# SCALING, PERTURBATIVE RENORMALIZATION AND ANALYTICITY FOR THE STANDARD MAP AND SOME GENERALIZATIONS

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ABSTRACT. We summarize some numerical works on the analyticity properties of the perturbative series (*e. g.* Lindstedt series) for some simple discrete-time hamiltonian systems like the so-called "standard map" and generalizations. We stress the relevance of our results for the development of a renormalization group picture based on perturbation theory.

## INTRODUCTION

The renormalization group picture of the transition to chaos in classical hamiltonian systems [1], [2] is usually based on a description of the dynamics in phase space, with the renormalization group acting as a flow on some space of hamiltonians. This approach is basically a non-perturbative one, and it proved successful in explaining many qualitative features of the transition.

On the other hand, it is still unclear how this picture affects the perturbative methods traditionally used since long time (like e. g. the Lindstedt series [3], [4]). In particular, many questions pertaining to the analytic structure of the invariant tori find their natural setting in the language and formalism of series expansions and perturbation theory.

We observe that a similar duality occurs in quantum field theory, where two approaches to renormalization coexist: one is perturbative and based on renormalized perturbation expansions, running coupling constants, and equation of Callan-Symanzik type, the other is non-perturbative and based on phase space cell decomposition and coarse-graining of the degrees of freedom. The two approaches largely overlap and complement each other (see [5] for a broad review and [6] for an approach directly pertinent to the problem at hand). The rôle of the propagators with non-integrable singularities at large momenta is played here by the "small denominators" which appear in Lindstedt-type expansions, making it difficult to control their convergence. We believe that an approach to renormalization based on perturbation theory would be very useful to explain the wealth of numerically observed phenomena and to shed some light on the related conjectures [7], [8], [9], [10], [11], [12].

In this review we summarize some numerical work on the analytic structure of the perturbation series for the simplest discrete-time hamiltonian systems exhibiting a transition between regular and chaotic motion, e. g. those given by the maps:

$$\begin{aligned} y_{n+1} &= y_n + \varepsilon f(x_n), \\ x_{n+1} &= x_n + y_{n+1}, \quad (x, y) \in \mathbb{T} \times \mathbb{R}, \end{aligned}$$

where $f(x)$ is a trigonometric polynomial with zero average; we also assume that $f(x)$ is an odd function of $x$, so that it may be expanded as a sines-only Fourier series.

The simplest case among the maps of this type *i. e.* $f(x) = \sin x$, is the so-called Chirikov-Greene standard map [13], [14]. The background literature on the standard map is huge and rather than summarizing it here we refer the interested reader to the review [15]. We just recall that besides being a "paradigmatic" model describing the transition to stochasticity in hamiltonian systems (like the scalar $\phi^4$ model has been used as a test ground in quantum field theory), the standard map has been used as a model in plasma physics [16], and in solid state physics to describe the ground states of the Frenkel-Kontorova model [17].

The standard map and its generalizations may be seen as perturbations of the linear map $(x, y) \mapsto (x' = x + y, y' = y)$, whose dynamics is given by rotations: $x_n = \omega n + x_0, y_n = \omega$. According to the standard (KAM) picture, if the "rotation number" $\omega$ is irrational and satisfies a diophantine condition, and if $|\varepsilon|$ is small enough, a homotopically nontrivial invariant curve $\mathcal{C}_{\varepsilon,\omega}$ exists, on which the dynamics is smoothly conjugated to the one of the linear, unperturbed map, *i. e.* rotations by $\omega$. In particular, if $\omega$ satisfies said diophantine condition, then there exists a function $u(\theta, \varepsilon, \omega)$, jointly analytic in $(\theta, \varepsilon)$ for $|\mathbf{Im}(\theta)| < \bar{\theta}, |\varepsilon| < \bar{\varepsilon}$ such that the invariant curve $\mathcal{C}_{\varepsilon,\omega}$ exists and is given in parametric form by:

$$\begin{aligned} x &= \theta + u(\theta, \varepsilon, \omega), \\ y &= \omega + u(\theta, \varepsilon, \omega) - u(\theta - \omega, \varepsilon, \omega), \end{aligned}$$

and in the new variable $\theta$ (the "linearizing variable") the dynamics is the same as the one of the linear map *i. e.* $\theta_n = n\omega + \theta_0$. A simple application of Birkhoff's theorem and of the implicit functions theorem shows also that the smothness class is the same for the curve $\mathcal{C}_{\varepsilon,\omega}$ and the function $u(\theta, \varepsilon, \omega)$, regarded as a function of $\theta$. $u(\theta, \varepsilon, \omega)$ and $\mathcal{C}_{\varepsilon,\omega}$ are therefore, in a certain sense, "the same thing".

