October 1994
LPTHE Orsay-94/29

SEMI CLASSICAL SUM RULES AND GENERALIZED COHERENT STATES

M. COMBESCURE*, **D. ROBERT****

* LPTHE Bâtiment 211, Université de Paris Sud, 91405 Orsay, France[1]

** Département de Mathématiques, Université de Nantes,
2 rue de la Houssinière, 44072 NANTES CEDEX 03, France [2]

Abstract

Various semi-classical sum rules are obtained for matrix elements of smooth observables using semi-classical estimates of time evolved coherent states together with stationary phase theorems.

[1]Laboratoire associé au Centre National de la Recherche Scientifique.
[2]URA 758 du Centre National de la Recherche Scientifique.

1 Introduction

Semi-classical sum rules are often encountered in the physics literature, in relationship with the so-called problem of quantum chaos [1, 5, 10, 12, 13, 14, 17]. In this respect, the statistical properties of transition probabilities (equal to the squared matrix elements of operators having a classical limit) are investigated, and in particular their fluctuation properties around a smooth mean part. Several sum rule formulas have been established, in connection with the more or less ergodic (or mixing) properties of the underlying classical dynamics. However these derivations are most of the time rather heuristic and in particular it is not always clear which "chaotic" properties of the classical dynamics are actually needed to obtain them (namely ergodicity or mixing properties). Moreover, the use of "coherent states" is often suggested, in connection with their intuitive link with the classical phase space picture, but their precise relevance and usefulness has never been established rigorously, up to now, to our knowledge.

On the other side, the mathematical literature has provided a new insight into the semi-classical behavior of classically chaotic systems [16, 8, 18, 19]. Most of these works rely on the pseudo-differential calculus [9] and microlocal analysis [15, 8] which allows a precise treatment of the dominant terms and of the remainder when $\hbar$ goes to zero. Recently a study of the classical limit of matrix elements of an observable has also been proposed, which tends to avoid the whole microlocal machinery by making use of the usual coherent states and the corresponding Bargmann representation [11]. However the problems to which this study applies are essentially those of a regular classical dynamics, and the particular way that $\hbar$ tends to zero in this approach reflects this particular situation via a "Bohr-Sommerfeld type condition".

Here we want to develop a method for obtaining semi-classical sum rules also by avoiding the whole pseudo-differential calculus, but instead by making use of precise semiclassical evolution estimates of generalized coherent states [6, 7, 3]. These estimates combined with the stationary phase theorems, allow a simple derivation of various sum rules in the semi-classical limit, with a precise estimate of the errors terms, and they can be applied to situations where the corresponding classical motion is chaotic. It is an alternative of our preceeding work [4], where similar results were obtained through the usual pseudo-differential calculus. This approach, has several advantages and drawbacks connected with the use of the coherent states, that we shall stress in the course of the paper.

2 An elementary sum rule for diagonal, matrix elements.

We shall consider a quantum system in $L^2(\mathbb{R}^n)$ where dynamics is governed by the Schrödinger Hamiltonian

$$P(\hbar) = -\hbar^2\Delta + V(x) \tag{2.1}$$

where Δ is the Laplacian in $L^2(\mathbb{R}^n)$, and $V(x)$ a real potential function in $\mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$|D^\alpha V(x)| \leq C(1+|x|^2)^{(\delta-|\alpha|)/2} \tag{2.2}$$

for suitable C and $\delta > 0$. We furthermore assume that

$(H.1)$ the spectrum of $P(\hbar)$ is purely discrete on I_{cl}, I_{cl} being an energy interval centered around E. We denote by E_j and φ_j respectively the eigenvalues and eigenstates of $P(\hbar)$ belonging to I_{cl}. Of course they depend on $\hbar$.

Let A be a self-adjoint operator in $L^2(\mathbb{R}^n)$ originating from a symbol $a(q,p)$ which is $\mathcal{C}^\infty(\mathbb{R}^{2n})$. We shall be interested in the matrix elements $A_{ij}(\hbar)$ of the operator A in the states φ_j, namely

$$A_{ij}(\hbar) :=< \varphi_i, A\varphi_j > \tag{2.3}$$

Then, under the assumption that the classical flow if ergodic on the energy shell $\sum_E = \{(q,p) \in \mathbb{R}^{2n} : \mathrm{p}^2 + \mathrm{V(q)} = \mathrm{E}\}$, it is known [16, 8, 2, 18] that for "almost all j", we have

$$\lim_{\hbar\searrow 0} A_{jj}(\hbar) = \int_{\sum_E} a \; d\sigma_E \tag{2.4}$$

where $d\sigma_E$ is the Liouville measure on $\sum_E$, and "almost all j" can be given a precise meaning independent of the operator A. Here we shall be interested to a semi-classical estimate of a sum like :

$$\sum_{j:E_j\in I_{cl}} A_{jj}(\hbar) g(\frac{E-E_j}{\hbar}) = \hat{g}(0)(2\pi\hbar)^{-n+1}(\int_{\sum_E} a d\sigma_E + O(\hbar)) \tag{2.5}$$

where g is a real function on $\mathbb{R}$ whose Fourier transform $\hat{g}$ is of compact support. This estimate is less accurate than property (2.4) and in particular it holds without ergodicity assumption on the classical flow. However it is interesting in itself, in particular as a preparation on a more sophisticated sum rule bearing on the variance of the statistical distribution of the series of non diagonal elements $\{A_{jh}(\hbar)\}_{jk}$, which will be discussed in the following section. The tools and estimates will be essentially the same in both sections, but we hope that they will be more transparent in the simpler case of (2.5).

Theorem 1 Let A be a self-adjoint operator in $L^2(\mathbb{R}^n)$ with symbol $a(q,p)$ being in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and let g be a real function on $\mathbb{R}$ whose Fourier transform $\hat{g}$ is of compact support, with support contained in $[-T,T]$. Let $I_{cl} \subset \mathbb{R}$ be a classical energy interval centered around E such that the spectrum of $P(\hbar) = -\hbar^2\Delta + V$ is purely discrete on I_{cl}, and denote by E_j and φ_j respectively the eigenvalues and eigenstates of $P(\hbar)$. Then if T is small enough, there exists h_T small enough such that

$$S(E) := \sum_{j:E_j\in I_{cl}} <\varphi_j, A\varphi_j> \quad g \quad (\frac{E-E_j}{\hbar}) = \hat{g}(0)(2\pi\hbar)^{-n+1}\{\int_{\Sigma_E} d\sigma_E(z)a_E(z) + O(\hbar)\} \tag{2.6}$$

$\forall \hbar \in [0, \hbar_T]$
where Σ_E is the energy surface $p^2 + V(q) = E$, with $\sigma_E(z)$ the Liouville measure on it, and where $a_E(z)$ is the restriction of the symbol $a(q,p)$ to Σ_E.

