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GAUSSIAN FLUCTUATION IN RANDOM MATRICES*

by

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Abstract Let $N(L)$ be the number of eigenvalues, in an interval of length L, of a matrix chosen at random from the Gaussian Orthogonal, Unitary or Symplectic ensembles of $\mathcal{N}$ by $\mathcal{N}$ matrices, in the limit $\mathcal{N} \to \infty$. We prove that $[N(L) - \langle N(L)\rangle]/\sqrt{\log L}$ has a Gaussian distribution when $L \to \infty$. This theorem, which requires control of all the higher moments of the distribution, elucidates numerical and exact results on chaotic quantum systems and on the statistics of zeros of the Riemann zeta function.

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Ensembles of $\mathcal{N}$ dimensional Gaussian Random Matrices (GRM) with invariances under the Orthogonal, Unitary or Symplectic groups corresponding to the GOE, GUE and GSE were introduced by Wigner and developed by Porter, Dyson, Mehta and others [1,2]. Wigner's inspired surmise that the statistics of eigenvalues of these GRM can be used to model the statistical properties of the observed spectra of complex nuclei turned out to be exactly right. There is indeed good agreement between the observed high energy level spacings, pair correlations and variance or Δ-statistics and those calculated analytically from the GRM in the limit $\mathcal{N} \to \infty$. Moreover, the GRM have been found to be the very robust "renormalization group fixed points" of a large class of RM [3] which play an important role in many areas of physics and mathematics [1–7].

In the present work we focus on the large L (long wavelength) behavior of the random variable $N(L)$ giving the number of eigenvalues of a GRM, chosen from any of the Gaussian ensembles, in an interval $(y, y+L)$: We always consider the $\mathcal{N} \to \infty$ limit when the distribution is translation invariant and use units in which the mean spacing is unity. It is well known that the variance of $N(L)$ grows like $\log L$ as $L \to \infty$. We prove that *all* the moments of $\xi(L) \equiv [N(L) - L]/\sqrt{\log L}$ approach for large L those of a Gaussian distribution which implies (weak) convergence of $\xi(L)$ to a Gaussian random variable. We shall discuss later the connection of our result with the statistics of energy levels of quantum systems with generic chaotic classical Hamiltonians and of the zeros of the Riemann zeta function [4,5,7].

It is a remarkable fact that the distribution of energy levels of the G(O,U,S)E are given by the Gibbs canonical distribution of the positions of charged point particles on the line interacting via the (two dimensional) repulsive logarithmic Coulomb potential, $v(r) = -\log r$, at reciprocal temperatures $\beta = 1, 2, 4$ respectively [1,2]. The particles with positions x_i, $i = 1, ..., \mathcal{N}$, on the real line, are confined by a uniform negative background, which produces a harmonic potential. The total potential energy of the system is

$$V_G(x_1, ..., x_{\mathcal{N}}) = \frac{1}{2}\sum_{i=1}^{\mathcal{N}} x_i^2 - \frac{1}{2}\sum_{i\neq j}^{\mathcal{N}} \log|x_i - x_j|. \tag{1}$$

In the corresponding circular ensembles of Dyson the x_i lie on a circle of length $\mathcal{N}$ and the energy is given by the second term on the r.h.s of (1) with distance measured in the plane. The canonical Gibbs measures corresponding to Gaussian and circular ensembles become equivalent in the thermodynamic limit, $\mathcal{N} \to \infty$, yielding the same k-point, $k = 1, 2, 3, ..$ correlation functions, for all $\beta > 0$ [1,2,8].

These infinite volume correlation functions are known explicitly for the "solvable" cases $\beta = 1, 2, 4$ corresponding to the GRM. Defining as usual $n_j(x_1, ..., x_j)$ as the joint density for j-tuples, the corresponding Ursell functions $U_k(x_1, ..., x_k)$ [9] are given by

$$U_1(x_1) \equiv n_1(x_1) \equiv 1 \tag{2}$$

$$U_2(x_1, x_2) \equiv n_2(x_1, x_2) - n_1(x_1)n_1(x_2) = n_2(x_1 - x_2) - 1$$

$$U_k(x_1, ..., x_k) \equiv \sum_G (-1)^{m-1}(m-1)! \prod_{j=1}^{m} n_{G_j}(\{x_{G_j}\})$$