The above mentioned diophantine condition on $\omega$ is best formulated in terms of its continued fraction expansion:

$$\frac{\omega}{2\pi} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_1, a_2, \ldots].$$

In fact, let $p_k/q_k$ be the sequence of rational approximants to $\omega/2\pi$ obtained by truncating its continued fraction expansion at the $k$-th place:

$$\frac{p_k}{q_k} = [a_1, a_2, \dots, a_k, \infty];$$

then the diophantine condition takes the form:

$$q_{k+1} = O(q_k^\gamma), \quad \gamma \geq 2.$$

It is useful to introduce a lagrangian formalism [18] by eliminating the variable $y$:

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon f(x_n). \tag{1}$$

The conjugation to a rotation may therefore be written as:

$$x = \theta + u(\theta, \varepsilon, \omega) \tag{2}$$

and the conjugating function $u$ satisfies the equation:

$$D_\omega^2 u = M_\varepsilon u, \tag{3}$$

where the operators $D_\omega^2$ and $M_\varepsilon$ act on functions of $\theta$ as follows:

$$\begin{aligned} D_\omega^2 \phi(\theta) &= \phi(\theta + \omega) - 2\phi(\theta) + \phi(\theta - \omega), \\ M_\varepsilon \phi(\theta) &= \varepsilon f(\theta + \phi(\theta)). \end{aligned}$$

The equation (3) is obtained by inserting (2) into (1) and using the fact that in the $\theta$ variable the dynamics is given by rotation by $\omega$. Imposing the condition that the average over $\theta$ of $u$ is 0 (which amounts to choosing an origin for the linearizing coordinate $\theta$), the solutions to (3) are formally unique.

The function $u$ has some simple properties which can be derived at once from its very definition. In particular, it is odd in $\theta$ since $f$ is and is $O(\varepsilon)$ as $\varepsilon \to 0$. Hence it can be (formally) expanded in Taylor series in $\varepsilon$ and each coefficient can be expanded as a sines-only Fourier series in $\theta$:

$$u(\theta, \varepsilon, \omega) = \sum_{n \geq 1} u_n(\theta, \omega)\varepsilon^n = \sum_{n \geq 1} \sum_{k \geq 1} c_{n,k}(\omega)\varepsilon^n \sin k\theta. \tag{4}$$

This is the so-called *Lindstedt series* and the study of its analyticity properties in the *three* variables $\theta$, $\varepsilon$, $\omega$ is the main focus of our work.

If the nonlinear term $f(x)$ contains only odd frequencies of $x$ in its Fourier expansion, then it is immediate to verify that in the Lindstedt series even orders in $\varepsilon$ contain only even frequencies in $\theta$ and odd orders contain only odd frequencies, so that:

$$u(\theta + \pi, \varepsilon, \omega) = u(\theta, -\varepsilon, \omega).$$

Finally, if the degree of the trigonometric polynomial $f$ (*i. e.* its highest frequency) is $p$, then the order $n$ in $\varepsilon$ contains only frequencies in $\theta$ up to order $pn$; for the standard map $p = 1$ and at order $n$ in $\varepsilon$ only frequencies of the same parity as $n$ and up to order $n$ may appear.

