Equilibrium States for Lattice Models of Hyperbolic Type

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Abstract

We study the structural stability of coupled map lattice models of hyperbolic type under certain metrics. We prove the existence of equilibrium states for continuous functions on lattice models under the conditions of weak interaction and translation invariance. We also study the uniqueness and ergodic properties of these equilibrium states for Hölder continuous functions.

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1. INTRODUCTION

Let M be a compact Riemannian manifold and $U \subset M$ an open set. Let $f : U \to M$ be a $C^{1+\alpha}$-diffeomorphism and let $\Lambda \subset U$ be a closed invariant *hyperbolic set* for f. This means that the tangent bundle TM has a splitting (*exponential splitting*) into subbundles over Λ: $TM = E^s \bigoplus E^u$, where E^s, E^u are both invariant under the differential Df, and for some $0 < \lambda < 1$,

$$\|Df^n v\| \leq C\lambda^n \|v\| \text{ for } n \geq 0, v \in E^s;$$

$$\|Df^{-n} w\| \leq C\lambda^n \|w\| \text{ for } n \geq 0, w \in E^u.$$

The hyperbolic set Λ is called *locally maximal* if there exists an open set $O \supset \Lambda$ such that $\Lambda = \cap_{n\in \mathbb{Z}} f^n(\bar{O})$. The dynamics on any compact locally maximal hyperbolic set is semiconjugate to a *subshift of finite type* and there exist invariant measures (*equilibrium states*) that have abundance of ergodic properties. Each of these measures is uniquely defined by a *Hölder continuous* function on Λ.

Starting from the diffeomorphism f, one can consider an space extended infinite dimensional dynamical system $F = \otimes_{i\in \mathbb{Z}} f$ and consider a small perturbation Φ of F on $\mathcal{M} = \otimes_{i\in \mathbb{Z}} M$. It is called a **lattice model**. The iteration of the map Φ generates a group of time translations. An important class of lattice models consists of those that also admit a group of *space translations* (see the definition below). Such models were considered in [BS] in the case when f is a one-dimensional expanding map, and in [PS] in the case when f possesses a *hyperbolic attractor* (i.e., there exists an open neighborhood $U \supset \Lambda$ such that $\overline{f(U)} \subset U$ and $\Lambda = \cap_{n \geq 0} f^n(U)$). In both cases, the authors constructed analogues of *SBR measures* that are equilibrium states for certain functions. In this paper, we consider the dynamics of lattice models in a more general situation when the dynamical system f possesses a hyperbolic set. We prove structural stability of lattice dynamical system F under different metrics and existence of equilibrium states for continuous functions. We also study the uniqueness and the ergodic properties of these measures with respect to both space and time translations for Hölder continuous functions. The above problems were also considered in [GR1], [GR2], [GR3], and [CR]. However, many statements in these papers are incorrect, the proofs presented in these papers have many gaps. The main result (Theorem 1) of the second paper in the series that claims the uniqueness of Gibbs states for Hölder continuous functions is actually wrong. A class of counterexamples is provided by the Ising model in statistical physics (see [MM]).

2. DEFINITION OF LATTICE MODELS

We begin with the description of lattice models. Define

$$\mathcal{M} = \otimes_{i \in \mathbb{Z}} M_i = \{\bar{x} \mid \bar{x} = (x_i), x_i \in M\},$$

where M_i is a copy of M.

We endow $\mathcal{M}$ with a *Finsler metric* so that $\mathcal{M}$ becomes a differentiable Banach manifold modeled on $(\mathbb{R}^n)^{\mathbb{Z}}$. Let $\bar{x} = (x_i) \in \mathcal{M}$ and (V_i, ϕ_i) be charts around $x_i \in M$ for each i. Then, a chart containing $\bar{x}$ will be: $(\otimes_{i \in \mathbb{Z}} V_i, \otimes_{i \in \mathbb{Z}} \phi_i)$. Let $\xi_i \in T_{x_i} M_i$. Then $\bar{\xi} = (\xi_i) \in T_{\bar{x}} \mathcal{M}$ and

$$\|\bar{\xi}\| = \sup_{i \in \mathbb{Z}} \|\xi_i\|.$$

Let $\bar{x}, \bar{y}$ be two points in $\mathcal{M}$. The distance on $\mathcal{M}$ induced by the Finsler metric is:

$$\rho(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}} d(x_i, y_i),$$

where d is the Riemannian distance on M.

The definition of hyperbolicity can be easily extended to Banach manifolds. Suppose that G is a C^1-diffeomorphism from an open set of a Banach manifold $\mathcal{N}$ with a Finsler metric and Δ_G is an invariant set under G. We say that Δ_G

is a (uniformly) hyperbolic set if the tangent bundle $T\mathcal{N}$ over Δ_G has a splitting: $T\mathcal{N} = E^s \oplus E^u$ with the following properties:

1) E^s, E^u *are invariant under* DG*;*
2) *for any continuous sections* v *valued of* E^s *and* w *of* E^u

$$\|DG^n(v)\| \leq C\lambda^n\|v\| \quad \text{and} \quad \|DG^{-n}(w)\| \leq C\lambda^n\|w\|$$

for some constants $C > 0$ *and* $0 < \lambda < 1$ *independent of* v *and* w*;*

3) *there exists* $b > 0$ *such that for any* $\bar{x} = (x_i)$*,*

$$\inf\{\|\xi - \eta\| : \xi \in E^s(\bar{x}), \|\xi\| = 1; \eta \in E^u(\bar{x}), \|\eta\| = 1\} \geq b.$$

Define $F = \otimes_{i\in\mathbb{Z}} f$. It is a diffeomorphism of $\mathcal{M}$ of class $C^{1+\alpha}$. The closed set $\Delta = \otimes_{i\in\mathbb{Z}}\Lambda_i$, $\Lambda_i = \Lambda$, is invariant under F and hyperbolic. For each point $\bar{x} = (x_i) \in \Delta$, the tangent space $T_{\bar{x}}\mathcal{M}$ admits a splitting: $T_{\bar{x}}\mathcal{M} = E^s(\bar{x}) \oplus E^u(\bar{x})$, where $E^s(\bar{x}) = \otimes_{i\in\mathbb{Z}} E^s(x_i)$ and $E^u(\bar{x}) = \otimes_{i\in\mathbb{Z}} E^u(x_i)$. The subbundles $E^s(\bar{x}), E^u(\bar{x})$ are invariant under the differential $DF = \otimes_{i\in\mathbb{Z}} Df$, and for any $\bar{v} = (v_i) \in E^s(\bar{x})$ and $\bar{w} = (w_i) \in E^u(\bar{x})$

$$\|DF^n(\bar{v})\| = \sup_{i\in\mathbb{Z}} \|Df^n(v_i)\| \leq C\lambda^n \sup_{i\in\mathbb{Z}} \|v_i\| = C\lambda^n\|\bar{v}\|$$

and

$$\|DF^{-n}(\bar{w})\| = \sup_{i\in\mathbb{Z}} \|Df^{-n}(w_i)\| \leq C\lambda^n \sup_{i\in\mathbb{Z}} \|w_i\| = C\lambda^n\|\bar{w}\|,$$

where C is a constant and $0 < \lambda < 1$. Without loss of generality, by switching to another equivalent Riemannian metric and replacing λ by a big constant $\lambda_1 < 1$ we may assume $C = 1$.

For each point $\bar{x} = (x_i)_{i\in\mathbb{Z}} \in \Delta$, let $V_i^s(x_i)$ and $V_i^u(x_i)$ be the local stable and unstable manifolds respectively at x_i. The local stable and unstable manifolds passing through $\bar{x}$ are

$$V^s(\bar{x}) = \otimes_{i\in\mathbb{Z}} V_i^s(x_i), \qquad V^u(\bar{x}) = \otimes_{i\in\mathbb{Z}} V_i^u(x_i).$$

One can obtain the global stable and unstable manifolds in a similar way.

The distance between two mappings on $\mathcal{M}$ is defined as usual: If G and G' are two C^1 diffeomorphism of $\mathcal{M}$, then

$$\mathrm{dist}_{C^0}(G, G') = \sup_{\bar{x}\in\mathcal{M}} \rho(G(\bar{x}), G'(\bar{x}));$$

$$\mathrm{dist}_{C^1}(G, G') = \sup_{\bar{x}\in\mathcal{M}} \rho(G(\bar{x})), G'(\bar{x})) + \sup_{\bar{x}\in\mathcal{M}} \|DG_{\bar{x}} - DG'_{\bar{x}}\|.$$

Let $G : \mathcal{M} \to \mathcal{M}$ be a map C^1-close to the identity *id*. We define the perturbation $\Phi = F \cdot G$. Let S be the left shift (*space translation*) on $\mathcal{M}$ defined by $(S(\bar{x}))_i = x_{i+1}$. The pair (Φ, S) is called a **lattice model**.

3. MAIN RESULTS

3.1 Structural Stability We first consider the problem of structural stability of the infinite dimensional hyperbolic system $(\mathcal{M}, F)$.

Theorem 1 *1) There exist a neighborhood $U(\Delta) \supset \Delta$ and $\delta_0 > 0$ such that if $\mathrm{dist}_{C^1}(\Phi, F) \le \delta_0$, then any subset $\Delta_\Phi \subset U(\Delta)$ invariant under Φ is uniformly hyperbolic.*

2) For any $\epsilon > 0$, there exists $0 < \delta < \delta_0$ such that if $\mathrm{dist}_{C^1}(\Phi, F) \le \delta$, then there is a unique homeomorphism $h : \Delta \to \mathcal{M}$ satisfying $\Phi \cdot h = h \cdot F$ with $\mathrm{dist}_{C^0}(h, id) \le \epsilon$.

This theorem is well-known for finite dimensional dynamical systems. It can also be stated for general hyperbolic systems on Banach manifolds that admit the partition of unity. The dynamical system $(\mathcal{M}, F)$ we consider does not admit such a property and the general result can not be applied in a direct way. However this system is a direct product of countable many copies of the same compact Riemannian manifold M. This special structure enables us to present the proof of the theorem that is modeled on the proof in the finite dimensional case in [KH] and [Sh].

