Applications of Quantum Probability to Classical Stochastics

Alexander M. Chebotarev

Quantum Statistics Department, Moscow State University,
Moscow 119899, Russia

New nonexplosion conditions for Markov processes are derived from the general operator form of the conservativity condition for a quantum dynamical semigroup.

1 Introduction

In this paper an operator-valued generalization of the Kolmogorov – Feller equation is discussed. The most remarkable feature of this generalization is that it also covers diffusion and Heisenberg equations as particular cases [L]. It is known that if intensity of jumps (or drift velocity, or diffusion coefficient)as a function on the phase space X of the corresponding Markov process is unbounded, then the probability measure of trajectories hitting singularities of the coefficients may be positive. If there exist a number of different ways to define the behaviour of the process after hitting a singular point then the formal evolution equation for the transition probability of the Markov jump process has a linear manifold of solutions. In the class of solutions $P(x,t|\Gamma)$ satisfying the initial condition $P(x,0|\Gamma) = I_\Gamma(x)$ there exists a minimal solution such that

$$P^{min}(x,t|X) < 1$$

(see [F]). A similar phenomenon exists for the operator-valued generalization of the Kolmogorov – Feller equation. Examples of Markov jump processes which explode or escape to infinity with positive probability are given in section 2 following [C1], [CF1]. In section 3 we describe an algebraic structure of operator-valued extensions of the basic evolution equations for Markov processes. Section 4 contains domain assumptions for coefficients of infinitesimal maps of semigroups in von Neumann algebras. In section 5 we introduce conditions which are necessary and sufficient for conservation of the total probability by the minimal solution for evolution equation [C2]. We call them *conservativity conditions*. If the minimal solution of the evolution equation is conservative (i.e. it preserves identity) then the corresponding Markov process is called *regular*. Section 6 contains a constructive form of sufficient conservativity conditions and give some applications to classical stochastic processes [C3], [CF2]. In Section 7 we present some new examples of regularity conditions derived for Markov processes from the general operator form of the conservativity condition for a quantum dynamical semigroup.

2 Conditions sufficient for the regularity of Markov jump processes

Consider a Markov jump process in a Polish space X with an abelian group operation $\circ$. We denote by $\kappa(x)$ the intensity of jumps from a point $x \in X$ and by $m(B|x)$ we denote the distribution of jumps from a point x:

$$\mathbf{P}\{value\, of\, the\, jump \in B,\ initial\, point\, of\, the\, jump = x\} = m(B|x).$$

We consider $\kappa(x)$, $m(B|x)$ as parameters of this Markov jump process. If the intensity of jumps is unbounded, then the probability measure of trajectories producing an infinite number of jumps can be positive. Hence the process escapes from any open set $D \in X$, where the intensity is bounded by a positive probability bounded below uniformly w.r.t. D.

Condition which is necessary and sufficient for regularity of such a process was proved originally in [GS]. The Markov jump process with parameters $\kappa(x)$, $m(B|x)$ is regular if and only if

$$\mathbf{P}\Big\{\sum_{n=1}^{\infty} \frac{1}{\kappa(x_n)} = \infty\Big\} = 1, \tag{2.1}$$

where $\{x_n\}_1^\infty$ is the Markovian chain with transition probability $m(B|x)$:

$$\mathbf{P}\{x_{n+1} \in B \circ x_n | x_n\} = m(B|x_n), \quad x_0 = x.$$

In nontrivial cases this condition is not constructive because it requires bounds for the joint distribution for functions of dependent random variables $x_n = q_n \circ \cdots \circ q_1 \circ x$, where q_k are random independent jumps. To derive constructive conditions, one can note that (2.1) is equivalent to the following:

$$\inf_{\alpha \in (0,1]} \lim_{m \to \infty} \mathbb{M} \frac{1}{1 + \Big(\sum\limits_{n=1}^{m} \frac{1}{\kappa(x_n)}\Big)^{\alpha}} = 0. \tag{2.2}$$

The proof immediately follows from the Kolmogorov inequality (see [C1], Lemma 3.2). The next observation has the form of nontrivial inequality:

$$\mathbb{M} \frac{1}{1 + \Big(\sum\limits_{n=1}^{m} \frac{1}{\kappa(x_n)}\Big)^{\alpha}} \leq \frac{1}{1 + \Big(\sum\limits_{n=1}^{m} \big(\frac{1}{\mathbb{M}\kappa(x_n)^{\alpha}}\big)^{\frac{1}{\alpha}}\Big)^{\alpha}} \tag{2.3}$$

([C1], Lemma 3.1). From this inequality we conclude that condition (2.2) holds if

$$\sup_{\alpha \in (0,1]} \lim_{m \to \infty} \Bigg(\sum_{n=1}^{m} \frac{1}{(\mathbb{M}\kappa(x_n)^{\alpha})^{\frac{1}{\alpha}}}\Bigg)^{\alpha} = \infty \tag{2.4}$$

([C1], theorem 3.1). So, condition (2.4) is sufficient for the regular behaviour of the Markov jump process with parameters κ, m. Case $\alpha < 1$ is essential when the function $\kappa(q)$ of a random jump $q : \mathbf{P}_x\{q \in B\} = m(B|x)$ does not have a first moment.

The necessary condition follows from the Jenssen inequality

$$\mathbb{M}\frac{1}{1+\left(\sum_{n=1}^{m}\frac{1}{\kappa(x_n)}\right)^{\alpha}} \geq \frac{1}{\left(1+\sum_{n=1}^{m}\mathbb{M}\frac{1}{\kappa(x_n)}\right)^{\alpha}}. \tag{2.5}$$

It follows from (2.5) that (2.2) holds only if the series diverges:

$$\sum_{n=1}^{\infty}\mathbb{M}\frac{1}{\kappa(x_n)} = \infty. \qquad (2.6).$$

This series is equal to the mean value of the time required to produce an infinite sequence of jumps.

Criteria (2.4), (2.6) can be easily applied if jumps have a space-homogeneous stable distribution. Being applied to a space-homogeneous Markov jump process with the Cauchy distributed jumps in $\mathbb{R}^n$

$$m(B|x) = \frac{1}{\pi^n}\int_B \prod_i \frac{dq_i}{(1+|q_i|^2)},$$

criteria (2.4) shows that all processes with parameters κ, m are regular whenever

$$\kappa(x) \leq C\,(1+|x|)\log(2+|x|).$$

If

$$\kappa(x) \geq C\,(1+|x|)\log^{1+\epsilon}(2+|x|), \quad \forall\epsilon > 0,$$

then process escapes to infinity with a positive probability. The proof for $n \geq 2$ was given in [C1].

For normally distributed jumps in $\mathbb{R}^n$

$$m(B|x) = \frac{1}{(2\pi)^{n/2}}\int_B e^{-\frac{|q|^2}{2}}dq$$

the Markov jump process is regular if

$$\kappa(x) \leq C\,(1+|x|)^2\log(2+|x|).$$

and unregular if

$$\kappa(x) \geq C\,(1+|x|)^2\log^{1+\epsilon}(2+|x|), \quad \forall\epsilon > 0,$$

for dimensions $n \geq 3$ (see also [C1]).