It is well known that, for real rotation numbers $\omega$, the convergence of the Lindstedt series is plagued by the presence of *small divisors*. This is because the operator $D_\omega$, acting on $2\pi$-periodic functions of $\theta$ with zero average, has eigenvalues $2i\sin(\pi k\omega)$ (with $k \in \mathbb{Z}, k \neq 0$), which accumulate to zero for $\omega$ irrational, while for rational values $\omega = p/q$ the null space of $D_\omega$ is actually the space of $2\pi/q$-periodic functions. Therefore the inverse operator of $D_\omega$ does not exist for rational values of $\omega$ (because of "resonances", which are reminiscent of "zero-modes" in quantum field theory), while for irrational values of $\omega$ the kernel of $D_\omega^{-1}$, which appears in the expressions of the coefficients $c_{n,k}(\omega)$, may lead to a divergent series since it grows arbitrarily large (as the eigenvalues of $D_\omega$ accumulate to 0). To control this small divisors, several approaches have been devised, like the super-convergent methods based on Newton's approximation scheme applied to the functional non-linear equation (3), leading to KAM theory [19], or the direct methods of refs. [20], [21], [22] which are based on the original approach of Siegel [23]. We recall also that a crucial rôle in the convergence of these series is played by cancellations among terms of a given order in $\varepsilon$. Using any direct method (à la Eliasson) involves explicitly exhibiting any such cancellations at any order.

As $\varepsilon$ grows, the invariant curve $\mathcal{C}_{\varepsilon,\omega}$ ceases to exist ("breaks up"). Accordingly, we define the (analytic) breakdown threshold $\varepsilon_c(\omega)$ as the sup of the set of those positive $\bar{\varepsilon}$ such that there exist analytic homotopically nontrivial invariant curves for each $\varepsilon \in [0, \bar{\varepsilon}]$. $\varepsilon_c(\omega)$ varies in a very irregular way as $\omega$ varies between 0 and $2\pi$ since it is 0 for rational values of $\omega/2\pi$ and nonzero for diophantine ones.

## Complex Analytic Structure of the Perturbative Series

It is natural to relate the analytic breakdown threshold $\varepsilon_c(\omega)$ to the radius of convergence (in $\varepsilon$) of the Lindstedt series:

$$\rho(\omega) = \inf_{\theta\in\mathbb{T}} \left( \limsup_{n\to\infty} |u_n(\theta,\omega)|^{1/n} \right)^{-1} . \tag{5}$$

Clearly $\rho(\omega) \leq \varepsilon_c(\omega)$, but it could well be that $\rho(\omega)$ is strictly less that the analytic breakdown threshold, depending on the *complex* analytic structure of the conjugating function $u$. In fact looking at the singularities of $u(\theta, \varepsilon, \omega)$ in the complex $\varepsilon$ plane on the circle of convergence for a fixed diophantine $\omega$ and at fixed $\theta$ and looking at how they depend on $\theta$ one can determine wether an analytic continuation is possible on the real axis beyond the radius of convergence itself: this fact would clearly imply that analytic invariant curves would exist for values of $\varepsilon$ *larger* than $\rho(\omega)$.

No analytic method is known for studying the analytic properties of the conjugating function $u$ *at the breakdown threshold*, *i. e.* where the singularities are. On the other hand, singularities of analytic functions can be located and studied numerically using Padé approximants, a well-known technique widely used in statistical mechanics and quantum theory. Padé approximants where also used in [24] to study the singularities of Birkhoff series.

We recall the essentials of the method, first used in this context in ref. [7].

The Padé approximant $[N_P, N_Q]$ of a function $f(z)$ analytic in a neighborhood of the origin in the complex plane is defined as a rational function $R(z) = P(z)/Q(z)$, $P$, $Q$ being polynomials in $z$ of degrees respectively equal to $N_P$, $N_Q$, such that the the first $N_P + N_Q + 1$ Taylor coefficients of $R$ are equal to their homologues of $f$. Since $P(z)$ and $Q(z)$ may be divided by a common constant factor without changin $R$, the number of independent unknown (the coefficients of the polynomials) is equal to the number of the equations (the number of the Taylor coefficients to be equaled to those of $f$), so the problem of their determination is (formally!) well-posed. A classical theorem tells us that for meromorphic $f$ the Padé approximants converge to $f$ as $N_P$, $N_Q$ tend to infinity, uniformly on compact sets in the analyticity domain of $f$. The singularities of $f$ can then be numerically found by determining the poles of $Q(z)$ for a sequence of Padé approximants of increasing order. Typically, many zero-pole ghost pairs appear, *i. e.* many zeros found in the denominator of $R(z)$ will cancel against a corresponding zero of the numerator: we take care of the problem by computing also the zeros of the numerator, and especially by computing the residues of each pole found; poles with a (numerically) negligible residue all have a zero at a negligible distance and are cancelled. In the case of more complicated singularities (like branching points, essential singularities or natural boundaries) less is known rigorously, though a rich numerical experience exists (see [25] for details and extensive references).

