CLOSEDNESS OF STAR PRODUCTS AND COHOMOLOGIES

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Dedicated to Bert Kostant with friendship and appreciation

Abstract

We first review the introduction of star products in connection with deformations of Poisson brackets and the various cohomologies that are related to them. Then we concentrate on what we have called "closed star products" and their relations with cyclic cohomology and index theorems. Finally we shall explain how quantum groups, especially in their recent topological form, are in essence examples of star products.

1. Introduction: Quantization

1.1 Geometry. The setting of classical mechanics in phase-space has long been a source of inspiration for mathematicians. But (according to writing on a wall of the UCLA mathematics department building) Goethe once said that "Mathematicians are like Frenchmen: they translate everything into their own language and henceforth it is something completely different". Being French mathematicians, we shall give here a flagrant illustration of that sentence, though not going as far as Bert Kostant's cofounder of geometric quantization (Jean-Marie Souriau) who derived symplectic formalism from the basic principles that are the core of the French "mécanique rationnelle" established by Lagrange.

The symplectic formalism is obvious in the Hamiltonian formulation on flat phase-space $\mathbb{R}^{2\ell}$, and prompted another French mathematician (Paulette Libermann) to systematize the notion of symplectic manifold. In parallel to these developments came the introduction of quantum mechanics (first called "mécanique ondulatoire" in France under de Broglie's influence). And then (which brings us close to our subject here) Dirac [1] introduced (both in the classical and in the quantum domain) his notion of "constrained mechanics", when external constraints restrict the degrees of freedom of phase-space. For mathematicians that is nothing but restricting phase-space to a Poisson submanifold of some $\mathbb{R}^{2\ell}$ [2]; this restriction permits a nice and compact formulation of classical mechanics, but was of little help in the quantum case where people needed some reference to the canonical formalism on flat phase-space, as examplified by Weyl quantization [3].

Then quite naturally Kostant (coming from representation theory of Lie groups) and Souriau (coming from symplectic formulation of classical mechanics) introduced independently [4] what is now called *geometric quantization*. The idea is to somehow select, via a polarization, a Lagrangian submanifold X of half the dimension of the symplectic manifold W so that locally W will look like T^*X and quantization can be done on $L^2(X)$; and in the meantine to work at the prequantization level on $L^2(W)$. That idea, quite efficient for many group representations, ran into serious problems (now well-known) on the physical side; in particular few observables were "quantizable" in that sense.

1.2 Deformations. The idea that the passage from one level of physical theory to a more evolved one is done through the mathematical notion of deformation became obvious only recently [5], but many people certainly felt that something of this kind must occur. On the space-time invariance level (Galilée to Poincaré to De Sitter) it is simple to formulate, because all objects are Lie groups. When interactions (nonlinearities) occur, it is more intricate (and requires the systematization of the notion of nonlinear representations [6]). For quantum theories, in spite of hints in several expressions (like "classical limit"), remained the quantum jump from functions to operators. Our approach, that started about 20 years ago [7-10] and is now often referred to as *"deformation quantization"*, showed that there is an alternative (a priori more general) and autonomous formulation in terms of "star" products and brackets, deformations (in Gerstenhaber's sense [12]) of the algebras of classical observables (Berezin [13] has independently written a parallel formulation in the complex domain, but not in terms of deformations).

The importance of algebras of observables (especially C^* algebras) in quantum theories has even spilled into geometry with the non-commutative geometry of Alain Connes [14] and its developments around algebraic index theory (generalizing the Atiyah-Singer index theorems for pseudo-differential operators), and with the exponential development of quantum groups [15]. In this paper, after a survey of the origins (section 2), we shall (in section 3) indicate that what we have called "closed star products" [16] permits a parallel treatment of the former, and (in section 4) that the latter, once formulated in a proper topological vector space context, are essentially examples of star products. In each case there are appropriate cohomologies to consider, e.g. cyclic for closed star products and bialgebra cohomologies for quantum groups, that are more specific than the traditional Hochschild (and Chevalley) cohomologies. An alternative name for our approach could thus be *"cohomological quantization"*. It stresses the importance of cohomology classes in all our approach and brings naturally to ideas like "cohomological renormalization" in field theory (when phase-space is infinite-dimensional) where "more finite" coycles can be obtained [17] by substracting "infinite" coboundaries from cocycles to define star products equivalent to (but different of) that of the normal ordering.

2. Star Products and Cohomologies.

2.1 Deformations of Poisson Brackets. Let W be a symplectic manifold of dimension 2ℓ, with (closed) symplectic 2-form ω; denote by Λ the 2-tensor dual to ω (the inner product by $-\omega$ defines an isomorphism μ between TW and T^*W that extends to tensors).