Proof: As in [4], we introduce a $\mathcal{C}_0^\infty(I_{cl})$ function χ, and denote by $A_\chi(\hbar)$ the self-adjoint operator :

$$A_\chi(\hbar) = \chi(P(\hbar))A\chi(P(\hbar)) \tag{2.7}$$

and we have

$$S(E) = \sum_{E_j\in I_{cl}} <\varphi_j, A_\chi(\hbar)\varphi_j> g(\frac{E-E_j}{\hbar}) + O(\hbar^\infty) \tag{2.8}$$

We then write

$$\begin{aligned}\sum_{E_j\in I_{cl}} \quad & <\varphi_j, A_\chi(\hbar)\varphi_j> g(\frac{E-E_j}{\hbar}) := S_\chi(E) \\ &= \sum_j \int \hat{g}(t)e^{it(E-E_j)/\hbar} <\varphi_j, A_\chi(\hbar)\varphi_j> dt \\ &= \int \hat{g}(t)e^{iEt/\hbar}Tr\{A_\chi U(t)\}dt\end{aligned} \tag{2.9}$$

where $U(t) = \exp[-itP(\hbar)/\hbar]$ is the quantum propagator associated to Hamiltonian $P(\hbar)$. We now introduce the coherent states φ_α ($\alpha = q + ip \in \mathbb{C}^n$, or equivalently $\alpha = (q,p) \in \mathbb{R}^{2n}$)

$$\varphi_\alpha := D(\alpha)\phi_0 \tag{2.10}$$

where $\phi_0(x) = \exp(-x^2/2\hbar)$ and

$$D(\alpha) = \exp{(i\frac{Q.p - P.q}{\hbar\sqrt{2}})} \qquad P = -i\hbar\nabla \tag{2.11}$$

Then it is known that, for any operator B which is in trace class :

$$TrB = (2\pi\hbar)^{-n} \int < \varphi_\alpha, B\varphi_\alpha > d\alpha \tag{2.12}$$

Thus, denoting $d\alpha = dqdp$:

$$S_\chi(E) = (2\pi\hbar)^{-n} \int \hat{g}(t)e^{iEt/\hbar} < \varphi_\alpha, A_\chi U(t)\varphi_\alpha > dtd\alpha \tag{2.13}$$

We now assume that the so-called "Anti-Wick quantization condition" has been used for the operator A_χ, namely

$$A_\chi = (2\pi\hbar)^{-n} \int a_\chi(q',p')P_{\alpha'}d\alpha' \tag{2.14}$$

where P_α is the operator of orthogonal projection over $\varphi_\alpha, \alpha' = (q',p') \in \mathbb{R}^{2n}$ and $a_\chi(x,\xi)$ has $a(x,\xi)\chi^2(P(x,\xi))$ as its principal symbol. Thus we get :

$$S_\chi(E) = (2\pi\hbar)^{-2n} \quad \int \quad \hat{g}(t)e^{iEt/\hbar}a_\chi(\alpha') < \varphi_\alpha, \varphi_{\alpha'} >< \varphi_{\alpha'}, U(t)\varphi_\alpha > d\alpha d\alpha' dt \tag{2.15}$$

Now we have the following elementary property of coherent states :

Lemma 1 $< \varphi_\alpha, \varphi_{\alpha'} >= \exp\{-\frac{|\alpha-\alpha'|^2}{2\hbar} - \frac{\alpha.\bar{\alpha}'-\bar{\alpha}.\alpha'}{2\hbar}\}$ (Note that the argument of the exponential is written as a real part plus a purely imaginary part.)

Proof of Lemma 1 Using the usual creation and annihilation operators

$$a = \frac{Q+iP}{\sqrt{2\hbar}}, a^+ = \frac{Q-iP}{\sqrt{2\hbar}} \qquad P = -i\hbar\nabla \tag{2.16}$$

and the Baker-Campbell-Hausdorff formula for

$$D(\alpha) = \exp\{\frac{1}{\sqrt{\hbar}}(\alpha.a^+ - \bar{\alpha}.a)\} = \exp(-\frac{|\alpha|^2}{2\hbar})\exp(\frac{\alpha.a^+}{\sqrt{\hbar}})\exp(-\frac{\bar{\alpha}.a}{\sqrt{\hbar}}) \tag{2.17}$$

we easily get (since $a\phi_0 = 0$)

$$\varphi_\alpha = \exp(-\frac{|\alpha|^2}{2\hbar}) \sum_{m_i\in N} \prod_{i=1}^{n} (\frac{\alpha_i}{\sqrt{\hbar}})^{m_i} \frac{1}{\sqrt{m_i!}} \varphi_{m_1} \otimes \ldots \otimes \varphi_{m_n} \tag{2.18}$$

φ_m being the m^{th} eigenstate of the 1-dimensional harmonic oscillator, and

$$< \varphi_\alpha, \varphi_{\alpha'} >= \exp[-\frac{|\alpha|^2 + |\alpha'|^2}{2\hbar}] \prod_{i=1}^{n} \sum_{0}^{\infty} (\frac{\bar{\alpha}_i\alpha'_i}{\hbar})^m \frac{1}{m!} \tag{2.19}$$

which yields the result by elementary algebra. (We recall the notation $A.B = \sum_{j=1}^{n} A_j.B_j$, A_j and B_j being c-numbers or operators). ∎

We now recall known results by Hagedorn and others [3, 6, 7] about the semiclassical estimate of the quantum evolution of coherent states. Assuming that the potential V has polynomially bounded derivatives, we give the result in a slightly different form than in the literature, since we need here an estimate on the behavior of the error term when α moves in phase space.

Let $m = (m_1, ..., m_n) \in \mathbb{N}$ be a multi index, and denote by $|m|$ and $m!$ the usual expressions :

$$\begin{aligned} |m| &= m_1 + ... + m_n && (2.20) \\ m\,! &= m_1\,!...m_n\,! && (2.21) \end{aligned}$$

If $\varphi_{m_i}(x_i)$ denotes the m_i^{th} normalized eigenstate of the one-dimensional harmonic oscillator of unit frequency, let us denote by $\phi_m(x)$ $(x \in \mathbb{R}^n)$ the following normalized state in $L^2(\mathbb{R}^n)$:

$$\phi_m(x) = \varphi_{m_1}(x_1)\varphi_{m_2}(x_2)...\varphi_{m_n}(x_n) \tag{2.22}$$

Now denote by $\alpha_t = (q_t, p_t) \in \mathbb{R}^{2n}$ the phase space point reached at time t by the classical trajectory starting at phase space point $(q, p) = \alpha$ at time 0, and by $S(t)$ the classical action along this trajectory. Let, as in [6], $c_m(t, \hbar)$ be the unique solution to the system of coupled ordinary differential equations :

$$\frac{dc_m}{dt} = \sum_{|j|=0}^{3(\ell-1)} \sum_{|k|=3}^{\ell+1} -i\hbar^{-1+|k|/2}(k!)^{-1} \tag{2.23}$$

$$(D^kV)(q_t)b_{m,k,j}(t)c_j(t, \hbar)$$

subject to the initial conditions : $c_o(0, \hbar) = 1$ and $c_m(0, \hbar) = 0$ for $|m| \geq 1$, where $b_{m,k,j}(t)$ are $\hbar$-independent, and are defined in [6]. Then we have the following result :

