Charge correlations for the two dimensional Coulomb gas*

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April 25, 1997

Abstract

This paper is a summary of mathematical results contained in [14] concerning integer charge correlations for the Coulomb gas/sine-Gordon system in two dimensions. For $\beta = T^{-1} < 8\pi$ and small activity z, the UV problem is considered in a finite volume. A new proof is given of the fact that the pressure $p^{>m}(\beta, z)$, renormalized up to order m in perturbation theory, is analytic in z for $\beta < \beta_m = 8\pi(1 - 1/m)$. Higher correlations are treated and proven to be analytic in z for all $\beta < 8\pi$. The mth threshold value β_m appears as the value at which the exponent of the short distance power law of the mth subleading contribution to any correlation changes nonanalytically. In the Kosterlitz-Thouless phase $\beta > 8\pi$, the IR problem is treated with a fixed UV cutoff. The existing framework for the pressure is extended to all higher correlations. For the two point function, it is shown that at length scales larger than $\mathcal{O}(|z|^{-1/(\beta/4\pi-2)})$ the free field power law $|x-y|^{-\beta/2\pi}$ at long distances crosses over to a slower power law $|x-y|^{-4}$. This verifies a conjecture of Fröhlich and Spencer [12].

1 Introduction

The two dimensional Coulomb gas/sine-Gordon system is a classic model in mathematical physics. There exists a large body of discussion of its special properties, including

*Talk presented at the Conference on Constructive Results on Field Theory, Statistical Mechanics and Condensed Matter Physics, Palaiseau, July 25-27, 1994.
†Research supported by the Natural Sciences and Engineering Research Council of Canada.

landmark papers such as that of Kosterlitz and Thouless [15], Coleman [6], Zamolodchikov [19], and Sklyanin et al. [18]. On the level of constructive quantum field theory, there is a somewhat smaller body of results: these include the works of [11], [12], [13], [3],[1],[16].

My aim here is to extend the constructive renormalization group (RG) programme developed in a series of papers [7],[9],[8],[10],[2] which began with a foundational paper [5] by Brydges and Yau. I shall explain how the BY method, with some adaptations based on ideas in [4], leads to a rather complete picture of perhaps the most important aspect of the model, namely the asymptotics of integer charge correlations, both in the ultraviolet (UV) and infrared (IR).

We consider the Coulomb gas with activity z at temperature $T = \beta^{-1}$ on a finite torus $\Lambda(M) = \mathbf{R}^2/(L^M\mathbf{Z})^2$ where $L > 1, M$ are integers. Let the Coulomb potential be the inverse Laplacian on $\Lambda(M)$ with a short distance cutoff at scales $\sim L^{-N}$:

$$v_{M,N}(x) = L^{-2M} \sum_{\substack{p\in\Lambda(M)^* \\ p\neq 0}} p^{-2} e^{ipx}\ e^{-L^{-2N}p^2}, \tag{1}$$

where $\Lambda(M)^*$ denotes the dual lattice $(2\pi L^{-M}\mathbf{Z})^2$. Then the system is described by the grand canonical partition function

$$Z_{CG}(\beta, z, v_{M,N}) = \sum_{n,\vec{q}} \int_{\Lambda(M)^n} d\vec{x}\ \frac{z^n}{n!}\ \exp(-\beta E_{n,v_{M,N}}(\vec{x};\vec{q}))$$

where the Coulomb energy is given by

$$E_{n,v_{M,N}}(\vec{x};\vec{q}) = \sum_{a<b} q_a q_b v_{M,N}(x_a - x_b)$$

This can be expressed equivalently in terms of the sine-Gordon model:

$$Z_{CG}(\beta, z, v_{M,N}) = Z_{SG}(2z, \beta v_{M,N})$$

where

$$Z_{SG}(\zeta, \beta v_{M,N}) = \int d\mu_{\beta v_{M,N}}(\phi) e^{\zeta\int :\cos\phi:}. \tag{2}$$

Here $\mu_{\beta v_{M,N}}(\phi)$ is the Gaussian measure with covariance $\beta v_{M,N}(x,y)$ on a certain Sobolev space $\mathcal{H}(\Lambda(M))$, perturbed by the interaction $e^{\zeta\int :\cos\phi:}$. The Wick dots denote Wick ordering with respect to μ.

I am interested in two problems. The first is the UV problem of taking the $N \to \infty$ limit for fixed volume parameter M. The Kosterlitz-Thouless argument suggests that this limit should exist for any $\beta < 8\pi$ and small activity $\zeta = 2z$. We have indeed proved this: apart from a finite number of vacuum energy counterterms, the Wick ordering in (2) is the only renormalization necessary.

The second aspect of the model is the IR problem of taking $M \to \infty$ while holding N fixed. For this, (2) is not quite the right starting point because the Coulomb potential

$v_{M,N}(x)$ diverges as $M \to \infty$. We should "zero" the Coulomb potential at the origin, i.e. take

$$w_{M,N}(x) = v_{M,N}(x) - v_{M,N}(0).$$

At the same time, it turns out to be necessary to suppress non-neutral charge configurations. Making use of the identities

$$\begin{aligned}\sum_{n,\vec{q}} \int_{\Lambda(M)^n} d\vec{x}\, \frac{z^n}{n!}\ \exp(-\beta v_{M,N}(0)(\sum_a q_a)^2)\ \exp(-\beta E_{n,w_{M,N}}(\vec{x};\vec{q}))& \\ = Z_{CG}(\beta, e^{\beta v_{M,N}(0)/2} z, v_{M,N})& \\ = \int d\mu_{\beta v_{M,N}}(\phi) e^{\zeta \int \cos\phi}.& \end{aligned} \tag{3}$$

and noticing that $v_{M,N}(0) = \mathcal{O}(N+M)\ \log L > 0$, we see that the SG model naturally leads to a suppression of non-neutral configurations. It appears, as indeed we shall show, that the *non-Wick-ordered* sine-Gordon model is an appropriate starting point for the IR problem. The difference between (2) and (3) is a divergent factor

$$\tau_{M,N} = e^{\beta v_{M,N}(0)/2} \sim L^{\beta(M+N)/4\pi} \tag{4}$$

multiplying the coupling constant ζ.