In our situation, one can derive some recursion relations for the coefficients $c_{n,k}$ by plugging the expansion (4) into (3) (see [7], [8]). The Fourier series can then be summed for each order $n$ in $\varepsilon$: we therefore are able to compute up to some (large) order $N$ the coefficients of the Taylor expansion in $\varepsilon$ of the function $u(\theta, \varepsilon, \omega)$, at fixed, given $\theta$ and $\omega$. We then compute some high order Padé approximants for $u$, for various values of $\theta$, and compute their poles, neglecting those with a vanishing (within numerical error) residue.

In ref. [7] the method sketched above was applied to the analysis of the complex analytic structure of the invariant curve with rotation number equal to the golden mean:

$$\omega = \gamma_1 = [1, 1, ...]$$

for the standard map ($f(x) = \sin x$ in (1)). The golden mean invariant curve for the standard map is supposed to break at $\varepsilon = 0.9761...$ ([14]), as numerical evidence shows. In ref. [7] Padé approximants of increasing order were computed, and an increasing number of poles distributed on a circle of radius comparable to the value of the breakdown threshold was found. This was regarded as a strong numerical evidence that (1) $\rho(\gamma_1) = \varepsilon_c(\gamma_1)$ and (2) a natural analyticity boundary exists on the circle of convergence in $\varepsilon$ of the Lindstedt series. This features appeared to be independent on the value of $\theta$ (except for the trivial cases $\theta = 0, \pi$, in which the Lindstedt series is identically zero). A similar calculation was performed for some other diophantine rotation numbers like the "silver number" $\gamma_2 = [2, 2, ...]$ and other quadratic irrationals, obtaining always a similar picture [26] (see Fig. 1 for the case $\omega = \gamma_2$ and $\theta = 1$).

These results lead us to conjecture that in the case of the standard map *for all diophantine rotation numbers $\omega$ the function $u(\theta, \varepsilon, \omega)$ has a natural boundary on the circle of convergence of its Taylor expansion in $\varepsilon$ for each fixed $\theta \neq 0, \pi$*; this implies that, for the standard map, $\rho(\omega) = \varepsilon_c(\omega)$ at least for diophantine $\omega$.

The situation is much clarified by *complexifying* the standard map and looking at the function $u$ for *complex values of $\omega$* [27], [12], [9]. In fact, when $\omega$ has a nonzero imaginary part the small denominators disappear, since the operator $D_\omega$ has a bounded inverse: $\|D_\omega^{-1}\| \leq \text{const}/\mathbf{Im}(\omega)$, and the convergence of the Lindstedt series my be proved directly by the contraction principle [12] since one can reformulate the problem as a fixed point one:

$$u(\theta, \varepsilon, \omega) = \varepsilon D_\omega^{-2} \sin(\theta + u(\theta, \varepsilon, \omega)).$$

The series (4) may now be regarded as an expansion of a function analytic in the three variables $\theta$, $\varepsilon$ and $\omega$. In this approach, we can state that only *one* KAM curve exists for the complexified map, and the usual, real KAM curves are obtained as limits of the complex one as $\omega$ tends to a real, diophantine value; their existence and regularity is determined by the existence and regularity of this limit.

Resonances will appear as singularities of $u(\theta, \varepsilon, \omega)$ for real, rational values of $\omega$. They form a natural boundary on the real $\omega$ axis. Following the earlier works of Kolmogorov and Arnol'd [28], we can prove that $u$, as a function of $\omega$, belongs to the class of Borel's *uniform monogenic functions*, *i. e.* functions with a dense, countable set of singularities which share most of the properties of ordinary , "weierstrassian", analytic functions [29].

In [9] we used Padé approximants to study numerically the analytic structure in $\varepsilon$ of $u(\theta, \varepsilon, \omega)$ for complex values of $\omega$. One can look at the structure of the singularities for large imaginary parts of $\omega$ and, on the other hand, for values of $\mathbf{Im}(\omega)$ which tend to zero, and in this second case it is important to differentiate the two cases $\mathbf{Re}(\omega)$ rational and $\mathbf{Re}(\omega)$ diophantine.

