SU-4240-566

February 1994

GRAVITY AND ELECTROMAGNETISM IN NONCOMMUTATIVE GEOMETRY

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Abstract

We present a unified description of gravity and electromagnetism in the framework of a Z_2 noncommutative differential calculus. It can be considered as a "discrete version" of Kaluza-Klein theory, where the fifth continuous dimension is replaced by two discrete points. We derive an action which coincides with the dimensionally reduced one of the ordinary Kaluza-Klein theory.

PACS. 04.20.Jb, 04.40. +c, 11.15. -q, 14.80.Hv

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Recently, Connes [1, 2] has proposed a new construction of noncommutative geometry where the basic objects are an algebra $\mathcal{A}$ (possibly noncommutative), a Hilbert space on which the algebra acts, and a "Dirac operator" on the Hilbert space. The algebra generalizes the idea of a manifold and the operator prescribes the rule for differentiation and provides the metric structure.

Within the general framework of Connes, different realizations of the geometry with different algebras and different Dirac operators have been proposed leading to various applications in particle physics [3]. One such version due to Connes and Lott [4], is based on regarding the space-time to be a direct product of a Riemannian manifold and a discrete "two-point" space. A gauge field on such a generalized space-time consists of the usual gauge field in the Riemannian manifold and a Higgs field as a part of the connection in the discrete internal space. Thus, it lends itself to a unified geometric description of the bosonic fields including both the ordinary gauge fields and Higgs fields. And what is remarkable is that the action constructed on such a generalized space contains a spontaneous symmetry-breaking Higgs potential.

An alternate way of picturing this generalized space-time is along the lines of Kaluza-Klein theories [5], in which the continuous extra fifth dimension describing the internal space is replaced by a finite number of discrete points. The advantage of this approach over the ordinary Kaluza-Klein theory is that there is no truncation of any physical modes, while in the latter an infinite number of massive modes is truncated. Since this is a new concept of space-time, it is natural to investigate its implication on gravity within the framework of noncommutative geometry. A first step in this direction was taken by Chamseddine, Felder and Fröhlich, who employed a vielbein and a connection in generalized Cartan structure equations. This has led to Einstein's gravity along with a massless scalar field coupled to gravity and a cosmological constant [6]. The Wodzicki residue has also been used to obtain gravity with a cosmological constant [7]. Since the analysis of these authors can be extended to a generalized space consisting of an arbitrary number of discrete points, the question arises regarding how the vector fields in the Kaluza-Klein theory disappear when we approximate the compact internal space by a finite number of discrete points. Does discretization eliminates the vector modes altogether?

This note addresses the above question. We show that in the simplest version of a generalized space-time with two discrete points, one can formulate a noncommutative geometry which includes a symmetric tensor, a vector and a scalar fields. In other words, we can construct a Kaluza-Klein type unified Einstein-Maxwell theory. In fact, in [7] a vector field was included in the connection and was shown that it dropped out completely. As we shall see, this does not happen if the vector fields are introduced in the vielbein [‡].

In this brief note, we shall limit the mathematical details to the necessary minimum and concentrate on the physical aspects of the model. More details will appear elsewhere [9].

We consider a noncommutative geometry *à la* Connes [2] built on a Z_2 algebra, $\Omega^0 = \mathcal{A} = C^\infty(\mathcal{M}) \otimes Z_2$, where $C^\infty(\mathcal{M})$ is the algebra of smooth real functions on the four dimensional manifold $\mathcal{M}$. That is to say, any element $F \in \Omega^0$ can be expressed as

$$F(x) = \tilde{f}_1(x)e + \tilde{f}_2(x)r \ , \tag{1}$$

where

$$e, r \in Z_2 : e^2 = e,\ er = re = r,\ r^2 = e \ . \tag{2}$$

We can realize this algebra $\mathcal{A}$ by mapping it into the algebra of 2×2 matrices,

$$\pi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \pi(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

[‡]After completing the work we received a preprint by Sitarz [8], which has some similar elements in the mathematical construction as ours, but the final result is not different from the previous ones for the same reason.

$$\pi(F) = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix} \;=\; \tilde{f}_1(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{f}_2(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{3}$$

where f_1, f_2 are obvious combinations of $\tilde{f}_1, \tilde{f}_2$.