The Poisson bracket can be written as $P(u,v) = i(\Lambda)(du \wedge dv)$ for $u, v \in N = C^\infty(W)$. Some of the results described here are valid when W is a Poisson manifold (where Λ is given, with Schouten autobracket $[\Lambda, \Lambda] = 0$ – the analogue of closedness for ω – but not necessarily everywhere nonzero); the dimension need not be even and a number of results hold when the dimension is infinite, which is the case of field theory (Segal and Kostant [8] were among the first to consider seriously infinite-dimensional symplectic structures). We shall however not enter here into specifics of these questions.

a. A deformation of the Lie algebra (N, P) is defined [12] by a formal series in a parameter $\nu \in \mathbb{C}$:

$$[u,v]_\nu = P(u,v) + \sum_{r=1}^{\infty} \nu^r \hat{C}_r(u,v), \qquad \text{for } u, v \in N \text{ (or } N[[\nu]]) \tag{1}$$

such that the new bracket satisfies the Jacobi identity, where the $\hat{C}_r$ are 2-cochains (linear maps from $N \wedge N$ to N). In particular $\hat{C}_1$ must be a 2-cocycle for the Chevalley cohomology $\hat{H}^*(N)$ of (N, P). As usual, equivalences of deformations are classified at each step by $\hat{H}^2(N)$ and the obstructions to extend a deformation from one step (in powers of ν) to the next are given by $\hat{H}^3(N)$ [11,12].

Moreover it can be shown that one gets consistent theories by restricting to differentiable (resp. 1-differentiable) cochains and cohomologies, when the $\hat{C}^r$ are restricted to be bidifferential operators (resp. bidifferential operators of order at most (1,1)). This has the advantage of giving finite-dimensional cohomologies with simple geometrical interpretation. In particular one has

$$\dim \hat{H}^2_{\text{diff}}(N) = 1 + \dim \hat{H}^2_{1.\text{diff}}(N) \quad \text{and} \quad \hat{H}^p_{1.\text{diff}}(N) = H^p(W, \mathbb{R}) \oplus H^{p-1}(W, \mathbb{R}) \tag{2}$$

if ω is exact (it can be smaller if not).

One can also restrict to differentiable cochains that are null on constants (n.c. in short), i.e. $\hat{C}_r(u,v) = 0$ whenever either u or v is constant, and get again a consistent theory. In that case $\hat{H}^2_{1.\text{diff,nc}}(N) = H^2(W, \mathbb{R})$, the de Rham cohomology. Similar results hold for $\hat{H}^3_{\text{diff}}(N)$, the obstructions space [19]; in particular $\hat{H}^3_{\text{diff,nc}}(N)$ is isomorphic to $H^1(W) \oplus H^3(W)$ (without the n.c. condition the space may be slightly larger). This explains that, when the third Betti number of W, $b_3 = \dim H^3(W)$, vanishes, J. Vey was able to trace the obstructions inductively into the zero-class of $\hat{H}^3(N)$ and show the existence of such deformed brackets (the condition $b_3 = 0$ is not necessary, as follows from the general existence theorems for star products that we quote later).

Replacing in all the above "differentiable" by "local" gives essentially the same results for the cohomology [19]. A related study, with useful by-products (one of which we shall need later) can be found in [20].

b. In the differentiable n.c. case one thus gets that, if $b_2 = \dim H^2(W) = 0$, there is (modulo equivalence) only one choice at each step, coming from the Chevalley cohomology class of the very special cocycle S^3_Γ given, on any canonical chart U of W, by

$$S^3_\Gamma(u,v)\big|_U = \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \Lambda^{i_3 j_3} (\mathcal{L}(X_u)\Gamma)_{i_1 i_2 i_3} (\mathcal{L}(X_v)\Gamma)_{j_1 j_2 j_3} \tag{3}$$

where $\mathcal{L}(X_u)$ is the Lie derivative in the direction of the Hamiltonian vector field $X_u = \mu^{-1}(du)$ defined by $u \in N$ and Γ is any symplectic connection (Γ_{ijk} totally skew-symmetric; $i, j, k = 1, ..., 2\ell$) on W. The Chevalley cohomology class of S^3_Γ is independent of the choice of Γ. On $\mathbb{R}^{2\ell}$ (with the trivial flat connection) it coincides with P^3, when we denote by P^r the r^{th} power of the bidifferential operator P. It is [9-11] the pilot term for the Moyal bracket [21] M, given by (1) where $(2r+1)!\hat{C}_r = P^{2r+1}$, i.e. the sinh function of P (the only [11] function of P giving a Lie algebra deformation). In the Weyl quantization procedure, M corresponds to the commutator of operators (when we take for deformation parameter $\nu = \frac{1}{2}i\hbar$).

2.2 Deformations of associative algebras. On N (or $N[[\nu]]$) we can consider the associative algebra defined by the usual (pointwise) product of functions. Its deformations are governed by the Hochschild cohomology $H^*(N)$, and here also it makes sense to restrict to local or differentiable (n.c. or not) cochains and cohomologies. All the latter cohomologies are in fact the same: $H^p(N) = \wedge^p(W)$, the contravariant totally skew-symmetric p-tensors on W; if b denotes the Hochschild coboundary operator, any (local, etc.) p-cocycle C is of the form $C = D + bE$ with $D \in \wedge^p(W)$ and E a (local, etc.) $(p-1)$ cochain.

a. In order to relate to the preceding theory and thereby reduce the (a priori huge) possibilities of choices, and also of course because this is the physically interesting case, we shall be interested only in deformations such that the corresponding commutator starts with the Poisson bracket P, what we call *"star products"*:

$$u * v = uv + \sum_{r=1}^{\infty} \nu^r C_r(u, v), \qquad u, v \in N \text{ (or } N[[\nu]]) \tag{4}$$

$$C_1(u, v) - C_1(v, u) = 2P(u, v) \tag{5}$$

where the cochains C_r are bilinear maps from $N \times N$ to N. In the local (etc.) case, one necessarily has $C_1 = P + bT_1$, and therefore any (local, etc.) star product is equivalent, via an equivalence operator $T = I + \nu T_1$, T_1 a differential operator, to a star product starting with $C_1 = P$. Note that, in the differentiable case, an equivalence operator $T = I + \sum_{r=1}^{\infty} \nu^r T_r$ between two star products is necessarily [11] given by a formal series of differential operators T_r (n.c. in the n.c. case). Star products are always *nontrivial* deformations of the associative algebra N because P is a nontrivial 2-cocycle for the Hochschild cohomology (a coboundary can never be a bidifferential operator of order (1,1)).