Now consider a configuration $\Sigma = \{\xi_1, \ldots, \xi_Q\}$ of Q external unit charges $\xi_a = (x_a, q_a)$, $q_a = \pm 1$, fixed at points x_a in $\Lambda(M)$. To this we associate a partition function Z^Σ where the Coulomb energies coming from the external charges are included. The *integer charge truncated correlations* are defined to be:

$$\begin{cases} S^{q_1}_{x_1} = Z^{-1} Z^{q_1}_{x_1} = (2z)^{-1} \times \text{density} & Q = 1 \\ S^{q_1 q_2}_{x_1 x_2} = Z^{-1} Z^{q_1 q_2}_{x_1 x_2} - Z^{-2} Z^{q_1}_{x_1} Z^{q_2}_{x_2} & Q = 2 \\ \qquad\qquad \text{etc.} & Q > 2 \end{cases}$$

Now, for the UV problem, we define a generating function

$$Z(\vec{\lambda}) = \int d\mu_{\beta v_{M,N}}(\phi) \prod_{a=1}^{Q} (1 + \lambda_a : e^{i q_a \phi(x_a)} :) e^{\zeta \int :\cos\phi:} \tag{5}$$

For the IR problem, we define

$$Z(\vec{\lambda}) = \int d\mu_{\beta v_{M,N}}(\phi) \prod_{a=1}^{Q} (1 + \lambda_a e^{i q_a \phi(x_a)}) e^{\zeta \int \cos\phi} \tag{6}$$

Then, in both cases, the integer charge truncated correlations are given by differentiating with respect to $\vec{\lambda}$:

$$\begin{cases} S^{q_1}_{x_1} = \left(\frac{\partial}{\partial\lambda_1}\right)\log Z(\vec{\lambda})\Big|_{\vec{\lambda}=0} & Q=1 \\ S^{q_1q_2}_{x_1x_2} = \left(\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\right)\log Z(\vec{\lambda})\Big|_{\vec{\lambda}=0} & Q=2 \\ \text{etc.} & Q>2 \end{cases} \cdot$$

Fröhlich and Seiler ([11], p 899), have observed that for the sine-Gordon quantum field theory, "... these fields (i.e. $:e^{iq\phi}:$) are actually much more natural than the field ϕ." Field correlations such as $\langle\phi(x)\phi(y)\rangle$ have been treated in the sine-Gordon model in [10].

In this review, I will describe how the recent refinements of the renormalization group (RG) in constructive quantum field theory can be extended to give a rather complete picture of integer charge correlations.

Sections 2 and 3 will describe how the main theorems described in §5,6 (Theorems 5.1 and 6.4) lead to detailed results on the behavior of correlations, for the UV problem with $\beta<8\pi$ and the IR problem for $\beta>8\pi$, respectively. I will discuss how these new results relate to and improve on previously existing results.

In section 4, I will sketch the Brydges and Yau RG method and show how it extends to correlation functions. Then in sections 5 and 6, I will state Theorems 5.1 and 6.4. These are the main technical results dealing with the UV problem and IR problem respectively. In §6, I provide a thumbnail sketch of the proof of the vacuum IR result (Theorem 6.1) to give a flavour of how these things go. For the complete proofs of the results on correlations, the reader is directed to the original paper [14].

2 Pressure and correlations for $\beta<8\pi$

In this temperature range, we concentrate on the UV limit $N\to\infty$ in a fixed volume $\Lambda(0)$ (equally we could work in infinite volume by introducing a mass term in the Gaussian measure). We fix the RG rescaling parameter L to be a large integer, take a parameter $0<\epsilon<1/2$, and take a complex activity parameter $\zeta=2z$ which is "small", i.e. $|\zeta|\le\bar{\zeta}(\beta,\epsilon,L)$.

The simplest consequence of the RG analysis of §5 is a formula (13) for the pressure as a sum over length scales L^{-i}, $i=0,1,\ldots,N$. In the $N\to\infty$ limit, this has the form

$$p(\zeta)=\sum_{i=0}^{\infty}\Omega_i(\zeta).$$

¿From Theorem 5.1, we find that each quantity Ω_i is analytic in $\zeta_i=L^{-2i}\tau_{i,0}\zeta\sim L^{(\beta/4\pi-2)i}\zeta$ with radius $\bar{\zeta}$, and is bounded there by

$$L^{2i}|\zeta_i|^{2-\epsilon}.$$

This result is somewhat stronger than the similar inequality (35) of [10].