where G is a partition of the indices $(1, 2, .., k)$ into m subgroups $G_1, .., G_m$ and $\{x_{G_j}\}$ are the x_i with indices in G_j. (In the GRM literature, $Y_k = (-1)^{k-1}U_k$ is usually called the k^{th} cluster function). The integrals, $\bar{U}_k$, of U_k over a k-dimensional cube having sides of length L are directly related to the cumulants $C_j(L), j = 1, ..., k$ of the random variable $N(L)$, the number of points (or eigenvalues) in an interval of length L, which we shall take for definiteness to be the interval $[-t, t]$, $L \equiv 2t$. Thus

$$\bar{U}_1 = \int_{-t}^{t} U_1(x_1)dx_1 = 2t = \langle N(L)\rangle = C_1(L) \tag{3}$$

$$\bar{U}_2 = \int_{-t}^{t}\int_{-t}^{t} dx_1 dx_2 U_2(x_1, x_2) \equiv \langle N(N-1)\rangle - \langle N\rangle^2 = \langle (N - \langle N\rangle)^2\rangle - \langle N\rangle$$

$$= C_2(L) - C_1(L)$$

$$\bar{U}_3 = C_3(L) - 3C_2(L) + 2C_1(L)\,, \text{ etc}$$

Using the generating function [9]

$$F(\mu) = \sum_{n=0}^{\infty} E_n(L)e^{n\mu}, \tag{4}$$

with $E_n(L)$ the probability of having exactly n particles in the interval L, we have,

$$\log F(\mu) = \sum_{n=1}^{\infty} \frac{C_n}{n!}\mu^n = \sum_{n=1}^{\infty} \frac{1}{n!}\bar{U}_n(e^{\mu} - 1)^n. \tag{5}$$

This gives

$$C_k = \left(\sum_{j=2}^{k-1} b_{k,j} C_j \right) + (-1)^k (k-1)! C_1 + \bar{U}_k \tag{6}$$

where

$$b_{n,j} = b_{n-1,j-1} - (n-1) b_{n-1,j}, \qquad 2 \le j \le n-1$$

$$b_{n,n} = -1 \qquad\qquad n \ge 2$$

The U_k take on a particularly simple form for the GUE ($\beta = 2$) [1,2],

$$U_k(x_1, ..., x_k) = (-1)^{k-1} \sum_{\mathrm{Perm}} \prod_{i=1}^{k} S(x_{i+1} - x_i) \tag{7}$$

where $S(x) = (\pi x)^{-1} \sin \pi x$ and $x_{k+1} = x_1$, so the indices are to be thought of as being on a circle. Using (6) one readily obtains [1,2]

$$C_2(L) = (\log L)/\pi^2 + O(1). \tag{8}$$

Note that for a system with short range interactions $C_2(L)$ would grow like L but the logarithmic interactions between the (charged) particles induce a sort of local crystalline order reducing the variance to $\log L$. It is this strong correlation which produces level repulsion between the eigenvalues and makes the large scale behavior of the fluctuations far from obvious. Defining now the normalized random variable

$$\eta(L) = (N(L) - L) \Big/ \sqrt{\frac{1}{\pi^2} \log L} \tag{9}$$

the $k^{\rm th}$ cumulant of $\eta(L)$ will be $c_k \equiv C_k/[\log L/\pi^2]^{\frac{k}{2}}$. As is well known c.f. [10], $\eta(L)$ will approach a Gaussian random variable (with mean zero and unit variance) as $L \to \infty$, if and only if all c_k, $k \geq 3$, go to zero, i.e. if $C_k = o([\log L/\pi^2]^{\frac{k}{2}})$. Using an induction argument based on the recurrence relation (6) and the equality (7), this corresponds to proving that

$$\begin{aligned} s_k(t) &\equiv \int_{-t}^{t} ... \int_{-t}^{t} dx_1..dx_k S(x_2 - x_1)S(x_3 - x_2)...S(x_1 - x_k) \\ &= 2t + o[(\log t)^{\frac{k}{2}}], k \geq 3 \end{aligned} \tag{10}$$

We shall actually prove that $s_k(t) = 2t + O(\log t)$ which implies that, for $k \geq 3$ $C_k(L) = O(\log L)$; in fact we believe that for $k \geq 3$, $C_k(L)$ stays bounded as $L \to \infty$, as suggested by the explicit asymptotic evaluation of the integrals.

$$\begin{aligned} s_3 &= 2t - \frac{3}{2\pi^2}\log t + O(1) \\ s_4 &= 2t - \frac{11}{6\pi^2}\log t + O(1) \end{aligned} \tag{11}$$

which gives, using (6), that C_3 and C_4 are of $O(1)$.