Another nontrivial example gives the so-called Azéma-Emery martingale (see [E1], [E2] for the basic properties) related to the noncommutative generalization of the Brownian motion (see [P] for details) with the infinitesimal map

$$Lf(x) = (\beta x)^{-2}(f(cx) - f(x) - \beta x\frac{d}{dx}f(x)), \quad c = 1+\beta, \quad \beta \neq 0, \quad x \in \mathbb{R}. \tag{2.7}$$

It describes scaled jumps with a singular drift. As we see later, this process is regular if the following series

$$\sum_{n=1}^{\infty} c^{2n}\xi_{n+1}\prod_{j=1}^{n}(1+\xi_j) = \sum_{n=1}^{\infty}\frac{1}{\kappa_n}, \tag{2.8}$$

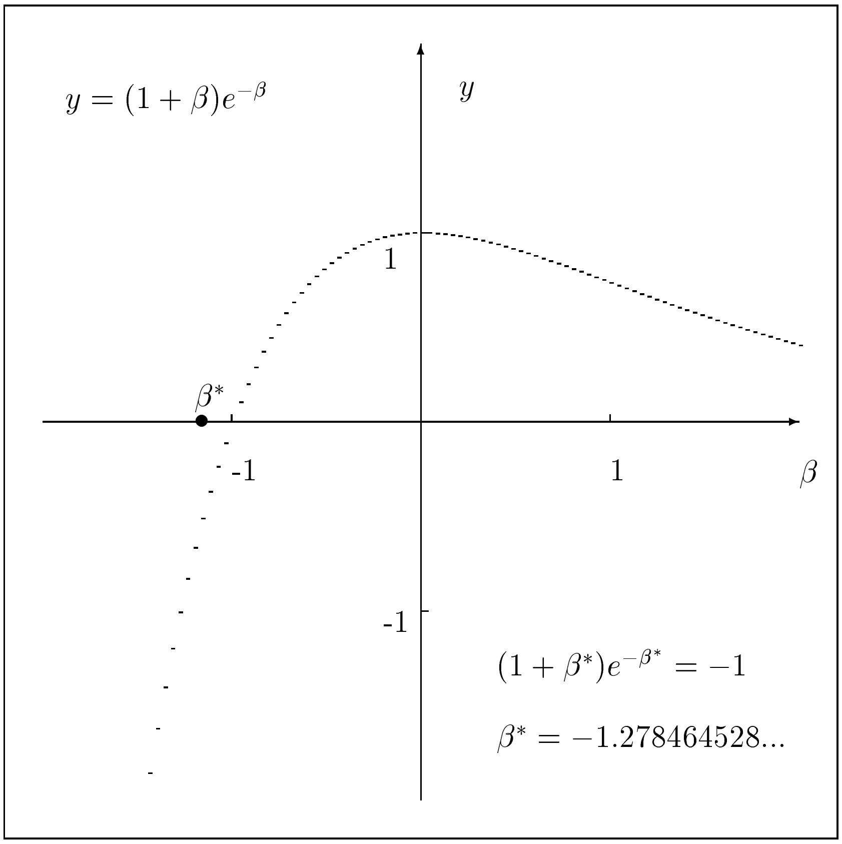
diverges with probability one, where ξ_j are independent random variables with distribution

$$\mathbf{P}\{\xi \in B\} = sign(\beta) \int_B d(1+\xi)^{\frac{1}{2\beta}},$$

where $\xi \in \mathbb{R}_+$ if $\beta < 0$, and $\xi \in [0,1]$ if $\beta > 0$. In this particular case $\kappa_n = (c^{2n}\xi_{n+1} \prod_{j=1}^{n}(1+\xi_j))^{-1}$ are positive random numbers. Since ξ_j are independent and identically distributed, then the regularity condition for the Azéma-Emery martingale is of the form

$$\sum_n \left(\mathbb{M}\kappa_n^\alpha\right)^{-\frac{1}{\alpha}} = \left(\mathbb{M}\xi^{-\alpha}\right)^{-\frac{1}{\alpha}} \sum_n \left((1+\beta)(1-2\alpha\beta)^{\frac{1}{2\alpha}}\right)^{2n} = \infty, \tag{2.9}$$

where $(1-2\alpha\beta) = \mathbb{M}(1+\xi)^{-\alpha}$. Since $\lim_{\alpha\to 0}(1-2\alpha\beta)^{\frac{1}{2\alpha}} = exp(-\beta)$, the series (2.9) diverges for all $\beta < \beta^*$, where β_* is the critical point which is unique solution for equation $(1+\beta)exp(-\beta) = -1$.



In [CF2] it was proved, that the Azéma-Emery martingale is not regular for $\beta > \beta^*$. So, the behaviour of the process at the critical point β^* is still unknown. Now we will introduce definitions which explain the origin of conditions (2.4), (2.6) and describe an operator-valued extension of these conditions.

3 Operator-valued extension of the Markov evolution

Consider a locally compact abelian algebra X with an invariant Haar measure μ and let $\mathcal{H} = \mathcal{L}_2(X,\mu)$. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all linear bounded operators in $\mathcal{H}$. The algebra A=$\mathcal{L}_\infty(X,\mu)$ of essentially bounded functions can be considered as an abelian subalgebra in $\mathcal{B}(\mathcal{H})$ consisting of multiplication by functions from $\mathcal{L}_\infty$. The transition probabilities $P(x,t|\Gamma)$ are natural elements of $\mathcal{L}_\infty$, the corresponding operators from A will be denoted by $P_t(I_\Gamma)$, where I_Γ is another element from A corresponding to multiplication by the indicator function $I_\Gamma(x)$. In this way

$$P_t(I_\Gamma)\psi(x) = P(x,t|\Gamma)\psi(x) \quad \psi(x) \in \mathcal{H}.$$

From this point of view the main linear operations over transition probabilities on $X = \mathbb{R}^n$ induce algebraic operations in $\mathcal{B}(\mathcal{H})$ as follows.

- *Shifts*: $P(x+q,t|\cdot) \Longleftrightarrow D_q^* P_t(\cdot) D_q$, where $D_q\psi(x) = \psi(x-q)$ and D_q^* is the adjoint unitar operator in $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n)$;
- *Scalings*: $P((1+c)x,t|\cdot) \Longleftrightarrow S_c^* P_t(\cdot) S_c$, where $S_c\psi(x) = (1+c)^{-\frac{n}{2}}\psi((1+c)^{-1}x)$ and $S_c^*\psi(x) = (1+c)^{\frac{n}{2}}\psi((1+c)^{\frac{n}{2}}x)$ is the adjoint unitar operator in $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n)$;
- *First derivatives*: $(b(x),\nabla)P(x,t|\cdot) \Longleftrightarrow i[H_b, P_t(\cdot)]$, where $H_b = -\frac{i}{2}((b(x),\nabla) + (\nabla,(b(x)))$ is the symmetrical first-order differential operator, and $[\cdot,\cdot]$ is the commutator. In the particular case $b(x) = (\beta_1^2(x),\cdots,\beta_n^2(x))$ the operator H has another equivalent form:

  $$H = H^* = -i\sum_k \beta_k(x)\frac{\partial}{\partial x_k}\beta_k(x),$$

  that is $H\psi(x) = -i\sum \beta_k(x)\left(\frac{\partial}{\partial x_k}\beta_k(x)\psi(x)\right)$.
- *Second derivatives*: $\frac{1}{2}(\nabla, A(x)\nabla)P(x,t|\cdot) \Longleftrightarrow \Phi(P_t(\cdot)) - \frac{1}{2}\{\Phi(I), P_t(\cdot)\} = \mathcal{L}(P)$, where $\{\cdot,\cdot\}$ is anticommutator and $\Phi(P) = -(\nabla, A^{\frac{1}{2}}(x)PA^{\frac{1}{2}}(x)\nabla)$.

