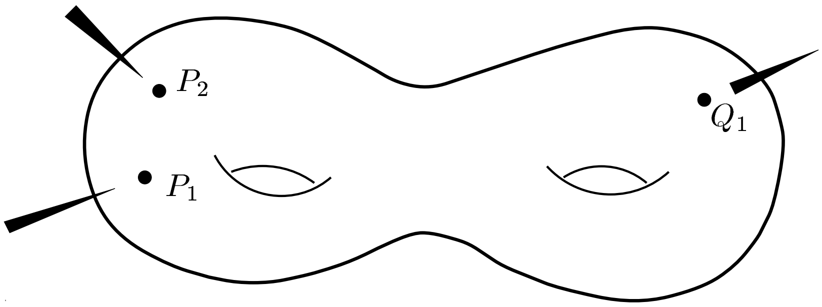

Fig. 3: An example of a generalized situation ($N = 3$, 2 in-points, 1 out-point)

Let me add that the many-point situation was also studied by Dick [3], by Guo, Na, Shen, Wang, Yu, Wu, Chang (see for example [6]), and by Bremner (3 points and genus 0) [2] but without introducing a grading. Only Sadov [14] did some work which is related to my approach.

2. The set-up and the involved algebras

For the following let X be a Riemann surface (or a nonsingular algebraic curve over an arbitrary algebraically closed field k with char $k = 0$), A a finite set of points which is divided into two disjoint subsets I and O, $A = I \cup O$, $\#I = k \geq 1$, $\#O = l \geq 1$, $N = k + l$. The elements of I are the "in-points", the elements of O are the "out-points". Let ρ be a meromorphic differential holomorphic outside A with exact pole order 1 at the points of A, (given) positive residues at I, (given) negative residues at O (of course, obeying $\sum_P \mathrm{res}_P(\rho) = 0$), and purely imaginary periods. Fix a point $Q \in X \setminus A$. Then the function

$$u(P) = \mathrm{Re} \int_Q^P \rho \tag{3}$$

is a well-defined harmonic function. The level lines

$$C_\tau = \{P \in X \setminus A \mid u(P) = \tau\}, \quad \tau \in \mathbb{R} \tag{4}$$

define a fibering of $X \setminus A$ (see Fig. 4).

Every level line separates the in- from the out-points. For $\tau \ll 0$ the level line C_τ is a disjoint union of deformed circles around the points in I. For $\tau \gg 0$ it is a disjoint union of deformed circles around the points in O. In the interpretation of string theory this τ might be interpreted as proper time of the string on the world sheet.

Let K be the canonical bundle, i.e. the bundle whose local sections are the local holomorphic differentials. For every $\lambda \in \mathbb{Z}$ we consider the bundle $K^\lambda := K^{\otimes\lambda}$, the bundle with local sections the forms of weight λ. After fixing a square root of the

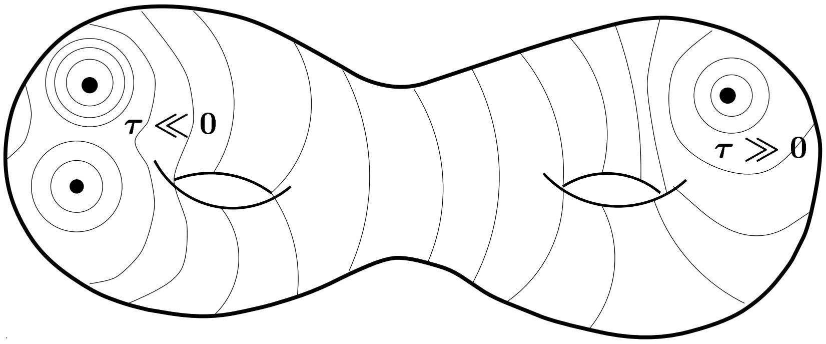

Fig. 4: Fibering by level lines

canonical bundle (a so-called theta characteristic) everything below can even be done for $\lambda \in \frac{1}{2}\mathbb{Z}$. For simplicity we consider here only the case of integer λ. We denote by $\mathcal{F}^\lambda(A)$ the vector space of global meromorphic sections of K^λ which are holomorphic on $X \setminus A$. Special cases are the differentials ($\lambda = 1$), the functions ($\lambda = 0$), and the vector fields ($\lambda = -1$). For the vector fields we also use the special notation $\mathcal{KN}(A)$.