The second crucial ingredient of noncommutative geometry is the Dirac operator. We will follow the idea of Connes to construct it in parallel with the ordinary commutative geometry. Let $\mathcal{E}_N$ $(N = \mu, 5)$ be a linearly independent 'tangent basis' which acts on Ω^0 by the commutator,

$$\mathcal{E}_N(F) = [\mathcal{E}_N, F]. \tag{4}$$

It has the following realization

$$\begin{aligned} \mathcal{E}_\mu &= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, \quad \mu = 1, \cdots, 4 , \\ \mathcal{E}_5 &= \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \end{aligned} \tag{5}$$

where m is a c-number with the dimension of mass.

It is easy to verify that $\mathcal{E}_N$ satisfies the Newton-Leibnitz rule,

$$\mathcal{E}_N(FG) = \mathcal{E}_N(F)G + F\mathcal{E}_N(G). \tag{6}$$

Hence, we can consider $\mathcal{E}_N$ as derivations in the Z_2- noncommutative geometry and denote them by $D_N \doteq \mathcal{E}_N$. The possibility to enlarge the 'tangent basis' by an outer automorphism D_5 enriches the structure of noncommutative geometry. Without it the geometry is of commutative character.

Working in the Hilbert space of spinors, Connes [2, 4] chose the 'cotangent basis' Γ^M of the extended Dirac matrices as follows

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}. \tag{7}$$

Hence, the Dirac operator in the Z_2-noncommutative geometry has the self-adjoint realization

$$D\!\!\!\!/ \doteq \Gamma^N D_N \equiv \begin{pmatrix} \partial\!\!\!/ & \gamma^5 m \\ \gamma^5 m & \partial\!\!\!/ \end{pmatrix} . \tag{8}$$

The choice (7) of Γ^5 leads to

$$\{\not{D}, \Gamma^5\} = 0 \ . \tag{9}$$

Since in this paper we are not working in the Hilbert space of spinors, we need an alternative realization for the 'cotangent basis' ϵ^N. We can choose the realization

$$\begin{aligned} DX^\mu &\equiv \epsilon^\mu \doteq \begin{pmatrix} dx^\mu & 0 \\ 0 & dx^\mu \end{pmatrix}, \quad \mu = 1, \cdots, 4 \ , \\ DX^5 &\equiv \epsilon^5 \doteq \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}, \end{aligned} \tag{10}$$

where θ is a Clifford element satisfying

$$\theta^2 = 1 \quad , \quad \theta dx^\mu = -dx^\mu \theta \ . \tag{11}$$

In our formalism θ replaces γ^5 when we are not working on the Hilbert space of spinors.

The exterior derivative operator D is given by

$$D \doteq DX^N D_N \equiv \begin{pmatrix} d & \theta m \\ \theta m & d \end{pmatrix} \ , \tag{12}$$

where d denotes the exterior derivative on $\mathcal{M}$. The exterior derivative acts on a function $F = (f_1, f_2) \in \Omega^0$ as follows:

$$DF \doteq DX^N D_N F = \begin{pmatrix} df_1 & \theta m(f_2 - f_1) \\ \theta m(f_1 - f_2) & df_2 \end{pmatrix}, \tag{13}$$

or in the Γ-realization

$$\not{D}F \doteq [\not{D}, F] = \begin{pmatrix} \not{\partial} f_1 & \gamma^5 m(f_2 - f_1) \\ \gamma^5 m(f_1 - f_2) & \not{\partial} f_2 \end{pmatrix}. \tag{14}$$

A 'vector field' $V \in \Lambda^1$ in the Z_2-noncommutative geometry can be defined as follows

$$V \doteq V^N D_N = \begin{pmatrix} v_1^\mu(x)\partial_\mu & m v_1(x) \\ -m v_2(x) & v_2^\mu(x)\partial_\mu \end{pmatrix}, \tag{15}$$

where V^N are elements of Ω^0.

A 'covector field' or 1-form $U \in \Omega^1$ is defined as

$$U \doteq DX^N U_N = \begin{pmatrix} dx^\mu u_{1\mu}(x) & \theta u_1(x) \\ \theta u_2(x) & dx^\mu u_{2\mu} \end{pmatrix}, \tag{16}$$

where

$$U_5 \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U \ , \tag{17}$$

and U^μ, U are element of Ω^0.