For example, in one-dimensional case we have:

$$i[H,P]\psi = \beta(\beta P\psi)' - P\beta(\beta\psi)' = \beta(\beta' P\psi + \beta P'\psi + \beta P\psi') - P\beta(\beta'\psi + \beta\psi') = (\beta)^2 P'\psi,$$

where H is the first-order differential operator described above, and

$$\mathcal{L}(P)\psi = -(aPa\psi')' + \frac{1}{2}(a^2(P\psi)')' + \frac{1}{2}P(a^2\psi')' = -a^2(P\psi'' + P'\psi') - 2aa'P\psi' +$$

$$+\frac{1}{2}a^2(P''\psi + 2P'\psi' + P\psi'') + aa'(P'\psi + P\psi') + \frac{1}{2}Pa^2\psi'' + Paa'\psi' =$$

$$= aa'P'\psi + \frac{1}{2}a^2P''\psi = \frac{1}{2}(a^2P')'\psi.$$

Multidimensional cases are similar to these examples.

The extension of the infinitesimal operator (2.7) to an operator map reduces to the cases considered above:

$$LP(x,t|\cdot) \Longleftrightarrow \mathcal{L}(P_t(\cdot)) = \Phi(P_t) - \frac{1}{2}\{\Phi(I), P_t\} + i[H, P_t],$$

$$\Phi(P) = \frac{1}{\beta x} S_c^* P S_c \frac{1}{\beta x}, \quad H = \frac{i}{2}(\frac{1}{\beta x}\partial_x + \partial_x \frac{1}{\beta x}\partial_x). \tag{3.1}$$

In what follows, we shall use closure of $\Phi(\cdot)$ and H as the notations for operator-valued coefficients of the infinitesimal map $\mathcal{L}$, and the dissipative operator $\frac{1}{2}\Phi(I) + iH$ will be denoted by G. Operator H is supposed to be symmetric. So formally $G^* = \frac{1}{2}\Phi(I) - iH$ and hence

$$\mathcal{L}(P) = \Phi(P) - G^*P - PG, \quad P \in \mathcal{B}(\mathcal{H}), \quad G = \frac{1}{2}\Phi(I) + iH. \tag{3.2}$$

Similar but more complicated computations put an infinitesimal operator L of the general Markov process

$$L = \frac{1}{2}(\nabla, A(x)\nabla) + (b(x), \nabla) + \int_{R^n\backslash\{0\}} \eta(dq)|C(q,x)|^2(D_q^* - 1 - (q,\nabla)) +$$

$$\int_{R\backslash\{0\}} \nu(dr)|E(r,x)|^2((1+r)^{-\frac{1}{2}}S_r^* - 1 - r(x,\nabla))$$

into correspondence with an infinitesimal map $\mathcal{L}(\cdot)$ such that $\mathcal{L}(P)(x) = LP(x)$ on the set of bounded smooth functions:

$$\mathcal{L}(B) = \Phi(B) - \frac{1}{2}\{\Phi(I), B\} + i[H, B], \;\; \Phi(\cdot) = \sum_0^4 \Phi_i(\cdot), \;\; \mathcal{H} = \sum_0^2 \mathcal{H}_i, \tag{3.3}$$

where

$$\begin{aligned}
H_0 &= \int_{0<|q|\leq 1} \mu(dq) C^*(q,\cdot)\left(\sin(\frac{1}{i}(\nabla,q)) - \frac{1}{i}(\nabla,q)\right) C(q,\cdot), \\
H_1 &= \int_{0<|r|\leq 1} \nu(dr) E^*(r,\cdot)\left(\frac{S_r^* - S_r}{2i} - \frac{r}{i}(x,\nabla)\right) E(r,\cdot), \\
H_2 &= \frac{1}{2i}\left((V(\cdot),\nabla) + (\nabla, V(\cdot))\right), \\
V(x) &= b(x) - \int_{|q|>1} \mu(dq) q |C(q,x)|^2 - x\int_{|r|>1} \nu(dr)|E(r,x)|^2, \\
\Phi_0(B) &= \int_{0<|q|\leq 1} \mu(dq) C(q,\cdot)^*(D_q^* - I) B (D_q - I) C(q,\cdot), \\
\Phi_1(B) &= \int_{0<|r|\leq 1} \nu(dr) E(r,\cdot)^*(S_r^* - I) B (S_r - I) E(r,\cdot), \\
\Phi_2(B) &= -(\nabla, A^{1/2}(x) B A^{1/2}(x)\nabla), \\
\Phi_3(B) &= \int_{|q|>1} \mu(dq) C(q,\cdot)^* D_q^* B D_q C(q,\cdot), \\
\Phi_4(B) &= \int_{|r|>1} \nu(dr) E(r,\cdot)^* S_r^* B S_r E(r,\cdot).
\end{aligned}$$

All maps denoted above by $\Phi_k(\cdot)$ belong to the class of so-called *completely positive* linear maps in operator algebra. A formal definition of a completely positive (CP) map has several equivalent forms. In the class of contractive maps the characteristic property of complete positivity is given by the inequality

$$Q(X^*X) \geq Q(X^*)Q(X)$$

(see [L]), where $X \in \mathcal{B}(\mathcal{H})$. The CP-property is equivalent to

$$\sum_{ij} (\psi_i, \Phi(X_i^* X_j)\psi_j) \geq 0$$

in the class of bounded maps for arbitrary finite sequences $\{\psi_i\} \in \mathcal{H},\ \{X_i\} \in \mathcal{B}(\mathcal{H})$ (see [S]).

A CP-map $Q(\cdot) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called *normal* if $l.u.b._n Q(X_n) = Q(l.u.b. X_n)$ for any bounded increasing sequence X_n from $\mathcal{B}(\mathcal{H})$. Here $l.u.b.$ means *the least upper bound.* The set of all normal bounded CP-maps on $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$ is described by the Kraus representation theorem:

$$\Phi(B) = \sum_k A_k^*\, B\, A_k, \quad \sum_k A_k^* A_k \in \mathcal{B}_+(\mathcal{H})$$

(see [K]).

A remarkable fact discovered by G.Lindblad in 1976 is that infinitesimal maps of a completely positive norm continuous semigroups have structure described by (3.2), where $\Phi(\cdot)$ is a completely positive map and H is a symmetrical operator. So the operator-valued extension of the Markov evolution equation has the form of the Lindblad's evolution equation:

$$\frac{d}{dt} P_t(B) = \mathcal{L}(P_t(B)), \quad B \in \mathcal{B}(\mathcal{H}),$$

where $\mathcal{L}$ is the infinitesimal map (3.2). A solution for the Lindblad equation is called *a dynamical semigroup.* $P_t(\cdot)$ is strongly continuous in $t \in \mathbb{R}_+$, normal, contractive and CP w.r.t. operator argument. A dynamical semigroup is called *conservative* if $P_t(I) = I$. This property generalizes total probability $P(x, t|X) = 1$ conservation law by solutions of the Markov evolution equations. Despite the formal property of the map $\mathcal{L}(\cdot)$

$$\mathcal{L}(I) = 0, \quad P_t(I) = I + \sum_k \frac{t^k}{k!} \mathcal{L}(\ldots \mathcal{L}(I)) = I,$$

which gives conservative solutions for equations with bounded coefficients, in the general case the conservativity of an evolution can be violated as well as the conservation law for total probability in the case of the Kolmogorov-Feller or diffusion equations with unbounded coefficients. It is known that the total probability is conserved by the Markov evolution if the minimal solution for the evolution equation enjoys this property [F]. We prove a similar result in the theory of dynamical semigroups.