The functions (i.e. the elements in $\mathcal{F}^0(A)$) operate by multiplication on $\mathcal{F}^\lambda(A)$. The vector fields (i.e. the elements in $\mathcal{KN}(A)$) operate by taking the Lie derivative on $\mathcal{F}^\lambda(A)$. In local coordinates the Lie derivative can be described as

$$L_e(g)_| = (e(z)\frac{d}{dz}) \,.\, (g(z)\,dz^\lambda) = \left(e(z)\frac{dg}{dz}(z) + \lambda\, g(z)\frac{de}{dz}(z)\right)\, dz^\lambda\ . \tag{5}$$

Here and in the following I will use the same symbol for the section of the bundle and its local representing function. By (5) $\mathcal{KN}(A)$ becomes a Lie algebra and the vector spaces $\mathcal{F}^\lambda(A)$ become Lie modules over $\mathcal{KN}(A)$ (i.e. $[L_e, L_f] = L_{[e,f]}$). The algebra $\mathcal{KN}(A)$ I call a generalized Krichever Novikov algebra.

For an arbitrary associative algebra R we obtain always a Lie algebra LR with same underlying vector space and taking the commutator $[a,b] = a\cdot b - b\cdot a$ as Lie product. Obviously, $L\mathcal{F}^0(A)$ is an abelian Lie algebra. We take the semi-direct product

$$\mathcal{D}^1(A) = L\mathcal{F}^0(A) \times \mathcal{KN}(A)$$

to obtain the Lie algebra of differential operators of degree ≤ 1 (which obey again the above regularity conditions). Its structure is given by

$$[\,(g,e),(h,f)\,] := (\,L_e h - L_f g\,,\,[e,f]\,), \tag{6}$$

and we have the exact sequence of Lie algebras

$$0 \longrightarrow L\mathcal{F}^0(A) \longrightarrow \mathcal{D}^1(A) \longrightarrow \mathcal{KN}(A) \longrightarrow 0\ . \tag{7}$$

Note that $\mathcal{KN}(A)$ is the subalgebra of differential operators of degree 1. The spaces $\mathcal{F}^\lambda(A)$ now become Lie modules over $\mathcal{D}^1(A)$.

If we want differential operators of arbitrary degree we have to do some universal constructions. I use the following constructions:

$$\mathcal{D}(A) = U\mathcal{D}^1(A)/J, \quad \text{resp.} \quad \mathcal{D}_\lambda(A) = U\mathcal{D}^1(A)/J_\lambda,$$

where $U\mathcal{D}^1(A)$ is the universal enveloping algebra of $\mathcal{D}^1(A)$ (with multiplication $\odot$ and unit $\mathbb{1}$), and J resp. J_λ are the following two-sided ideals

$$\begin{aligned} J &:= (\ a \odot b - a \cdot b,\ \mathbb{1} - 1 \mid a, b \in \mathcal{F}^0(A)\ ), \\ J_\lambda &:= (\ a \odot b - a \cdot b,\ \mathbb{1} - 1,\ a \odot e - a \cdot e + \lambda L_e(a) \mid a, b \in \mathcal{F}^0(A), e \in \mathcal{KN}(A)\ ). \end{aligned}$$

All $\mathcal{F}^\lambda(A)$ are modules over $\mathcal{D}(A)$, but only $\mathcal{F}^\lambda(A)$ is a module over $\mathcal{D}_\lambda(A)$. It is easy to check that every element D of $\mathcal{D}(A)$ resp. of $\mathcal{D}_\lambda(A)$ operates as differential operator on $\mathcal{F}^\lambda(A)$. I call the elements of $\mathcal{D}(A)$ the coherent differential operators and the elements of $\mathcal{D}_\lambda(A)$ the differential operators on $\mathcal{F}^\lambda(A)$. Because $X \setminus A$ is affine every algebraic differential operator can be represented by such an element D [4].

3. The grading

It is possible to introduce a grading in $\mathcal{F}^\lambda(A)$ in such a way that the homogeneous subspaces $\mathcal{F}^\lambda_n(A)$ of degree n are finite-dimensional and that the Lie module structure is an almost graded one. Of course, this is a crucial step. Nevertheless I want to skip the details here because they can be found in [15]. Essentially the grading is given by the zero order at the points in I. To give an example: Let $k = l$, $\lambda \neq 0, 1$, and $I = \{P_1, P_2, \dots, P_k\}$, $O = \{Q_1, Q_2, \dots, Q_k\}$ be points in generic positions. Then there is for every $n \in \mathbb{Z}$ and every $p = 1, \dots, k$ up to multiplication with a scalar a unique element $f^\lambda_{n,p} \in \mathcal{F}^\lambda(A)$ with

$$\begin{aligned} \operatorname{ord}_{P_i}(f^\lambda_{n,p}) &= (n + 1 - \lambda) - \delta_{i,p}, \qquad && i = 1, \dots, k, \\ \operatorname{ord}_{Q_i}(f^\lambda_{n,p}) &= -(n + 1 - \lambda), && i = 1, \dots, k-1, \\ \operatorname{ord}_{Q_k}(f^\lambda_{n,p}) &= -(n + 1 - \lambda) + (2\lambda - 1)(g - 1)\ . \end{aligned} \tag{8}$$