In the following theorem we study the topological properties of map Φ on the hyperbolic set Δ_Φ.

Theorem 2 *Assume that $f|_\Lambda$ is topologically mixing and Λ is locally maximal. Then $\Phi|_{\Delta_\Phi}$ is topologically mixing and Δ_Φ is locally maximal.*

3.2 Short Range Mappings The main goal of this article is to investigate the metric properties of the above infinite dimensional hyperbolic systems. In the finite dimensional case, one uses the *thermodynamic formalism* [Bo] [Ru] to study the ergodicity of maps on hyperbolic sets. The extension of this formalism to infinite dimensional case is tricky since the hyperbolic set Δ is not compact. We overcome this problem by introducing the following family of metrics on the manifold $\mathcal{M}$.

We define the metric ρ_q, $0 < q < 1$, as follows: for $\bar{x}, \bar{y} \in \mathcal{M}$

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i\in\mathbb{Z}} q^{|i|} d(x_i, y_i).$$

For different $0 < q < 1$, the metrics ρ_q induce the same compact topology on $\mathcal{M}$. Thus in this new topology the set Δ is compact. However, the set Δ is no longer hyperbolic in the usual sense with respect to the coarser metric ρ_q. The set $\mathcal{M}$ is not a differential manifold and the maps F and Φ, while being continuous, need not be differentiable.

We will restrict our perturbations to a smaller class of maps that are called **short range mappings**. This concept arises naturally from statistical mechanics. The short range mappings were first considered by Bunimovich and Sinai in the one-dimensional case [BS] (see also [KF]), and were later investigated by Pesin and Sinai in a more general situation [PS] (see also [GR1] [CR]). Our definition of a short range mapping below follows that in [PS].

Let Y be a subset of $\mathcal{M}$.

Definition *1) A mapping $G : Y \to \mathcal{M}$ is called* **a short range mapping** *if G has the form $G = (\cdots, G_{-n}, \cdots, G_{-1}, G_0, G_1, \cdots, G_n, \cdots)$, where $G_i, Y \to M_i, i \in \mathbb{Z}$, satisfies the following condition: for any fixed $k \in \mathbb{Z}$ and any points $\bar{x} = (x_j), \bar{y} = (y_j) \in Y$ with $x_j = y_j$ for all $j \in \mathbb{Z}, j \neq k$, we have*

$$d(G_i(\bar{x}), G_i(\bar{y})) \leq C\theta^{|i-k|} d(x_k, y_k),$$

where C and θ are constants and $0 < \theta < 1$. We shall call θ the **decay constant** *of G.*

2) A mapping G is called **shift invariant** *if $G \cdot S = S \cdot G$.*
Let $G = (\cdots, G_{-n}, \cdots, G_{-1}, G_0, G_1, \cdots, G_n, \cdots)$. One can easily see that G is shift invariant if and only if $G_i = G_0 \cdot S^i, i \in \mathbb{Z}$, where S is the left shift on $\mathcal{M}$. If G is shift invariant, then G is short ranged with a decay constant θ if and only if

$$d(G_0(\bar{x}), G_0(\bar{y})) \leq C\theta^{|k|} d(x_k, y_k)$$

for $\bar{x} = (x_j), \bar{y} = (y_i) \in Y$ with $x_j = y_j$ for all $j \in \mathbb{Z}, j \neq k$.

We formulate some basic properties of short range mappings.

Lemma 1 1) Let $G : \mathcal{M} \to \mathcal{M}$ be a short range mapping with a decay constant θ. Then it is Lipschitz continuous as a map from $(\mathcal{M}, \rho_q)$ to itself for any $q > \theta$.

2) If G is a Lipschitz continuous map from $(\mathcal{M}, \rho_q)$ to $(\mathcal{M}, \rho_{q_1})$, with some $0 < q_1 < 1$, then it is short ranged with the decay constant θ equal to q.

Lemma 2 Let G be a C^1-diffeomorphism from an open set $\mathcal{U} \subset \mathcal{M}$ onto its image. Then, the following statements are equivalent.

1) The mapping G is short ranged with a decay constant θ.

2) The differential of G at $\bar{x}$, $D_{\bar{x}}G : T_{\bar{x}}\mathcal{M} \to T_{\bar{x}}\mathcal{M}$ is a short range linear mapping with the same decay constant θ which is independent of $\bar{x}$.

3) The bundle map DG is short ranged with the same decay constant θ.

Lemma 3 For any $0 < \theta < 1$, there exists $\epsilon > 0$ such that if $G : \mathcal{M} \to \mathcal{M}$ is a short range $C^{1+\alpha}$-diffeomorphism with the decay constant θ and $\mathrm{dist}_{C^1}(G, id) < \epsilon$, then G^{-1} is also a short range mapping.

Lemma 4 For any $\epsilon > 0$ and $0 < \theta < q < 1$ there exist δ such that if G is a $C^{1+\alpha}$ shift invariant short range mapping on $\mathcal{M}$ with the decay constant θ and $\mathrm{dist}_{C^1}(G, id) \leq \delta$, then G is Lipschitz continuous on $\mathcal{M}$ in the metric ρ_q with a Lipschitz constant $L \leq 1+\epsilon$.

Note that F^{-1}, the inverse of the unperturbed map F is Lipschitz continuous in the metric ρ_q on the local unstable manifold $\otimes V^u(\bar{x})$ with a Lipschitz constant $L = \lambda < 1$. As an easy corollary of the previous lemma, one can see that the inverse of the perturbed map $\Phi = F \cdot G$ Φ^{-1} is also Lipschitz continuous in the metric ρ_q with a Lipschitz constant $L = (1+\epsilon)\lambda$ on $\otimes V^u(\bar{x})$. We will fix an ϵ so that $(1+\epsilon)\lambda < 1$.

In Theorem 1, we proved that $(\Delta, \mathcal{F})$ and (Δ_Φ, Φ) are conjugate hyperbolic systems. One can prove that the conjugacy h is Hölder continuous. However, we need another property of h that is related to the metric ρ_q. The following theorems, which use the above lemmas, show that when the perturbation is short ranged, the conjugacy h is continuous in the metric ρ_q, and that if G is C^2, the conjugacy map h is actually Hölder continuous in the metric ρ_q.

Theorem 3 Let $\Phi = F \cdot G$ be the perturbation of F. If the map G is short ranged and sufficiently C^1-close to the identity, then the conjugacy h is also continuous in the metric ρ_q, hence, it is a homeomorphism between Δ and Δ_Φ. Moreover, if G is shift invariant, so is h.

Theorem 4 For any $0 < \theta < 1$ there exists $\delta > 0$ such that if G is a C^2-shift invariant short ranged mapping with the decay constant θ and $\mathrm{dist}_{C^1}(G, id) \leq \delta$, then the conjugacy map h is Hölder continuous in the metric ρ_q, $0 < q < 1$.

3.3 Existence of Equilibrium States Let Ω be a compact metric space, τ a $\mathbb{Z}^2-$action on Ω induced by a pair of commutative homeomorphisms. Let $\mathcal{U} = \{U_i\}, \mathcal{B} = \{B_i\}$ be covers of Ω. The cover $\mathcal{U} \vee \mathcal{B}$ consists of the sets $U_i \cap B_j$. For a finite set $X \subset \mathbb{Z}^2$, define

$$\mathcal{U}^X = \vee_{x \in X} \tau^{-x} \mathcal{U}.$$

Definitions [Ru] *1. A $\mathbb{Z}^2-$action τ is said to be* **expansive** *if there exists $\epsilon > 0$ such that for any $\xi, \eta \in \Omega$,*

$$d(\tau^x \xi, \tau^x \eta) \leq \epsilon \text{ for all } x \in \mathbb{Z}^2 \text{ implies } \xi = \eta$$

for any $\xi, \eta \in \Omega$.

*2. A Borel measure μ on Ω is said to be τ-***invariant** *if μ is invariant under both homeomorphisms. We denote the set of all τ-invariant measure on Ω by $I(\Omega)$.*

3. Let $\mu \in I(\Omega)$ and $\mathcal{U} = \{U_i\}$ be any finite Borel partition of Ω . Define

$$H(\mu, \mathcal{U}) = -\sum_i \mu(U_i) \log \mu(U_i)$$

and

$$h_\tau(\mu,\mathcal{U}) = \lim_{(n,m)\to\infty} \frac{1}{nm} H(\mu,\mathcal{U}^{X_{nm}}) = \inf_{n,m} \frac{1}{nm} H(\mu,\mathcal{U}^{X_{nm}}),$$

where $X_{nm} = \{(i,j) \in \mathbb{Z} \mid |i| \le n, |j| \le m\}$. *The* **entropy** *of* μ *is defined to be*

$$h_\tau(\mu) = \sup_{\mathcal{U}} h_\tau(\mu,\mathcal{U}) = \lim_{diam\mathcal{U}\to 0} h_\tau(\mu,\mathcal{U}).$$

4. Let $\mathcal{U}$ *be any finite open cover of* Ω, φ *a continuous function on* Ω, *and* X *a finite subset of* $\mathbb{Z}^2$. *Define*

$$Z_X(\varphi,\mathcal{U}) = \min_{\{B_j\}} \Big\{ \sum_j \exp\Big[\inf_{\xi\in B_j} \sum_{x\in X} \varphi(\tau^x\xi) \Big] \Big\},$$

where the minimum is taken over all subcovers $\{B_j\}$ *of* $\mathcal{U}^X$. *Let*

$$P_\tau(\varphi,\mathcal{U}) = \mathop{lim\,sup}_{(n,m)\to\infty} \frac{1}{nm} \log Z_{X_{nm}}(\varphi,\mathcal{U}).$$

Then

$$P_\tau(\varphi) = \lim_{diam\mathcal{U}\to 0} P_\tau(\varphi,\mathcal{U}) = \sup_{\mathcal{U}} P_\tau(\varphi,\mathcal{U})$$

is called the **topological pressure** *of* φ.