The case when the cochains C_r are differentiable and odd or even together with r (what we call the *parity condition*, $C_r(u, v) = (-1)^r C_r(v, u)$) is simpler and we considered it first [11]. However the parity condition is not always needed, and the differentiability condition is sometimes not completely satisfied. This is especially the case when one deals with what we call "star representations" of (semi-simple) Lie groups G, by star products on coadjoint orbits. There, the orbits being given by polynomial equations in the vector space of the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of G, the C_r will in general be bi-pseudodifferential operators on the orbits. It would thus be of interest to introduce another category of star products, when the cochains are algebraic functions of bidifferential operators; the related cohomologies would probably be not very different from the differentiable case ones. On the other hand, restricting to 1-differentiable cochains is not of much interest here (by opposition to the Lie algebra case [2]) since one then looses all connection to quantum theories because the cochain S^3_Γ is lost (the order of differentiation is either 1 or unbounded).

b. From (3) one gets a deformed bracket by taking the commutator $\frac{1}{2\nu}(u*v-v*u)$, which gives a Lie algebra deformation (1) with $2\hat{C}_{r-1}(u,v) = C_r(u,v) - C_r(v,u)$. In contradistinction with the Lie case, the Hochschild cohomology spaces are always huge but the choices for star products will be much more limited because of the associated Lie algebra deformations.

In particular when the C_r are differentiable and satisfy the parity condition, the C_{2r+1} being n.c. (what is called a "weak star product"), any star product is equivalent [22] to a "Vey star product", one for which the $r!C_r$ have the same principal symbol as P^r_Γ, the r^{th} power of the Poisson bracket expressed with covariant derivatives ∇ relative to some symplectic connection Γ, i.e. on a local chart U (with summation convention on repeated indices):

$$P^r_\Gamma(u,v)\big|_U = \Lambda^{i_1j_1}\ldots\Lambda^{i_rj_r}\nabla_{i_1\ldots i_r}u\nabla_{j_1\ldots j_r}v, \quad u,v\in N. \tag{6}$$

For such products the most general form of the first terms are

$$C_2 = P^2_\Gamma + bT_{(2)} \text{ and } C_3 = S^3_\Gamma + \Lambda_2 + 3\hat{b}T_{(2)}, \tag{7}$$

where $T_{(2)}$ is a differential operator of order at most 2, $\hat{b}$ denotes the Chevalley coboundary operator and Λ_2 is a 2-tensor, image (under μ^{-1}) of a closed 2-form [19,22]. A somewhat general expression can also be given for C_4 [19], but it is much more complicated. For higher terms no explicit formula was published (Jacques Vey knew more or less how to do it for C_5) but there exists an algorithmic construction in terms of a symplectic connection that gives a class of examples term by term [23].

What happens here (assuming the parity condition) is that the Lie algebra (i.e. the odd cochains) determines inductively the star product from which it originates; the only freedom is the possible addition, at each even level, of multiples of uv to the cochains (and to the equivalence operators). The parity condition is of course satisfied at level 0 and can (by equivalence) be assumed at level 1 for star products, but the Lie algebra will in general (except when $b_2=0$) give enough informations only on the odd part of the cochains C_r.

2.3. Existence, uniqueness and examples of star products.

a. Existence. On $\mathbb{R}^{2\ell}$ with the flat symplectic connection one has the Moyal star product and bracket:

$$u *_M v = \exp(\nu P)(u,v) \qquad \mathrm{M}(u,v) = \sinh P(u,v). \tag{8}$$

The idea is to take such star products M_α on Darboux charts U_α for any symplectic W and to glue them together. This cannot be done brutally (when $b_3 \neq 0$ the topology of the manifold hits back). But $N[[\nu]]$ can be viewed as a space of flat sections in the bundle of formal Weyl algebras on W (a Weyl algebra is generated by the canonical commutation relations $[x^i,x^j] = 2\nu\Lambda^{ij}I$); a flat connection on that bundle is algorithmically constructed

[23] starting from any symplectic connection on W. Pulling back the multiplication of sections gives a star product [24], which can also be seen as obtained by the juxtapositions of star products $T_\alpha M_\alpha$ on each U_α, when the equivalence operators T_α are such that all $T_\alpha M_\alpha$ and $T_\beta M_\beta$ coincide on $U_\alpha \cap U_\beta$ (the Darboux covering is chosen locally finite). These star products can be taken differentiable n.c. (d.n.c. in short) and satisfying the parity condition.