Proposition 2 : Assume V has polynomially bounded partial derivatives. Then there exists $\delta' > 0$, and for any given $T \in \mathbb{R}$ and $\ell \in \mathbb{N}$ a positive constant C such that :

$$\sup_{t\in[-T,T]} ||U(t)\varphi_\alpha - e^{i\delta(t)/\hbar}\phi_{\alpha,t}|| \leq C\hbar^{\ell/2}(1 + |\alpha|^2)^{\delta'} \tag{2.24}$$

where $\phi_{\alpha,t}$ is a superposition of "squeezed states" of the following form :

$$\phi_{\alpha,t} = D(\alpha_t)T(B_t) \sum_{|m|=0}^{3(\ell-1)} c_m(t, \hbar)\phi_m \tag{2.25}$$

B_t being a $n \times n$ complex hermitian matrix defined in [3], $\delta(t) = S(t) - \frac{q_t.p_t - q.p}{2}$, and the operator $T(B)$ is defined below ((2.29))

We do not give here the detailed proof of Proposition 2, since it is a simple extension of the estimates of ref.[3, 6] : recall that it is obtained through a Taylor expansion of $V(x)$ around the classical trajectory at time t, up to order 2, and that the error term in this Taylor expansion is dealt with through the Duhamel formula ; then using estimate (2.2) for the higher order derivatives of V, it is not difficult to extract an explicit factor $(1 + |\alpha|^2)^{\delta'}$ in the R.H.S. of (2.24).

We now plug estimate (2.24) into (2.15) for a suitable value of ℓ that we shall determine for our purposes :

$$\begin{aligned} S_\chi(E) &= (2\pi\hbar)^{-2n} \int \hat{g}(t) e^{iEt/\hbar} a_\chi(\alpha') < \varphi_\alpha, \varphi_{\alpha'} > \\ & \qquad e^{i\delta(t)/\hbar} < \varphi_{\alpha'}, \phi_{\alpha,t} > dt d\alpha d\alpha' + T_\chi(E) \end{aligned} \tag{2.26}$$

where, using Lemma 1 and Proposition 2 with $\ell = 4$:

$$\begin{aligned} |T_\chi(E)| &\leq (2\pi\hbar)^{-2n} C\hbar^2 ||\hat{g}||_{L_1} \int |a_\chi(q', p')| e^{-\frac{(q-q')^2}{4\hbar} - \frac{(p-p')^2}{4\hbar}} \\ & \qquad (1 + |\alpha|^2)^\delta dp dq dp' dq' \end{aligned} \tag{2.27}$$

We shall use the estimate :

$$(1 + |\alpha|^2)^\delta \leq C_\delta (1 + q'^2)^\delta (1 + p'^2)^\delta (1 + (q - q')^2)^\delta (1 + (p - p')^2)^\delta$$

which yields

$$\begin{aligned} |T_\chi(E)| &\leq C'(2\pi\hbar)^{-2n+2} \int dq' dp' |a_\chi(q', p')| (1 + q'^2)^\delta (1 + p'^2)^\delta \\ & \times \int dq dp (1 + q^2)^\delta (1 + p^2)^\delta \exp\left(-\frac{q^2 + p^2}{4\hbar}\right) \end{aligned}$$

But the integral in (q', p') is finite due to the compact support property of a_χ in $\mathbb{R}^{2n}$. Furthermore there exists a constant C such that the integral over (q, p) is dominated by $C(2\pi\hbar)^n$ for $\hbar$ small. Therefore we get

$$|T_\chi(E)| \leq C''(2\pi\hbar)^{-n+2} \tag{2.28}$$

We shall now estimate the first term in (2.26). Then we shall use the very simple properties of the "squeezed states" $\phi_{\alpha,t}$ to study their contribution to the first term of (2.26). The very interest of the squeezed states formalism is that their scalar product with various coherent states reduce to very simple expressions using simply the algebra of the multidimensional creation and annihilation operators a^+ and a. We first state :

Lemma 3 $T(B)\phi_0 = [\det(\mathbb{1} - B^\star B)]^{1/4} e^{\frac{1}{2}a^+.Ba^+}\phi_0$ where B is any symmetric $n\times n$ matrix with $B^\star B < \mathbb{1}$, and $T(B)$ is the unitary transformation (with inverse $T(-B)$) defined by :

$$T(B) = \exp \frac{1}{2}(a^+.\Gamma a^+ - a.\Gamma^\star a) \tag{2.29}$$

where $\Gamma = (\mathrm{Argtanh}|B|)U$ and $B = |B|U$ is the polar decomposition of B.

Proof of Lemma 3 We shall use the simple transformation properties of a and a^+ through $T(B)$ (see [3]) :

$$T(B)aT(-B) = (\mathbb{1} - BB^\star)^{-1/2}(a - Ba^+) \tag{2.30}$$
$$T(B)a^+T(-B) = (\mathbb{1} - B^\star B)^{-1/2}(a^+ - B^\star a) \tag{2.31}$$

We define

$$K_0 = \frac{1}{4}(a^+.a + a.a^+) \tag{2.32}$$

We simply deduce from (2.30-2.32) that

$$\begin{aligned} T(B)K_0T(-B) &= \frac{1}{4}[(a^+ - B^\star a).(\mathbb{1} - BB^\star)^{-1}(a - Ba^+) \\ &+ (a - Ba^+).(\mathbb{1} - B^\star B)^{-1}(a^+ - B^\star a)] \end{aligned} \tag{2.33}$$

But since ϕ_0 is the ground state of the n-dimensional harmonic oscillator, we know that

$$K_0\phi_0 = \frac{n}{4}\phi_0 \tag{2.34}$$

and therefore $\Psi_0 = T(B)\phi_0$ is an eigenstate of the R.H.S. of (2.33) with eigenvalue $\frac{n}{4}$. Furthermore this eigenvalue is non degenerate, since the ground state of K_o is so ; therefore it is enough to show that $\exp(\frac{1}{2}a^+.Ba^+)\phi_0$ is such an eigenstate, and we shall deduce the result since $[\det(\mathbb{1} - B^\star B)]^{1/4}$ is just a normalizing factor. Let us denote by Λ the operator in square bracket in the R.H.S. of (2.33). We have to check :

$$e^{-1/2a^+.Ba^+}\Lambda a^{1/2a^+.Ba^+}\phi_0 = n\phi_0 \tag{2.35}$$

We shall use

$$e^{-1/2a^+.Ba^+}ae^{1/2a^+.Ba^+} = a + Ba^+$$

from which we deduce

$$(a - Ba^+)e^{1/2a^+.Ba^+} = e^{1/2a^+.Ba^+}a$$

and since $a\phi_0 = 0$, the first term of Λ applied to $e^{1/2a^+.Ba^+}\phi_0$ is simply zero. Therefore

$$e^{-1/2a^+.Ba^+}\Lambda e^{1/2a^+.Ba^+} = a.e^{-1/2a^+.Ba^+}(\mathbb{I} - B^\star B)^{-1}(a^+ - B^\star a)e^{1/2a^+.Ba^+}$$
$$= a.(\mathbb{I} - BB^\star)^{-1}a^+ - a.e^{-1/2a^+.Ba^+}(\mathbb{I} - B^\star B)^{-1}B^\star e^{1/2a^+.Ba^+}(a + Ba^+)$$