4 Main assumptions

If $P_t(\cdot)$ is a solution of the Lindblad evolution equation with bounded coefficients, then it also satisfy the following integral equation, which one can consider as a Duhamel equation:

$$P_t(B) = W_t^* B W_t + \int_0^t ds W_{t-s}^* \Phi(P_s(B)) W_{t-s}, \tag{4.1}$$

where $W_t = exp(-Gt)$ is a contractive semigroup in $\mathcal{H}$ with generator $G = \frac{1}{2}\Phi(I) + iH$. The sequence $P_t^{(n)}(B)$ such that $P_t^{(0)}(B) = 0$,

$$P_t^{(n)}(B) = W_t^* B W_t + \int_0^t ds W_{t-s}^* \Phi(P_s^{(n-1)}(B)) W_{t-s}, \tag{4.2}$$

increases monotonically and is uniformly bounded w.r.t. arbitrary bounds for $\Phi(\cdot)$:

$$P_t^{(n)}(B) \leq ||B|| I.$$

We use this observation to construct the minimal solution of the Lindblad equation with unbounded coefficients.

Suppose that there exists a strongly continuous contractive semigroup W_t with the infinitesimal operator $-G$ such that for some $N \geq 2$

$$G\phi = iH\phi + \frac{1}{2}\Phi(I)\phi, \quad H\phi = H^*\phi \quad \forall \phi \in dom\, G^N, \tag{4.3}$$

where H is a densely defined symmetric operator and $\Phi(I)$ is a positive selfadjoint operator such that $\Phi(I) \geq I$. Suppose also that

$$ess\, dom\, \Phi(I) \subseteq dom\, G^N \subseteq dom\, H \cap \Phi(I) \tag{4.4}$$

$$dom\, G \subseteq \Phi(I)^{\frac{1}{2}}. \tag{4.5}$$

Note that $\mathcal{L}(\cdot)$ is invariant under shifts of the coefficient $\Phi(\cdot) \to \Phi(\cdot) + \lambda I$, so our assumption $\Phi(I) \geq I$ is not restrictive. If

$$i[\Phi(I), H] \geq -c\Phi(I) \quad on\ dom\, G^N \tag{4.6}$$

then (4.5) holds also. (4.4) is The most restictive assumption is (4.4). Anyway it holds if H is relatively bounded w.r.t. $\Phi(I)$.

It is known that there exists one to one correspondence between densely defined bounded from below closable quadratic forms and selfadjoint operators bounded below . We shall denote by $\Phi_*(B)[\cdot]$ the quadratic form generated by the selfadjoint operator $\Phi(B)$ and by $[\![\Phi_*(B)]\!]$ - the selfadjoint operator corresponding to the closed quadratic form $\Phi_*(B)[(\cdot)]$.

Let $\mathbb{B} \in \mathcal{H}$ be a dense Banach subspace in $\mathcal{H}$ such that

$$\mathbb{B} \subseteq dom\, \Phi_*(I), \quad || \cdot ||_{\mathbb{B}} \geq || \cdot ||_{\mathcal{H}} \tag{4.7}$$

$$W_t\ is\ uniformly\ bounded\ and\ strongly\ continuous\ in\ \mathbb{B} \tag{4.8}$$

$$\Phi_*(\cdot) \in CPn_*(\mathbb{B}), \tag{4.9}$$

where $CPn_*(\mathbb{B})$ is a completion of $CPn_*(\mathcal{H})$ w.r.t. locally convex topology induced by seminorms

$$p_{A,B}(\Phi_*) = \sup_{X \in A,\, \phi \in B} |\Phi_*(X)[\phi]|. \tag{4.10}$$

Here $\mathcal{A} \in \mathcal{B}(\mathcal{H}), B \in \mathbb{B}$ are absolutely convex compact subsets and $CPn(\mathcal{H})$ is a set of completely positive normal maps, $CPn_*(\mathcal{H})$ is the positive cone of the corresponding quadratic forms:

$$Q_*(B)[\phi] = (\phi, Q(B)\phi), \quad B \in \mathcal{B}(\mathcal{H}), \quad \phi \in \mathcal{H}.$$

The main examples of A and B give absolutely convex linear spans of compact sets

$$\{W_{t-s}\phi\}_{s\in[0,t],\, \phi\in\mathcal{H}} \in \mathcal{H}, \qquad \{P_s(X)\}_{s\in[0,t],\, X\in\mathcal{B}(\mathcal{H})} \in \mathcal{B}(\mathcal{H}).$$

The main examples of $\mathbb{B}$ are $dom\, G^N$ and $dom\, \Phi(I)^{\frac{1}{2}}$ endowed with the graphic norms. If for example $\mathbb{B} = domG^N$ then (4.7) follows from (4.6), and (4.8) always holds . We need then to check only (4.9) which becomes the only restrictive assumption.

If $\mathbb{B} = dom\, \Phi(I)^{\frac{1}{2}}$, then $\Phi_*(\cdot) \in CPn_*(\mathbb{B})$, and we need to need to check (4.8) which holds if we require the relative bound (4.6).

Here are some important examples of maps $\Phi_*(\cdot) \in CPn_*(\mathbb{B})$ for $\mathbb{B} = dom\, \Phi(I)^{\frac{1}{2}}$.

(i)

$$\Phi_*(B)[\phi] = \sum_1^\infty (a_k\phi, Ba_k\phi),$$

where $\{a_k\}$ is a sequence of closed operators with a dense joint domain $\mathcal{D} \subseteq \cap_k dom\, a_k$ such that $\sum_k ||a_k\phi||^2 < \infty \quad \forall \phi \in \mathcal{D}$.

(ii)

$$\Phi_*(B)[\phi] = \int_\Omega \mu(d\omega)(a_\omega\phi, Ba_\omega\phi),$$

where $\{a_\omega\}$ is a sequence of closed operators with a dense joint domain $\mathcal{D} \subseteq \cap_\omega dom\, a_\omega$ such that $||Xa_\omega\phi|| \in \mathcal{L}_2(\Omega, \mu) \quad \forall X \in \mathcal{B}(\mathcal{H}) \quad \forall \phi \in \mathcal{D}$.

These examples include the following maps:

(1)

$$\Phi_*(B)[\phi] = (A^{\frac{1}{2}}(\cdot)\nabla\phi, BA^{\frac{1}{2}}(\cdot)\nabla\phi),$$

where $A(x)$ is a matrix with a.e. finite measurable coefficients and $A(x) > 0$ uniformly on each compact set in $\mathbb{R}^n$;

(2)

$$\Phi_*(B)[\phi] = \int_{\mathbb{R}^n} \mu(dq)(D_qC(\cdot,q)\phi(\cdot), BD_qC(\cdot,q)\phi(\cdot)),$$

where $\int_{\mathbb{R}^n} \mu(dq)|C(\cdot,q)|^2 = \kappa(x) < \infty$ on each compact set in $\mathbb{R}^n$.

(3) $\Phi_*(B)[\phi] = \frac{1}{\beta x} S_c^* B S_c \frac{1}{\beta x}$, where $\beta = 1 + c$, and S_c is defined by (2.7),(3.1).

Theorem 4.1. *If assumptions (4.3-4.5),(4.7-4.9) hold, then*

1°. *There exist the minimal dynamical semigroup $P_t^{min}(\cdot)$ such that*

$$P_t^{min}(B) = l.u.b. P_t^{(n)}(B) \quad \forall B \in \mathcal{B}_+(\mathcal{H})$$

.