5. A measure $\mu \in I(\Omega)$ *is called an* **equilibrium state** *for* φ *on* Ω *with respect to a* $\mathbb{Z}^2$ *action* τ *if*

$$P_\tau(\varphi) = h_\tau(\mu) + \int \varphi d\mu.$$

Proposition 1 [Ru] *If* τ *is an expansive* $\mathbb{Z}^2$*-action on a compact metric space, then there exists an equilibrium state for any continuous function* φ.

The existence of equilibrium states on Δ_Φ for continuous functions φ follows directly from the above proposition and Theorems 3 since the $\mathbb{Z}^2-$action induced by (Φ, S) is expansive on (Δ_Φ, ρ_q).

Theorem 5 *Let* τ *be the* $\mathbb{Z}^2-$*action on* Δ_Φ *induced by* Φ *and* S, *where* $\Phi = F\cdot G$ *and* G *is short ranged, shift invariant and sufficiently* C^1-*close to identity. Then for any* $0 < q < 1$ *and any continuous function* φ *on* (Δ_Φ, ρ_q) *there exists an equilibrium state* μ_φ *for* φ *with respect to* τ. *The measure* μ_φ *does not depend on* q.

3.4 Uniqueness of Equilibrium States. Following the general scheme that was used in the finit-dimensional case, we study the uniqueness of equilibrium states for Hölder continuous functions using symbolic dynamics. First we construct a **Markov partition** for Φ on Δ_Φ. Since (Δ_Φ, Φ) is conjugate to (Δ, F), we construct a Markov partition on Δ and then transform it to a Markov partition on Δ_Φ using the conjugating map h.

We shall assume that f is topologically mixing and Λ is locally maximal. For any $\epsilon > 0$, there exists a Markov partition of "size" ϵ of Λ for f: $\Lambda = \cup_{i=1}^m R_i$. This means that

1) each set R_i is a "*rectangle*", i.e., for any $x, y \in R_i$, $V^s(x) \cap V^u(y) \in R_i$, $diam R_i < \epsilon$, R_i is the closure of its interior, i.e., $R_i = \overline{(int R_i)}$.

2) $R_i \cap R_j = \partial R_i \cap \partial R_j$, where ∂R_i denotes the boundary of R_i.

3) if $x \in R_i$ and $f(x) \in int R_j$, then $f(V^s(x, R_i)) \subset V^s(f(x), R_j)$; if $x \in R_i$ and $f^{-1}(x) \in int R_j$, then $f^{-1}(V^u(x, R_i)) \subset V^u(f(x), R_j)$, where $V^s(x, R_i) = V^s(x) \cap R_i$ and $V^u(x, R_i) = V^u(x) \cap R_i$.

The transfer matrix $A = (a_{ij})_{1 \le i,j \le m}$ associated with the Markov partition is defined by

$$a_{ij} = 1 \text{ if } f(int R_i) \cap int R_j \neq \emptyset;$$

$$a_{ij} = 0 \text{ if } f(int R_i) \cap int R_j = \emptyset.$$

Let (Σ_A, σ) be the associated subshift of finite type, where σ is the left shift on Σ_A. For each $\xi \in \Sigma_A$, the set $\cap_n f^{-n}(R_{\xi(n)})$ contains a single point and the map $\pi : \Sigma_A \to \Lambda$ defined by $\pi \xi = \cap_n f^{-n}(R_{\xi(n)})$ is a semiconjugacy between f and σ on . Denote $m^{\mathbb{Z}} = \otimes_{i \in \mathbb{Z}} \{1, 2, \cdots, m\}$, $R_{(n_i)} = \{\bar{x} : \bar{x} = (x_i), x_i \in R_{n_i}, n_i \in \{1, 2, \cdots, m\}\}$. Then $\Delta = \cup_{(n_i) \in m^{\mathbb{Z}}} R_{\xi(n)}$ and the collection of subsets of Δ, $\{R_{(n_i)} : (n_i) \in m^{\mathbb{Z}}\}$, has the following properties:

1) each $R_{(n_i)}$ is a closed set in both the Finsler topology and the product topology and for any $\bar{x}, \bar{y} \in \Delta$, $V^s(\bar{x}) \cap V^u(\bar{y}) \in R_{(n_i)}$.

2) $F(V^s(\bar{x}, R_{(n_i)})) \subset V^s(F(\bar{x}), R_{(m_i)})$ and $F^{-1}(V^u(F(\bar{x}), R_{(n_i)}))$
$\subset V^u(\bar{x}, R_{(m_i)})$, where $V^{s,u}(\bar{x}, R_{(n_i)}) = V^{s,u}(\bar{x}) \cap R_i$.

3) $S(V^{s,u}(\bar{x}, R_{(n_i)}) = V^{s,u}(S\bar{x}, R_{S(n_i)})$.

We call such a collection a **Markov partition** of Δ for F.

Let $\Sigma_A^{\mathbb{Z}} = \otimes_{i \in \mathbb{Z}} \Sigma_A$ be endowed with the usual product topology (*Tykhonov topology*). For any $0 < q < 1, \bar{\xi} = (\xi_i), \bar{\eta} = (\eta_i) \in \Sigma_A^{\mathbb{Z}}$ define the distance $\rho_q(\bar{\xi}, \bar{\eta}) = sup_{i,j \in \mathbb{Z}} q^{|i|+|j|} |\xi_i(j) - \eta_i(j)|$. With the above metric compatible with its topology $\Sigma_A^{\mathbb{Z}}$ is a compact metric space. Let σ^t and σ^s denote the two shift maps on $\Sigma_A^{\mathbb{Z}}$, i.e., for $\bar{\xi} = (\xi_i) \in \Sigma_A^{\mathbb{Z}}$, $\xi_i = \xi_i(j) \in \Sigma_A$,

$$(\sigma^t \bar{\xi})_i(j) = \xi_i(j+1) \qquad \text{and} \qquad (\sigma^s \bar{\xi})_i = \xi_{i+1}.$$

We define the map $\bar{\pi}$: $\Sigma_A^{\mathbb{Z}} \to \Delta = \otimes_i \Lambda$ by

$$\bar{\pi} : \bar{\xi} = (\xi_i) \to (\pi(\xi_i)), \quad \xi_i \in \Sigma_A.$$

The map $\bar{\pi}$ enables one to study the uniqueness and ergodic properties of the equilibrium state for a continuous function φ on (Δ, ρ_q) by studying the same properties of the equilibrium state for the corresponding continuous function $\varphi \cdot \bar{\pi}$ on $\Sigma_A^{\mathbb{Z}}$.

We introduce now the concept of **Gibbs states** for Hölder continuous functions on $\Sigma_A^{\mathbb{Z}}$ and describe the relation between Gibbs states and an equilibrium states.

Let φ be a Hölder continuous function on $\Sigma_A^{\mathbb{Z}}$. Any element of $\Sigma_A^{\mathbb{Z}}$ is called a *configuration.* For any subset $X \subset \mathbb{Z}^2$, let $\Omega_X := \{\xi_X |$ there exists $\bar{\xi} \in \Sigma_A^{\mathbb{Z}}$ such that $\bar{\xi}|_X = \xi_X\}$ and $X^C = \mathbb{Z}^2 \setminus X$. Elements of Ω_X are also called the restriction of configurations to X.

For each finite subset $X \subset \mathbb{Z}^2$, define the function $p_X(\bar{\xi})$ on $\Sigma_A^{\mathbb{Z}}$ by

$$p_X(\bar{\xi}) = \frac{1}{\sum_{\bar{\eta}, \bar{\eta}|_{X^C} = \bar{\xi}|_{X^C}} \exp(\sum_{x \in \mathbb{Z}^2} \varphi(\tau^x \bar{\eta}) - \varphi(\tau^x \bar{\xi}))},$$

where τ^x denote the shift $(\sigma^t)^i \cdot (\sigma^s)^j$ and $x = (i, j)$.

Definition [Ru] *A probability measure μ on $\Sigma_A^{\mathbb{Z}}$ is called a* **Gibbs state** *for φ if for any finite subset $X \subset \mathbb{Z}^2$,*

$$\mu_X(\xi_X) = \int_{\Omega_{X^C}} p_X(\bar{\xi}) d\mu_{X^C},$$

where μ_X and μ_{X^C} are the probability measures on Ω_X and Ω_{X^C} that are induced by natural projections respectively.

A transfer matrix A is called *aperiodic* if there is a positive integer N such that every entry of the matrix A^N is positive.

Proposition 2 [Ru] *If the transfer matrix A is aperiodic, then μ is an equilibrium state for φ if and only if it is an invariant Gibbs state.*

Thus, the uniqueness and mixing property for equilibrium states becomes the problem of the uniqueness and mixing property for invariant Gibbs states provided that the function φ is Hölder continuous. This problem has been extensively studied in statistical mechanics in more general settings. It is not true in general that for a Hölder continuous function the Gibbs state is unique. The well-known Ising model provides a class of counterexamples (see [MM]). The conjecture is that uniqueness and mixing property hold for those functions defined in Theorem 7 (2). The potential functions constructed in [BS] and [PS] are of this type. The following proposition is a well-known result that is used to prove Theorem 8 (2).

Proposition 3 [D] [Si] [MJ] *Let $\Omega^{\mathbb{Z}^2}$ be the fullshift on the lattice $\mathbb{Z}^2$, the invariant Gibbs state is unique for any Hölder continuous function on $\Omega^{\mathbb{Z}^2}$ with a sufficiently small Hölder constant. The Gibbs state is mixing with respect to both horizontal and vertical shifts.*

In this paper we obtain some partial results in this direction. By Theorem 4 the conjugating map h between (Δ_Φ, ρ_q) and (Δ, ρ_q) is Hölder continuous for some $0 < q < 1$. Since all metrics ρ_q, $0 < q < 1$ are Hölder equivalent, h is actually

Hölder continuous for any $q,\ 0<q<1$. Thus it suffices to consider the uniqueness of equilibrium states for Hölder continuous functions on (Δ,ρ_q) with respect to the $\mathbb{Z}^2-$action τ induced by (F,S). We denote the $\mathbb{Z}^2-$action on $\Sigma_A^{\mathbb{Z}}$ by τ^*.