Since we apply this expression to ϕ_0, and since $a\phi_0 = 0$, we obtain

$$\begin{aligned} e^{-1/2a^+.Ba^+}\Lambda e^{1/2a^+.Ba^+}\phi_0 &= a.(\mathbb{I} - BB^\star)^{-1}a^+\phi_0 \\ &- a.e^{-1/2a^+.Ba^+}(\mathbb{I} - B^\star B)^{-1}B^\star B e^{1/2a^+.Ba^+}a^+\phi_0 \\ &= a.(\mathbb{I} - BB^\star)^{-1}a^+\phi_0 - a.(\mathbb{I} - BB^\star)^{-1}BB^\star a^+\phi_0 \\ &= aa^+\phi_0 = n\phi_0 \end{aligned}$$

This completes the proof of Lemma 3 (the proof of $||e^{1/2a^+.Ba^+}\phi_0||^2 = [\det(\mathbb{I} - B^\star B)]^{-1/2}$ is left to the reader). ∎

We shall now establish the following result :

Lemma 4 m being an arbitrary multi index, there exists a polynomial $P_m(\hbar^{-1/2}, \alpha, B)$ in $\hbar^{-1/2}$ of degree $|m|$ such that

$$\begin{aligned} < D(\alpha)\phi_0, T(B)\phi_m >= [\det(\mathbb{I} - B^\star B)]^{1/4} \\ \exp\left(-\tfrac{|\alpha|^2}{2\hbar} + \tfrac{\bar{\alpha}.B\bar{\alpha}}{2\hbar}\right)P_m(\hbar^{-1/2}, \alpha, B) \end{aligned} \tag{2.36}$$

Furthermore the coefficients of $P_m(\hbar^{-1/2}, \alpha, B)$ are continuous with respect to α and B, and we have :

$$P_m(\hbar^{-1/2}, 0, B) = 0 \text{ if } |m| \text{ is odd,}$$

and if $|m|$ is even, all coefficients of $\hbar^{-p/2}$ in $P_m(\hbar^{-1/2}, 0, B)$ are zero except that of $p = 0$.

Proof : The proof is quite easy, although it involves some cumbersome calculations. For simplicity, we give it in the case $n = 2$, $m = (1, 0)$ only, and we leave the general case to the interested reader.

Clearly we have, if $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$:

$$\begin{aligned} D(\alpha)\phi_0 &= e^{-|\alpha|^2/2\hbar} \sum_{\substack{m_1', m_2' \\ \in N}} \left(\tfrac{\alpha_1}{\sqrt{\hbar}}\right)^{m_1'} \left(\tfrac{\alpha_2}{\sqrt{\hbar}}\right)^{m_2'} \frac{1}{\sqrt{m_1'! m_2'!}} \varphi_{m_1'} \otimes \varphi_{m_2'} \\ T(B)\phi_0 &= [\det(\mathbb{I} - B^\star B)]^{1/4} \sum_{\substack{m_1, m_2, m_3 \\ \in N}} \frac{\sqrt{(2m_1+m_3)!(2m_2+m_3)!}}{m_1! m_2! m_3!} \\ &\quad \left(\tfrac{B_{11}}{2}\right)^{m_1} \left(\tfrac{B_{22}}{2}\right)^{m_2} (B_{12})^{m_3} \varphi_{2m_1+m_3} \otimes \varphi_{2m_2+m_3} \end{aligned}$$

Since $T(B)a_1^+T(-B) = [(1\!\!1 - B^\star B)^{-1/2}(a^+ - B^\star a)]_1$ (where by C_1 we denote the first component of the bi-vector C) we have

$$\begin{aligned}
< & D(\alpha)\phi_0, T(B)a_1^+\phi_0 >=< D(\alpha)\phi_0, [(1\!\!1 - B^\star B)^{-1/2}(a^+ - B^\star a)]_1 T(B)\phi_0 > \\
= & [\det(1\!\!1 - B^\star B)]^{1/4}[(1\!\!1 - B^\star B)^{1/2}\frac{\bar\alpha}{\sqrt{\hbar}}]_1 \exp\{-\frac{|\alpha]^2}{2\hbar} + \frac{\bar\alpha.B\bar\alpha}{2\hbar}\} \\
\times & \sum_{\substack{m_1,m_2,m_3\\ \in N}} (\frac{\bar\alpha_1}{\sqrt{\hbar}})^{2m_1+m_3}(\frac{\bar\alpha_2}{\sqrt{\hbar}})^{2m_2+m_3}\frac{(\frac{B_{11}}{2})^{m_1}(\frac{B_{22}}{2})^{m_2}(B_{12})^{m_3}}{m_1!m_2!m_3!} \\
= & [\det(1\!\!1 - B^\star B)]^{1/4}[(1\!\!1 - B^\star B)^{1/2}\frac{\bar\alpha}{\sqrt{\hbar}}]_1 \exp\left(-\frac{|\alpha|^2}{2\hbar} + \frac{\bar\alpha.B\bar\alpha}{2\hbar}\right)
\end{aligned}$$

Now we come to the last claims of Lemma 4, by letting $\alpha = 0$:

$$\begin{aligned}
< \phi_0, T(B)\phi_m > &= < T(-B)\phi_0, \phi_m >= \overline{< \phi_m, T(-B)\phi_0 >} \\
&= [\det(1\!\!1 - B^* B)]^{1/4}\overline{< \phi_m, e^{-1/2a^+.Ba^+}\phi_0 >}
\end{aligned}$$

using Lemma 3. Clearly this is independent of $\hbar$, and vanishes if $|m|$ is odd. ∎

We now deduce a simple expression for the scalar product entering (2.26) :

$$\begin{aligned}
< \varphi_{\alpha'}, \phi_{\alpha,t} >= [\det(1\!\!1 - B_t^\star B_t)]^{1/4} \exp\{\tfrac{1}{2\hbar}(\bar\alpha'.\alpha_t - \alpha'.\bar\alpha_t) \\
-\tfrac{|\alpha'-\alpha_t|^2}{2\hbar} + \tfrac{1}{2\hbar}(\bar\alpha' - \bar\alpha_t).B_t(\bar\alpha' - \bar\alpha_t)\} \textstyle\sum_{|m|=0}^{3(\ell-1)} c_m(t,\hbar) \\
\times P_m(\hbar^{-1/2}, \alpha' - \alpha_t, B_t)
\end{aligned} \tag{2.37}$$

where we have used the following formula :

$$D(\alpha)D(-\alpha') = \exp\{\frac{1}{2\hbar}(\bar\alpha.\alpha' - \alpha.\bar\alpha')\}D(\alpha - \alpha')$$

Moreover, according to Lemma 1 :

$$< \varphi_\alpha, \varphi_{\alpha'} >= \exp\{\frac{1}{2\hbar}(\bar\alpha.\alpha' - \alpha.\bar\alpha') - \frac{|\alpha - \alpha'|^2}{2\hbar}\}$$