When a small imaginary part is addedd to a real diophantine $\omega$ we see that the natural boundary breaks and radial lines of singularities of the Padé approximants appear, which accumulate towards the radius of convergence of the series. As $\mathbf{Im}(\omega) \to 0$ the number of lines increases and they tend to form the natural boundary at $\mathbf{Im}(\omega) = 0$ (see Fig. 2). The radius of convergence of the series appears to depend continuously on $\mathbf{Im}(\omega)$.

The nature of these lines is better understood by looking at the behavior of the singularities structure as $\mathbf{Im}(\omega) \to 0$ *above a resonance*, *i. e.* as $\omega/2\pi$ tends to a rational value $p/q$ from *e. g.* the upper complex half plane. In this case the series converges thanks to the small imaginary part given to $\omega$ but the radius of convergence tends to 0 as $\mathbf{Im}(\omega) \to 0$; if the imaginary part is small enough (*i. e.* if we are close enough to the resonance $p/q$ so that to isolate its contribution from the neighboring ones) we see that there are in the singularities structure of the Padé approximants exactly $2q$ lines of poles interlaced with zeros equally spaced around the origin, which accumulate towards the radius of convergence of the series (see Fig. 3). This kind of lines of poles interlaced with zeros are typical manifestation in the Padé approximants of branching points of the approximated function. We therefore conjecture that each resonance (*i. e.* each rational value of $\omega/2\pi$) contributes $2q$ branching points to the singularities structure in $\varepsilon$ of $u$; as the imaginary part of $\omega$ decreases and $\omega$ tends to a diophantine value, more resonances of higher and higher order contribute an increasing number of singularities to the singularities structure of $u$, creating the natural boundary for real values of $\omega$.

In [8] Padé approximants were applied to carry a similar analysis to that of [7] for generalizations of the standard map in which the nonlinear term $f(x)$ contains higher frequencies than just $\sin x$. We still found natural boundaries, but their shape now is no longer circular (see *e. g.* Fig. 4 for the case $f(x) = \sin x + 1/20 \sin 2x$, $\omega/2\pi = \gamma_1$ and Fig. 5 for the case $f(x) = \sin x + 1/50 \sin 5x$, $\omega/2\pi = \gamma_3 = [3, 3, ...]$; in both cases $\theta = 1$).

Note that, thanks to the noncircular shape of the natural boundaries, *now the radius of convergence in $\varepsilon$ of the Lindstedt series (4) $\rho(\omega)$ may be strictly less than the breakdown threshold $\varepsilon_c(\omega)$.*. This is indeed the case illustrated in Fig. 5: $\rho(\gamma_1)$ is given by the distance to the origin of the closest singularity, that is about 0.8, while the breakdown threshold as $\varepsilon$ grows through real, positive values $\varepsilon_c(\gamma)$ is given by the intersection of the natural boundary with the real, positive $\varepsilon$ axis, that is about 1.1. To relate the shape of the natural boundary to the frequencies appearing in the Fourier expansion of the nonlinear term $f(x)$ and to the arithmetic properties of $\omega$ remains an open problem, whose solution would also explain under which circumstances the $\rho(\omega)$ and $\varepsilon_c(\omega)$ differ and why.

## Scaling of the radius of convergence

According to the picture sketched above, the various real KAM curves of different real diophantine rotation number are limit as $\omega$ tends to a real diophantine value of the same mathematical object, the function $u(\theta, \varepsilon, \omega)$. On the other hand, as $\omega$ tends to a resonance $p/q$ from the complex plane we saw that the radius of convergence of the series tends to zero. Two questions naturally arise: (1) does it tend to zero according to some simple scaling law? (2) if so, does the *whole* function $u(\theta, \varepsilon, \omega)$ tend to some "simple" limit as $\omega$ tends to a resonance and $\varepsilon$ is suitably rescaled in order to keep the radius of convergence fixed?

These questions are important for at least two reasons. First, the limit function alluded in (2) above would completely characterize the nature of the singularities corresponding to the resonance considered; eventually it should be possible to write the conjugating function $u$ as a sum of contributions coming from its singularities only.