2°. *$P_t^{min}(B)$ satisfies the integral Lindblad equation*

$$P_t^{min}(B) = W_t^* B W_t + \left[\!\left[\int_0^t ds W_{t-s}^* \Phi_*(P_s^{min}(B)) W_{t-s} \right]\!\right]. \tag{4.10}$$

3°. *$P_t(B) - P_t^{min}(B) \geq 0, \quad \forall B \geq 0$ for any other dynamical semigroup $P_t(B)$ which satisfies (4.10).*

4°. *If $P_t^{min}(\cdot)$ is conservative then $P_t^{min}(\cdot)$ is the only solution of equation (4.10) in the class of dynamical semigroups.*

Brakets $[\![\cdot]\!]$ in (4.10) are used to correspond a bounded operator to densely defined quadratic form

$$\int_0^t ds \Phi_*(P_s^{min}(B))[W_{t-s}\psi] \leq ||B||(||\phi||^2 - ||W_t\psi||^2) \qquad \forall \psi \in dom \Phi(I).$$

5 Necessary and sufficient conditions for conservativity of a dynamical semigroup

The integral equations (4.2) or (4.10) for a dynamical semigroup $P_t(\cdot)$ imply the corresponding integral form of the resolvent equation

$$X = A_\lambda(B) + Q_\lambda(X), \quad X = \int_0^\infty e^{-\lambda t} P_t(B)\, dt, \quad \lambda \in \mathbb{R}_+, \tag{5.1}$$

where

$$A_\lambda(B) = \int_0^\infty e^{-\lambda t} W_t^* B W_t\, dt, \qquad Q_\lambda(B) = \left[\!\left[\int_0^\infty e^{-\lambda t} W_t^* \Phi_*(B) W_t\, dt \right]\!\right].$$

Values of the infinitesimal map $\mathcal{L}$ on the range of the resolvent map $\mathcal{R}$ consists of bounded operators. Since the range $\mathcal{R}$ of the resolvent map does not depend on the parameter λ then the explicit equation for the range of the minimal resolvent map follows from (5.1) and from the definition of the minimal dynamical semigroup $P_t^{min}(\cdot)$:

$$\mathcal{R}^{min} = \left\{ X : X = \int_0^\infty e^{-t} P_t(B)\, dt = \sum_0^\infty Q^n(A(B)) \quad \forall B \in \mathcal{B}(\mathcal{H}) \right\}, \tag{5.2}$$

where $A(\cdot) = A_1(\cdot)$, $Q(\cdot) = Q_1(\cdot)$. Hence $I \in dom\, \mathcal{L}^{min}$ if and only if $I \in \mathcal{R}^{min}$. The conservativity of the minimal dynamical semigroup depends on this property.

Theorem 5.1.(see [C2], [C3]) *Under the assumptions of Theorem 4.1 the following are equivalent:*

1. $P_t^{min}(\cdot)$ *is conservative.*

2. $I \in \mathcal{R}^{min}$.

3. $Q^n(I) \to 0$ *strongly as* $n \to \infty$.

4. *Equation* $Q(B) = B$ *does not have solutions in* $\mathcal{B}_+(\mathcal{H})$.

5. *Equation* $\mathcal{L}(B)_* = B_*$ *does not have positive bounded solutions on dom* G^N.

Condition 3 is the most efficient. Consider an application of this condition to the regularity problem for the Azéma-Emery martingale. Now we are able to explain the origin of the series (2.8). As we observed it in Section 3 the infinitesimal operator generated by this process can be extended to the infinitesimal map $\mathcal{L}(\cdot)$ of the dynamical semigroup with coefficients

$$\Phi(B) = \frac{1}{\beta x} S_c^* B S_c \frac{1}{\beta x}, \quad H = \frac{i}{2}\left(\frac{1}{\beta x}\partial_x + \partial_x \frac{1}{\beta x}\partial_x\right).$$

Hence $G = \frac{1}{2}\Phi(I) + iH = \frac{1}{2x^2}\frac{1+\beta}{\beta^2} - \frac{1}{\beta x}\frac{\partial}{\partial x}$, and semigroups $W_t = exp(-Gt)$, $W_t^* = exp(-G^* t)$ can be constructed explicitly by method of characteristics as solutions for the first order linear PDEs:

$$(W_t f)(x) = \left(1 + \frac{2t}{\beta x^2}\right)^{-\frac{1+\beta}{4\beta}} f\left(x\sqrt{1 + \frac{2t}{\beta x^2}}\right) I_{\Gamma_t}(x)$$

$$(W_t^* f)(x) = \left(1 - \frac{2t}{\beta x^2}\right)^{\frac{1-\beta}{4\beta}} f\left(x\sqrt{1 - \frac{2t}{\beta x^2}}\right) I_{\Gamma_{-t}}(x),$$

where $\beta \in \mathbb{R}\backslash\{0\}$, $\Gamma_t = \left\{x \in \mathbb{R} : x^2 \geq -\frac{2t}{\beta}\right\}$ and $I_{\Gamma_t}(\cdot)$ is the indicator function of Γ_t.

In this case, the map Q leaves invariant the space $L_\infty(\mathbb{R}, dx)$ and, since $1 \in L_\infty(\mathbb{R}, dx)$ it is sufficient for us to know how does Q act on bounded functions.

A direct computation shows that, for $f \in L_\infty(\mathbb{R}, dx)$ $\quad (Qf)(x) =$

$$\int_0^\infty dt e^{-t}\left(1 - \frac{2t}{\beta x^2}\right)^{\frac{1-\beta}{4\beta}} \frac{f(cx(1-\frac{2t}{\beta x^2})^{\frac{1}{2}})}{\beta^2 x^2 (1 - \frac{2t}{\beta x^2})}\left(1 + \frac{2t}{\beta x^2(1-\frac{2t}{\beta x^2})}\right)^{-\frac{1+\beta}{4\beta}} I_{\Gamma_{-t}}(x) I_{\Gamma_t}\left(x(1 - \frac{2t}{\beta x^2})^{\frac{1}{2}}\right)$$

Note that $I_{\Gamma_t}(x\sqrt{1 - \frac{2t}{\beta x^2}}) \equiv 1$ and, when $\beta < 0$, we have $I_{\Gamma_{-t}}(x) \equiv 1$. In the case $\beta < 0$, with the change of variables $s = -\frac{2t}{\beta x^2}$ we obtain

$$(Qf)(x) = \int_0^\infty e^{\frac{\beta s x^2}{2}} f\left(cx\sqrt{1+s}\right) d(1+s)^{\frac{1}{2\beta}} = E e^{\frac{\beta \xi x^2}{2}} f(cx\sqrt{1+\xi}),$$

where ξ is a positive random variable with the following distribution

$$P(\xi \leq s) = 1 - (1+s)^{\frac{1}{2\beta}} \qquad (s \geq 0).$$

In the general case we have

$$Q^n(1) = \mathbb{M} \exp \frac{\beta x^2}{2} \left\{ \xi_n + \xi_{n-1}(1+\xi_n)c^2 + \cdots + \xi_1(1+\xi_2)\cdots(1+\xi_n)c^{2(n-1)} \right\}$$

It is convenient to replace the numbering of random variables to obtain the explicit expression for $Q^n(I)$:

$$Q^n(1) = \mathbb{M} \exp \frac{\beta x^2}{2} \left\{ \sum_{k=1}^{n} c^{2k} \xi_{k+1} \prod_{\ell=1}^{n} (1+\xi_\ell) \right\} \tag{5.3}$$

Now it follows from (5.3) that $Q^n(1) \longrightarrow 0$ as $n \to \infty$ if and only if

$$P \left\{ \sum_{k=1}^{\infty} c^{2k} \xi_{k+1} \prod_{\ell=1}^{k} (1+\xi_\ell) = \infty \right\} = 1.$$

This condition coincides with our previous requirement of divergence of (2.8). Now let us consider a constructive form of conditions sufficient for the conservativity.