Earlier proofs of existence were done first in the case $b_3 = 0$, then for $W = T^*X$ with X parallelizable, and shortly afterwards for any symplectic (or regular Poisson) manifold; but that proof was essentially algebraic, while we now see better the underlying geometry.

b. Uniqueness. Formula (7) is very instructive on what happens. Indeed whenever we have two Lie algebra deformations equivalent to some order, after making them identical to that order by an equivalence, the difference of the cochains is a cocycle of the form given by C_3 in (7), in the d.n.c. case of course. Therefore, assuming the parity condition, we have at each step at most $1 + b_2$ choices modulo equivalence, and we see exactly where the second de Rham cohomology enters.

Moreover, in the d.n.c. case, *when $b_2 = 0$, the Moyal-Vey product is unique* even without the parity condition. Indeed one can choose a star product satisfying the parity condition (denote its cochains by C'_r). Any other d.n.c. star product (with cochains C_r) can then step by step be made equal to the chosen one by an argument (due to S. Gutt) similar to those of [19,22]: at the first step where $C_k - C'_k$ is nonzero, it is of the form $D_k + bE_k$ with D_k a closed, thus exact, 2-form: $D_k = dF_k$ (here $D_k(u, v)$ is defined as $D_k(X_u \otimes X_v)$, and similarly $F_k(u) \equiv F_k(X_u)$). The equivalence will then be extended to the next order by $I - \nu^{k-1}F_k - \nu^k E_k$.

c. Examples. The various orderings considered in physics are the inverse image of the product (or commutator) of operators in $L^2(\mathbb{R}^\ell)$ under the Weyl mappings

$$N \ni\mapsto \Omega_w(u) = \int \tilde{u}(\xi, \eta)\exp(i(P\xi + Q\eta)/\hbar)w(\xi, \eta)\omega^\ell \tag{9}$$

where $\tilde{u}$ is the inverse Fourier transform of u, P and Q are operators satisfying the canonical commutation relations $[P_\alpha, Q_\alpha] = i\hbar\delta_{\alpha\beta}(\alpha, \beta = 1, ..., \ell)$, w is a weight function, $2\nu = i\hbar$ and the integral is taken in the weak operator topology. Normal ordering corresponds to the weight $w(\xi, \eta) = \exp(-\frac{1}{4}(\xi^2 + \eta^2)$, standard ordering (the case of the usual pseudodifferential operators in mathematics) to $w(\xi, \eta) = \exp(-\frac{i}{2}\xi\eta)$ and Weyl (symmetric) ordering to $w = 1$. Only the latter is such that $C_1 = P$ (e.g. standard ordering starts with the first half of P) but they are all mathematically equivalent via the Fourier transform of w. (Physically they give different spectra, when we define the star spectrum as indicated in the next paragraph, for the image of most classical observables; in fact two isospectral star products are identical [25]).

Other examples can be obtained from these products by various devices. For instance one can restrict to an open submanifold (like $T^*(\mathbb{R}^\ell - \{0\})$), quotient it under the action of a group of symplectomorphisms and restrict to invariant functions; one can also transform by equivalences, or look (cf. below) for G-invariant star products. A variety of physical systems can thus be treated in an autonomous manner [11]. The simplest of course is the harmonic oscillator, which relates marvelously to the metaplectic group (dear to Bert Kostant). But other systems, such as the hydrogen atom, have also been treated from the beginning – which is not the case of geometric quantization.

An essential ingredient in physical applications is an autonomous spectral theory, with the spectrum defined as the support of the Fourier-Stieltjes transform of the star exponential $\mathrm{Exp}(tH) = \sum_{n=0}^{\infty} \frac{1}{n!}(tH/i\hbar)^{*n}$, where the exponent $*n$ means the n^{th} star power. Such a spectrum can even be defined in cases when operatorial quantization would give nonspectrable operators (e.g. symmetric with different deficiency indices). The notion of trace is also important here, and will bring us to closed star products. Interestingly enough, the trace of the star exponential of the harmonic oscillator was already obtained in 1960 by Julian Schwinger [26] (within conventional theory, of course).

d. Groups. $\mathbb{R}^{2\ell}$ is the generic coadjoint orbit of the Heisenberg group in $\mathcal{H}_\ell^* = \mathbb{R}^{2\ell+1}$; the uniqueness of Moyal parallels the uniqueness theorem of von Neumann, but this goes much further. For any Lie group G with Lie algebra $\mathcal{G}$ one has an autonomous notion of star representation. Every $x \in \mathcal{G}$ can be considered as a function on $\mathcal{G}^*$ and restricted to a function $u_x \in N(W)$ on a G-orbit (or a collection of orbits) W, so that $P(u_x, u_y)$ realizes the bracket $[x, y]_{\mathcal{G}}$. If we now take a star product on W for which $[u_x, u_y]_\nu = P(u_x, u_y)$, what we call a *$G$-covariant star product*, the map $x \mapsto \frac{1}{2}\nu^{-1}u_x$ will define a representation of the enveloping algebra $\mathcal{U}(\mathcal{G})$ in $N[\nu^{-1}, \nu]]$, the space of formal series in ν and ν^{-1} (polynomial in the latter) with coefficients in N, endowed with the star product. This will give a *representation* of G in $N[[\nu^{-1}, \nu]]$ by the star exponential:

$$G \ni e^x \mapsto E(e^x) = \mathrm{Exp}(x) = \sum_{n=0}^{\infty} (n!)^{-1}(u_x/2\nu)^{*n}. \tag{10}$$