Secondly, it is well known that the radius of convergence of the Lindstedt series for the standard map is related to a purely arithmetic function called *Brjuno's function* in [30]. Brjuno's function $B(t)$ may be defined in various ways, *e. g.* by the relation:

$$B(t) = -\log t + \omega B(t^{-1}) \tag{6}$$

for all irrational $t$ in the interval $[0, 1/2]$, with the boundary conditions:

$$B(t) = B(t+1) = B(-t).$$

$B(t)$ depends *only* on the number-theoretical properties of $t$, and in particular it may be easily shown to be finite if and only if $t$ satisfies the so-called *Brjuno's condition* [31]:

$$\sum_k \frac{\log q_{k+1}}{q_k} < \infty, \tag{7}$$

where $\{q_k\}$ is the sequence of the denominators of the convergents ot the continued fraction expansion of $t$.

It was proved by Yoccoz in [30] that the difference of the logarithm of the Siegel radius with $B(\omega/2\pi)$ is bounded of all irrational values of $\omega/2\pi$. One may pose the same *interpolation* problem for the standard map and for its generalizations. In particular, for the standard map one may ask whether there is a positive number $\beta$ and a bounded function $C(\omega)$ such that:

$$\rho(\omega) = C(\omega) e^{-\beta B(\omega/2\pi)}. \tag{8}$$

It was actually announced in [21], [32] that Brjuno's condition is sufficient for the convergence of the Lindstedt series for the standard map.

An analogous result was conjectured in [11] for the semi-standard map, which is the *complex* map obtained by choosing $f(x) = e^{ix}$ in equation (1). In [11] it was proved by elementary majorant and minorant series arguments that for this model the radius of convergence of the Lindstedt series $\rho(\omega)$ is greater than zero if and only if $B(\omega/2\pi)$ is finite, *i. e.* $\omega$ satisfies Brjuno's condition (7). More recently Davie [33] proved, always for the semi-standard map, that actually the following estimate holds:

$$|\log \rho(\omega) + 2B(\omega/2\pi)| \leq 176$$

thus generalizing Yoccoz's theorem and giving a positive answer to the conjecture stated in [11]. A stronger result, supported by numerical evidence, would be to prove that the interpolation (8) holds for the semi-standard map, with $C(\omega)$ a continuous positive function of $\omega \in [0, 2\pi]$.

The interpolation (8) implies a scaling properties for the radius of convergence $\rho(\omega)$ as $\omega$ tends to a resonance. In fact, let $\{\omega_n\}$ be a sequence of diophantine rotation numbers which tends to $2\pi p/q$ as $n \to \infty$; then, by simple calculations and by elementary properties of the continued fraction expansions, we have that $B(\omega_n/2\pi) \to \infty$ as $-(1/q)\log|\omega_n/2\pi - p/q|$ (see [12] for details). It follows that:

$$\rho(\omega_n) \sim \exp(-\beta B(\omega_n/2\pi)) \sim |\omega_n/2\pi - p/q|^{\frac{\beta}{q}}. \tag{9}$$

In [10] we computed numerically the "scaling exponent" for the radius of convergence (5) of the standard map using Padé approximants to compute $\rho(\omega)$ on a sequence of *complex* values of $\omega$ tending to $2\pi p/q$.

Some results are summarized in Table 1: we fitted the values of $\rho(2\pi p/q + i\eta)$ against $\eta$, for $p/q = 0/1, 1/2, 1/3, 1/5, 2/5$, using the formula $\rho(2\pi p/q + i\eta) = K(p/q)\eta^{\beta/q}$, with $\eta$ ranging between $10^{-1}$ and $10^{-5}$. The results are consistent with the conjecture that $\beta = 2$ (the order of the recurrence in (1), like $\beta = 1$ for Siegel's radius). This suggests that the limit:

$$u^{(p/q)}(\theta, \varepsilon) = \lim_{\eta \to 0} u(\theta, \eta^{2/q}\varepsilon, 2\pi p/q + i\eta) \tag{10}$$

exists and is jointly analytic in $\varepsilon$, $\theta$ for $|\varepsilon|$, $|\mathbf{Im}(\theta)|$ small enough.

| $\mathbf{Re}(\omega)$ | $\hat{\beta}$ |
|---|---|
| 0 | 2.01 |
| 1/2 | 2.04 |
| 1/3 | 2.03 |
| 1/5 | 2.00 |
| 2/5 | 2.02 |