This can be shown either by using Riemann-Roch type arguments or by explicit constructions. After fixing a coordinate at the points P_i the scalar constant can be fixed. The elements $f^\lambda_{n,p}$ for $n \in \mathbb{Z}$ and $p = 1, \dots, k$ are a basis of $\mathcal{F}^\lambda(A)$. The homogeneous elements are defined to be the elements of the space

$$\mathcal{F}^\lambda_n(A) := \langle\ f^\lambda_{n,p} \mid p = 1, \dots, k\ \rangle, \quad \text{resp.} \quad \mathcal{D}^1_n(A) := \langle\ f^0_{n,p}, f^{-1}_{n,p} \mid p = 1, \dots, k\ \rangle\ .$$

The basis elements obey the important duality relation (after a suitable fixing of the scalar)

$$\frac{1}{2\pi\,\mathbf{i}} \int_{C_\tau} f^{\lambda}_{n,p} \cdot f^{1-\lambda}_{m,r} = \delta_{n,-m} \cdot \delta_{p,r}, \tag{9}$$

where C_τ is any non-singular level line. In fact, the grading depends on the numbering of the points Q_i but the induced filtration depends only on the partition $A = I \cup O$. Another essentially different partition induces a non-equivalent filtration. Note that the duality (9) is true in all cases not only in the example discussed above.

An almost graded structure means that the homogeneous subspaces are finite-dimensional and that there are constants K and L not depending on n and m such that

$$[\,\mathcal{D}^1_n(A), \mathcal{D}^1_m(A)\,] \subseteq \bigoplus_{h=n+m+K}^{n+m+L} \mathcal{D}^1_h(A), \qquad \mathcal{D}^1_n(A) \,.\, \mathcal{F}^{\lambda}_m(A) \subseteq \bigoplus_{h=n+m+K}^{n+m+L} \mathcal{F}^{\lambda}_h(A)\ . \tag{10}$$

In our situation $K = 0$ and there exist explicit formulas for L which depend on the genus g and on $N = k + l$. In the Virasoro situation ($g = 0$, 2 points) $L = 0$ and everything reduces to the well-known graded situation. Here I like to stress the fact, that the conditions (10) are necessary to construct semi-infinite wedge representations (see Section 5).

4. Central extensions

As it is well-known it is not possible to obtain representations of the Witt algebra with certain properties one is interested in (irreducible, unitary highest weight representations). Such representations exist only for the central extension of the Witt algebra, the Virasoro algebra. Hence, it is necessary to study central extensions also for the above algebras. Central extensions can be given by 2-cocycles (with respect to Lie algebra cohomology). Such defining cocycles are known for the Virasoro situation. Unfortunately these cocycles are not defined invariantly. Only after adding some suitable counterterms, we obtain objects which make sense on arbitrary Riemann surfaces.

(**1**) Let R be a holomorphic projective connection[1] , then

$$\chi(e,f) = \frac{1}{24\pi\,\mathbf{i}} \int_{C_\tau} \left(\frac{1}{2}(e'''f' - e'f''') - R \cdot (e'f - ef') \right) dz \tag{11}$$

for $e, f \in \mathcal{KN}(A) \subseteq \mathcal{D}^1(A)$, defines a nontrivial 2-cocycle of $\mathcal{KN}(A)$. By pull-back via (7) we obtain a cocycle for $\mathcal{D}^1(A)$. Recall, we represent e by $e(z)\frac{d}{dz}$ and the prime means derivative of $e(z)$ with respect to z. This cocycle is a generalization to the many-point situation of the cocycle introduced by Krichever and Novikov [10].

[1] see the appendix for the definition

(**2**) For the abelian Lie algebra $L\mathcal{F}^0(A)$ a non-trivial 2-cocycle is defined by

$$\gamma(g,h) = \frac{1}{2\pi\,\mathbf{i}} \int_{C_\tau} g\,dh \tag{12}$$

for $g,h \in \mathcal{F}^0(A)$. The centrally extended Lie algebra is the generalized Heisenberg algebra. For applications see for example [8]. It is easy to show that γ can be extended to a cocycle on $\mathcal{D}^1(A)$ via $\gamma((g,e),(h,f)) := \gamma(g,h)$.