Therefore, the first term in the R.H.S. of (2.26) can be rewritten as

$$(2\pi\hbar)^{-2n}\int \hat g(t)e^{\phi(t,\alpha,\alpha')/\hbar}b(\hbar,\alpha,\alpha',t)dtd\alpha d\alpha' \tag{2.38}$$

where b is a polynomial in $\hbar^{1/2}$ and $\hbar^{-1/2}$ with coefficients being C^∞ in the other variables, and where

$$\begin{aligned}
\phi(t,\alpha,\alpha') &= iEt + iS(t) - \tfrac{i}{2}(q_t.p_t - q.p) + \tfrac{1}{2}(\bar\alpha'.(\alpha_t - \alpha) - \alpha'.(\bar\alpha_t - \bar\alpha)) \\
&\quad -\tfrac{|\alpha-\alpha'|^2}{2} - \tfrac{|\alpha_t-\alpha'|^2}{2} + \tfrac{1}{2}(\bar\alpha' - \bar\alpha_t).B_t(\bar\alpha' - \bar\alpha_t)
\end{aligned} \tag{2.39}$$

Writing, for definiteness

$$b(\hbar, q, p, q', p', t) = \sum_{j=-N}^{N'} \hbar^{+j/2} b_j(q, p, q', p', t) \tag{2.40}$$

we are now in a position to apply the stationary phase method for any fixed j between $-N$ and N' to the following integral with phase ϕ, and amplitude $\hat{g}(t) b_j(q, p, q', p', t)$:

$$(2\pi\hbar)^{-2n} \int \hat{g}(t) e^{\phi(t,\alpha,\alpha')/\hbar} b_j(q, p, q', p', t) dt dq dp dq' dp' \tag{2.41}$$

Since $\alpha = \frac{q+ip}{\sqrt{2}}$, and similarly for α' and α_t, we get :

$$\bar{\alpha}'.(\alpha_t - \alpha) - \alpha'.(\bar{\alpha}_t - \bar{\alpha}) = i\{q'.(p_t - p) - p'.(q_t - q)\}$$

and therefore

$$\begin{aligned} 2\nabla_{q'}\phi(t, \alpha, \alpha') &= i(p_t - p) - (q' - q) - (q' - q_t) + B_t[q' - q_t - i(p' - p_t)] \\ 2\nabla_{p'}\phi(t, \alpha, \alpha') &= -i(q_t - q) - (p' - p) - (p' - p_t) - iB_t[q' - q_t - i(p' - p_t)] \end{aligned}$$

Assuming that $\nabla_{q'}\phi(t, \alpha, \alpha') = \nabla_{p'}\phi(t, \alpha, \alpha') = 0$, we get by separating the real and imaginary parts :

$$q + q_t - 2q' + ReB_t(q' - q_t) + ImB_t(p' - p_t) = 0 \tag{2.42}$$

$$p_t - p + ImB_t(q' - q_t) - ReB_t(p' - p_t) = 0 \tag{2.43}$$

$$p + p_t - 2p' + ImB_t(q' - q_t) - ReB_t(p' - p_t) = 0 \tag{2.44}$$

$$q - q_t - ReB_t(q' - q_t) - ImB_t(p' - p_t) = 0 \tag{2.45}$$

Combining (2.42) and (2.45) together, and (2.42-2.43), respectively, we obtain :

$$\begin{cases} q - q' &= 0 \\ p - p' &= 0 \end{cases} \tag{2.46}$$

which, inserted into (2.42-2.43), yields :

$$\begin{cases} (\mathbb{I} - ReB_t)(q_t - q) & -ImB_t(p_t - p) &= 0 \\ -ImB_t(q_t - q) & +(\mathbb{I} + ReB_t)(p_t - p) &= 0 \end{cases} \tag{2.47}$$

But (2.47) can be rewritten, in matrix form, as :

$$\begin{pmatrix} \mathbb{I} - ReB_t & -ImB_t \\ -ImB_t & \mathbb{I} + ReB_t \end{pmatrix} \begin{pmatrix} q_t & - & q \\ p_t & - & p \end{pmatrix} = 0 \tag{2.48}$$

which implies

$$\begin{cases} q &= q_t \\ p &= p_t \end{cases} \tag{2.49}$$

since $\mathbb{I} - B_t^\star B_t$ is non-singular for any t; (see [3]). Now differentiating $\phi(t,\alpha,\alpha')$ with respect to t, and taking it at the points α and α' satisfying (2.46,2.49), we obtain

$$\begin{aligned} \partial_t \phi(t,\alpha,\alpha')|_{\alpha=\alpha'=\alpha_t} &= i[E+\dot{S}-\dot{q}_t.p_t] \\ &= i[E-H(q,p)] \end{aligned} \tag{2.50}$$

Therefore the stationarity of the phase implies that (q,p) lies on the energy surface σ_E defined by $H(q,p)=E$, and that the classical trajectory starting at (q,p) at time zero is closed at time t (see(2.49)). Assuming that $supp\hat{g}$ is small enough, so that the only possibility on $\sum_E$ to have a closed trajectory at time t is that $t=0$, we are left, in the $(4n+1)$-dimensional space of variable (t,q,p,q',p') with a $(2n-1)$-dimensional manifold given by :

$$\{t=0,(q',p')=(q,p),(q,p)\in\Sigma_E\} := M_E$$

We are now in a position to apply Theorem 7.7.12 of Hörmander [9] ; note that the phase $\phi(t,\alpha,\alpha')$ is complex, but that the manifold M_E where it is stationary is real.

Note also that the support in (q,p) is compact, due to the localization (2.7), and that we can restrict also the support in (q',p') to a compact K, up to errors of order $0(\hbar^N)$ with N large, since for $(q',p')\in\mathbb{R}^{2n}\backslash\mathrm{K}$, the non-stationary phase estimates can be applied. We have to check that the Hessian, $Hess\ \phi$, restricted on the manifold M_E has rank $4n+1-(2n-1)=2n+2$. It is a tedious, but simple calculation to check that it has the

$$Hess\ \phi|_{M_E} = \begin{pmatrix} -2|z|^2 & z & -iz & -\bar{z} & -i\bar{z} \\ {}^t z & -\mathbb{I} & 0 & \mathbb{I} & 0 \\ -i\ {}^t z & 0 & -\mathbb{I} & 0 & \mathbb{I} \\ -{}^t\bar{z} & \mathbb{I} & 0 & -\mathbb{I} & 0 \\ -i\ {}^t\bar{z} & 0 & \mathbb{I} & 0 & -\mathbb{I} \end{pmatrix}$$

where $\mathbb{I}$ (resp. 0) is the $n\times n$ identity (resp. nul) matrix. Here z denotes the number in $\mathbb{C}^n$

$$z := \frac{1}{2}(\nabla_p H + i\nabla_q H)(q,p) = \frac{1}{2}(\dot{q}_t - i\dot{p}_t) \tag{2.51}$$

Now, since E is a regular value of the classical flow, $|z|\neq 0$, and it is easy to see that the rank of $Hess\ \phi|_{M_E}$ is simply $2(n+1)$.