6 Review of conditions sufficient for the conservativity of a minimal dynamical semigroup

As was noticed in [C2] for a given CP-map Q, there exists a positive selfadjoint operator $S \in \mathcal{C}(\mathcal{H})$, $S \geq I$ such that

$$Q(I) = (I + S^{-1})^{-1}, \tag{6.1}$$

which plays essential role in what follows. In particular $S = \kappa(\cdot)$=\{*multiplication by intensity of jumps*\} for the map Q corresponding to an extension of the Kolmogorov-Feller evolution equation, and $S = -(\nabla, A(\cdot)\nabla)$ for an extension of the symmetrical diffusion equation with diffusion coefficients $A(x) = \{A_{ij}(x)\}$.

For any contractive CP-map Q and for any sequence of positive bounded and invertable operators $X_1, \ldots, X_n$ such that operators $Q(I)^{-1}, Q(X_k)^{-1}$, $k \in \{1, \ldots, n\}$ exist, we have proved the inequality

$$Q((I + X_1^{-1} + \ldots + X_n^{-1})^{-1}) \leq (Q(I)^{-1} + Q(X_1)^{-1} + \ldots + Q(X_n)^{-1})^{-1} \tag{6.2}$$

(see [C2]).

Formal iterations of (6.1), (6.2) give: $\quad Q^n(I) =$

$$= Q^{n-1}((I+S^{-1})^{-1}) \leq Q^{n-2}((Q(I)^{-1} + Q(S)^{-1})^{-1}) = Q^{n-2}((I + S^{-1} + Q(S)^{-1})^{-1}) \leq$$

$$\leq Q^{n-3}((Q(I)^{-1} + Q(S)^{-1} + Q^2(S)^{-1})^{-1}) \leq \cdots \leq (I + S^{-1} + Q(S)^{-1} + \cdots + Q^n(S)^{-1})^{-1}.$$

Hence $Q^n(I) \to 0$, $n \to \infty$ in the strong sense if

$$(I + S^{-1} + Q(S)^{-1} + \cdots + Q^n(S)^{-1})^{-1} \to 0 \tag{6.3}$$

also converges strongly.

Regularized version of (6.3) is given by the following lemma.

Lemma 6.1.[C2,C3] *A quantum dynamical semigroup with the contractive CP-map $Q(\cdot)$ corresponding to the infinitesimal map $\mathcal{L}(\cdot)$ is conservative if*

$$\sup_n \inf_{\lambda\in(0,1]} \sum_1^n (\phi, Q_\lambda^n(S_\lambda)\phi)^{-1} = \infty \tag{6.4}$$

$\forall\phi$ from an arbitrary dense subset $D \in \mathcal{H}$, where $S_\lambda = S(1+\lambda S)^{-1} \in \mathcal{B}_+(\mathcal{H})$.

The operator S in inequality (6.4) can be replaced by the operator $\Phi(I)$ under the following restriction.

Lemma 6.2.[C2,C3] *Let*

$$i(H\phi, \Phi(I)^{-1}\phi) - i(\Phi(I)^{-1}\phi, H\phi) \geq -c||\phi||^2 \quad \forall\phi \in dom\, G^N \tag{6.5}$$

or $i[H, \Phi(I)^{-1}] \geq -cI$ on $dom\, G^N$. Then $S \leq (c+2)\Phi(I)$, and $S_\lambda \leq (c+2)\Phi_\lambda(I)$.

The last simplification follows from the assumption described in the next theorem.

Theorem 6.3. *Suppose that $\sup\limits_{\lambda\in(0,1]} \Phi_*(\Phi_\lambda(I))[\phi] < \infty, \quad \forall\phi \in dom\, G^N$. If there exists a constant $c \in \mathbb{R}$ such that for any $\lambda \in (0,1]$*

$$\mathcal{L}(\Phi(I))[\phi] = \sup_{\lambda\in(0,1]} \Phi_*(\Phi_\lambda(I))[\phi] - (G\phi, \Phi(I)\phi) - (\Phi(I)\phi), G\phi) \leq c\Phi_*(I)[\phi], \tag{6.6}$$

then $\Phi(I) - Q(\Phi(I)) \geq -(c-1)Q(I)$ and the series (6.4) diverges.

Under the assumptions of this theorem we have:

$$Q^n(\Phi_\lambda(I)) \leq Q(Q\ldots(Q(\Phi_{\lambda_1}(I)))_{\lambda_2}\ldots)_{\lambda_n}\, |_{\sum\lambda_k=\lambda} \quad \leq$$

$$\leq Q(Q\ldots(Q(\Phi_{\lambda_2}(I) + (c-1)I))\ldots)_{\lambda_n} \leq \ldots \leq \Phi_{\lambda_n}(I) + (c-1)nI.$$

Hence $(\phi, Q^n(S_\lambda)\phi) \leq (\phi, (\Phi(I) + (c-1)nI)\phi$ and $Q^n(I) = O(\log n)^{-1}$. Thus, we obtain the following result.

Theorem 6.4. *Let the assumptions (4.3)–(4.5), (4.7)–(4.9), (6.4), (6.6) be fulfilled. Then the minimal dynamical semigroup is conservative.*

Let an infinitesimal map $\mathcal{L}(\cdot)$ be the extension (3.3) of an infinitesimal operator L of a Markov semigroup. Then the Markov evolution is regular if the corresponding minimal dynamical semigroup is conservative.

7 Examples

(1) Let $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n)$, $\quad H = -\frac{i}{2}((b(x), \nabla) + (\nabla, (b(x)))$ and

$$\Phi(B) = \int_{\mathbb{R}^n} m(dq) C^*(\cdot, q) D_q^* B D_q C(\cdot, q),$$

where m is a σ-finite measure on $\mathbb{R}^n$ and $C(\cdot,\cdot) : \mathbb{R}^{2n} :\to \mathbb{C}$ is a measurable function such that $C(\cdot, q) \in \mathcal{L}_2(\mathbb{R}^n, m)$ and

$$\kappa(x) = \int m(dq)|C(\cdot, q)|^2$$

is a smooth locally bounded function. Suppose that there exists the family of trajectories $q_\tau : \dot{q}_\tau = b(q_\tau)$, $q_0 = y$, such that for each point $x \in \mathbb{R}^n$ there exist a finite or infinite return time from infinity $T_\infty(x)$ such that

$$\tilde{q}_0 = x, \ \dot{\tilde{q}}_\tau = -b(\tilde{q}_\tau), \lim_{t\to T_\infty(x)} |\tilde{q}_t| = \infty$$

for reversed trajectories $\tilde{q}_\tau$. Denote by $q_0(x,t)$ a point such that $q_t = x$ if $q_0 = q_0(x,t)$.

Suppose that for any compact set $K \subseteq \mathbb{R}^n$ there exists $t_0 = t_0(K) > 0$ sufficiently small such that the set $K_{t_0} = \cup_{t\le t_0}\{x : q_0(x,t) \in K\}$ is also compact, where $q_0(x,t)$ denotes a point such that $q_t = x$ if $q_0 = q_0(x,t)$. We assume, that the Jacobian

$$D(x,t) = det\left(\frac{\partial q_0(x,t)}{\partial x}\right)$$

is a smooth function on $K \times t_0(K)$ for each compact K.

Theorem 7.1. *Let there exist a constant $c \in \mathbb{R}_+$ such that*

(a) $\kappa(x) \le c\kappa(q_0(x,t)) \quad \forall t < T_\infty(x)$,

(b) $\sup_x \kappa(x)^{-2}(b(x),\nabla)\kappa(x) < \infty$,

(c) $\sup_x \left(\int \mu(x,dq)(\kappa(x+q)-\kappa(x)) + (b(x),\nabla)log(\kappa(x))\right) < \infty$,

(d) $|div\, b(x)| \le c(1+\kappa(x))|^m$ *for some* $m > 0$,

(e) $\int \mu(x,dq)\kappa(x+q) \le c(1+\kappa(x))^{2m-1}$,

where

$$\mu(x,B) = \frac{1}{\kappa(x)}\int_B m(dq)|C(x,q)|^2$$

is a measurable family of probability measures. Then the minimal dynamical semigroup is conservative.