The definition of the fifth component of the 1-form by the rule (17) is implied by the natural requirement that the element DF be a 1-form.

It is possible to construct higher differential forms and a differential algebra from the following definition of the wedge product

$$\begin{aligned} DX^\mu \wedge DX^\nu &\doteq \begin{pmatrix} dx^\mu \wedge dx^\nu & 0 \\ 0 & dx^\mu \wedge dx^\nu \end{pmatrix} \equiv -DX^\nu \wedge DX^\mu, \\ DX^5 \wedge DX^\mu &\doteq \begin{pmatrix} \theta dx^\mu & 0 \\ 0 & \theta dx^\mu \end{pmatrix} \equiv -DX^\mu \wedge DX^5, \\ DX^5 \wedge DX^5 &\doteq 0 \end{aligned} \tag{18}$$

Alternately, we could have postulated $DX^5 \wedge DX^5 \neq 0$ and recover the construction of Coquereaux et al. [3]. Our construction treats "X^5" as an 'even' element on an equal footing with the space-time coordinates "X^μ".

A general p-form $W_p \in \Omega^p$ is defined as

$$W_p \doteq DX^{N_1} \wedge ... \wedge DX^{N_p} W_{N_1 ... N_p}. \tag{19}$$

By generalizing the rule (17), we can express the components $W_{N_1....N_p}$ in the form,

$$W_{N_1....N_p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^r W_{\nu_1....\nu_{p-r}}, \tag{20}$$

where r=0,1 ; ν_i is the i-th index, which is different from 5 and $W_{\nu_1...\nu_{p-r}} \in \Omega^0$. If the index 5 appears more than once, the component is zero.

The exterior derivative $DW_p \in \Omega^{p+1}$ of a p-form $W_p \in \Omega^p$ and the wedge product $W_{1p} \wedge W_{2q} \in \Omega^{p+q}$ of a p-form $W_{1p} \in \Omega^p$ and a q-form $W_{2q} \in \Omega^q$ are defined to be

$$\begin{aligned} DW_p &= DX^M \wedge DX^{N_1} \wedge .... \wedge DX^{N_p} D_M W_{N_1 ... N_p}, \\ W_{1p} \wedge W_{2q} &= DX^{N_1} \wedge ... \wedge DX^{N_p} \wedge DX^{N_{p+1}} \wedge ... \wedge DX^{M_{p+q}} W_{1N_1 ... N_p} W_{2N_{p+1} ... N_{p+q}} . \end{aligned} \tag{21}$$

We have the following essential properties for the exterior derivative,

$$\begin{aligned} & D^2 W_p = 0 \ , \qquad \forall \, p \ , \\ & D(W_p \wedge W_q) = DW_p \wedge W_q + (-1)^p W_p \wedge DW_q \ . \end{aligned} \tag{22}$$

The noncommutative character of our geometry is reflected in the fact that $W_p \wedge W_q$ and $W_q \wedge W_p$ are not related in general to each other by a simple factor as in the case of ordinary commutative geometry.

Although in what follows the geometrical objects we construct resemble those of ordinary geometry, their noncommutative character dictates strictly the order.

Next we introduce an orthonormal basis of vielbein $\{E^A\}$ $(A = a, 5)$. As a direct generalization of vielbein, E^A are 1-forms in the Z_2-noncommutative geometry, $E^A \doteq DX^M E^A_M$, whose general expression is as follows,

$$\begin{aligned} E^a &\doteq \begin{pmatrix} e_1^a & \theta f_1^a \\ \theta f_2^a & e_2^a \end{pmatrix}, \quad a = 1, \cdots, 4 \ , \\ \\ E^5 &\doteq \begin{pmatrix} a_1 & \theta \phi_1 \\ \theta \phi_2 & a_2 \end{pmatrix}, \end{aligned} \tag{23}$$

where e_1^a, e_2^a are vielbein on $\mathcal{M}$, a_1, a_2 are 1-forms on $\mathcal{M}$ and $f_1^a, f_2^a, \phi_1, \phi_2$ are real function on $\mathcal{M}$.