The star product is said *G-invariant* if $[u_x, v]_\nu = P(u_x, v)\ \ \forall v \in N$, i.e. if the geometric action of G on W defines an automorphism of the star product. A whole theory of star representations has been developed (see e.g. [27] for an early review). By now it includes an autonomous development of nilpotent and solvable groups (in a way adapted to the Plancherel formula), with a correspondence (via star polarizations) with the usual Kirillov and Kostant theories [28]; there the orbits are (in the simply connected case) symplectomorphic to some $\mathbb{R}^{2\ell}$, and the Moyal product can be lifted to the orbits. Star representations have also been obtained for compact groups and for several series of representations of semi-simple Lie groups [29] (including the holomorphic discrete series and some with unipotent orbits); the cochains C_r are here in general pseudodifferential. Integration over W of the star exponential (a kind of trace) will give a scalar-valued distribution on G that is nothing but the character of the representation.

3. Closed Star Products

3.1 Trace and closed star products. Existence.

a. For Moyal product (weight $w = 1$ in (9)) one has, whenever $\Omega_1(u)$ is trace-class:

$$\mathrm{Tr}(\Omega_1(u)) = (2\pi\hbar)^{-\ell} \int u\omega^\ell \equiv \mathcal{T}_M(u) \tag{11}$$

while for other orderings (like the standard ordering S) this formula is true only modulo higher powers of $\hbar$: $\mathrm{Tr}(\Omega_S(u)) = \mathcal{T}_M(u) + O(\hbar^{1-\ell})$. But for all of them the abovedefined $\mathcal{T}_M$ has the property of a trace, i.e.

$$\mathcal{T}(u * v) = \mathcal{T}(v * u). \tag{12}$$

One even has (for the Moyal product, because of the skew-symmetry of the Λ^{ij}) that $\mathcal{T}_M(u *_M v) = \mathcal{T}_M(uv)$. All these star products are what we call [16] *strongly closed*:

$$\int C_r(u,v)\omega^\ell = \int C_r(v,u)\omega^\ell \qquad \forall r \text{ and } u,v \in N. \tag{13}$$

The existence proofs of [24] can be made such that the star products constructed are strongly closed. When $b_2 = 0$ the uniqueness (modulo equivalence) of star products shows that all d.n.c. star products are equivalent to a closed one (which exists). But there is more: *All differentiable null on constants star products are strongly closed.*

As was pointed out to us by P. Lecomte, this is a simple consequence of a result of M. De Wilde ([20], Prop. 2.1.(iii)) which shows that $(C_r(u,v) - C_r(v,u))\omega^\ell = d\alpha$ (where α is some $2\ell - 1$ form, depending on r, u, v). Integration with the Liouville measure (an integral invariant) absorbs the effects of the 2-cohomology. This is somewhat related to a recent result by O. Mathieu [30] that gives an often (not always, but always in degree 2) satisfied necessary and sufficient condition for the existence of harmonic forms on compact symplectic manifolds, and thereby counterexamples to a conjecture by Brylinski.

b. Another way to look at the question is to consider the algebra $N[[\nu]]$ endowed with star product and to restrict (in order to get finite integrals) to $\mathcal{D}[[\nu]]$, where $\mathcal{D}$ denotes the C^∞ functions with compact support on the manifold W. A trace on $\mathcal{D}[[\nu]]$ is then defined as a $\mathbb{C}[[\nu]]$-linear map $\mathcal{T}$ into $\mathbb{C}[\nu^{-1}, \nu]]$ satisfying (12). In the d.n.c. case it has been shown by Tsygan and Nest [31] that there exists (up to a factor) a unique trace on $\mathcal{D}[[\nu]]$.

If one takes a locally finite covering of W by Darboux charts U_α (all the intersections of which are either empty or diffeomorphic to $\mathbb{R}^{2\ell}$) and a partition of unity (ρ_α) subordinate to it, this trace can be defined, in a consistent way [31], by

$$\mathcal{T}(u) = \sum_\alpha \mathcal{T}_\alpha(T_\alpha(\rho_\alpha * u)) \tag{14}$$

where $\mathcal{T}_\alpha$ is the Moyal trace (11) on the image of U_α in a standard $\mathbb{R}^{2\ell}$ by coordinate maps, and T_α an equivalence that maps the given star product restricted to U_α into the Moyal product of the standard $\mathbb{R}^{2\ell}$ (any self-equivalence of the Moyal product on $\mathbb{R}^{2\ell}$ preserves the Moyal trace). It is thus of the form

$$\mathcal{T}(u) = (4i\pi\nu)^{-\ell} \int_W (Tu)\omega^\ell, \qquad \text{with } T = I + \sum_{r=1}^{\infty} \nu^r T_r, \tag{15}$$

the T_r being differential operators, and this T transforms the initial product into an equivalent one that is obviously closed. A more careful look at the construction of T shows that in fact (in the d.n.c. case) the original star product is already closed.

c. In the last construction, the d.n.c. assumption seems somewhat less crucial than in subsection **a**, but relaxing it is not a trivial matter because it involves going beyond the differentiable case for the Hochschild cohomology and (for the n.c. assumption) because it is needed that 1 be a unity for the star algebra. For more general star products the difference between general, closed and strongly closed star products should appear.