Because of analyticity, we can immediately derive bounds on the power series about $\zeta = 0$. Write

$$p(\zeta) = \sum_{i=0}^{m-1} \zeta^i p^{(i)} + p^{\geq m}$$

where

$$\begin{aligned} \zeta^m p^{(m)} &= \frac{1}{2\pi i} \oint \frac{ds}{s^{m+1}} p(s\zeta) \\ p^{\geq m} &= \frac{1}{2\pi i} \oint \frac{ds}{s^m (s-1)} p(s\zeta) \end{aligned}$$

and similar expressions for $\Omega^{(m)}, \Omega^{\geq m}$. For each Ω_i, we can bound the integral over the contour $|s| = (L^{-2i}\tau_{i,0}|\zeta|)^{-1}\bar{\zeta} \gg 1$, and conclude

Theorem 2.1 *1. For any even integer $m \geq 2$, the mth order contribution to the pressure is bounded by*

$$|p^{(m)}| \leq \bar{\zeta}^{2-m-\epsilon} \sum_{i=0}^{\infty} L^{[2-m(2-\beta/4\pi)]i} \tag{7}$$

which converges for $\beta < \beta_m = 8\pi(1 - 1/m)$.

2. The mth renormalized pressure $p^{\geq m} = \sum_{a=m}^{\infty} p^{(m)}$ is bounded by

$$|p^{\geq m}| \leq \; 2 \; |\zeta|^m \; \bar{\zeta}^{2-m-\epsilon} \sum_{i=0}^{\infty} L^{[2-m(2-\beta/4\pi)]i}$$

which converges for $\beta < \beta_m$.

Thus for $\beta \in [\beta_m, \beta_{m+2})$, the interaction needs only a vacuum renormalization up to counterterms of order m. This result is an independent and very much simplified proof of what is essentially Theorem 1.0 of [17].

A similarly comprehensive picture holds for the general truncated integer charge correlations. Let Σ be a configuration of $Q > 1$ external charges where the interparticle separations are all of order L^{-I} for some integer $I > 0$, i.e.

$$L^{-(I+1)} < dist(x_i, x_j) < L^{-I}$$

for all $i \neq j$. The RG analysis of §5 leads to an expansion

$$S^{\Sigma} = \sum_{i=0}^{I} D_{\Sigma}\Omega_i.$$

Theorem 5.1 implies that the quantities $D_\Sigma \Omega_i$ are analytic functions of ζ_i with radius $\bar\zeta$, bounded there by

$$Q!(c' L^{(\beta/4\pi)})^{(i-I)} \left(cL^{[\beta/4\pi+\epsilon(2-\beta/4\pi)]I}\right)^Q$$

for some quantities $c = c(\beta, \epsilon, L)$ and $c' = \mathcal{O}(1)$. By Cauchy inequalities, the mth order perturbative contribution is bounded by

$$|S_\Sigma^{(m)}| \leq Q! \left(\frac{|\zeta|}{\bar\zeta}\right)^m c^Q \; L^{[\beta/4\pi+\epsilon(2-\beta/4\pi)]IQ} \sum_{i=0}^{I} (c' L^{(\beta/4\pi)})^{(i-I)} L^{-(2-\beta/4\pi)im}$$

Thus

Theorem 2.2 *There is a number $C_2(\beta, \epsilon, L)$ such that for any integer $m \geq 0$*

$$|S_\Sigma^{(m)}|, |S_\Sigma^{>m}| \leq C_2^Q Q! \left(\frac{|\zeta|}{\bar\zeta}\right)^m d(\Sigma)^{-\Delta(\Sigma)}$$

where $d(\Sigma) = L^{-I}$ and the exponent of the short distance power law is

$$\begin{cases} \frac{(Q-1)\beta}{4\pi} + \epsilon Q(2 - \frac{\beta}{4\pi}) & \text{for } \beta \leq \beta_{m+1} \\ \frac{Q\beta}{4\pi} - m(2 - \frac{\beta}{4\pi}) + \epsilon Q(2 - \frac{\beta}{4\pi}) & \text{for } \beta > \beta_{m+1} \end{cases} \tag{8}$$

This result is new. In fact, I have not been able to find any perturbative or constructive statement in the literature on the short distance asymptotics of general correlations. A direct comparison with perturbative bounds obtained by extending the tree expansion technique of [1] suggests that the above exponents for each $S^{(m)}$ are in fact sharp if we take $\epsilon = 0$.

Example: Two-point function, $\beta \in (\beta_8, \beta_9] = (7\pi, 72\pi/9]$:

$$\begin{cases} S^{(0)} \sim d(\Sigma)^{-[2\beta/4\pi]} \\ S^{(2)} \sim d(\Sigma)^{-[2\beta/4\pi-2(2-\beta/4\pi)]} \\ S^{(4)} \sim d(\Sigma)^{-[2\beta/4\pi-4(2-\beta/4\pi)]} \\ S^{(6)} \sim d(\Sigma)^{-[2\beta/4\pi-6(2-\beta/4\pi)]} \\ S^{(\geq 8)} \sim d(\Sigma)^{-[\beta/4\pi+2\epsilon(2-\beta/4\pi)]} \end{cases}$$

In general, for $\beta \in (\beta_m, \beta_{m+1}]$ we shall see a finite linear sequence of exponents, which at the mth level stabilize at the value $\frac{(Q-1)\beta}{4\pi}$. We can see that the thresholds β_m are values at which the exponent of the short distance power law of the mth order two-point function changes non-smoothly. This crossover phenomenon can be expected on the basis of a perturbative analysis. Note that correlations remain smooth even at the thresholds.