Table 1

We also looked at the scaling of the radius of convergence of the Lindstedt series for some maps with a more complex nonlinear term $f(x)$ (*e. g.* $f(x) = \sin x + \alpha \sin 2x$). The picture here is much more complex to understand, since we no longer have the interpolation formula with Brjuno's function. As $\omega \to 2\pi p/q$ again the radius of convergence tends to zero as a power of $|\omega - 2\pi p/q|$, but the exponent is no longer so easily related to the order $q$ of the resonace; in fact, due to the presence of higher frequencies in the nonlinear term $f(x)$ a much more complicated structure of the small denominators occurs in the Lindstedt series especially when $q$ is a multiple of some of the frequencies of $f$. Table 2 summarizes some preliminary results in the case

$f(x) = \sin x + 1/20 \sin 2x$, which suggest that in this case the scaling of the radius of convergence is given by:

$$\rho(\omega) \approx |\omega/2\pi - p/q|^{\beta/\lceil q/2 \rceil},$$

where $\lceil x \rceil$ denotes the lowest integer greater or equal to $x$ and again $\beta = 2$.

| $\mathbf{Re}(\omega)$ | $\hat{\beta}$ |
|---|---|
| 0 | 2.001 |
| 1/2 | 2.000 |
| 1/3 | 1.998 |
| 1/5 | 2.008 |
| 2/5 | 2.015 |

Table 2

It is interesting to ask whether there are "higher order Brjuno's functions" that interpolate the radius of convergence for these maps justifying these scaling laws and how they are obtained in terms of the continued fraction expansion of the rotation number $\omega$ and in terms of the frequencies that appear in the Fourier expansion of the nonlinear term $f(x)$. This would shed some light on the mechanism that leads to the non-circular natural boundaries observed in ref. [8] and therefore on the discrepancy between the radius of convergence and the breakdown threshold that shows up in some of these maps.

## Conclusions

A first partial attempt at a classical analogue of "perturbative renormalization" has been made by Bogomol'nyi [34]. He considered the Hénon-Heiles hamiltonian [35]:

$$H(p_1, p_2, x_1, x_2) = \frac{1}{2}\left(p_1^2 + p_2^2 + x_1^2 + x_2^2 + 2x_1^2 x_2 - \frac{2}{3}x_2^3\right),$$

which was investigated in the context of galactic dynamics. The Hénon-Heiles model is resonant and non-integrable, and therefore it does not admit a single-valued suppelentary integral $I(p_1, p_2, x_1, x_2)$ independent on the energy. Nevertheless, canonical perturbation theory allows to construct a formal series:

$$\sum_{n=2}^{\infty} I_n(p_1, p_2, x_1, x_2), \tag{11}$$

where $I_n$ is a homogeneous polynomial of degree $n$ in its arguments, which is a formal solution of the first-integral equation:

$$\{I, H\} = 0, \tag{12}$$

where $\{,\}$ denotes Poisson's brackets.

Perturbation theory [36], [37] here consists in the use of adapted variables - called *Birkhoff's variables* - and normal form theory to solve the recursion relations for the polynomials $I_n$ obtained substituting the formal series (11) into equation (12) and projecting on the space of homogeneous polynomials of increasing degree. The series constructed in this way will be divergent but asymptotic to a function $I(p_1, p_2, x_1, x_2)$ which has a square root singularity at each periodic orbit $\mathcal{P}$ of the system. Since the set of periodic orbits is dense, the series expansion (11) will diverge.

To determine the precise form of $I(p_1, p_2, x_1, x_2)$ in a neighborhood of a given periodic orbit $\mathcal{P}$ one has to linearize the hamiltonian flow and compute the monodromy matrix $M^{\mathcal{P}}$. Bogomol'nyi showed that the most singular part $I_{\text{sing}}^{(\mathcal{P})}$ of $I$ in a neighborhood of the periodic orbit $\mathcal{P}$ is given by:

$$I_{\text{sing}}^{(\mathcal{P})} = \sqrt{(I - I^{(\mathcal{P})})_E - \Delta^{(\mathcal{P})}\left(1 + \frac{\Delta^{(\mathcal{P})}}{4}\right) A^{(\mathcal{P})}}, \tag{13}$$

where $I^{(\mathcal{P})}$ is the value of $I$ on the periodic orbit, $\left(I - I^{(\mathcal{P})}\right)_E$ is the variation of $I$ keeping the energy $E$ fixed, $\Delta^{(\mathcal{P})} = \text{Trace}(M^{\mathcal{P}}) - 4$ and $A^{(\mathcal{P})}$ is a non-vanishing regular function of $p_1, p_2, x_1, x_2$.