Consider some particular cases.

1°. Let $b(x) \equiv 0$. Then $q_0(x,t) \equiv x$ and hence only one restriction from (a-e) survives:

$$\sup_x \int_{\mathbb{R}^n} \mu(x,dq)(\kappa(x+q)-\kappa(x)) < \infty.$$

2°. Let $n = 1$, $b(x) = -\frac{1}{\alpha}|x|^{\alpha+1}$. Then

$$q_0(x,t) = \frac{x}{(1+|x|^\alpha t)^{\frac{1}{\alpha}}}, \quad T_\infty(x) = |x|^{-\alpha},$$

$$D(x,t) = (1-|x|^\alpha t)^{-\frac{1+\alpha}{\alpha}}, \quad T_0(K) = (diam\, K)^{-\alpha}.$$

In this case the assumption (a) of Theorem 7.1 is fulfilled for $c = 1$. The assumption (b) is satisfied if the scalar product of the vector field $b(x)$ and the gradient of the intensity κ is not positive, i.e. the drift is opposite to the direction of larger intensities of jumps .

The assumption (d) will be fulfilled if $\kappa(x) = \kappa_0(1+|x|^\beta)$, $\beta \geq 0$, for all $m > \frac{\alpha}{\beta}$. The same time, assumption (c) is fulfilled if 1° takes place and drift is opposite to the direction of larger intensities $\kappa(x)$.

Let $\beta > 1$ and $\mu(x, B)$ is such that

$$\int \mu(x, dq) f(q) = f(x + a\, sign(x)),\ a > 0$$

i.e. the jumps push trajectories to infinity and the drift moves it back to the origin. Then assumption (e) is satisfied if

$$\sup_{x \geq 0} \left(\kappa_0 (x+a)^\beta - x^\beta) - \frac{1}{\alpha} \frac{x^{\alpha+\beta}}{1+x^\beta} \right) < \infty. \tag{7.1}$$

Now from the expansion

$$(x+a)^\beta = x^\beta (1 + \frac{\beta\alpha}{x} + O(x^{-2})) \qquad |x| > 1$$

we conclude that (7.1) is fulfilled and **for all** $\alpha > \beta - 1$, **and for any** κ_0, α **if** $\alpha = \beta - 1$, $a\alpha\kappa_0 \leq 1$ **the Markov process with the intensity of jumps** $\kappa(x) = \kappa_0(1 + |x|^\beta)$, $\beta \geq 0$ **and the drift velocity** $b(x) = -\frac{1}{\alpha}|x|^{\alpha+1}$ **is regular, i.e. the drift resists successfully the critical behaviour of one sided jumps of by** a.

(2) Let $\mathcal{H}$ be $\mathcal{L}_2(\mathbb{R}^n, dm)$ with an absolutely continuous measure m:

$$(\phi, \psi) = \int_{\mathbb{R}^n} \phi(x)^* \psi(x) b(x) dx, \quad m(b) = \int_B b(x) dx,$$

where $b(x)$ is a positive measurable function from $\mathcal{L}_1^{loc}(\mathbb{R}^n)$. So the adjoint operator $\partial^* = \partial_x^*$ has components

$$\partial_k^* = -\frac{1}{b(x)} \frac{\partial}{\partial x_k} b(x), \qquad k = 1, \dots, n$$

and each positive operator $B \in \mathcal{L}_2(\mathbb{R}^n)$ corresponds to the positive operator $b(x)^{-1}B \in \mathcal{L}_2(\mathbb{R}^n, dm)$.

Consider the CP-map $\Phi_*(\cdot)$ related to a Markov symmetrical diffusion process in $\mathbb{R}^n$ with diffusion matrix $A(x) = a^2(x)$:

$$\Phi_*(\cdot) = \sum_1^n (a_k(\cdot)\partial_k \psi(\cdot), B a_k(\cdot) \partial_k \psi(\cdot)), \tag{7.2}$$

where $\{a_k(x)\}_{k=1}^n$ are smooth positive functions and $a_k(\cdot)$ operators of multiplication by the function $a_k(x)$. We suppose that the functions a_k, b are locally bounded and nondegenerate:

$$\lambda^{-1} \leq a_k(x) \leq \lambda, \quad \lambda^{-1} \leq b(x) \leq \lambda, \quad \lambda = \lambda(D), \quad \forall x \in D,$$

for any compact set $D \subset \mathbb{R}^n$.

It is known (see [Fu], [I]) that in this case

- *The quadratic form $\Phi_*(I)$ is closable in $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n, dm)$;*

– *$C_0^\infty(\mathbb{R}^n) = ess\, dom\, \Phi_*(I)$;*

– *There exists a Markov jump process $\{\xi_t, T_\infty, P_x\}$ with a random escape to infinity time $T_\infty(\omega)$ such that*

$$\lim_{t\to T_\infty(\omega)} |\xi_t(\omega)| = \infty,$$

$$\lim_{t\to 0} \frac{1}{t}\left(\int_{\mathbb{R}^n} \mathbf{P}_x\{\xi_t(\omega)\in dy, t < T_\infty(\omega)\} f(y) - f(x)\right) = Lf(x) \quad \forall f \in C_0^\infty(\mathbb{R}^n),$$

where L is the second order negative partial derivative operator

$$L = -\frac{1}{2}[\![\Phi_*(I)]\!] = \frac{1}{2b(x)}\sum_1^n \frac{\partial}{\partial x_k} b(x) a_k^2(x)\frac{\partial}{\partial x_k}$$

corresponding to the minimal extension (the Friedrichs extension) of the quadratic form $\Phi_*(I)$.

As was proved in [Kh], [I], the necessary and sufficient condition for the regular behaviour of the Markov diffusion process (i.e., $\mathbf{P}\{T_\infty(\omega) = \infty\} = 1$) is equivalent to nonexistence of a bounded positive solutions for the spectral problem

$$LP(x) = P(x).$$

If the regularity is violated then the positive bounded function

$$P(x) = \mathbb{M}_x e^{-T_\infty(\omega)}$$

is such solution.

Denote by $\widehat{P}$ the multiplication by $P(\cdot)$ operator. Note that $LP(\cdot) = \mathcal{L}(\widehat{P})$. Hence $\mathcal{L}(\widehat{P}) = \widehat{P}$ if $LP(x) = P(x)$. This means a violation of the condition 5 (theorem 5.1) which is necessary for the conservativity of the minimal dynamical semigroup. If the equation $\mathcal{L}(\widehat{P}) = \widehat{P}$ does not have any positive bounded solutions then the equation $LP(x) = P(x)$ also has no such solutions, and the Markov diffusion process is regular. Consider conditions of the theorem 6.4 for this particular case:

$$\mathcal{L}(B) = \Phi(B) - \frac{1}{2}\left(\Phi(I)B + B\Phi(I)\right).$$

As was pointed out above, $C_0^\infty(\mathbb{R}^n) = ess\, dom\, \Phi_*(I)$ and hence

$$C_0^\infty(\mathbb{R}^n) = ess\, dom\, \Phi(I)^2 = ess\, dom\, \Phi(\Phi(I)).$$