Lemma 6.1 *The semiconjugacy $\bar\pi$ between Δ and $\Sigma_A^{\mathbb{Z}}$ has the following properties:*

1) $\bar\pi$ is surjective and Hölder continuous from $\Sigma_A^{\mathbb{Z}}$ to (Δ,ρ_q) for any $0<q<1$ (it is actually Lipschitz continuous).

2)

$$\bar\pi\cdot\sigma^t=F\cdot\bar\pi,\qquad \bar\pi\cdot\sigma^s=S\cdot\bar\pi,\ \text{i. e.,}\ \ \bar\pi\cdot\tau^*=\tau\cdot\bar\pi,$$

where σ^t and σ^s are two shift maps on $\Sigma_A^{\mathbb{Z}}$.

3) $\bar\pi$ is injective outside the set $\bar\pi^{-1}(\mathcal{B})$, where $\mathcal{B}$ denotes the boundary of the Markov partition of Δ.

Lemma 6.2 *1) Let φ be a Hölder continuous function on Δ and let $\varphi^*=\varphi\cdot\bar\pi$. Then $P_{\tau^*}(\varphi^*)\geq P_\tau(\varphi)$.*

2) Assume that μ^* is a τ^* invariant measure on $\Sigma_A^{\mathbb{Z}}$ and measure μ on Δ satisfies $\mu(E)=\mu^*(\bar\pi^{-1}(E))$ for any Borel set E of Δ. Then $h_\tau(\mu)\leq h_\tau(\mu^*)$. Furthermore, if $\mu^*(\bar\pi^{-1}(\mathcal{B}))=0$ then, $h_\tau(\mu)=h_\tau(\mu^*)$.

Theorem 6 *Let φ be Hölder continuous on Δ. Assume that $\mu^*(\bar\pi^{-1}(\mathcal{B}))=0$ for any equilibrium states μ^* for $\varphi^*=\varphi\cdot\bar\pi$. Then,*

1) the measure μ defined by $\mu(E)=\mu^*(\bar\pi^{-1}(E))$ is an equilibrium state on Δ for φ and $P_{\tau^*}(\varphi^*)=P_\tau(\varphi)$;

2) if μ is an equilibrium state on Δ for φ then there exists an equilibrium state μ^* for $\varphi^*=\varphi\cdot\bar\pi$ on $\Sigma_A^{\mathbb{Z}}$ with the property $\mu(E)=\mu^*(\bar\pi^{-1}(E))$ for any Borel set $E\subset\Delta$.

Theorem 7 *1) Assume μ^* is an equilibrium state ergodic with respect to the vertical shift σ^t for a Hölder continuous function on $\Sigma_A^{\mathbb{Z}}$. Then $\mu^*(\bar\pi^{-1}(\mathcal{B}))=0$.*

2) For any $0<q<1, 0<\alpha<1$, and any aperiodic matrix A there exists a small constant c such that if ψ is a Hölder continuous function on $\Sigma_A^{\mathbb{Z}}$ satisfying the following condition

$$|\psi(\bar\xi)-\psi(\bar\eta)|\leq c\rho_q^\alpha(\bar\xi,\bar\eta)$$

for any $\bar\xi=(\xi_i),\bar\eta=(\eta_i)\in\Sigma_A^{\mathbb{Z}}$ with $\xi_0=\eta_0$, then $\mu^*(\bar\pi^{-1}(\mathcal{B}))=0$ holds for any equilibrium state μ^* for ψ.

Corollary *Let φ be a Hölder continuous function on Δ satisfying the condition*

$$|\varphi(\bar x)-\varphi(\bar y)|\leq c\rho_q^\alpha(\bar x,\bar y),$$

where $\bar x=(x_i),\bar y=(y_i)\in\Delta, x_0=y_0$, and c is sufficiently small. Then the equilibrium state for φ is unique if and only if the equilibrium state for $\varphi^*=\varphi\cdot\bar\pi$ is unique on $\Sigma_A^{\mathbb{Z}}$.

Theorem 8 *1) There is a unique measure μ on Δ_Φ such that the entropy $h_\tau(\mu)$ attains its supremum and this measure of maximum entropy is mixing and positive on open sets.*

2) Assume that the symbolic presentation for (Φ, S) is given by the fullshift on $\mathbb{Z}^2$ (e.g. Λ is a Smale horseshoe). If φ is a Hölder continuous function with a sufficiently small Hölder constant on (Δ_Φ, ρ_q), then the equilibrium state for φ is unique, mixing with respect to τ and positive on open sets.

4. PROOFS

Proof of Theorem 1 The first statement of the theorem follows from the classic graph transform technique. The proof of the second statement is a modification of the proof in the finite dimensional case [KH] [Sh] and is an immediate consequence of the following shadowing lemma:

Shadowing Lemma *There exist a neighborhood $U(\Delta) \supset \Delta$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$, there is an $\epsilon > 0$ with the following property:*
If $\Phi = F \cdot G : U(\Delta) \to \mathcal{M}$, is a diffeomorphism with G ϵ_0-close to id in the C^1-topology, $Y \subset U(\Delta)$, is a subset and $Q : Y \to Y$, a homeomorphism with

$$\mathrm{dist}_{C^0}(Q, \Phi) = \sup_{y \in Y}(Q(y), \Phi(y)) \leq \epsilon,$$

then, there exists a unique $h \in C^0(Y, U(\Delta))$ such that $h \cdot Q = \Phi \cdot h$ with $\mathrm{dist}_{C^0}(h, id) <$ *δ.*

Proof Define a map

$$\mathcal{F} : C^0(Y, U(\Delta)) \to C^0(Y, \mathcal{M}) \tag{1.1}$$

by $\beta \longmapsto \Phi \cdot \beta \cdot Q^{-1}$. For $y \in Y$, $\mathcal{F}\beta(y) = \Phi \cdot \beta(Q^{-1}(y))$. We wish to show that $\mathcal{F}$ has a unique fixed point near the identity.

Set

$$C^0_s(Y, T\mathcal{M}) = \{v \in C^0(Y, T\mathcal{M}) \ : \ v(y) \in T_{\bar{y}}\mathcal{M}, y \in Y\}.$$

We denote by $\mathcal{I}$ the identity embedding of Y into $\mathcal{M}$. Let $B_\gamma(\mathcal{I})$ be the ball in $C^0(Y, U(\Delta))$ centered at $\mathcal{I}$ with radius γ. Consider the map $\mathcal{A} : B_\gamma(\mathcal{I}) \to \mathcal{C}'_f(\mathcal{Y}, \mathcal{TM})$ defined by

$$\mathcal{A}\beta(\bar{y}) = (\exp^{-1}_{y_i} \beta_i(\bar{y}))_{i \in \mathbb{Z}}. \tag{1.2}$$

The map $\mathcal{A}$ is well-defined when γ is small and is a homeomorphism onto $B_\gamma(0) \subset C^0_s(Y, T\mathcal{M})$, where $B_\gamma(0)$ is the ball centered at the zero section 0 with radius γ.

Define

$$\mathcal{F}' = \mathcal{A} \cdot \mathcal{F} \cdot \mathcal{A}^{-1} : \ B_\gamma(0) \to C^0_s(Y, T\mathcal{M}). \tag{1.3}$$

If $v \in B_\gamma(0)$ is a fixed point of $\mathcal{F}'$, then $\mathcal{A} \cdot \mathcal{F} \cdot \mathcal{A}^{-1} v = v$. Hence, the preimage of v, $\mathcal{A}^{-1} v \in B_\gamma(\mathcal{I})$, is a fixed point of $\mathcal{F}$ and the proof is then completed.

To show that $\mathcal{F}'$ has a fixed point in $B_\gamma(0)$, we note that $C^0_s(Y, T\mathcal{M})$ is a Banach space and the map $\mathcal{F}'$ is differentiable in $B_\gamma(0)$. In fact, $D\mathcal{F}'$ is Lipschitz in v since the exponential map and its inverse are smooth. By the hyperbolicity of F, without making any major changes we can similarly claim the following (Lemma 18.1.4 in [KH]).

Claim *There exist a neighborhood* $U(\Delta) \supset \Delta, \epsilon_0, \epsilon > 0$, *and* $R > 0$ *independent of* Y, Q, *so that*

$$\|(D\mathcal{F}')|_0 - Id)^{-1}\| < R, \ \textit{where} \ \mathrm{dist}_{C^1}(F, F') < \epsilon_0, \mathrm{dist}_{C^0}(Q, F') < \epsilon.$$

Then, we define a map $\mathcal{K} : B_\gamma(0) \to C^0_I(Y, T\mathcal{M})$ by

$$\mathcal{K}v = -((D\mathcal{F}')|_0 - Id)^{-1} \cdot \mathcal{H}v, \tag{1.4}$$

where $\mathcal{H}v = \mathcal{F}'v - (D\mathcal{F}')|_0 v$. By estimating the Lipschitz constant of $\mathcal{K}$ near 0 and $\|\mathcal{K}(0)\|$, we can see that if ϵ_0, ϵ are chosen small enough, the mapping $\mathcal{K}$ is a contraction in a smaller ball $B_{\gamma_0}(0) \subset B_\gamma(0) \subset C^0_s(Y, T\mathcal{M})$ and $\mathcal{K}(B_{\gamma_0}(0)) \subset B_{\gamma_0}(0)$. Thus it has a unique fixed point in $B_{\gamma_0}(0)$. Obviously, v is also a fixed point of $\mathcal{F}'$. This finishes the proof of the shadowing lemma.

Proof of Theorem 2 Since (Δ, F) and (Δ_Φ, Φ) are topologically conjugate we need only to prove the theorem for (Δ, F). By assumption $f|_\Lambda$ is topologically mixing and Λ is locally maximal. One can directly see that Δ is locally maximal. To show that $F|_\Delta$ is topologically mixing, it suffices to show that the global unstable manifold of any periodic point $\bar{x} \in \Delta$ is dense in Δ and that the periodic points of F are dense in Δ.