A star product is said *closed* if (13) is supposed only for $r \leq \ell$, i.e. if the coefficient of ν^ℓ in $(u*v-v*u)$ for all $u,v \in \mathcal{D}[[\nu]]$ has vanishing integral. The reason for this definition is obvious from a glance at (11) if one remembers that in the algebraic index theorems of A. Connes [14], indices of operators are expressed as traces, and that star products permit an alternative and autonomous treatment of operator algebras directly on phase-space. Note that all star products on 2-dimensional manifolds are closed (because of (5)).

3.2 Closed star products, cyclic cohomology and index theorems.

a. Cyclic cohomology was introduced by A. Connes in connection with trace formulas for operators (the dual theory of cyclic homology was developed independently and in a different context by B. Tsygan) [32]. The motivation is that cyclic cocycles are higher analogues of traces and thus make easier the computation of the index as the trace of some operator by giving it an algebraic setting.

Let $\mathcal{A}$ be an algebra, $\mathcal{A}=\mathcal{D}[[\nu]]$ to fix ideas (but the notion of cyclic cohomology can be defined abstractly). To every $u \in \mathcal{A}$ one can associate $\tilde{u} \in \mathcal{A}^*$ defined by

$$\mathcal{A}^* \ni \tilde{u} : \mathcal{A} \ni v \mapsto \int uv\omega^\ell. \tag{16}$$

$\mathcal{A}$ acts on $\mathcal{A}^*$ by $(x\phi y)(v)=\phi(yvx)$, with $\phi \in \mathcal{A}^*$ and $v,x,y \in \mathcal{A}$. The map $u \mapsto \tilde{u}$ defines a map $C^p(\mathcal{A},\mathcal{A}) \to C^p(\mathcal{A},\mathcal{A}^*)$ compatible with the (Hochschild) coboundary operator b, and we can restrict to the space of cyclic cochains $C^p_\lambda \subset C^p(\mathcal{A},\mathcal{A}^*)$, those that satisfy the cyclicity condition

$$\tilde{C}(u_1,...,u_p)(u_0)=(-1)^p\tilde{C}(u_2,...,u_p,u_0)(u_1). \tag{17}$$

The *cyclic cohomology* of $\mathcal{A}$, $HC^p(\mathcal{A})$, is defined as the cohomology of the complex (C^p_λ, b). Now, on the bicomplex $C^{n,m}=C^{n-m}(\mathcal{A},\mathcal{A}^*)$ for $n \geq m$ (defined as $\{0\}$ for $n<m$), b is of degree 1 and one can define another operation B of degree -1 that anticommutes with it ($bB=-Bb, B^2=0=b^2$). We refer to [14] and [16] for a precise definition in the general case; when $\gamma(1;u,v)=\int C(u,v)\omega^\ell$ is a normalized 2-cochain $\gamma=\tilde{C} \in C^2(\mathcal{A},\mathcal{A}^*)$, B can be defined by $B\gamma(u,v)=\gamma(1;u,v)-\gamma(1;v,u)$. This bicomplex permits to compute the cyclic cohomology (at each level p) by :

$$HC^p(\mathcal{A})=(\mathrm{ker}b \cap \mathrm{ker}B)/b(\mathrm{ker}B). \tag{18}$$

Obviously the closedness condition at order 2 of a star product (4,5) is expressed by $B\tilde{C}_2=0$. By standard deformation theory [11] we know that the Hochschild 2-cocycle C_1 determines a 3-cocycle E_2 that has to be of the form bC_2 (if the deformation extends to order 2) and therefore (if the star product is closed at order 2) $\tilde{E}_2 \in \mathrm{ker}b \cap \mathrm{ker}B$. Since modifying $\tilde{C}_2$ by an element of ker$B$ will not affect closedness, and since [11] the same story can be shifted step by step to any order, (18) shows us that the *obstructions to existence* of a C_r yielding a star product closed at order $r \geq 2$ are given by $HC^3(\mathcal{A})$. Similarly, like in [11], the obstructions to extend an equivalence of closed star products from one step to the next are classified by $HC^2(\mathcal{A})$. *Cyclic cohomology replaces Hochschild cohomology* for closed star products, and this will become especially important when the n.c. assumption is not satisfied.

b. Character, and index theorems. As in the Banach algebra case [33], which is a specification of the general framework developed by A. Connes [14], we see that here, when $*$ is closed, the formula:

$$\varphi_{2k}(u_0, ..., u_{2k}) = \tau(u_0 * \theta(u_1, u_2) * ... * \theta(u_{2k-1}, u_{2k})) \tag{19}$$

where $\ell \leq 2k \leq 2\ell$ (otherwise it is necessarily 0),

$$\theta(u_1, u_2) = u_1 * u_2 - u_1 u_2 = \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2) \tag{20}$$

is a quasi-homomorphism (that measures the noncommutativity of the $*$-algebra and is also a Hochschild 2-cocycle) and τ is a trace defined by

$$\tau(u) = \int u_\ell \omega^\ell, \qquad u = \sum_{r=0}^{\infty} \nu^r u_r \in \mathcal{D}[[\nu]], \tag{21}$$

defines the *components of a cyclic cocycle* φ in the (b, B) bicomplex on $\mathcal{D}$ that is called the *character* of the closed star product. In particular

$$\varphi_{2\ell}(u_0, ..., u_{2\ell}) = \int u_0 du_1 \wedge ... \wedge du_{2\ell}, \qquad u_k \in \mathcal{D}. \tag{22}$$