3 Results for $\beta > 8\pi$

In this low temperature range, we consider the IR problem, with a fixed UV cutoff at the unit distance scale. The volume is taken to be $\Lambda(M)$, where M is arbitrarily large. The main theorem for this problem is stated in §6: Here I describe the picture of the pressure and correlations which follows from it. Again, the parameter ζ is chosen "small", i.e. $|\zeta| \leq \bar{\zeta}(\beta, \epsilon, L)$.

Apology: In order to be consistent with the UV picture, scale labels i, j are negative in the IR regime and correspond to large length scales $L^{|i|}$.

Theorem 3.1 *The finite volume pressure is given by a sum over scales:*

$$p(\Lambda(M)) = \sum_{j=0}^{M} \Omega_{-j}.$$

Each term $\Omega_{-j}(\zeta)$ is analytic for $|\zeta| < \bar{\zeta}$, and bounded there by $L^{-2|j|} D\delta^{|j|}$, where $\delta < 1/2$ and D is a small constant. Therefore the pressure itself is analytic in ζ and since it vanishes to first order in ζ, is bounded by $\mathcal{O}(|\zeta|^2)$, uniformly in M.

Now consider the general truncated integer charge correlations. As in §2, I suppose that the configuration Σ is such that the interparticle separations are all of the same order $L^{|I|}$ for some negative integer I. The theorem implies that there is a crossover size

$$L^{|I_0|} \sim |\zeta|^{\frac{-1}{\beta/4\pi - 2}}$$

where the power law decay rate shifts. At distances shorter than this scale, the power is that of the free field $|x-y|^{-\beta/2\pi}$; a larger scales, the decay is the faster $|x-y|^{-4}$.

The RG leads to an expansion

$$S^{\Sigma} = \sum_{j=|I|}^{M} D_{\Sigma}\Omega_{-j}.$$

Theorem 6.1 implies that each $D_\Sigma \Omega_{-j}$ is analytic in ζ with radius $\bar{\zeta}$, and bounded by

$$\begin{cases} Q!(c' L^{-2})^{|i-I|} \left(c_1 L^{-\beta|I|/4\pi}\right)^{Q} & \text{if } |I| \leq |I_0| \\ Q!(c' L^{-2})^{|i-I|} \left(c_2 |\zeta| L^{-2|I|}\right)^{Q} & \text{if } |I| \geq |I_0| \end{cases}$$

for some quantities $c_i = c_i(\beta, \epsilon, L)$ and $c' = \mathcal{O}(1)$.

Thus in all cases, the sum over j converges uniformly in M.

Theorem 3.2 *The truncated correlation of a configuration Σ of $Q > 1$ points with separation $d(\Sigma) = L^{|I|}$ is bounded above by*

$$c^Q Q! d(\Sigma)^{-\Delta(\Sigma)}$$

where

$$\Delta(\Sigma) = \begin{cases} \frac{\beta}{4\pi} Q & \text{if } |I| \le |I_0| \\ 2Q & \text{if } |I| \ge |I_0| \end{cases}$$

The value of c is

$$\begin{cases} c_1 & \text{if } |I| \le |I_0| \\ c_2|\zeta| & \text{if } |I| \ge |I_0| \end{cases}$$

This result verifies a conjecture which goes back to [12]. They state that the two-point correlator should decay at large distances like the dipole-dipole correlation in a dipole gas, averaged over the dipole direction (i.e. like $|x|^{-4}$).

If we combine these upper bounds with an analysis of low order perturbation theory, we presumably obtain the same power laws as lower bounds on correlations. For example, an explicit calculation should show that the second order contribution to the two point correlator has the exact power law $|x|^{-4}$, and that for small ζ the higher order contributions are negligible.

4 RG maps for charge correlations

The remainder of the paper will sketch the method by which the results of §2,3 are proved. We consider in more detail the UV problem $N \to \infty$ in a finite volume $\Lambda(0)$, for $\beta < 8\pi$. We take a configuration $\Sigma = \{\xi_1, \dots, \xi_Q\}$ of Q external unit charges $\xi_a = (x_a, q_a)$, $q_a = \pm 1$, located at points x_a in $\Lambda(0)$. The truncated Q-point correlation function is defined to be:

$$S^{\Sigma} = \frac{\partial^Q}{\partial\lambda_1 \dots \partial\lambda_Q}|_{\vec{\lambda}=0} \log Z(\vec{\lambda}), \quad \vec{\lambda} = (\lambda_1, \dots, \lambda_Q),$$

where

$$Z(\vec{\lambda}) = \int d\mu_{\beta v_{N,0}}(\phi) \prod_{a=1}^{Q} (1 + \lambda_a \tau_{N,0} e^{iq_a\phi(L^N x_a)}) \mathcal{Z}_N^N(\phi). \tag{9}$$

The functional integral has been rewritten on the rescaled volume $\Lambda(N) = L^N\Lambda(0)$. The "Boltzmannian" or "Gibb's Factor" $\mathcal{Z}_N^N$ is given by

$$\mathcal{Z}_N^N(\phi) = \exp[\zeta_N \int_{\Lambda(N)} \cos\phi],$$

where $\zeta_N = L^{-2N}\tau_{N,0}\zeta$ with

$$\tau_{N,0} = e^{\beta v_{N,0}(0)/2} \sim L^{\beta N/4\pi}$$

being just the Wick-ordering constant. The truncated 2-point function, for example, is

$$S^{q_1q_2}_{x_1x_2} = \tau^2_{N,0} \left\langle e^{iq_1\phi(L^N x_1)} e^{iq_2\phi(L^N x_2)} \right\rangle_{\beta,N,\zeta} - \tau^2_{N,0} \left\langle e^{iq_1\phi(L^N x_1)} \right\rangle_{\beta,N,\zeta} \left\langle e^{iq_2\phi(L^N x_2)} \right\rangle_{\beta,N,\zeta}$$