Let $\bar{x} = (x_i)$ be a periodic point of F and let $W^u(x_i)$ denote the global unstable manifold for f at the point $x_i \in M_i$. Then $\bigotimes W^u(x_i)$ is the global unstable manifold for F at the point $\bar{x} = (x_i)$, which is clearly dense in Δ since $W^u(x_i)$ is dense in Λ.

Since $f|_\Lambda$ is hyperbolic and topologically mixing, the periodic points of f are dense in Λ. By the compactness of Λ, one can further see that for any $\epsilon > 0$, there exists an integer $N(\epsilon) > 0$ such that in the ϵ-neighborhood of any $x \in \Lambda$, there is a periodic point with a prime period less than $N(\epsilon)$.

To show that periodic points of F are dense in Δ, let us take a point $\bar{x} = (x_i) \in \Delta$ and its ϵ-neighborhood $B_\epsilon(\bar{x}) = \bigotimes B_\epsilon(x_i)$. Pick a periodic point z_i in $B_\epsilon(x_i)$ with the prime period less that $N(\epsilon)$ for each i. Then, $\bar{z} = (z_i)$ is a periodic point of F in $B_\epsilon(\bar{x})$. Its prime period is less than $N(\epsilon)!$.

Proof of Lemma 1 We first fix $i \in \mathbb{Z}$ and let $\bar{x}, \bar{y} \in \mathcal{M}$ with $x_j = y_j, j \neq k$ and $0 < \theta < q < 1$. If $i \neq k \in \mathbb{Z}$, then

$$q^{|i|} d(G_i(\bar{x}), G_i(\bar{y})) = q^{|i|} d(G_0(S^i \bar{x}), G_0(S^i \bar{y})) \leq C q^{|i|} \theta^{|k-i|} d(x_k, y_k)$$

$$\leq C q^{|i|} q^{|k-i|} (\frac{\theta}{q})^{|k-i|} d(x_k, y_k) \leq C q^{|k|} (\frac{\theta}{q})^{|k-i|} d(x_k, y_k).$$

If $i = k$, then

$$q^{|i|} d(G_i(\bar{x}), G_i(\bar{y})) \leq C q^{|k|} d(x_k, y_k).$$

Thus, for every fixed i and arbitrary $\bar{x}, \bar{y} \in \mathcal{M}$, we have

$$q^{|i|} d(G_i(\bar{x}), G_i(\bar{y})) \leq C(1 + \sum_{k \neq i} (\frac{\theta}{q})^{|k-i|}) \sup_{n \in \mathbb{Z}} q^{|n|} d(x_n, y_n)$$

$$= C(1 + \frac{2\theta}{q - \theta}) \sup_{n \in \mathbb{Z}} q^{|n|} d(x_n, y_n).$$

Therefore, we have $\rho_q(G(\bar{x}), G(\bar{y})) \leq C(1 + \frac{2\theta}{q-\theta}) \rho_q(\bar{x}, \bar{y})$.

The proof of the second part of the lemma is trivial. As an easy corollary of Lemma 1 one can see that if both G, G' are short ranged, then the composition $G \cdot G'$ is also short ranged.

Proof of Lemma 2 (1) $\Rightarrow$ (2): Without loss of generality we assume G is a C^1 diffeomorphism of $\mathcal{M}$. The tangent space $T_{\bar{x}}\mathcal{M}$ at any point $\bar{x} = (x_i)$ is a product space $\bigoplus T_{x_i} M_i$ which is equipped with a Finsler metric. We write G in the coordinate form: $G = (\cdots G_{-n}, \cdots, G_{-1}, G_0, G_1, \cdots, G_n, \cdots)$. Then the differential DG can be expressed as

$$DG = (\cdots DG_{-n}, \cdots, DG_{-1}, DG_0, DG_1, \cdots, DG_n, \cdots).$$

Without loss of generality, let us consider the linear mapping $DG_0 : \bigoplus T_{x_i} M_i \to T_{y_0} M$, where $(\bar{y}) = (y_i) = G(\bar{x})$. Let $\bar{\xi} = (\xi_i) \in \bigoplus T_{x_i} M_i$ be any tangent vector satisfying $\xi_i = 0$ except $i = k$ for some fixed $k \in \mathbb{Z}$. Consider a smooth curve $\gamma(t)$ in M_k passing through x_k with $\frac{d\gamma(t)}{dt}|_{t=0} = \xi_k$. Then $\Gamma(t) : t \to (\cdots, x_{k-1}, \gamma(t), x_{k+1}, \cdots)$ is a smooth curve in $\mathcal{M}$ passing through $\bar{x}$ with $\frac{d\Gamma(t)}{dt}|_{t=0} = \bar{\xi}$. Thus

$$\|D_{\bar{x}} G_0(\bar{\xi})\| = \|dG_0(\Gamma(t))/dt\| = \lim_{t \to 0} \frac{\|G_0(\Gamma(t)) - G_0(\bar{x})\|}{|t|}$$

$$\leq C\theta^{|k|} \lim_{t \to 0} \frac{\|\gamma(t) - x_k\|}{|t|} = C\theta^{|k|} \|\xi_k\|.$$

(2) $\Rightarrow$ (1): For any integer $k \in \mathbb{Z}$ and two points $\bar{x} = (x_i), \bar{y} = (y_i) \in \mathcal{M}$ with $x_i = y_i$ for $i \neq k$, let $\Gamma(t)$ be a geodesic connecting x_k and y_k in the compact manifold $M_k = M$, $t \in [0, l]$. $\alpha : t \to (\cdots, x_i, \cdots, x_{k-1}, \Gamma(t), x_{k+1}, \cdots)$ defines a smooth curve from $[0, l]$ to $\mathcal{M}$. $d(G_0(\bar{x}), G_0(\bar{y}))$

$$= d(G_0(\alpha(0)), G_0(\alpha(1))) \leq \max_t \{\|DG_0|_{\alpha(t)}\xi_k\| \cdot \|D\Gamma(t)\|\} d(x_k, y_k), \tag{2.1}$$

where ξ_k is a unit tangent vector in $T\mathcal{M}$ with every coordinate zero except the kth. By assumption, $\|DG_0|_{\alpha(t)}\xi_k\| \leq C\theta^{|k|}$. Then the inequality (2.1) means that G is short ranged with a decay constant θ.

(1) + (2) $\Leftrightarrow$ (3) is obvious.

Proof of Lemma 3 By Lemma 2, the properties of linear short range mappings can then be passed on to those differentiable mappings between Banach manifolds. Let $l^{\mathbb{Z}}$ be the Banach space

$$\{x = (x_i) \ : \quad x_i \in \mathbb{R}^m, i \in \mathbb{Z}, \sup_{i \in \mathbb{Z}} \|x_i\| < \infty\} \tag{3.1}$$

with the norm $\|x\| = (x_i)$ is $\sup_{i \in \mathbb{Z}} \|x_i\|$. Any linear operator A of $l^{\mathbb{Z}}$ can be expressed as an infinite matrix in the usual weak* basis: $A = (a_{ij}); -\infty < i, j < \infty$, where a_{ij} are linear operators of $\mathbb{R}^m$. The operator A is said to be shift invariant if $a_{ij} = a_{i+1j+1}; -\infty < i, j < \infty$ and A is said to be short ranged if $\|a_{ij}\| \leq C\theta^{|i-j|}$ for some constant $C > 0$ and $0 < \theta < 1$. To prove Lemma 3, we need only to show the following statement.

Claim Let A be an invertible linear operator on $l^{\mathbb{Z}}$ and I the identity. For any $C > 0$, and $0 < \theta < \theta' < 1$ there exists $\epsilon > 0$ such that if A is a short range operator with constants C and θ and $\|A - I\| \leq \epsilon$, then the inverse A^{-1} is also a short ranged mapping with the decay constant θ'.

Proof For convenience we prove the claim when $m = 1$. Let us first express the operator A in a form of an infinite matrix (a_{ij}) under the usual weak* basis. Then A has the short range property if and only if $|a_{ij}| \leq C\theta^{|i-j|}$, where θ is the decay constant and C is the other constant in the definition of short range mappings. It is easy to see that such an infinite matrix also defines a bounded linear operator of the Banach space:

$$l^{\mathbb{Z}}_{\theta'} = \{x = (x_i) \ : \ x_i \in \mathbb{R}, i \in \mathbb{Z}, \sup_{i \in \mathbb{Z}} (\theta')^{|i|} |x_i| < \infty\}$$

endowed with the norm $\|x\|_{\theta'} = \sup_{i \in \mathbb{Z}} (\theta')^{|i|} |x_i|$. We estimate the norm of $I - A$ in the Banach space $l^{\mathbb{Z}}_{\theta'}$.

$$\|(I - A)(x_j)\|_{\theta'} = \sup_{i \in \mathbb{Z}} (\theta')^{|i|} |(1 - a_{ii})x_i + \sum_{j \neq i} a_{ij} x_j|$$

$$\leq \sup_{i\in\mathbb{Z}} (\theta')^{|i|}(|(1-a_{ii})x_i| + \sum_{j\neq i} |c_{ij}|\theta^{|i-j|}|x_j|), \tag{3.2}$$

where $c_{ij}\theta^{|i-j|} = a_{ij}$ and $|c_{ij}| \leq C$. Thus,

$$\|(I-A)(x_j)\|_{\theta'} \leq \sup_{i\in\mathbb{Z}} (|(1-a_{ii})|\theta'^{|i|}|x_i| + \sum_{j\neq i} |c_{ij}|\frac{\theta^{|i-j|}}{(\theta')^{|j|-|i|}}(\theta')^{|j|}|x_j|)$$

$$\leq \sup_{j\in\mathbb{Z}} (\theta')^{|j|}|x_j| \ \sup_{i\in\mathbb{Z}} (|1-a_{ii}| + \sum_{j\neq i} |c_{ij}|\frac{\theta^{|i-j|}}{(\theta')^{|j|-|i|}}).$$