The difference $\Phi(I)^2 - \Phi(\Phi(I)) = -\mathcal{L}(\Phi(I))$ is the second-order symmetric differential operator:

$$\Phi(I)^2 - \Phi(\Phi(I)) = -\mathcal{L}(\Phi(I)) = \frac{1}{b(x)}\sum_{j,k=1}^n \frac{\partial}{\partial x_j} b(x) a_j(x) A_{j,k}(x) a_k(x)\frac{\partial}{\partial x_k},$$

where $A(x)$ is the matrix with coefficients

$$A_{j,j}(x) = a_j(x)\frac{\partial^2}{\partial x_j^2}\left(a_j(x)\right) + a_j(x)\frac{\partial}{\partial x_j}\left(a_j(x)\frac{\partial}{\partial x_j}\ln b(x)\right) +$$

$$+a_j^2(x)\frac{\partial^2}{\partial x_j^2}(\ln b(x)) - \sum_{k\neq j}\frac{1}{a_j(x)b(x)}\frac{\partial}{\partial x_k}\left(a_k^2(x)b(x)\frac{\partial}{\partial x_k}a_j(x)\right),$$

$$A_{k,j}(x) = \frac{a_k(x)}{a_j(x)}\frac{\partial}{\partial x_k}\left(\frac{1}{b(x)}\frac{\partial}{\partial x_j}a_j^2(x)b(x)\right) \quad (k \neq j). \tag{7.3}$$

Since $H \equiv 0$, our Theorem 6.4 gives the following condition, which is sufficient for the conservativity of the minimal dynamical semigroup:

$$\mathcal{L}(\Phi(I)) = -\frac{1}{b(x)}\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}b(x)a_j(x)A_{j,k}(x)a_k(x)\frac{\partial}{\partial x_k} \leq$$

$$\leq c\Phi(I)) = -\frac{c}{b(x)}\sum_1^n\frac{\partial}{\partial x_k}b(x)a_k^2(x)\frac{\partial}{\partial x_k}. \tag{7.4}$$

The operator $\sum\frac{\partial^2}{\partial x_k^2}$ is negative. Hence, (7.4) implies

Theorem 7.2. *The symmetrical diffusion with the formal generator*

$$L = \frac{1}{2b(x)}\sum_1^n\frac{\partial}{\partial x_k}b(x)a_k^2(x)\frac{\partial}{\partial x_k}$$

is regular and the corresponding minimal dynamical semigroup is conservative if

$$\sup_{x\in\mathbb{R}^n} A(x) \leq cI \tag{7.5}$$

for some $c \in \mathbb{R}$, where $A(x)$ is matrix with coefficients $A_{k,j}(x)$ given by equations (7.3).

This condition seems to be unknown. In the spherically symmetrical case it can be compared with the results of Khas'minskii [Kh] and McKean [MK]. They proved that the symmetrical diffusion in $\mathbb{R}^n$ with the diffusion coefficient $a^2(r)$ is regular if and only if the following integrals diverge:

$$\int_0^\infty dr\int_r^\infty\frac{dr}{a^2(r)} = \infty \quad (n=1), \qquad \int_0^\infty\frac{rdr}{a^2(r)} = \infty \quad (n>2). \tag{7.6}$$

It follows from this,that up to multipliers of order $O(\ln(\ln r)^{1+\epsilon})$ the upper bound on the rate of increase of the coefficient $a^2(x)$ gives function $a^2(x) = r^2\ln(2+r)$. It is known that the symmetrical diffusion process with coefficient $a^2(x) = r^2\ln^{1+\epsilon}(2+r)$ explodes with a positive probability.

In order to apply (7.5) to this problem we put

$$a_1(r) = a(r), \quad \partial_1 = \frac{\partial}{\partial r}, \quad a_j(r) = 0,\, i>1, \quad b(r) = r^{n-1}.$$

Then $A_{i,j}(r) = 0$ for all $i, j > 1$ and

$$A_{11} = a(r)a(r)'' + \frac{n-1}{r}a(r)a(r)' - \frac{n-1}{r^2}a^2(r).$$

Hence **a symmetrical diffusion in $\mathbb{R}^n$ with a smooth locally bounded diffusion coefficient $a(r)^2$ is regular if** $\sup_{r>1}\left(a(r)a(r)'' + \frac{n-1}{r}a(r)a(r)' - \frac{n-1}{r^2}a^2(r)\right) < \infty$.

For example, if $a(r) = r\ln^{\frac{1}{2}} r$ then $A_{11}(r) = \frac{n}{2} - (4\ln r)^{-1} < \infty$ as $r \to \infty$. This gives the regularity of the symmetrical diffusion with the smooth diffusion coefficient $a(r)$ such that $a(r) \le c(1+r)\ln^{\frac{1}{2}}(2+r)$. On the other hand, if $a(r) = r\ln^{\frac{1}{2}+\epsilon} r$, then $A_{11}(r) = O(\ln^{2\epsilon} r) \to \infty$ as $r \to \infty$. Hence, up to multipliers of the order $O(\ln^{\epsilon})$ the condition (7.5) gives the same result as (7.6).

8 Bibliography

[C1] Chebotarev A.M. Conditions sufficient for regularity of the Markov jump processes. Probability theory and its applications, 1988, V.33, N.1, P.21-35.

[C2] Chebotarev A.M. Necessary and sufficient conditions for conservativeness of dynamical semigroups. Journal of Soviet Mathematics, 1991, V.56, P.2697-2719.

[C3] Chebotarev A.M. Sufficient conditions of the conservatism of a minimal dynamical semigroup. Mathematical Notes, 1993, V.52, N3-4, P. 1067-1077.

[CF1] Chebotarev A.M., Fagnola F. Sufficient conditions for conservativity of quantum dynamical semigroup. Journal of Functional Analyse, vol. 113, N 1, 1993, p. 131 - 153.

[CF2] Chebotarev A.M., Fagnola F. On a remarkable quantum dynamical semigroup (to appear in *Sém. Prob.*, 1994).

[E1] Emery M. On the Azéma martingales. *Sém. Prob.* **XXIII**, LNM 1372, 1989, P.67–87.

[E2] Emery M. Sur les martingales d'Azéma. *Sém. Prob.* **XXIV** (1990).

[I] Ichihara K. Explosion problems for symmetric diffusion processes. Lecture Notes in Math. Springer-Verlag, 1986, V. 1203, P.75-89.

[GS] Gikhman I.I., Skorokhod A.V. Introduction to the theory of random processes. Scripta Technica, W.B.Saunders Co., Philadelphia, 1969.

[F] Feller W. An introduction to probability theory and its applications, v.1-2, John Wiley & Sons, 1977.

[Fu] Fukushima M. Dirichlet forms and Markov processes. Kodanasha, Tokyo, 1980.

[K] Kraus K. Annals of Physics, New-York, 1971, V.64, p.311-331

[Kh] Khas'minskii R.Z. Ergodic problems of recurrent diffusion processes and stabilization of solutions of the Cauchy problem for a parabolic equation. Probability theory and its applications, 1960, V.5, N.1, P.196-214.

[L] Lindblad G. On the generators of quantum dynamical semigroups. Commun. Math. Phys., 1976, V.48, N2, P. 119-130.

[MK] McKean H.P. (Jr.) Stochastic Integrals. Academic Press, N.-Y., 1969.

[P] Parthasarathy K.R. Azéma martingales and quantum stochastic calculus. In: Proc. R.C.Bose Symp. (ed. R.R. Bahadur) Wiley Eastern, New Delhi, 1990, P.551-569.

[S] Stinespring W.F. Positive functions on C^* –algebras. Proc.of Amer. Math. Soc., 1955, V.6, N2, P. 211-218.