Let us denote by $d\sigma_E(q,p)$ the Liouville measure on Σ_E. Now theorem (7.7.12) of Hörmander applied to (2.41) yields :

$$\begin{aligned} (2.41) &= (2\pi\hbar)^{-2n}.(2\pi\hbar)^{n+1}\left\{\hat{g}(0)\int_{\Sigma_E} d\sigma_E(q,p)b_j(q,p,q,p,0)+O(\hbar)\right\} \\ &= (2\pi\hbar)^{-n+1}\left\{\hat{g}(0)\int_{\Sigma_E} d\sigma_E(q,p)b_j(q,p,q,p,0)+O(\hbar)\right\} \end{aligned} \tag{2.52}$$

Now we shall make clear that the only non vanishing contribution in (2.52) comes from $j = 0$, namely that

$$\left\{ \begin{array}{lcll} b_j(q,p,q,p,0) & = & 0 & j \neq 0 \\ b_0(q,p,q,p,0) & = & a_\chi(q,p) & \end{array} \right. \tag{2.53}$$

¿From Lemma 4 with $\alpha = 0$, it is clear that the only j's contributing to the stationary phase estimate come from multi indices m with $|m|$ even. Therefore from (2.40) we get

$$b(\hbar,q,p,q,p,t) = \sum_{j=0}^{N} \hbar^{j/2} b_j(q,p,q,p,t)$$

where only non-negative powers of $\hbar^{1/2}$ enter the RHS, and where $b_j(q,p,q,p,t)$ is the coefficient of $\hbar^{j/2}$ in the sum

$$[\det(\mathbb{I} - B_t^* B_t)]^{1/4} \sum_{\substack{|m|=0 \\ |m| even}}^{8} c_m(t,\hbar) P_m(1,0,B_t) a_\chi(q,p)$$

If we now let $t = 0$, according to the stationary phase condition, we get (since $B_o = 0$ and $c_m(0,\hbar) = \delta_{|m|,o}$)that

$$\begin{array}{lcl} b_j(q,p,q,p,0) & = & 0 \qquad \text{if } j \neq 0, \text{ and} \\ b_o(q,p,q,p,0) & = & P_o(1,0,0)\ a_\chi(q,p) = a_\chi(q,p) \end{array}$$

But since $a_\chi(q,p)|_{\sum_E} = a_E(z)$ due to the support property of the function χ, we get the result of Theorem 1 from ((2.52, 2.53)) together with estimate (2.28). ∎

<u>Remark 1</u> Nowhere in this proof the estimates are restricted to a small time interval $[-T,T]$, but only to finite T. Therefore, through this method, we can hope in principle to get the first periodic-orbit contributions to $S(E)$ by taking $\hat{g}$ with larger and larger support property.
<u>Remark 2</u> By the same method we can get a full asymptotic expansion in $\hbar$ for $S(E)$; namely proposition 2 and the stationary phase method give a complete expansion in $\hbar$ up to arbitrary order.

3 A sum rule for the variance of the statistical distribution of the matrix elements $A_{jk}(\hbar)$

In [16] Wilkinson proposes the following definition for the variance :

$$S_{f,g}(\hbar,E,\Delta E) = \sum_{\substack{E_j(\hbar), \\ E_k(\hbar) \in I_{cl}}} |A_{jk}(\hbar)|^2 f_\hbar[E - \frac{1}{2}(E_j(\hbar) + E_k(\hbar))]$$

$$g_\hbar[\Delta E - (E_j(\hbar) - E_k(\hbar))] \tag{3.1}$$

where $f_\hbar$ and $g_\hbar$ are Gaussian regularizations of the Dirac δ-distribution. For technical convenience, as in our previous article [3], we choose for f and g smooth functions on $\mathbb{R}$, compactly supported in Fourier variable, and we define $f_\hbar(u) = \frac{1}{\hbar} f(\frac{u}{\hbar})$. We shall prove the following : (recall that $\hat{f}(v) = \int e^{-iuv} f(u) du$ is the Fourier transform of f).

<u>Theorem 2</u> Let us assume that $Supp\hat{f} \subset]-T_0, T_0[$ with $T_0 > 0$ small enough, and that $Supp\hat{g}$ is compact. Then we have, as $\hbar \searrow 0$

$$S_{f,g}(\hbar, E, \Delta E) = \hbar^{-n-1}\{\hat{f}(0) \int \hat{g}(t) e^{it\Delta E/\hbar} C_a(E,t) dt + O(\hbar^{1/2})\} \tag{3.2}$$

where $C_a(E,t)$ is the classical autocorrelation function

$$C_a(E,t) = \int_{\sum_E} a(z) a(\phi^t(z)) d\sigma_E(z) \tag{3.3}$$

ϕ^t being the classical flow corresponding to $P(\hbar)$.

Proof of theorem 2 : Using the definition of the Fourier transform, and the definition of the trace of an operator, we easily get :

$$\begin{aligned} S_{f,g}(\hbar, E, \Delta E) = (2\pi\hbar)^{-2} \quad & \int \hat{f}(s) \hat{g}(t) e^{it\Delta E/\hbar} e^{isE/\hbar} Tr\{AU(t+\tfrac{s}{2}) \\ & AU(\tfrac{s}{2} - t)\} ds dt \end{aligned} \tag{3.4}$$

where $U(s) = \exp\{-isP(\hbar)/\hbar\}$ is the quantum propagator for the Hamiltonian $P(\hbar)$. Now we shall use formula (2.12) relating the trace of an operator to the integral of its expectation value in the coherent states, together with formula (2.14) relating an operator to its symbol via the Anti-Wick quantization condition, and we obtain :

$$\begin{aligned} Tr\{AU(t+\frac{s}{2}) AU(-t+\frac{s}{2})\} = (2\pi\hbar)^{-3n} & \int d\alpha d\alpha' d\alpha'' a(\alpha') a(\alpha'') \\ < \varphi_\alpha, \varphi_{\alpha'} >< \varphi_{\alpha'}, U(t+\frac{s}{2}) \varphi_{\alpha''} > & < U(t-\frac{s}{2}) \varphi_{\alpha''}, \varphi_\alpha > \end{aligned} \tag{3.5}$$

where $d\alpha$ denotes $d^n q d^n p \qquad (\alpha = (q,p) \in \mathbb{R}^{2n})$. Now we shall use Lemma 2, with a suitable value of ℓ, namely $\ell = 2(n+2)$.