Obviously, $|1-a_{ii}| \leq \epsilon$ when $\|A-I\| \leq \epsilon$. Note that

$$\frac{\theta^{|i-j|}}{(\theta')^{|j|-|i|}} \leq (\frac{\theta}{\theta'})^{|i-j|}$$

and $\sum_{j\neq i} |c_{ij}|\theta^{|i-j|} \leq \epsilon$. Let us choose an integer $K > 0$ such that $2C\sum_{n=K+1}^{\infty}(\frac{\theta}{\theta'})^n \leq 1/4$ and $0 < \epsilon < (1/4)\theta^K$. Then,

$$\sum_{0<|i-j|\leq K} |c_{ij}|\frac{\theta^{|i-j|}}{(\theta')^{|j|-|i|}} \leq \epsilon/(\theta')^K < (1/4).$$

Therefore,

$$\sup_{i\in\mathbb{Z}} (|1-a_{ii}| + \sum_{j\neq i} |c_{ij}|\frac{\theta^{|i-j|}}{(\theta')^{|j|-|i|}}) < \epsilon + 1/4 + 1/4 < 1.$$

Thus $\|(I-A)\|_{\theta'} < 1$, which implies that A is invertible on $l_{\theta'}^{\mathbb{Z}}$. It is easy to see that the boundedness of A^{-1} on $l_{\theta'}^{\mathbb{Z}}$ implies that A^{-1} is a short range mapping on $l^{\mathbb{Z}}$ with the decay constant θ'. If we denote the entries of the infinite matrix representing A^{-1} by b_{ij}, then there exists a constant C' such that $|b_{ij}| \leq C'(\theta')^{|i-j|}$. One can also see that the constant C' depends only on constants C, θ, and θ'.

Remark. As an easy corollary of Lemma 3, one can show that if F and F^{-1} are both short ranged diffeomorphisms of $\mathcal{M}$ and the C^1 distance between F and G is sufficiently small, then G^{-1} is also short ranged.

Proof of Lemma 4 For any $|i-k| \geq k_0, k_0 \in I\!N$ fixed, $0 < q < 1$, and $\bar{x}, \bar{y}$ with $x_j = y_j, j \neq k$,

$$q^{|i|}d(G_i(\bar{x}), G_i(\bar{y})) = q^{|i|}d(G_0(S^i\bar{x}), G_0(S^i\bar{y})) \leq Cq^{|i|}\theta^{|k-i|}d(x_k, y_k)$$

$$\leq Cq^{|i|}q^{|k-i|}(\frac{\theta}{q})^{|k-i|}d(x_k, y_k) \leq Cq^{|k|}(\frac{\theta}{q})^{|k-i|}d(x_k, y_k).$$

For $i = k$,

$$q^{|i|}d(G_i(\bar{x}), G_i(\bar{y})) = q^{|i|}d(G_0(S^i\bar{x}), G_0(S^i\bar{y})) \leq q^{|k|}(1+\delta)d(x_k, y_k)$$

For $0 < |i-k| < k_0$,

$$q^{|i|}d(G_i(\bar{x}), G_i(\bar{y})) = q^{|i|}d(G_0(S^i\bar{x}), G_0(S^i\bar{y})) \leq q^{|i|}\delta d(x_k, y_k)$$

$$= q^{|i|-|k|}\delta q^{|k|}d(x_k, y_k) \leq q^{-|i-k|}\delta q^{|k|}d(x_k, y_k).$$

Therefore, for every fixed $i \in \mathbb{Z}$ and for arbitrary $\bar{x}, \bar{y} \in \mathcal{M}$, we have

$$q^{|i|}d(G_i(\bar{x}), G_i(\bar{y})) \leq (1 + \delta(\sum_{l=0}^{k_0-1} q^{-l}) + \sum_{|k-i|\geq k_0} (\frac{\theta}{q})^{|k-i|}) \sup_{i\in\mathbb{Z}} q^{|i|}d(x_i, y_i)$$

$$= [1 + \delta(q^{-k_0} - 1)(q^{-1}-1)^{-1} + 2(\frac{\theta}{q})^{k_0}(\frac{q}{q-\theta})] \sup_{i\in\mathbb{Z}} q^{|i|}d(x_i, y_i).$$

We first fix k_0 so that

$$2(\frac{\theta}{q})^{k_0}\frac{q}{q-\theta} \leq \epsilon/2,$$

and then choose δ so that

$$\delta(q^{-k_0} - 1)(q^{-1}-1)^{-1} \leq \epsilon/2.$$

Thus, we have $\rho_q(G(\bar{x}), G(\bar{y})) \leq (1+\epsilon)\rho_q(\bar{x}, \bar{y})$. By the virtue of Lemma 3 one can further show that δ can be chosen so small that the inverse G^{-1} is also Lipschitz continuous in the metric ρ_q with a Lipschitz constant less than $1+\epsilon$.

Proof of Theorem 3 We prove a modified version of the shadowing lemma.

Modified Shadowing lemma *There exist a neighborhood $U(\Delta) \supset \Delta$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$, there is an $\epsilon > 0$ with the following property:*
If $\Phi = F \cdot G : U(\Delta) \to \mathcal{M}$, is a short ranged diffeomorphism with G ϵ_0-close to id in the C^1-topology, $Y \subset U(\Delta)$, is an subset and $Q:\ Y \to Y$, a homeomorphism with both Q and Q^{-1} short ranged and

$$\mathrm{dist}_{C^0}(Q, \Phi) = \sup_{y\in Y}(Q(y), \Phi(y)) \leq \epsilon,$$

then, there exists a unique $h \in C^0(Y, U(\Delta))$ such that $h \cdot Q = \Phi \cdot h$ with $\mathrm{dist}_{C^0}(h, id) < \delta$. The map h is also continuous in the metric ρ_q.

Proof Following the the proof of the shadowing lemma and using lemmas 1 - 3, one can easily check that the short range property is preserved at all steps of this proof.

To complete the proof of Theorem 3, let us first choose $Y = \Delta$, $Q = F$. By applying the modified shadowing theorem we obtain a mapping $h : \Delta \to U(\Delta)$, satisfying $h \cdot F = \Phi \cdot h$. h is ρ_q continuous. $\Delta_\Phi = h(\Delta)$ is invariant under Φ. Then, we choose $Y = h(\Delta)$, $Q = \Phi$, and $\Phi = F$ in the modified shadowing lemma. Since $\mathrm{dist}_{C^1}(\Phi, F) < \epsilon_0$, Lemma 3 indicates that Φ^{-1} is also short ranged. Thus, there exists β such that $\beta \cdot \Phi = F \cdot \beta$. Finally, when we choose $Q = F$ and $\Phi = F$, we have

$$(\beta \cdot h) \cdot F = \beta \cdot \Phi \cdot h = F \cdot (\beta \cdot h).$$

By the uniqueness, we conclude that $\beta \cdot h = id$. Thus, both $h, h^{-1} (= \beta)$ are continuous in the metric ρ_q.

Proof of Theorem 4 The proof is divided into three major steps.

Step 1 We first prove that Hölder continuity for the conjugacy h follows from the following statement:

There exist constants $C > 0, \delta_1 > 0$, and $0 < \alpha < 1$ such that for any $\bar{y} \in V^u(\bar{x}) \cap \Delta$ or $\bar{y} \in V^s(\bar{x}) \cap \Delta$ with $\rho(\bar{x}, \bar{y}) \leq \delta_1$

$$\rho_q(h(\bar{x}), h(\bar{y})) \leq C \cdot \rho_q^\alpha(\bar{x}, \bar{y}), \tag{4.1}$$

where $V^s(\bar{x}), V^u(\bar{x})$ are the local stable and unstable manifolds at $\bar{x}$ respectively.

In fact, since Δ is locally maximal, it has a local product structure, i.e., there exists $\delta_0 > 0$ such that for any two points $\bar{x}, \bar{y} \in \Delta$ with $\rho(\bar{x}, \bar{y}) < \delta_0$, $V^s(\bar{x}) \cap V^u(\bar{y})$ consists of a single point $\bar{z} = [\bar{x}, \bar{y}]$ and $\bar{x} \in V^s(\bar{z}), \bar{y} \in V^u(\bar{z})$. Now, let us assume that the previous statement is true. We can choose δ_0 sufficiently small so that $\rho(\bar{x}, \bar{z}) < \delta_1$ and $\rho(\bar{y}, \bar{z}) < \delta_1$ whenever $\rho(\bar{x}, \bar{y}) < \delta_0$. Thus,

$$\rho_q(h(\bar{z}), h(\bar{x})) \leq C\rho_q^\alpha(\bar{z}, \bar{x}), \quad \rho_q(h(\bar{z}), h(\bar{y})) \leq C\rho_q^\alpha(\bar{z}, \bar{y}).$$

Hence, for any $\bar{x}, \bar{y} \in \Delta$ with $\rho(\bar{x}, \bar{y}) < \delta_0$, we have

$$\begin{aligned} \rho_q(h(\bar{x}), h(\bar{y})) &\leq C[\rho_q^\alpha(\bar{z}, \bar{x}) + \rho_q^\alpha(\bar{z}, \bar{y})] \\ &= C\{[\sup_i q^{|i|} d(x_i, z_i)]^\alpha + [\sup_i q^{|i|} d(y_i, z_i)]^\alpha\} \\ &\leq C'[\sup_i q^{|i|} d(x_i, y_i)]^\alpha = C'\rho_q^\alpha(\bar{x}, \bar{y}). \end{aligned}$$

The last inequality holds because stable and unstable manifolds are transversal uniformly over $i \in \mathbb{Z}$ and x_i, y_i.