To prove (10) we make use of the fact [1] that

$$s_k(t) = \sum_{i=1} \lambda_i^k(t) = \mathrm{Tr}\,\mathbf{S}^k(t) \tag{12}$$

where the $\lambda_i(t)$ are the eigenvalues of the integral operator $\mathbf{S}(t), (\mathbf{S}f)(x) = \int_{-t}^{t} dy S(x-y)f(y)$. It is known [1] (and can be easily proven) that the spectrum of $\mathbf{S}$ lies in $[0, 1]$. We can now use an induction argument to prove that $s_k(t) = 2t + O(\log t)$. This is so for $k = 1, 2$ (also for $3, 4$), and for $k \geq 2$ we have

$$\mathrm{Tr}\, \mathbf{S}^{k+1} = \mathrm{Tr}\, \mathbf{S}^{\mathbf{k}} - \mathrm{Tr}\left[\mathbf{S}^{k-1}(\mathbf{S} - \mathbf{S}^2)\right]. \tag{13}$$

The first term is of the desired form by the induction assumption while the terms in the parenthesis are positive operators, and $\|\mathbf{S}^{k-1}\| \leq 1$ so it can be taken out of the product yielding $\mathrm{Tr}\, \mathbf{S}^{k+1} = \mathrm{Tr}\, \mathbf{S}^{\mathbf{k}} + O(\log t)$ and the proof is complete.

Our results readily extend to show that if we divide up the real line into a union of intervals of length L, let $N_j(L)$ be the particle number in $[jL, (j+1)L]$ and set $\eta_j(L) = [N_j(L) - L]/\sqrt{C_2(L)}$, $j \in Z$ then the $\{\eta_j(L)\}$ approach, as $L \to \infty$, jointly Gaussian random variables with mean zero and covariances $\langle \eta_j \eta_k \rangle = \delta_{j,k} - \frac{1}{2}\delta_{j\pm 1,k}$.

To prove the results for the GOE, $\beta = 1$, we use an identity conjectured by Dyson and proved by Gunson [11] (we thank Freeman Dyson for pointing this out to us). According to this identity, superimposing two noninteracting Coulomb gases, say blue and red, in the circular ensemble at reciprocal temperature $\beta = 1$ and then looking only at alternate particles, e.g. at all the odd (or even) ones, yields the distribution at $\beta = 2$. Considering now the number of particles in an interval of length L gives

$$N_{total}(L) = N^{(1)}_{blue}(L) + N^{(1)}_{red}(L) = N^{(2)}_{odd}(L) + N^{(2)}_{even}(L) = 2N^{(2)}(L) + (0, \pm 1) \tag{14}$$

where the superscripts $(1,2)$ stand for the random variables obtained from the ensembles with $\beta = 1, 2$, and N_{blue}, N_{red} are independent. This shows immediately that in the infinite $\mathcal{N}$ limit, the variables $N^{(2)}(L)$ and $N^{(1)}(L)$ normalized by the square root of their variances have the same asymptotic behavior. Taking $\langle N^{(1)}_{blue}(L)\rangle = \langle N^{(1)}_{red}(L)\rangle = \langle N^{(1)}(L)\rangle = L$ we have

$$\frac{1}{2}\Big\langle \big[(N^{(1)}_{blue}(L) - L) + (N^{(1)}_{red}(L) - L)\big]^2 \Big\rangle = \langle (N^{(1)}(L) - L)^2\rangle = \\ 2\langle (N^{(2)}(L) - L)^2\rangle \sim \frac{2\log L}{\pi^2} \tag{15}$$

giving the well known variance of the GOE [1].

For the GSE, $\beta = 4$, we use the equality between the statistics of its eigenvalues and the odd eigenvalues of the GOE [1]. This again leads to Gaussian asymptotics with a variance given by $\frac{1}{2\pi^2}\log L + O(1)$. It seems very reasonable to expect and one can give strong heuristic arguments, based on the "long wavelength response" of "Coulomb" systems, that the Gaussian nature of the fluctuations, with variances $(2/\pi^2\beta)\log L$, holds for all β. (We thank Bernard Jancovici for pointing this out to us, see also [8].)

Using more detailed information on the spectrum of $\mathbf{S}$ (see [12]), it follows that $s_k(t) = 2t - \pi^{-2}\sum_{j=1}^{k-1} j^{-1}\,\log t + o(\log t)$ which using (6) implies that $C_k = o(\log t)$ for all $k \geq 3$. (We are indebted to Harold Widom for this information). Widom also noted that our proof does not make any use of the specific form of $\mathbf{S}$. It only uses the property spec$(S) \in [0,1]$ and the fact that $Tr(\mathbf{S}-\mathbf{S}^2) \to \infty$ (as $t \to \infty$). The conclusion therefore holds for a larger class of matrix models [3].