We shall establish that estimate (2.24)with $\ell = 2(n+2)$ is sufficient to control the correction terms in (3.5).

$$S_{f,g}(\hbar, E, \Delta E) = (2\pi\hbar)^{-3n-2} \int \hat{f}(s)\hat{g}(t)e^{it\Delta E/\hbar + isE/\hbar + i\delta(t+\frac{s}{2})/\hbar}$$
$$\times \quad e^{+i\delta(-t+\frac{s}{2})/\hbar}a(\alpha')a(\alpha'') < \varphi_\alpha, \varphi_{\alpha'} >$$
$$\times < \varphi_{\alpha'}, \phi_{\alpha'',t+\frac{s}{2}} >< \phi_{\alpha'',t-\frac{s}{2}}, \varphi_\alpha > dsdtd\alpha d\alpha' d\alpha'' + R(\hbar, t) \qquad (3.6)$$

where

$$|R(\hbar, t)| \leq C(2\pi\hbar)^{-3n-2}\int d\alpha d\alpha' d\alpha'' |a(\alpha')||a(\alpha'')|| < \varphi_\alpha, \varphi_{\alpha'} > | \qquad (3.7)$$

$$\times(\hbar^{2n+4}(1+|\alpha|^2)^\delta(1+|\alpha''|^2)^\delta + \hbar^{n+2}(1+|\alpha|^2)^\delta| < \varphi_{\alpha'}, \phi_{\alpha'',t+\frac{s}{2}} > |$$
$$+\hbar^{n+2}(1+|\alpha''|^2)^\delta| < \varphi_{\alpha''}, \phi_{\alpha,-t+\frac{s}{2}} > |)$$

But

$$| < \varphi_\alpha, \varphi_{\alpha'} > | = e^{-\frac{|\alpha-\alpha'|^2}{2\hbar}} \qquad (3.8)$$

and

$$\sup_{\substack{s\in \mathrm{supp}\hat{f} \\ t\in \mathrm{supp}\hat{g}}} | < \varphi_{\alpha'}, \phi_{\alpha'',-t+\frac{s}{2}} > | \leq C \qquad \sup_{\substack{s\in \mathrm{supp}\hat{f} \\ t\in \mathrm{supp}\hat{g}}} | < \varphi_{\alpha''}, \phi_{\alpha,-t+\frac{s}{2}} > | \leq C \qquad (3.9)$$

so that, by similar estimates as in section 2, we simply get :

$$|R(\hbar, t)| \leq C(2\pi\hbar)^{-2n-2}.\hbar^{n+2} = C(2\pi\hbar)^{-n} \qquad (3.10)$$

Therefore we are simply left with the first term of equation (3.6), that we shall estimate via stationary phase theorems.

Using Lemma 4, we shall now prove Theorem 2 via the stationary phase theorem [[9], Theorem 7.7.12] ; namely, as in Section 2, the integration support in α' and α'' can be taken to be compact using the regularized operator $A_\chi(\hbar)$ defined by (2.7), and the integration support in α can also be taken as a compact one using non-stationary phase estimates. For each fixed t and multi index k and k', we have to estimate integrals like

$$(2\pi\hbar)^{-3n-2}\int \ ds \ d\alpha \ d\alpha' \ d\alpha'' \ \hat{g}(s) \ a(\alpha') \ a(\alpha'') \ P_k(\hbar^{-1/2}, \alpha' - \alpha''_{t+\frac{s}{2}}, B_{t+\frac{s}{2}})$$

$$\bar{P}_{k'}(\hbar^{-1/2}, \alpha - \alpha''_{t-\frac{s}{2}}, B_{t-\frac{s}{2}}) \exp\{\phi(s, t, \alpha, \alpha', \alpha'')/\hbar\} \qquad (3.11)$$

where the (complex) phase $\phi(s,t,\alpha,\alpha',\alpha'')$ has the following form

$$\begin{aligned}
&\phi(s,t,\alpha,\alpha',\alpha'') = isE + iS(\alpha'', t+\frac{s}{2}) - iS(\alpha'', t-\frac{s}{2}) + \frac{i}{2}(q.p' - q'.p) \\
&-\frac{1}{4}(q-q')^2 - \frac{1}{4}(p-p')^2 - \frac{i}{2}(q''_{t+\frac{s}{2}}.p''_{t=\frac{s}{2}} - q''_{t-\frac{s}{2}}.p''_{t-\frac{s}{2}}) \\
&-\frac{1}{4}(q'-q''_{t+\frac{s}{2}})^2 - \frac{1}{4}(p'-p''_{t+\frac{s}{2}})^2 - \frac{1}{4}(q-q''_{t-\frac{s}{2}})^2 - \frac{1}{4}(p-p''_{t-\frac{s}{2}})^2 \\
&+\frac{i}{2}(q'.p''_{t+\frac{s}{2}} - q''_{t+\frac{s}{2}}.p') - \frac{i}{2}(q.p''_{t-\frac{s}{2}} - q''_{t-\frac{s}{2}}.p) \\
&+\frac{1}{2}(\bar{\alpha}' - \bar{\alpha}''_{t+\frac{s}{2}}).B_{t+\frac{s}{2}}(\alpha' - \alpha''_{t+\frac{s}{2}}) + \frac{1}{2}(\bar{\alpha} - \bar{\alpha}''_{t-\frac{s}{2}}).B_{t-\frac{s}{2}}(\alpha - \alpha''_{t-\frac{s}{2}})
\end{aligned} \tag{3.12}$$

We can calculate

$$\left.\begin{array}{lcl}
2\nabla_q\phi &=& q' + q''_{t-\frac{s}{2}} - 2q + i(p' - p''_{t-\frac{s}{2}}) + \bar{B}_{t-\frac{s}{2}}[q - q''_{t-\frac{s}{2}} + i(p - p''_{t-\frac{s}{2}})] \\
2\nabla_p\phi &=& p' + p''_{t-\frac{s}{2}} - 2p - i(q' - q''_{t-\frac{s}{2}}) + i\bar{B}_{t-\frac{s}{2}}[q - q''_{t-\frac{s}{2}} + i(p - p''_{t-\frac{s}{2}})] \\
2\nabla_{q'}\phi &=& q + q''_{t+\frac{s}{2}} - 2q' - i(p - p''_{t+\frac{s}{2}}) + B_{t+\frac{s}{2}}[q' - q''_{t+\frac{s}{2}} - i(p' - p''_{t+\frac{s}{2}})] \\
2\nabla_{p'}\phi &=& p + p''_{t+\frac{s}{2}} - 2p' + i(q - q''_{t+\frac{s}{2}}) - iB_{t+\frac{s}{2}}[q' - q''_{t+\frac{s}{2}} - i(p' - p''_{t+\frac{s}{2}})]
\end{array}\right\} \tag{3.13}$$

Writing now the stationary phase condition, and separating real and imaginary parts, we get 8 equations (in $\mathbb{R}^n$) which yield :

$$\left.\begin{array}{ll}
q' & = q = q''_{t-\frac{s}{2}} = q''_{t+\frac{s}{2}} \\
p' & = p = p''_{t-\frac{s}{2}} = p''_{t+\frac{s}{2}}
\end{array}\right\} \tag{3.14}$$

We have used, like in the proof of Lemma 4, that

$$\mathbb{1} - B^\star_{t\pm\frac{s}{2}} B_{t\pm\frac{s}{2}} \text{is not singular}$$