Now let us consider the Hölder continuity along the global stable and unstable manifolds. Let d' be the induced metric on the global stable manifold $W^s(x)$ at any point $x \in \Lambda$. Let l be a fixed positive number. For any $y \in W^s(x)$ with

$d'(x,y) < l$, there exists a positive integer $N(l)$ which solely depends on l and λ such that $d(f^{N(l)}(x), f^{N(l)}(y)) \leq \delta_0$. Then,

$$\rho_q(h(\bar{x}), h(\bar{y})) = \rho_q(hF^{-N(l)}F^{N(l)}(\bar{x}), hF^{-N(l)}F^{N(l)}(\bar{y}))$$

$$= \rho_q(\Phi^{-N(l)}hF^{N(l)}(\bar{x}), \Phi^{-N(l)}hF^{N(l)}(\bar{y})) \leq L^{N(l)}\rho(hF^{N(l)}(\bar{x}), hF^{N(l)}(\bar{y}))$$

$$\leq CL^{N(l)}\rho_q^{\alpha}(\bar{x}, \bar{y}),$$

where L is the Lipschitz constant of Φ^{-1} in the metric ρ. Along the global unstable manifolds, we have a similar estimation.

In general, by the compactness and the topological mixing property of Λ there exists $l \geq 0$ such that for arbitrary two points $\bar{x}, \bar{y} \in \Delta$,, one can find a fixed integer $k \in I\!N$ and a sequence of points $\bar{x}_0 = \bar{x}, \bar{x}_1, \cdots, \bar{x}_k = \bar{y} \in \Delta$ with the following properties:

1) $\bar{x}_{2n+1} \in W^u(\bar{x}_{2n}), n = 0, 1, 2, \cdots$ and $\bar{x}_{2n} \in W^s(\bar{x}_{2n-1}), n = 1, 2, \cdots$.

2) $d'((\bar{x}_n)_i, (\bar{x}_{n+1})_i) < l,\ n = 0, 1, \cdots, k-1;\ i \in Z\!\!Z$.

Thus,

$$\rho(h(\bar{x}), h(\bar{y}) \leq \sum_n \rho_q^{\alpha}(h(\bar{x}_n), h(\bar{x}_{n+1}))$$

$$\leq C'' \sum_n \rho_q^{\alpha}(\bar{x}_n, \bar{x}_{n+1}) \leq kC''\rho_q^{\alpha}(\bar{x}, \bar{y}).$$

This proves the Hölder continuity of the conjugacy h in the metric ρ_q when the statement is assumed true.

To prove the statement itself, we need only to show that it is true when $\bar{y} \in V^u(\bar{x})$ by the symmetry. One can easily prove the Hölder continuity of the conjugacy h in the Finsler metric by applying the usual technique [KH]. However, this kind of technique fails when we deal with the metric ρ_q. We shall show that h satisfies (4.1) by examining the procedure in which the map h is constructed.

Let $\bar{x} \in \Delta$. Then $h(\bar{x}) \in \Delta_\Phi$ is hyperbolic for Φ and there exist the local stable and unstable manifolds $V^s_\Phi(h(\bar{x})), V^u_\Phi(h(\bar{x}))$ passing through $h(\bar{x})$. For any $\bar{y} \in V^u(\bar{x}) \cap \Delta$ with $\rho(\bar{x}, \bar{y}) < \delta_1$, $h(\bar{y}) \in V^u_\Phi(h(\bar{x})) \cap \Delta_\Phi$. Note that $h(\bar{y})$ is exactly the unique intersecting point of $V^s_\Phi(h(\bar{y}))$ and $V^u_\Phi(h(\bar{x}))$. By the short range property of the perturbation and the transversality of the stable and unstable manifolds, we see that if one can show the unstable manifold $V^s_\Phi(h(\bar{y}))$ varies Hölder continuously in the metric ρ_q with respect to $\bar{y}$, then the Hölder continuity of h in the metric ρ_q follows.

Step 2 We consider the exponential splitting of the tangent bundle at $\bar{x}$ under map F:

$$T_{\bar{x}}\mathcal{M} = E^u_{\bar{x}} \oplus E^s_{\bar{x}},\ n = 0, 1, 2, \cdots. \qquad DFE^u_{\bar{x}} = E^u_{F(\bar{x})};\ DFE^s_{\bar{x}} = E^s_{F(\bar{x})}.$$

Let $B_\gamma(\bar{x})$ denote the γ-ball in $T_{\bar{x}}\mathcal{M}$ centered at $\bar{x}$ and $B^u_\gamma(\bar{x}) = B_\gamma(\bar{x}) \cap E^u_{\bar{x}}$ $B^s_\gamma(\bar{x}) = B_\gamma(\bar{x}) \cap E^s_{\bar{x}}$. When γ is small and Φ is sufficiently close to F the following map

$$\mathcal{F}_{\bar{x}} : B_\gamma(\bar{x}) \to T_{F(\bar{x})}\mathcal{M}, \ n = 1, 2, \cdots$$

$$\mathcal{F}_{\bar{x}}\bar{\xi} = \exp^{-1}_{F(\bar{x})} \cdot \Phi \cdot \exp_{\bar{x}}(\bar{\xi})$$

is well defined. Now let us consider the C^1-manifold $\widetilde{V}^s_\Phi(h(\bar{x})) = \exp^{-1}_{\bar{x}} V^s_\Phi(h(\bar{x}))$ in $B_\gamma(\bar{x})$. Since $\mathrm{dist}_{C^0}(h, id)$ is small, the manifold $\widetilde{V}^s_\Phi(h(\bar{x}))$ can be obtained by applying the graph transform technique. Namely, we consider maps

$$\varphi_{\bar{x}}(t) : B^s_\gamma(\bar{x}) \to B^u_\gamma(\bar{x}), \ n = 0, 1, \cdots$$

such that

$$\{(t, \varphi_{\bar{x}}(t)) : t \in B^s_\gamma(\bar{x})\} = \widetilde{V}^u_\Phi(h(\bar{x}))$$

is invariant under map $\mathcal{F}_{\bar{x}}$. For any $\bar{\xi} = (u, v) \in B_\gamma(\bar{x}) \subset B^s_\gamma(\bar{x}) \oplus B^u_\gamma(\bar{x})$, the map $\mathcal{F}_{\bar{x}}$ can be written as

$$\mathcal{F}_{\bar{x}} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = D\mathcal{F}_{\bar{x}} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_{1,\bar{x}}(u, v) \\ g_{2,\bar{x}}(u, v) \end{pmatrix},$$

By the hyperbolicity of Φ, the differential $D\mathcal{F}_{\bar{x}} = D\exp^{-1}_{F(\bar{x})} D\Phi D \exp_{\bar{x}}$ can be written as follows.

$$D\mathcal{F}_{\bar{x}} = \begin{pmatrix} P_{11}(\bar{x}) & P_{12}(\bar{x}) \\ P_{21}(\bar{x}) & P_{22}(\bar{x}) \end{pmatrix}$$

where $P_{11}(\bar{x}) : E^s_{\bar{x}} \to E^s_{F(\bar{x})}$, $P_{21}(\bar{x}) : E^s_{\bar{x}} \to E^u_{F(\bar{x})}$, $P_{12}(\bar{y}) : E^u_{\bar{x}} \to E^s_{F(\bar{x})}$, and $P_{22}(\bar{y}) : E^u_{\bar{x}} \to E^u_{F(\bar{x})}$ are all linear maps.

Let $(u, v) = (t, \varphi_{\bar{x}}(t)), \ n = 0, 1, \cdots$. The invariance under $\mathcal{F}_{\bar{x}}$ requires that $\varphi_{\bar{x}}(t)$ satisfies

$$\begin{aligned} &\varphi_{F(\bar{x})}[P_{11}(\bar{x})t + P_{12}(\bar{x})\varphi_{\bar{x}}(t) + g_{1,\bar{x}}(t, \varphi_{\bar{x}}(t))] \\ &= P_{21}(\bar{x})t + P_{22}(\bar{x})\varphi_{\bar{x}}(t) + g_{2,\bar{x}}(t, \varphi_{\bar{x}}(t)). \end{aligned} \tag{4.2}$$

We consider the space $\mathcal{S}$ of all continuous maps from $B^s_\gamma(\bar{x})$ to $B^u_\gamma(F(\bar{x}))$ with $\bar{x}$ as a parameter

$$\mathcal{S} : \{\phi_{\bar{x}}(t) | \phi_{\bar{x}}(t) : B^s_\gamma(\bar{x}) \to B^u_\gamma(F(\bar{x})); \phi_{\bar{x}}(t) \text{ is continuous in } \bar{x} \text{ and } t\}$$

With the norm $\|\phi_{\bar{x}}(t)\| = \sup_{\bar{x}\in\Delta} \sup_{t\in B^s_\gamma} \|\phi_{\bar{x}}(t)\|$, this space is a Banach space. Define the operator $\mathcal{T}$ from $\mathcal{S}$ into itself by

$$(\mathcal{T}\phi_{\bar{x}})|_{\bar{x}}(t) = P^{-1}_{22}(\bar{x})(\phi_{F(\bar{x})}(P_{11}(\bar{x})t + P_{12}(\bar{x})\phi_{\bar{x}}(t) + g_{1,\bar{x}}(t, \phi_{\bar{x}}(t))))$$

$$-P_{22}^{-1}(\bar{x}) \cdot P_{21}(\bar{x})t - P_{22}^{-1}(\bar{x}) \cdot g_{2,\bar{x}}(t, \phi_{\bar{x}}(t)).$$

By the usual argument, one can see that $\mathcal{T}$ is a contraction in a neighborhood of 0 and the unique fixed point $\varphi_{\bar{x}}(t)$ satisfies (4.2). In fact, $\varphi_{\bar{x}}(t) = lim_{m\to\infty}\mathcal{T}^m 0$, where 0 is the zero map from $B^s_\gamma(\bar{x})$ to $B^u_\gamma(F(\bar{x}))$.