(**3**) There is another cocycle of $\mathcal{D}^1(A)$ which connects $L\mathcal{F}^0(A)$ with $\mathcal{KN}(A)$. Let T be an affine meromorphic connection[2] with at most one pole at a point of O then

$$\beta(e,g) = -\beta(g,e) = \frac{1}{2\pi\,\mathbf{i}} \int_{C_\tau} (eg'' + T\cdot eg')dz \tag{13}$$

for $e=(0,e)$ and $g=(g,0)\in\mathcal{D}^1$ defines a 2-cocycle by obvious extension.

In any of the cases above the choice of another connection does not change the cohomology class of the cocycle. Hence, the equivalence class of the central extension does not depend on the chosen connection. Obviously, the cocycle is independent on the value of τ used to fix the integration curve C_τ. Of course, the above expressions would also define a cocycle if we integrate along any other closed curve on $X\setminus A$. By considering only integration along C_τ we obtain cocycles which are local in the sense of Krichever and Novikov with respect to the grading of the algebra $\mathcal{D}^1(A)$ induced by the partition of A. A cocycle is called local if there are constants M and H such that for every homogeneous $e,f\in\mathcal{D}^1(A)$

$$\psi(e,f)=0 \qquad \text{if } \deg(e)+deg(f)>H, \quad \text{or } \deg(e)+deg(f)<M. \tag{14}$$

In our situation we have $H=0$, and M depends on the genus and the number of points in A. Why is locality important? It allows us to extend the almost grading to the centrally extended algebra by defining the central element t to have degree zero. Recall that the structure equations of the central extension

$$0 \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathcal{D}}^1(A) \xrightarrow{r} \mathcal{D}^1(A) \longrightarrow 0 \tag{15}$$

defined by the cocycle ψ is given by

$$[\Phi(e),\Phi(h)] = \Phi([\,e,h\,]) + \psi(e,h)\cdot t, \tag{16}$$

where Φ is a linear splitting map of r (resp. a lift of the basis elements in $\mathcal{D}^1(A)$) and t is the image of 1 in $\widehat{\mathcal{D}}^1(A)$.

[2] see the appendix for the definition

Remarks. **1.** For $g=0, N=2$ everything reduces to the well-known case. In particular, we obtain again the 3 linearly independent cocycles for $\mathcal{D}^1(\{0,\infty\})$ studied by Arbarello-DeConcini-Kac-Procesi [1]. They showed that $H^2(\mathcal{D}^1(\{0,\infty\}))$ is 3-dimensional. Hence every one-dimensional central extension is defined up to equivalence by a certain linear combination of the above given ones.

2. For the general situation this is not true anymore. Different partitions of the points will yield different level lines and in general non-equivalent central extensions. But the new cocycle is not local anymore with respect to the first partition. Hence,

Conjecture. *(a) Every local cocycle of $\mathcal{D}^1(A)$ is cohomologous to a linear combination of the above 3 cocycles (11), (12), (13).*
(b) Every local cocycle of $\mathcal{KN}(A)$ is cohomologous to the cocycle (11).

3. If we consider $\mathcal{D}(A)$ or $\mathcal{D}_\lambda(A)$ as Lie algebras then $\mathcal{D}^1(A)$ is a Lie subalgebra. But it is not clear (and in fact not true) whether any of the above 3 cocycles can be extended to the whole of $\mathcal{D}_\lambda(A)$. In the Virasoro situation Radul introduced for $\mathcal{D}_0(A)$ a "canonical cocycle" [12] (which of course is an extension of a certain linear combination of the above three cocycles). It was shown by Li [11] that this is in fact the only one which could be extended. Using semi-infinite wedge representations I was able to show that there is a central extension $\widehat{\mathcal{D}}_\lambda(A)$ (as Lie algebra !) of $\mathcal{D}_\lambda(A)$. The defining cocycle ψ_λ restricted to $\mathcal{D}^1(A)$ is local. If we accept the above conjecture than it has to be cohomologous to a certain linear combination of the above cocycles. In this case it can be calculated as

$$[\psi_\lambda]=[\chi]+\frac{2\lambda-1}{2c_\lambda}[\beta]+\frac{-1}{c_\lambda}[\gamma], \tag{17}$$

where $c_\lambda:=-2(6\lambda^2-6\lambda+1)$ denotes the famous expression from Mumford's formula and [..] the cohomology class.