The local statistics of the eigenvalues $\epsilon_j, j = 1,2,...,\infty$, $0 \leq \epsilon_1 \leq \epsilon_2 \leq ...$ of a classically chaotic quantum Hamiltonian (CQH), such as the geodesic flow on a (non-arithmetic) surface of constant negative curvature or the Sinai billiard appear to coincide at high energies with those obtained from the GRM [4]. More precisely, if we consider the energy levels of a generic CQH, suitably scaled so that the mean distance between levels is unity, in an interval $(y, y+L)$ then their statistics, obtained by letting y vary uniformly in some interval (T_1, T_2) will coincide, for $T_2 \to \infty$, with that obtained from one of the standard GRM ensembles when the matrix size tends to infinity. Our result then predicts a Gaussian distribution of the fluctuations in the number of levels $N(L)$ when $L \to \infty$. A numerical check of this for some CQH will require using energy levels in a (scaled) energy interval L with $1 \ll L \ll (T_2 - T_1)$, T_1 large.

It is also interesting , as emphasized by Berry [4], to consider in addition to the local statistics of quantum levels also their global statistics. These correspond, in our context, to fluctuations in the number of levels in an interval (y, x) whose length $L(x) = x - y(x)$ is not fixed but grows with x as x varies in the interval (T_1, T_2) with $T_2 \to \infty$. This includes in particular the case $y = 0$, $L(x) = x$ corresponding to the fluctuations in the number of eigenvalues less than x. This quantity, normalized by the square root of its variance, was conjectured in [5] (where it is denoted by $N_{fl}(x)$) to have a Gaussian distribution as $x \to \infty$ for all CQH. If true this would be a general characterization of CQH and distinguish them from integrable systems where it was found rigorously that the global distribution is non-Gaussian [13]. Quite generally, it was shown by Berry [5] that when $L(x) > L_{\max}(x)$, the variance of $N(L(x))$ saturates for $L > L_{\max}$. Berry also found that $N_{fl}^2(x)$ (averaged over some interval containing many eigenvalues but very small compared to x) grows for billiard systems like $(2\pi)^{-2}\log x$.

As already noted, the distribution of eigenvalues in the GRM is translation invariant, when the matrix size $\mathcal{N}$ goes to infinity, so there is no analog of $L_{\max}$ in our considerations. One can however consider fluctuations in $N(L)$ for an interval $L(\mathcal{N})$ which contains a number of eigenvalues small compared to $\mathcal{N}$ but goes to infinity when $\mathcal{N} \to \infty$, e.g. in the circular ensemble we could have $L(N) \sim N^{\gamma}, \gamma < 1$ or even like $\alpha\mathcal{N}, \alpha << 1$. For the Coulomb system with neutralizing background it is also possible to consider semi-infinite systems with various boundary conditions and/or nonuniform background. Some such systems have been considered in [8] and we believe that our results about Gaussian behavior would extend also to these systems which might model some of the saturation features of CQH.

We turn finally to the (non trivial) zeros of the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ which are, according to the Riemann Hypothesis, of the form $z_n = \frac{1}{2} + i\gamma_n$.

As pointed out by Berry [4] there are reasons to expect similarities between the statistics of the γ_n and of energy levels of CQH. In fact Montgomery [14] proved that the pair correlation function of the γ_n agrees with that of the GUE, Eq. (7). Numerical calculations by Odlyzko [7] give striking evidence that the nearest neighbor level spacing distribution of the γ_n is, for large n, indeed the same as that obtained from the GUE. In a very interesting recent paper Rudnick and Sarnak [7] greatly extended the results of Montgomery by showing that the n-point correlation functions of these zeros converge, on a large class of test functions to those of the GUE. Moreover the normalized global fluctuation in these zeros corresponding to $N_{fl}(x)$, was shown by Selberg [15] to have a Gaussian distribution. The same arguments imply that the local fluctuation in their number in an interval $(y, y+L)$ averaged over (T_1, T_2) and properly scaled becomes Gaussian when $T_2 \to \infty$, followed by $L \to \infty$ (we are indebted to Peter Sarnak for explaining this to us). The results proved here thus fit completely with the picture of the statistics of Riemann zeros being in the same "universality class" as that of the GUE.

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