When $2\ell = 4$, a simple computation shows that the other component is $\varphi_2 = \tilde{C}_2$ and then $b\varphi_2 = -\frac{1}{2}B\varphi_4$. But $HC^2(\mathcal{D}) = Z_2(W, \mathbb{C}) \oplus \mathbb{C}$ (where Z_2 denotes the closed 2-dimensional currents on W). Therefore the *integrality condition* $< \varphi, K_0(\mathcal{D}) > \subset \mathbf{Z}$, necessary to have a deformation of $\mathcal{D}$ to the algebra of compact operators in a Hilbert space, what is traditionally called a quantization, *has no reason to be true* for a general closed star product.

Now the pseudodifferential calculus on $W = T^*X$, with X compact Riemannian, gives [16] a closed star product, the character of which coincides with the character given by the trace on pseudodifferential operators, and therefore satisfies the integrality condition (up to a factor). The Atiyah-Singer index formula can then be recovered in an autonomous manner also in the star product formulation [14,16,23]. But the algebraic index formulas are valid in a much more general context [14,31,33]. It is therefore natural to expect that the character of a general star product should make it possible to define a *continuous index.*

4. Star Products and Quantum Groups

4.1 Topological Algebras. The notion of quantum group has two dual aspects: the modification of commutation relations in Lie and enveloping algebras, and deformations of algebras of functions on a group (with star products). The latter gives by duality a coproduct deformation. In both cases *Hopf algebra* structures are considered, but there is a catch: except for finite-dimensional algebras, the algebraic dual of a Hopf algebra is not a Hopf algebra. The best way to circumvent this (largely overlooked) difficulty is to topologize the algebras in a proper way.

a. Deformations revisited. Let A be a topological algebra. By this we mean an associative, Lie or Hopf algebra, or a bialgebra, endowed with a locally convex topology for which all needed algebraic laws are continuous. For simplicity we fix the base field to be the complex numbers $\mathbb{C}$. Extending it to the ring $\mathbb{C}[[\nu]]$ gives the module $\tilde{A} = A[[\nu]]$, on which we can consider various algebraic structures.

A *deformation* of an algebra A is a topologically free $\mathbb{C}[[\nu]]$-algebra $\tilde{A}$ such that $\tilde{A}/\nu\tilde{A} \approx A$. For associative or Lie algebras this means that there exists a new product or bracket satisfying (4) or (1) (resp.). For a bialgebra (associative algebra A with coproduct $\Delta : A \to A \otimes A$ and the obvious compatiblity relations), denoting by $\otimes_\nu$ the tensor product of $\mathbb{C}[[\nu]]$-modules, one can identify $\tilde{A}\hat{\otimes}_\nu\tilde{A}$ with $(A\hat{\otimes}A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some operator topology (projective for Fréchet nuclear topology e.g.), we similarly have a deformed coproduct

$$\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty} \nu^r D_r, \quad D_r \in \mathcal{L}(A, A\hat{\otimes}A) \tag{23}$$

and here also an appropriate cohomology can be introduced [34-36]. In the case of Hopf algebras, the deformed algebras will have the same unit and counit, but in general not the same antipode. As in the algebraic theory [12], *equivalence* of deformations has to be understood here as isomorphism of $\mathbb{C}[[\nu]]$-topological algebras (the isomorphism being the identity in degree 0 in ν), and a deformation is said *trivial* if it is equivalent to that obtained by base field extension.

b. The required objects. In the beginning Kulish and Reshetikhin [37] discovered a strange modification of the $\mathcal{G} = s\ell(2)$ Lie algebra, where the commutation relation of the two nilpotent generators is a sine in the semi-simple generator instead of being a multiple of it, which requires some completion of the enveloping algebra $\mathcal{U}(\mathcal{G})$. This was developed in the first half of the 80's by the Leningrad school of L. Faddeev, systematized by V. Drinfeld who developed the Hopf algebraic context and coined the extremely effective (though somewhat misleading) term of *quantum group* [15], and from the enveloping algebra point of view by Jimbo [38]. Shortly afterwards, Woronowicz [39] realized these models in the context of the noncommutative geometry of Alain Connes [14] by matrix pseudogroups, with coefficients in C^* algebras satisfying some relations.

Let us take (for simplicity) a Poisson Lie group, a Lie group G with compatible Poisson structure i.e. a Poisson bracket P on $N = C^\infty(G)$, considered as a bialgebra with coproduct defined by $\Delta u(g, g') = u(gg'), g, g' \in G$, and satisfying

$$\Delta P(u, v) = P(\Delta u, \Delta v) \qquad \text{where} \ \ u, v \in N. \tag{24}$$

The enveloping algebra $\mathcal{U}(\mathcal{G})$ can be identified with distributions with (compact) support at the identity of G, thus is part of the topological dual N' of N. But we shall need more than N' for the quantized universal enveloping algebra $\mathcal{U}_\nu(\mathcal{G})$, some infinite series in the Dirac δ and its derivatives, thus shall have to restrict to a subalgebra H of N. When G is compact we shall take the space H of G-finite vectors for the regular representation.