4. It is possible to define higher genus analogues of untwisted Kac-Moody algebras. For this let $\mathfrak{g}$ be a Lie algebra with $<.,.>$ an invariant, symmetric, bilinear form on $\mathfrak{g}$ (i.e. $<[x,y],z>=<x,[y,z]>$). $\widehat{\mathfrak{g}}=(\mathfrak{g}\otimes\mathcal{F}^0(A))\oplus\mathbb{C}c$ is now a Lie algebra if we define

$$[x\otimes f, y\otimes g]=[x,y]\otimes f\cdot g-<x,y>\gamma(f,g)\cdot c, \qquad [c,\widehat{\mathfrak{g}}]=0 \tag{18}$$

where γ is the cocycle introduced in (12). For $g=0,\ A=\{0,\infty\}$ they coincide exactly with the untwisted affine Lie algebras. For $N=2$ the same construction has been given by Krichever and Novikov [10] and if additionally $g=1$ they were examined in detail by Sheinman [19].

5. Further results

Here space does not permit to explain the semi-infinite wedge representations in detail. Roughly speaking, the semi-infinite wedge spaces $\mathcal{H}^\lambda(A)$ are "restricted $\frac{\infty}{2}$ external forms" constructed from the elements of $\mathcal{F}^\lambda(A)$. The elements of $\mathcal{H}^\lambda(A)$ are called semi-infinite forms of weight λ. To extend the action of $\mathcal{D}_\lambda(A)$ (resp. $\mathcal{KN}(A)$, or $\mathcal{D}^1(A)$) on $\mathcal{F}^\lambda(A)$ to $\mathcal{H}^\lambda(A)$ one has to "regularize the action" and hence has to allow for central extensions of these Lie algebras. The regularization is done by embedding the algebras via their action on $\mathcal{F}^\lambda(A)$ into $\overline{gl}(\infty)$, the algebra of both-sided infinite matrices with only finitely many diagonals. Here the almost graded structure is crucial. For the algebra $\overline{gl}(\infty)$ the regularization and central extension is well-known [9]. Pulling back the action and the cocycle gives a regularized action for a centrally extended algebra of $\mathcal{D}_\lambda(A)$ (resp. of $\mathcal{KN}(A)$, or of $\mathcal{D}^1(A)$). The pulled back cocycle is local. Assuming the conjecture to be true it can be written in the form (17).

If we consider only the algebra $\mathcal{KN}(A)$ and if we require a coherent action of $\widehat{\mathcal{KN}}(A)$ on every $\mathcal{H}^\lambda(A)$ we obtain that the central element operates as $(-2)(6\lambda^2-6\lambda+1)\cdot \mathit{identity}$.

In this manner one obtains highest weight, resp. Verma module representations of the centrally extended algebras.

Let me just give a brief summary what else has been done. It is possible to define an action of $\mathcal{F}^\lambda(A)$ and $\mathcal{F}^{1-\lambda}(A)$ on $\mathcal{H}^\lambda(A)$ by wedging, resp. contracting the semi-infinite forms. By this one obtains a Clifford algebra structure (or a $b-c$ system in physicist's language). There are interesting compatibility relations of the $\widehat{\mathcal{D}}^1(A)$ action and the $b-c$ action. In addition there is a natural pairing between right semi-infinite forms of weight λ and left semi-infinite forms of weight $1-\lambda$ induced by the pairing (9).

In the case of the torus ($g=1$) explicit calculations have been done in [13] for the 2, 3 and 4 point case. In these cases also degenerations of the Riemann surface have been studied [17]. Here it is advantageous to use the language of Algebraic Geometry. It is interesting to note that starting from a two-point situation one is forced to study many-point situations on Riemann surfaces of lower genus.

Appendix: Affine and projective connections

Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha) = h(z_\alpha)$. A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ resp. $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) projective (resp. affine) connection if it transforms as

$$R_\beta(z_b)\cdot (h')^2 = R(z_\alpha) + S(h), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 \tag{19}$$

the Schwartzian derivative, resp.

$$T_\beta(z_b)\cdot h' = T(z_\alpha) + \frac{h''}{h'}\ . \tag{20}$$

Here $'$ denotes differentiation with respect to the coordinate z_α. Note that the difference of two affine (projective) connections is always a usual (quadratic) differential.

Lemma. *(a) There is always a holomorphic projective connection.*
(b) Given a point P on X there is always a meromorphic affine connection which is holomorphic outside P and has there at most a pole of order 1.

For a proof of (a) see for example [7] or [5]. A proof of (a) and (b) can also be found in [16].

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