As in the usual Riemannian geometry, we still have a degree of freedom to choose the following forms for vielbein without any loss of generality:

$$E^a \doteq \begin{pmatrix} e_1^a & 0 \\ 0 & e_2^a \end{pmatrix}, \quad a = 1, \cdots, 4 \ ,$$

$$E^5 \doteq \begin{pmatrix} a_1 & \theta\phi_1 \\ \theta\phi_2 & a_2 \end{pmatrix}. \tag{24}$$

In this paper, we are particularly interested in the self-adjoint vielbein

$$E^a \doteq \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} = DX^\mu e^a_\mu$$

$$E^5 \doteq \begin{pmatrix} a & \theta\phi \\ \theta\phi & a \end{pmatrix} = DX^\mu a_\mu + DX^5 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi(x). \tag{25}$$

while the general case will be treated elsewhere [9][§].

Having a vielbein we can construct a metric tensor G. We will think of G as a functional [2] $G : \Omega^1 \times \Omega^1 \longrightarrow \mathcal{A}$, such that

$$G(UF, WH) = F^\dagger G(U, W)H \ , \quad \forall \ U, W \in \Omega^1 \ , \ F, H \in \Omega^0. \tag{26}$$

In the E^A-basis, the metric is taken to be

$$G(E^A, E^B) = \eta^{AB} \ ,$$
$$\eta^{AB} = diag(-1, 1, 1, 1, 1) \ . \tag{27}$$

In the DX^M-basis we will have

$$G^{MN} = G(DX^M, DX^N) = E^M{}_A \eta^{AB} E^N{}_B \ , \tag{28}$$

where $E^M{}_A$ are the inverses of $E^A{}_M$.

It is worth noting that this metric is symmetric with the particular vielbein (25), but is not in the general case allowed by Z_2-noncommutative geometry.

[§]In the Eq.(25), by setting $a = 0$ we obtain the vielbein used by Chamseddine et al. [6] as a particular case. It is worth noting that, if we use the vielbein (25) in a Dirac operator approach to gravity [7], the Dirac operator is still self-adjoint and obeys Eq.(8), provided we take for the Γ^5 the appropriate "curved" one.

Following Connes [2], we define the connection through a covariant derivative ∇

$$\nabla : \Omega^1 \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \ ,$$

$$\nabla(UF) = (\nabla U)F + U \otimes DF \ . \tag{29}$$

Here the tensor product $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is generated by elements $\{U_1 \otimes U_2; U_1, U_2 \in \Omega^1\}$ with the relation $U_1 F \otimes U_2 = U_1 \otimes F U_2$ for any $F \in \Omega^0$.

A connection is equivalently given by a set of connection one-forms ${\Omega^A}_B \in \Omega^1$, the relation being

$$\nabla E^A = E^B \otimes {\Omega^A}_B \ . \tag{30}$$

The Cartan structure equations define torsion and curvature of a given connection as follows

$$T^A = DE^A - E^B \wedge {\Omega^A}_B \ , \tag{31}$$

$${R^A}_B = D{\Omega^A}_B + {\Omega^A}_C \wedge {\Omega^C}_B \ , \tag{32}$$

where T^A and ${R^A}_B$ are 2-forms.

As in the case of ordinary Riemannian geometry, we can impose the torsion free condition $T^A = 0$. Then the structure equation (31) reduces to

$$DE^A = E^B \wedge {\Omega^A}_B \ , \tag{33}$$

The connection is said to be a Levi-Civita one it it is also metric compatible, that is if it satisfies

$$D(G(U,W)) = \tilde{G}(\nabla U, W) + \tilde{G}(U, \nabla W) \ , \ \forall \ U, W \in \Omega^1 \ . \tag{34}$$

Here $\tilde{G}$ is the extension of the metric given by

$$\tilde{G}(U_1 \otimes U_2, W) = U_2^\dagger G(U_1, W) \ ,$$

$$\tilde{G}(U, W_1 \otimes W_2) = G(U, W_1) W_2 \ , \quad \forall \ \ U, U_1, U_2 \ W, W_1, W_2 \ \in \Omega^1 \ , \tag{35}$$

where † denotes the adjoint.