We define the following operators on linear functions of $\vec{\lambda}$: for each $\sigma \subset \{1,\ldots,Q\}$, let

$$\begin{aligned} \lambda_\sigma &= \prod_{a\in\sigma} \lambda_a && (10)\\ D_\sigma\,\cdot &= \prod_{a\in\sigma} \frac{\partial\,\cdot}{\partial\lambda_a}\Big|_{\vec{\lambda}=0} && (11)\\ P_\sigma &= \lambda_\sigma\, D_\sigma && (12)\end{aligned}$$

The identity operator on linear functions of $\vec{\lambda}$ can be written

$$P = \sum_\sigma P_\sigma = P_\emptyset + P_>.$$

For any Σ, the correlation function S^Σ is given by

$$D_\Sigma \log Z.$$

Our analysis is thus concerned with generating a convergent formula for $Z(\vec{\lambda})$.

Now I shall give a brief sketch of the RG maps developed for the partition function in the sequence of papers [5],[7],[10]. The reader should ideally refer to the last of these papers for a complete description.

The BY method generates a sequence of measures $d\nu_i$ on certain function spaces over the volumes $\Lambda(i)$, for $i = 0,1,\ldots,N$. The measures of interest are always weak perturbations

$$d\nu_i(\phi) = d\mu_{\beta v_{i,0}}(\phi)\ \mathcal{Z}_i(\Lambda(i),\phi)$$

of a specific sequence of Gaussian measures $d\mu_{\beta v_{i,0}}$ whose covariances $\beta v_{i,0}(x,y)$ are given by (1). The Gibb's factor is taken in the form of a polymer expansion:

$$\begin{aligned} \mathcal{Z}_i^N(\Lambda(i),\phi) &= \sum_{X_1,\ldots,X_L} \prod_j K_i(X_j,\phi)\\ &\equiv \mathcal{E}xp[\Box + K_i](\Lambda(i),\phi)\end{aligned}$$

where the sum is over disjoint collections of "polymers", a polymer X being a union of closed unit squares with corners lying on the integer lattice in $\Lambda(i)$. The "$\mathcal{E}xp$" notation may be thought of as shorthand for the polymer expansion. The collection of polymer activities $K = \{K_i(X,\phi), X \text{ a polymer}\}$ is called an *analytic functional.*

The analytic functionals are to be regarded as lying in Banach spaces $\mathcal{B}_i$, whose norms $\|\cdot\|_i$ are given weights parametrized by certain quantities:

1. the large field parameter $\kappa_i > 0$;

2. the large set parameter $A > 0$;

3. the large derivative parameters $\mathbf{h}_i = (h_{0i}, h_{1i})$.

Now the RG map $\mathcal{R}_i$ is a specific nonlinear functional which takes a ball in $\mathcal{B}_i$ into $\mathcal{B}_{i-1}$. $\mathcal{R}$ is quite naturally the composition of three maps $\mathcal{S}\circ\mathcal{E}\circ\mathcal{F}$, called scaling, extraction and fluctuation. Taken all together, the map $\mathcal{R}$ has the following defining property. Let the *fluctuation covariance* be defined by the Fourier components $c_i(p) = v_{i,0}(p) - L^2 v_{i-1,0}(Lp)$ for $i > 1$ and $c_1(p) = v_{1,0}(p)$ for $i = 1$. Then for any $K_i \in \mathcal{B}_i$ small enough, $K_{i-1} = \mathcal{R}(K_i)$ is such that

$$\mathcal{E}xp[\Box + K_{i-1}](\Lambda(i-1),\phi) = e^{\Omega_i}\mu_{\beta c_i} * \mathcal{E}xp[\Box + K_i](\Lambda(i),\phi_L)$$

Here, $\mu *$ denotes Gaussian convolution, Ω_i is a certain carefully chosen constant, and ϕ_L denotes the rescaled field $\phi_L(x) = \phi(L^{-1}x)$. This defining property implies in particular that

$$\int d\mu_{\beta v_{i,0}}\mathcal{Z}_i = e^{\Omega_i}\int d\mu_{\beta v_{i-1,0}}\mathcal{Z}_{i-1}$$

The central idea of the RG is that by iteration starting from the measure with

$$\mathcal{Z}_N^N(\phi) = \exp\zeta_N\int_{\Lambda(N)}\cos\phi = \mathcal{E}xp[\Box + K_N^N](\phi),$$

the UV regularized partition function can be written

$$Z_{SG}(\zeta,\beta,v_{0,N}) = \left(\prod_{j>i}^{N} e^{\Omega_j^N}\right)\int d\mu_{\beta v_{i,0}}\mathcal{Z}_i^N$$

for any integer $i = 1, 2, \ldots, N$, or for $i = 0$

$$Z_{SG}(\zeta,\beta,v_{0,N}) = \left(\prod_{j>0}^{N} e^{\Omega_j^N}\right)\mathcal{Z}_0^N(\Lambda(0),\phi = 0).$$

The regularized pressure is

$$p(\zeta,\beta,v_{0,N}) = \sum_{j=1}^{N}\Omega_j^N + \log\mathcal{Z}_0^N. \tag{13}$$