Using these conditions, it iq not difficult to perform the derivation with respect to time s, which yields :

$$\partial_s\phi(s,t,\alpha,\alpha',\alpha'') = iE + i\partial_s S(\alpha, s) \tag{3.15}$$

since

$$S(\alpha'', t+\frac{s}{2}) - S(\alpha'', t-\frac{s}{2}) = S(\alpha, -t+\frac{s}{2}) + S(\alpha'', t+\frac{s}{2}) = S(\alpha, s) \tag{3.16}$$

(since α'' is the phase space point reached by the classical flow by evoluting from phase space point α during time $-t+\frac{s}{2}$) and $S(\alpha,s)$ in the classical action along the closed orbit starting at α and reaching α after time s. Keeping t fixed, the stationary phase condition in the variables $(s,\alpha,\alpha',\alpha'')$ yields, in addition to (3.14) that

$$E = H(\alpha) = H(\alpha'') \tag{3.17}$$

due to the conservation of energy along the classical trajectory under consideration. But, if $supp\hat{f}$ is small enough, the only possibilitty for $s \in supp\hat{f}$ to satisfy simultaneously (3.14) and (3.17) is that

$$s \;=\; 0$$

Therefore, we are left in the space $\mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ for variables $(s, \alpha, \alpha', \alpha'')$ with a $2n-1$ dimensional manifold

$$M'_E \begin{cases} \alpha \;=\; \alpha' = \alpha'' \\ \alpha'' \;\in\; \Sigma_E \\ s \;=\; 0 \end{cases}$$

and we can check, as in Section 2, that the Hessian of ϕ restricted to M'_E has rank $6n+1-(2n+1) = 4n+2$ We now look at the various amplitude terms in the stationary phase integrals, reminding that the $c_k(t,\hbar)$ are polynomials in $\hbar^{1/2}$ satisfying (see [6] , [7])

$$|c_0(t,\hbar) - 1| \leq C$$

and

$$|c_k(t,\hbar)| \leq C\hbar^{r/2}$$

for an integer r satisfying $3(r-1) < |k| \leq 3r$. Using the second statement of Lemma 4, we see that, at the stationary phase point, the amplitude is of the form : $a(\alpha'')a(\alpha''_{t-\frac{s}{2}}) + h(t,s,\alpha'',\hbar)$ where $h(t,s,\alpha'',\hbar)$ is a polynomial in $\hbar^{1/2}$ with smooth coefficients and $\sup_{s,t,\alpha''} |h(t,s,\alpha'',\hbar)| = O(\hbar^{1/2})$. Thus the first term of the RHS of (3.6) becomes, as $\hbar \to 0$:

$$S_{f,g}(\hbar, E, \Delta E) = (2\pi\hbar)^{-n-1}\{\hat{f}(0)\int dt\hat{g}(t)e^{it\Delta E/\hbar}C_a(t) + O(\hbar^{1/2})\} \qquad (3.18)$$

where

$$C_a(t) = \int_{\sum_E} a(\alpha)a(\alpha_t)d\sigma_E(\alpha)$$

which completes the proof of theorem 2. ∎

Remark : Here, unlike in Theorem 1, we get a correction term which in $O(\hbar^{1/2})$ instead of the wished $O(\hbar)$ term.

However we expect, in principle, that suitable cancellations between $O(\hbar^{1/2})$ terms actually occur, allowing to show that the coefficient of $\hbar^{1/2}$ is zero, and more generally that so are the coefficients of $\hbar^{m/2}$ (with m odd), so that a complete asymptotic expansion in $\hbar$ appears in theorem 2 (see our results in [4]).

4 Concluding remark

The use of coherent states in this approach, and more precisely of the estimate of Lemma 2 have imposed us that the differential term in the operator $P(\hbar)$ be of the form $-\hbar^2\Delta$. Namely the spirit of such estimates is to take a Taylor expansion of the Hamiltonian around a classical phase space point $\alpha_t = (q_t, p_t)$ belonging to some classical orbit, and to control the time evolution of a corresponding "toy hamiltonian" $P^2 + \omega(t)Q^2$ (where $P = -i\hbar\nabla$, and Q is multiplication by x as usual). But this toy hamiltonian, being quadratic both in P and Q evolves quantum coherent states into their cousins the squeezed states (or generalized squeezed states) given by (2.25). This explains the relevance of such generalized squeezed states in our semi-classical time-dependent estimates.

However we do not think that this restriction to kinetic energy of the form $-\hbar^2\Delta$ is really imposed by the "coherent states" method, for the following reason : by extending Hagedorn's results, we can hope that the "squeezed states approach" can be generalized to more general $\hbar$-pseudodifferential Hamiltonians (this extension is presently under study). This has the advantage of keeping the parallelism with the WKB approach more transparent.

On the other hand, note that the usual semi-classical estimates of the quantum propagator via the microlocal approach requires, in its simpler form, the time domain to be very small, because of turning points problems, and an iteration procedure is needed to get finite time intervals. Here, on the contrary, the time interval is not restricted, and can be any $[-T, T], T > 0$. The statement, in Theorems 1 and 2 that the support of $\hat{g}$ (resp. $\hat{f}$) is small is only imposed in order to get the "Weyl term" in the RHS of (2.6) and (3.2), and not the contribution of other periodic orbits which appears naturally through the stationary phase theorem. We have not done it explicitely here, for the sake of simplicity, but we want to stress that it could be possible in principle to obtain them by taking $\hat{g}$(resp.$\hat{f}$) with larger support property.

References

[1] Berry M.V., J. Phys. A 18, 15-27 (1985).

[2] Colin de Verdière Y. Commun. Math. Phys. 102, 497-502 (1985).

[3] Combescure M., J. Math. Phys. 33, 3870-3880 (1992).

[4] Combescure M., Robert D., Distribution of Matrix Elements and level spacings for classically chaotic systems, Nantes preprint (1994).

[5] Feingold A., Peres A., Phys. Rev. A 34, 591-595 (1986).

[6] Hagedorn G., Ann. Inst. Henri Poincaré 42, 363-374 (1985).

[7] Hagedorn G., Ann. Phys. 135, 58-70 (1981).

[8] Helffer B., Martinez A., Robert D., CMP 102, 497-502 (1985).

[9] Hörmander L., The analysis of linear partial differential operators I, Springer-Verlag (1983).

[10] Marshalek E., da Providência J., Phys. Rev. C 7, 2281-2293 (1973).

[11] Paul T., Uribe A., Ann. Inst. Henri Poincaré 59, 357-381 (1993).

[12] Prosen T., Robnik M., J. Phys. A : Math. Gene. 26 L319-L326 (1993).

[13] Robbins J.M., Berry M.V., J. Phys. A. : Math. Gene. 25 L961-L965 (1992).

[14] Robbins J.M., Berry M.V., Proc. R. Soc. Lond. A 436 631-661 (1992).

[15] Robert D., Autour de l'approximation semi-classique, Birkhäuser PM 68 (1987).

[16] Schnirelman A.I., Upehi Math. Nauk. 29, 181-182 (1974).

[17] Wilkinson M., J. Phys. A. Math. Gene. 20 2415-2423 (1987).

[18] Zelditch S., Duke Math. J. 55, 919-941 (1987).

[19] Zelditch S., J. Funct. Anal. 94, 415-436 (1990).