If one expands (13) into powers of $\Delta^{(\mathcal{P})}$ and retains only the first non-trivial term one finds:

$$I_{\text{sing}}^{(\mathcal{P})} \approx -\frac{1}{2}\Delta^{(\mathcal{P})} \frac{A^{(\mathcal{P})}}{(I - I^{(\mathcal{P})})_E}$$

and then, summing over all periodic trajectories one obtains:

$$I_{\text{sing}} \approx -\frac{1}{2} \sum_{\text{periodic orbits } \mathcal{P}} \Delta^{(\mathcal{P})} \frac{A^{(\mathcal{P})}}{(I - I^{(\mathcal{P})})_E}. \tag{14}$$

This should be a representation of the function whose asymptotics series expansion (11) is given by canonical perturbation theory. Clearly $I_{\text{sing}}^{(\mathcal{P})}$ plays here the rôle of our $u^{(p/q)}(\theta, \varepsilon)$ defined in (10) and (14) is analogous to the expansion of $u$ in terms of the $u^{(p/q)}(\theta, \varepsilon)$ that we are looking for.

Apart from the absence of mathematical rigor in Bogomol'nyi's argument, the main shortcoming of his method is the need to compute the values $\Delta^{(\mathcal{P})}$, $A^{(\mathcal{P})}$ and $I^{(\mathcal{P})}$, which depend on the position in phase space of the periodic orbit (*i. e.* on the action values corresponding to them). On the other hand, Lindstedt series only depend on frequencies, and therefore seem to be a better candidate for a perturbative renormalization argument.

Next, we observe that for the standard map, if in (4) we set $\omega = 2\pi p/q + i\eta$, $\varepsilon = \bar{\varepsilon}\eta^{2/q}$, ($\eta \in \mathbb{R}$, $\eta > 0$) and let $\eta \to 0$, then it is possible to prove that each term in the Lindstedt series has a finite limit and the resulting (formal) series is not identically zero; in the case of the semi-standard map we could also prove that the resulting series is convergent for all rational values of $\omega/2\pi$ and compute explicitly its sum (see [38] for details).

The general picture that emerges is therefore the following: only one object, the conjugating function $u(\theta, \varepsilon, \omega)$, describes the breakdown phenomenon at *all* rotation numbers; as $\omega$ tends to a sufficiently irrational rotation number (*e. g.* as $\omega$ tends to a number satisfying Brjuno's condition in the case of the standard map) and $\varepsilon$ is small enough, $u$ tends to a limit which is jointly analytic in a neighborhood of the origin in the complex $\varepsilon$ plane times a neighborood of the real axis in the complex $\theta$ plane; as $\omega/2\pi$ tends to a rational value, $u$ also has a limit provided we rescale $\varepsilon$ according to the scaling law satisfied by the radius of convergence of the Lindstedt series (5). The analytic structure of these limits in the complex $\varepsilon$ plane explains the nature of the singularities observed in [9] for $u$ when the rotation number is close to a resonance and determines their nature. Furthermore, as the behavior of an analytic function is determined by its singularities, we conjecture that it is possible to expand the conjugating function $u$ for *all* values of $\omega$ in terms of its limits at resonances, obtaining for $u$ an expansion analogous to Bogomol'nyi expansion (14) for the first-integral of the Hénon-Heiles hamiltonian.

Finally, we whish to stress the analogy of the mechanism of eliminating the divergencies from the Lindstedt series by adding a small imaginary part to the rotation number with the procedure of regularizing the perturbative series of a quantum field theory by modifying the propagators in such a way to eliminate the divergences which occur in the continuum limit. The rôle of the propagators which behave in a singular way at large momenta is played here by the small denominators. The analogy may be pushed further: in fact, Gallavotti has shown in [39] that the "tree graphs" in the work of [20], which label each term in each coefficient $c_{n,k}(\omega)$ of the Lindstedt series (4), are the Feynmann graphs of a tree-level approximation to a simple quantum field theory, with interaction depending on the perturbative term and with propagator given, in our case, by the kernel of the operator $D_\omega^{-2}$.

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All the calculation described here have been performed on several DEC VAX/VMS computer systems at several locations.

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