Now the UV problem can be solved by controlling the limit

$$\mathcal{Z}_i = \lim_{N\to\infty}\mathcal{Z}_i^N$$

Now, following a similar idea introduced in [4], we can develop the analogous treatment for the λ-dependent partition function, by extending the activities K_N^N to λ-dependent activities so that

$$\mathcal{E}xp[\Box + K_N^N(\vec{\lambda})](\Lambda(N)) = \prod_{a=1}^{Q}(1 + \lambda_a\tau_{N,0}e^{iq_a\phi(L^N x_a)})\mathcal{Z}_N^N(\phi). \tag{14}$$

There is an essential distinction between vacuum and non-vacuum activities: $P_\emptyset K_N^N(X)$ is translation invariant, whereas $P_\Sigma K_N^N(X)$ is zero unless X contains all of the points in the set $L^N\Sigma$. This "pinning" is preserved under the RG and leads at each scale to a good powercounting factor of L^{-2} for non-vacuum activities.

We define a new renormalization group map

$$\mathcal{R}_P = \mathcal{S}\circ\mathcal{E}_P\circ\mathcal{F} : K_i \to K_{i-1} = \mathcal{R}_P(K_i)$$

such that

$$\int d\mu_{v_{i,0}}\ \mathcal{E}xp(\square + K_i(\vec{\lambda}))(\Lambda(i)) = P\left(e^{\Omega_i}\ \int d\mu_{v_{i-1,0}}\ \mathcal{E}xp(\square + K_{i-1}(\vec{\lambda})\right)(\Lambda(i-1)).$$

It turns out that the only modifications needed to define $\mathcal{R}_P$ are to the extraction step $\mathcal{E}_P$, and lead to no new difficulties.

Applying this iteration N times to the formula (9) leads to an expansion for $Z(\vec{\lambda})$:

$$Z(\vec{\lambda}) = P\left(\left(\prod_{i=1}^{N} e^{\Omega_i^N}\right)\mathcal{E}xp(\square + K_0^N)(\Lambda(0))\right)$$

Since K_0^N is defined on a unit block,

$$\mathcal{E}xp(\square + K_0^N)(\Lambda(0)) = 1 + P_\emptyset K_0^N(\Delta) + P_> K_0^N(\Delta,\vec{\lambda}) = Pe^{\Omega_0^N(\vec{\lambda})}$$

where

$$\Omega_0^N(\vec{\lambda}) = P\log(1 + P_\emptyset K_0^N(\Delta) + P_> K_0^N(\Delta,\vec{\lambda})).$$

Then

$$Z(\vec{\lambda}) = P\left(\prod_{i=0}^{N} e^{\Omega_i^N}\right)$$

which gives the desired formula for any truncated correlation:

$$S^\Sigma = \sum_{i=0}^{N}\left(D_\Sigma\Omega_i^N\right). \tag{15}$$

5 The main result for $\beta < 8\pi$

We note a simplifying property of the expansion (15): since extractions are only made from "small" sets, $D_\Sigma\Omega_i^N = 0$ if $L^i\Sigma$ is not contained in some small set. This means that for two or more distinct points Σ, the expansion (15) has a finite number of terms which depends on the geometry of the points.

For simplicity, we make an assumption on the configuration Σ that the interparticle distances are all of the order L^{-I} for some integer I:

$$L^{-(I+1)} < dist(x_i, x_j) < L^{-I}$$

for all $i < j$. Then the upper limit of the sum (15) is $i = I$.

Norm parameters are chosen as in the vacuum case. The only change is to take a larger parameter $A = L^4$ in the large set regulator: this leads a smaller maximum allowed value $\bar{\zeta}$ of $|\zeta|$.

The main theorem now generalizes our earlier theorem ([10], Theorem 4.4) on the vacuum activities to the many point activities:

Theorem 5.1 *There is a number $\bar{\zeta}(\epsilon, L, \beta)$ and a geometric constant c_1 such that the following properties hold, uniformly in the UV cutoff N, and hence in the $N \to \infty$ limit for all integers $0 \leq j \leq i$:*

1. *(analyticity) The activities $D_\Sigma K_i^N$ and extracted parts $D_\Sigma \Omega_i^N$ are analytic functions of ζ_j for $|\zeta_j| \leq \bar{\zeta}$.*

2. *(vacuum activities) The vacuum activities $D_\emptyset K_i^N$ and extracted parts $D_\emptyset \Omega_i^N$ satisfy the bounds:*

$$\begin{aligned} \|D_\emptyset K_i^N\|_{i-j} &\leq |\zeta_i|^{1-\epsilon} && (16) \\ |D_\emptyset \Omega_i^N| &\leq L^{2i}|\zeta_i|^{2-\epsilon} && (17) \end{aligned}$$

3. *(one point function) For any $\Sigma = \{(x, q)\}$,*

$$\begin{aligned} \|D_\Sigma K_i\|_{i-j} &\leq 2AL^{\beta i/4\pi}\ e^{h_{0i-j}} && (18) \\ |D_\Sigma \Omega_i^N| &\leq 2L^{\beta i/4\pi}\ e^{h_{00}} && (19) \end{aligned}$$

4. *(multi-point functions) For any Σ with $|\Sigma| = Q > 1$*

$$\|D_\Sigma K_i\|_{i-j} \leq \begin{cases} Q!\left(2AL^{\beta i/4\pi}\ e^{h_{0i-j}}\right)^Q & \textit{if } i \geq I \\ Q!(c_1 L^{(\beta/4\pi)})^{(i-I)}\left(2AL^{\beta I/4\pi}\ e^{h_{0I-j}}\right)^Q & \textit{if } i \leq I \end{cases}$$

   and

$$|D_\Sigma \Omega_i^N| \leq \begin{cases} 0 & \textit{if } i > I \\ Q!(c_1 L^{(\beta/4\pi)})^{(i-I)}\left(2AL^{\beta I/4\pi}\ e^{h_{0I}}\right)^Q & \textit{if } i \leq I \end{cases} \qquad (20)$$

Proof: The proof, given in [14], follows with moderate changes the proof of the vacuum result in [10]. □

This result leads rather quickly to the claimed bounds of §2.