It is natural to look for topologies [40] such that both aspects will be in full duality, i.e. reflexive topologies. We also would like to avoid having too much problems with tensor product topologies, that can be quite intricate; for instance we need to identify $C^\infty(G \times G) = N \hat{\otimes} N$.

When G is a general Lie group, N is Fréchet nuclear with dual $\mathcal{E}'$, the distributions with compact support, but $\mathcal{D}$ (with dual $\mathcal{D}'$, all distributions) is only a (LF)-space, also nuclear. There is no simple candidate to replace the G-finite vectors of the compact case (the most likely are the analytic vectors for the regular, or a quasi-regular, representation). In the following we shall thus from now on restrict to the original setting of G compact.

4.2. Compact Topological Quantum Groups [35].

a. The Classical Objects. For a compact Lie group G we shall consider the following topological bialgebras (in fact, Hopf algebras):

$$H = \sum_{\rho \in \hat{G}} \mathcal{L}(V_\rho), \qquad \text{and its dual } H' = \prod_{\rho \in \hat{G}} \mathcal{L}(V_\rho) \supset \mathcal{D}'. \tag{25}$$

Here V_ρ is the isotypic component of type $\rho \in \hat{G}$ in the Peter-Weyl decomposition of the (left or right) regular representation of G in $L^2(G)$. H is also called the space of coefficients, since it is the space of all coefficients of unitary irreducible representations (each isotypic representation being counted with its multiplicity, equal to its dimension). The enveloping algebra $\mathcal{U} = \mathcal{U}(\mathcal{G})$ is imbedded in H' by

$$\mathcal{U} \ni x \mapsto i(x) = (\rho(x)) \in H'. \tag{26}$$

This imbedding has a dense image in H'; the image is in fact in $\mathcal{D}'$ but is not dense for the $\mathcal{D}'$ topology. The product in $\mathcal{D}'$ is the convolution of distributions, the coproduct satisfies $\Delta(g) = g \otimes g$ for $g \in G$ (identified with the Dirac δ at g), and the counit is the trivial representation. Since the objects will be the same in the "deformed" case, only some composition laws being modified (exactly like in deformation quantization, of which it is in fact an example), we have discovered the initial group G *"hidden"* (like a hidden classical variable) in the compact quantum groups.

All these algebras are what we call *well-behaved*: the underlying topological vector spaces are nuclear and either Fréchet or (DF). The importance of this notion comes from the fact that *the dual A' of a well-behaved A is well-behaved, and the bidual $A'' = A$.*

b. The deformations. First let us mention that duality and deformations work very well in the setting of well-behaved algebras: if $\tilde{A}$ is a deformation of A well-behaved, its $\mathbb{C}[[\nu]]$-dual $\tilde{A}^*_\nu$ is a deformation of A^* and two deformations are equivalent iff their duals are equivalent deformations.

In view of the known models of quantum groups, we select a special type of bialgebra deformations, those we call [34,35] *preferred*: deformations of N or H with *unchanged coproduct*. Here this is not a real restriction because any coassociative deformation of H or$N$ can (by equivalence) be made preferred (with a quasicocommutative and quasiassociative product).

This follows by duality from the fact that for $\mathcal{D}'$ or H', any associative algebra deformation is trivial, that these bialgebras are rigid (in the bialgebra category) and that any associative bialgebra deformation with unchanged product has coproduct $\tilde{\Delta}$ and antipode $\tilde{S}$ obtained from the undeformed structures by a similitude (expressing the "quasi-" properties):

$$\tilde{\Delta} = \tilde{P}\Delta\tilde{P}^{-1}, \quad \tilde{S} = \tilde{a}S\tilde{a}^{-1} \tag{27}$$

for some $\tilde{P} \in (A\hat{\otimes}A)[[\nu]]$ and $\tilde{a} \in A[[\nu]]$, with $A = \mathcal{D}'$ or H'. When associative, the product is a *star product* that can (by equivalence [41]) be transformed into a (noninvariant) star product $*'$ satisfying $\Delta(u *' v) = \Delta u *' \Delta v,\ u, v \in N$.

This general framework can be adapted to the various models. We refer to [35] for a thorough discussion. The Drinfeld and Faddeev-Reshetikhin-Takhtajan models fall exactly into this framework. An essential tool is the *Drinfeld isomorphisms* $\tilde{\varphi}$, (algebra) isomorphisms between a Hopf deformation $\mathcal{U}_\nu$ of $\mathcal{U}$ and $\mathcal{U}[[\nu]]$, which give (27). (Two Drinfeld isomorphisms give equivalent preferred deformations of H that extend to preferred Hopf deformations of N).

The Jimbo models [38] are somewhat aside because their classical limit is not $\mathcal{U}$ but $\mathcal{U}(\mathcal{G})$ extended by Rank(G) parities, and this makes out of the deformed algebras nontrivial deformations.

Finally we are now in position to have a good formulation of the *quantum double* [42]. If A (resp. A') denote H' or $\mathcal{D}'$ (resp. H or N) or their deformed versions, the double is $D(A) = A'\bar{\otimes}A$; its dual is $D(A)' = A\hat{\otimes}A', D(A)'' = D(A)$ and all these algebras are rigid.

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