By using the fact that we have a self-adjoint vielbein, condition (34) gives

$$\Omega^A{}_C \eta^{CB} + \eta^{AC} \Omega^B{}_C = 0 \ . \tag{36}$$

The structure equation (33) and condition (36) determine the connection 1-forms $\Omega^A{}_B$ uniquely. From $\Omega^A{}_B$ we can derive the curvature 2-forms $R^A{}_B$ and their components $R^A{}_{BCD}$, $R^A{}_B = E^C \wedge E^D R^A{}_{BCD}$.

After a little algebra, the scalar curvature $R = R^A{}_{BAD} \eta^{BD}$ is found to be

$$R = R_4 - 2\frac{\Box \phi}{\phi} - \frac{1}{4} \Psi_{ab} \Psi^{ab} \ , \tag{37}$$

where

$$\begin{aligned}
&R_4 \quad \text{is the } 4-\text{dimensional Ricci scalar} \\
&\Box = \nabla_\mu \partial_\mu \ , \quad \nabla_\mu \ \text{ is the } 4-\text{dimensional covariant derivative} \\
&\Psi_{ab} = e^\mu_a e^\nu_b [(\partial_\mu a_\nu - \partial_\nu a_\mu) + a_\mu \frac{\partial_\nu \phi}{\phi} - a_\nu \frac{\partial_\mu \phi}{\phi}] \ .
\end{aligned} \tag{38}$$

We can redifine the vector field $a_\mu \to \phi a_\mu$ and obtain

$$R = R_4 - \frac{2\Box \phi}{\phi} - \frac{1}{4} \phi^2 f_{\mu\nu} f^{\mu\nu} \ , \tag{39}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

The expression for the scalar curvature R in (39) is identical to that in Kaluza-Klein theory [5], when one retains only the zero-modes in the expansion with respect to the fifth coordinate and assumes that the fields are independent of this coordinate. We could then obtain the action by a suitable integration of $\sqrt{-det|G|}R$. Here $det|G|$ denotes the determinant of our generalized metric defined above and is given as follows

$$\begin{aligned}
det|G| \ &\doteq \ \frac{1}{5!} \epsilon_{N_1 N_2 N_3 N_4 N_5} \epsilon_{M_1 M_2 M_3 M_4 M_5} G^{N_1 M_1} G^{N_2 M_2} G^{N_3 M_3} G^{N_4 M_4} G^{N_5 M_5} \\
&= \ \frac{1}{4!} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} G^{\nu_1 \mu_1} G^{\nu_2 \mu_2} G^{\nu_3 \mu_3} G^{\nu_4 \mu_4} G^{55} \equiv det|g| \phi \ \mathbf{1}
\end{aligned} \tag{40}$$

where $det|g|$ is the determinant of the 4-dimensional metric and ϵ are fully antisymmetric Levi-Civita tensors. If we take the trace of the 2×2 matrix as the integration over the discrete coordinate, we obtain

$$S \sim \int d^4x \sqrt{-det|g|}\phi R, \tag{41}$$

which reproduces the Kaluza-Klein action up to a proportional constant. The parameter m replaces the radius of the compactified circle in the fifth dimension of the Kaluza-Klein theory[¶]. Thus, in the model based on the vielbein (25), we have the dimensionally reduced Kaluza-Klein theory with massless tensor, vector and scalar fields. The most general vielbein of Eq.(24) yields a full Kaluza-Klein theory, with a finite number of fields. The field content of the full theory contains a pair of tensors, a pair of vectors and a pair of scalar fields without imposing any truncation condition. In each pair, one field is massless and the other is its massive excitation. The inconsistencies of the Kaluza-Klein theory that arise when one truncates the spectrum by including only a finite number of massive modes are absent in the noncommutative geometric approach. These matters and further details of mathematical formalism will be presented elsewhere [9].

Acknowledgments.

This work was supported in part by the U.S. Department of Energy under contract number DE-FG02-85ER40231.

The work of G.L. is partially supported by the Italian 'Ministero dell' Università e della Ricerca Scientifica'.

G.L. is grateful to Prof. A.P. Balachandran for his kind hospitality in Syracuse. He also wishes to thank him, L. Chandar, E. Ercolessi, A. Momen and P. Teotonio for many useful discussions on noncommutative geometry.

[¶]Our action differs from the action obtained previously [6, 8, 7] by a factor of ϕ even if we set $a = 0$ to have the same vielbein.

N.A.V. is grateful to Professor A.Zichichi for a World Laboratory scholarship.

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