6 Sketch of the method and results for $\beta > 8\pi$

For the Kosterlitz-Thouless phase $\beta > 8\pi$, one takes the initial measure $d\nu_0$ to be a weak perturbation of the Gaussian measure on a large volume $\Lambda(M)$ with covariance $v_{M,0}$ (with short distance cutoff $N = 0$). The RG map is defined in a similar way to the map defined for $\beta < 8\pi$. For each negative integer $i = 0, -1, \dots, -(M-1)$, it takes activities K_i on volume $\Lambda(M+i)$ to activities on volume $\Lambda(M+i-1)$. There is one important difference in the definition of $\mathcal{R}$: the extraction step removes a factor $e^{\delta\sigma_i \int_{\Lambda(M+i)} (d\phi)^2}$ from $\mathcal{E}xp[\Box + K_i]$, in addition to the vacuum extraction Ω_i. These wave function renormalizations are accumulated in the covariances

$$\hat{v}_i(p)^{-1} = p^2(e^{p^2} + \sigma_i)$$

where $\sigma_i = \sum_{j=1}^{|i|} \delta\sigma_{-j}$. Changing the normalization of the Gaussian measure leads to an additional contribution Ω' to the vacuum term Ω_i. To adequately control the extra renormalization cancellations, it is now necessary to consider functionals of $(\phi, \partial_i\phi, \partial_i\partial_j\phi)$, and the corresponding derivative regulators $\mathbf{h} = (h_0, h_1, h_2)$.

The following parameters lead to a contractive estimate for $\mathcal{R}$. Here, c_2, c_3, c_4, c_5 are certain $\mathcal{O}(1)$ geometric constants. We fix $\epsilon > 0$ and a large integer L so that

$$\delta \equiv c_2 \max(L^{2-(1-\epsilon)\beta/4\pi}, L^{-2}) < 1/2.$$

We take the regulators as follows:

1. G_i as before, with
$$\kappa_i = c_3(1 + \sum_{j=1}^{|i|} \delta^j);$$
2. Γ as before, with constant $A = \mathcal{O}(L^{2+\epsilon})$;
3. $\mathbf{h}_i = h_i(c_4 L, L, L^2)$ where $h_i = c_5\beta(2 - \sum_{j=1}^{|i|} \delta^j)$.

Note that $\frac{1}{2}h_0 < h_\infty < h_i \le h_0$ and $\kappa_0 \le \kappa_i < \kappa_\infty < 2\kappa_0$.

Theorem 6.1 *There is a constant $\bar{\zeta}$ depending on the above parameters such that for $|\zeta| < \bar{\zeta}$ the sequence of activities $K_0, K_{-1}, \dots, K_{-M}$ is defined, as well as the associated constants $\Omega_i, \delta\sigma_i$. They satisfy the following bounds, uniformly in the volume parameter M, for $i = 0, -1, \dots, -M$:*

$$\begin{aligned} \|K_i\|_i &\le D\delta^{|i|}; \\ |\Omega_i| &\le L^{-2|i|} D\delta^{|i|}; \\ |\delta\sigma_i| &\le h_{1i}^{-2} D\delta^{|i|}, \end{aligned}$$

where $D = Ae^{2c_4c_5\beta L}|\zeta|$.

Remark: This theorem is a restatement of Theorem 5.1, the main result of [7], which incorporates the corrected regulators described in the erratum contained in [10].

The theorem is a consequence of the following two propositions on the IR RG map.

Proposition 6.2 *Suppose the wave function renormalization satisfies $|\sigma_i| < \epsilon$. Then the constants and regulators chosen above are such that the linearization of $\mathcal{R}$ at $K = 0$ satisfies the following bound:*

$$\|\mathcal{R}_1 K\|_{i-1} \leq \delta/2\|K\|_i.$$

Proposition 6.3 *Suppose the wave function renormalization satisfies $|\sigma_i| < \epsilon$. Then there is a number $\bar{K}$ (depending on β, L, ϵ) and a geometric constant c_6 such that $\mathcal{R}$ is an analytic functional of K for $\|K\|_i \leq \bar{K}\delta^{|i|}$. For such K,*

$$\|\mathcal{R}(K)\|_{i-1} \leq c_6 L^2 \|K\|_i.$$

The extractions satisfy the bounds

$$\begin{aligned} |\Omega(K)| &\leq L^{-2|i|}\|K\|_i \\ |\delta\sigma(K)| &\leq h_{1i}^{-2}\|K\|_i. \end{aligned}$$

Proof of Theorem 6.1: We begin the induction from $i = 0$ to $i = -M$ by noting that

$$\|K_0\|_0 \leq 2Ae^{h_{00}}|\zeta| \equiv 2D.$$

Next, we write

$$\mathcal{R}(K) = \mathcal{R}_1 K + \oint \frac{ds}{s^2(s-1)}\mathcal{R}(sK)$$

and estimate the two terms. The first term is bounded by $\delta/2\|K\|_i$, by Proposition 6.2. The second term can be taken over the contour $|s| = \bar{K}/D \gg 1$, and by Proposition 6.3 has the bound

$$(c_6 L^2 \bar{K}\delta^{|i|})(D/\bar{K})^2$$

which is less than $D\delta^{|i|}/2$ if D (i.e. $|\zeta|$) is taken small enough. One can check that $|\sigma_i| < \epsilon$ remains satisfied. □

The decay of correlations at large distance is governed by the least irrelevant eigenvalue of the linearized RG map. Wave function terms decay like L^{-2} and induce a $|x-y|^{-4}$ power law in the two-point function. Charge q terms decay like $L^{-\beta|q|/4\pi}$: the charge one terms induce a power law $|x-y|^{-\beta/2\pi}$. Certainly for $\beta > 8\pi$ it is the wave function terms which dominate the asymptotics. However, the picture is complicated, because wave function terms enter only in second order perturbation theory. This means that for small coupling constant there is a crossover at a scale $I \sim I_0$ with

$$I_0 \sim \frac{1}{(\log L)(2-\beta/4\pi)} \log(L^2 e^{h_{00}}|\zeta|)$$

For $|I| \leq |I_0|$ one sees the power law $|x-y|^{-\beta/2\pi}$ induced by the charge 1 terms, while for $|I| > |I_0|$ one sees the $|x-y|^{-4}$ law of the wave function terms.

All of this is a consequence of the following theorem on general correlations, whose proof, given in [14], amounts to a moderate extension of the above vacuum result.

Theorem 6.4 *There is a number $\bar{\zeta}(\epsilon, L, \beta)$ and a geometric constant c_1 such that the following properties hold, uniformly in the volume $\Lambda(M)$, and hence in the thermodynamic limit for all $i = 0, -1, \ldots, -M$:*

1. *(analyticity) The activities $D_\Sigma K_i$ and extracted parts $D_\Sigma \Omega_i$ are analytic functions of ζ for $|\zeta| \leq \bar{\zeta}$.*

2. *(one point function) For any $\Sigma = \{(x,q)\}$,*

$$\|D_\Sigma K_i\|_i \leq \begin{cases} 2AL^{-\beta|i|/4\pi}\ e^{h_{0i}} & \text{if } |i| \leq |I_0| \\ 2AL^2L^{-2|i|}\ e^{2h_{0i}}|\zeta| & \text{if } |i| \geq |I_0| \end{cases} \tag{21}$$

$$|D_\Sigma \Omega_i^N| \leq \begin{cases} 2AL^{-\beta|i|/4\pi}\ e^{h_{0i}} & \text{if } |i| \leq |I_0| \\ 2AL^2L^{-2|i|}\ e^{2h_{0i}}|\zeta| & \text{if } |i| \geq |I_0| \end{cases} \tag{22}$$

3. *(multi-point functions) Let Σ be a configuration of $Q > 1$ points with linear separation of order $L^{|I|}, I < 0$.*

   (a) Suppose $|I| \leq |I_0|$. Then

$$\|D_\Sigma K_i\|_i \leq \begin{cases} Q!\left(2AL^{-\beta|i|/4\pi}\ e^{h_{0i}}\right)^Q & \text{if } |i| < |I| \\ Q!(c_1L^{-2})^{|i-I|}\left(2AL^{-\beta|I|/4\pi}\ e^{h_{0I}}\right)^Q & \text{if } |i| \geq |I| \end{cases}$$

   and

$$|D_\Sigma \Omega_i^N| \leq \begin{cases} 0 & \text{if } |i| < |I| \\ Q!(c_1L^{-2})^{|i-I|}\left(2AL^{-\beta|I|/4\pi}\ e^{h_{0I}}\right)^Q & \text{if } |i| \geq |I| \end{cases} \tag{23}$$

   (b) Suppose $|I| \geq |I_0|$.

$$\|D_\Sigma K_i\|_i \leq \begin{cases} Q!\left(2AL^{-\beta|i|/4\pi}\ e^{h_{0i}}\right)^Q & \text{if } |i| < |I_0| \\ Q!\left(2AL^2L^{-2|i|}\ e^{2h_{0i}}|\zeta|\right)^Q & \text{if } |I_0| \leq |i| \leq |I| \\ Q!(c_1L^{-2})^{|i-I|}\left(2AL^2L^{-2|I|}\ e^{2h_{0I}}|\zeta|\right)^Q & \text{if } |i| \geq |I| \end{cases}$$

and

$$|D_\Sigma \Omega_i^N| \le \begin{cases} 0 & \textit{if } |i| < |I| \\ Q!(c_1 L^{-2})^{|i-I|} \left(2AL^2 L^{-2|I|}\ e^{2h_{0I}}|\zeta|\right)^Q & \textit{if } |i| \ge |I| \end{cases} \tag{24}$$

7 Acknowledgements

I am indebted to G. Benfatto, whose insight and key observations precipitated the present work, and to G. Keller and D. Brydges for telling me at an early stage about their approach to correlations. I would also like to thank V. Rivasseau and K. Gawedzki for their interest, comments and support, and to acknowledge the support of the following institutions where some of this work was done: Institut des Hautes Etudes Scientifiques, France; Ecole Polytechnique, France; and Università di Roma II.

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