Now, let us fix a point $\bar{x}$. We wish to show that $\exp_{\bar{y}} \varphi_{\bar{y}}(t)$ Hölder continuously depends on $\bar{y}$ when $\bar{y} \in V^u(\bar{x})$ is close to $\bar{x}$. For any $\bar{y}$ sufficiently close to $\bar{x}$, we may consider all maps $\varphi_{\bar{y}}(t)$ as maps from $B^s_\gamma(\bar{x})$ to $B^u_\gamma(\bar{x})$ and rewrite the transformation $\mathcal{F}_{\bar{y}}$ into a matrix form using the exponential splitting at $\bar{x}$. For any $\bar{\xi} = (u, v) \in B_\gamma(\bar{x}) \subset B^s_\gamma(\bar{x}) \oplus B^u_\gamma(\bar{x})$,

$$\mathcal{F}_{\bar{y}} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = D\mathcal{F}_{\bar{y}} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_{1,\bar{y}}(u,v) \\ g_{2,\bar{y}}(u,v) \end{pmatrix}.$$

We have that

$$D\mathcal{F}_{\bar{y}} = \begin{pmatrix} A_{11}(\bar{y}) & A_{12}(\bar{y}) \\ A_{21}(\bar{y}) & A_{22}(\bar{y}) \end{pmatrix}$$

Note that $D\mathcal{F}_{\bar{y}} = D\exp^{-1}_{F(\bar{y})} \cdot D\Phi \cdot D\exp_{\bar{y}} = D\exp^{-1}_{F(\bar{y})} \cdot DF \cdot DG \cdot D\exp_{\bar{y}}$. Since the diffeomorphism G is C^2 and short ranged, by the virtue of Lemma 4, DG is Lipschitz continuous in both the Finsler metric and the metric ρ_q with Lipschitz constants arbitrarily close to one. This implies that for any fixed $\bar{y}$ both $A_{11}(\bar{y})$ and $A_{22}^{-1}(\bar{y})$ are contracting maps in the metric ρ_q for $\theta < q \leq 1$. The contracting constants will be denoted by k_1 and k_2 respectively. We use K_1 and K_2 to denote the Lipschitz constants of $A_{11}^{-1}(\bar{y})$ and $A_{22}(\bar{y})$ respectively.

The operator $\mathcal{T}$ can also be expressed locally as

$$\mathcal{T}\phi_{\bar{y}}|_{\bar{y}}(t) = A_{22}^{-1}(\bar{y})(\phi_{F(\bar{y})}(A_{11}(\bar{y})t + A_{12}(\bar{y})\phi_{\bar{y}}(t) + g_{1,\bar{y}}(t, \phi_{\bar{y}}(t))))$$

$$-A_{22}^{-1}(\bar{y}) \cdot A_{21}(\bar{y})t - A_{22}^{-1}(\bar{y}) \cdot g_{2,\bar{y}}(t, \phi_{\bar{x}}(t)),$$

and $\varphi_{\bar{y}}(t) = lim_{m\to\infty}\mathcal{T}^m 0$. We want to show that

$$\sup_{t\in B^s_\gamma(\bar{x})} \|(\varphi_{\bar{y}_1}(t)) - (\varphi_{\bar{y}_2}(t))\|_q \leq C \cdot {\rho_q}^\alpha(\bar{y}_1, \bar{y}_2). \tag{4.3}$$

We prove that (4.3) holds for $\mathcal{T}^m 0$, $m = 0, 1, 2, \cdots$ by induction. Then, by taking the limit of this sequence, we have the desired result. For convenience, we write $\mathcal{T}^m 0 = \varphi^m_{\bar{y}}(t)$.

For $m = 0$, $(t, \varphi^0_{\bar{y}}(t))$ is the stable subspace of $T_{\bar{y}}\mathcal{M}$. So, (4.3) is obviously satisfied since the exponential splitting for DF is Hölder continuous in the metric ρ_q. Assume (4.3) is true for $m = n - 1$. Then,

$$\varphi^n_{\bar{y}}(t) = A_{22}^{-1}(\bar{y})(\varphi^{n-1}_{F(\bar{y})}(A_{11}(\bar{y})t + A_{12}(\bar{y})\varphi^{n-1}_{\bar{y}}(t) + g_{1,\bar{y}}(t, \varphi^{n-1}_{\bar{y}}(t))))$$

$$-A_{22}^{-1}(\bar{y}) \cdot A_{21}(\bar{y})t - A_{22}^{-1}(\bar{y}) \cdot g_{2,\bar{y}}(t, \varphi_{\bar{y}}^{n-1}(t)).$$

Let us denote

$$\Sigma_j = A_{11}(\bar{y}_j)t + A_{12}(\bar{y}_j)\varphi_{\bar{y}_j}^{n-1}(t) + g_{1,\bar{y}_j}(t, \varphi_{\bar{y}_j}^{n-1}(t)),\ j = 1, 2$$

Thus,

$$\begin{aligned} \|\varphi_{\bar{y}_1}^n(t) - \varphi_{\bar{y}_2}^n(t)\|_q &\le \|A_{22}^{-1}(\bar{y}_1)\varphi_{F(\bar{y}_1)}^{n-1}(\Sigma_1) - A_{22}^{-1}(\bar{y}_2)\varphi_{F(\bar{y}_2)}^{n-1}(\Sigma_2)\|_q \\ &+\|A_{22}^{-1}(\bar{y}_1)A_{21}(\bar{y}_1)t - A_{22}^{-1}(\bar{y}_2)A_{21}(\bar{y}_2)t\|_q \\ &+\|A_{22}^{-1}(\bar{y}_1) \cdot g_{2,\bar{y}_1}(t, \varphi_{\bar{y}_1}^{n-1}(t)) - A_{22}^{-1}(\bar{y}_2) \cdot g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_2}^{n-1}(t))\|_q. \end{aligned} \tag{4.4}$$

By assumption there exist constants $C > 0$, $0 < \alpha < 1$ such that

$$\|\varphi_{\bar{y}_1}^{n-1}(t) - \varphi_{\bar{y}_2}^{n-1}(t)\|_q \le C \cdot {\rho_q}^{\alpha}(\bar{y}_1, \bar{y}_2)$$

for all $t \in B_\gamma^s(\bar{x})$. In the last step we estimate those three terms involved in (4.4).

Step 3 We first consider $g_{1,\bar{y}}(t, \varphi_{\bar{y}}^{n-1}(t))$ and $g_{2,\bar{y}}(t, \varphi_{\bar{y}}^{n-1}(t))$. Note that

$$\begin{pmatrix} g_{1,\bar{y}}(u, v) \\ g_{2,\bar{y}}(u, v) \end{pmatrix} = (\mathcal{F}_{\bar{y}} - D\mathcal{F}_{\bar{y}}) \begin{pmatrix} u \\ v \end{pmatrix}.$$

and for each fixed $\bar{y}$, $\mathcal{F}_i - D\mathcal{F}_{\bar{y}}$ is a C^1, shift invariant, and short ranged map with a Lipschitz constant ϵ_1, which can be taken arbitrarily small when γ, the radius of the ball $B_\gamma(\bar{x})$, is small. By similar arguments used in proving Lemma 4, one can show that $g_{j,\bar{y}}, j = 1, 2$ are also Lipschitz in the metric ρ_q and the Lipschitz constant in the metric ρ_q, denoted by ϵ_2, approaches zero when $\epsilon_1 \to 0$. Moreover, since $\Phi = F \cdot G$ is C^2, by Lemma 2, we have $g_{j,\bar{y}}$ is also Lipschitz with respect to $\bar{y}$ in the metric ρ_q. We use L_1 to denote a common bound for these two Lipschitz constants.

There exists a constant $\epsilon_3 > 0$ such that $\|A_{12}(\bar{y})\|_q \le \epsilon_3$ and $\|A_{21}(\bar{y})\|_q \le \epsilon_3$ for any $\bar{y}$ close to $\bar{x}$. In fact ϵ_3 can be taken arbitrary small. Also one can easily see that $A_{ij}(\bar{y})$ are all Lipschitz continuous with respect to $\bar{y}$ in metric ρ_q. We will use K_3 to denote the common bound of all four Lipschitz constants.

Therefore,

$$\|A_{22}^{-1}(\bar{y}_1) \cdot g_{2,\bar{y}_1}(t, \varphi_{\bar{y}_1}^{n-1}(t)) - A_{22}^{-1}(\bar{y}_2) \cdot g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_2}^{n-1}(t))\|_q$$

$$\begin{aligned} &\le \|A_{22}^{-1}(\bar{y}_1) \cdot g_{2,\bar{y}_1}(t, \varphi_{\bar{y}_1}^{n-1}(t)) - A_{22}^{-1}(\bar{y}_1) \cdot g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_1}^{n-1}(t))\|_q \\ &+ \|A_{22}^{-1}(\bar{y}_1)[g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_1}^{n-1}(t)) - g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_2}^{n-1}(t))\|_q \\ &+ \|[A_{22}^{-1}(\bar{y}_1) - A_{22}^{-1}(\bar{y}_2)]g_{2,\bar{y}_2}(t, \varphi_{\bar{y}_2}^{n-1}(t))\|_q \end{aligned}$$

$$\le k_2 L_1 \rho_q(\bar y_1, \bar y_2) + k_2\epsilon_2 \|\varphi^{n-1}_{\bar y_1}(t) - \varphi^{n-1}_{\bar y_2}(t)\|_q + L_2 K_3 \rho_q(\bar y_1, \bar y_2)$$

$$\le k_2\epsilon_2 C \rho^\alpha_q(\bar y_1, \bar y_2) + L'_1 \rho_q(\bar y_1, \bar y_2)$$

where L_2 denotes the bound of the ρ_q norms of $g_{j,y}, j = 1,2$ in the metric ρ_q and $L'_1 = k_2 L_1 + L_2 K_3$. For the second term in (4.4) we have

$$\|A^{-1}_{22}(\bar y_1)A_{21}(\bar y_1)t - A^{-1}_{22}(\bar y_2)A_{21}(\bar y_2)t\|_q$$

$$\|A^{-1}_{22}(\bar y_1)A_{21}(\bar y_1)t - A^{-1}_{22}(\bar y_1)A_{21}(\bar y_2)t\|_q + \|[A^{-1}_{22}(\bar y_1) - A^{-1}_{22}(\bar y_2)]A_{21}(\bar y_2)t\|_q$$

$$\le k_2 K_3 \rho_q(\bar y_1, \bar y_2) + K_2\epsilon_3 \rho_q(\bar y_1, \bar y_2).$$

Finally, we estimate the first term in (4.4):

$$\|A^{-1}_{22}(\bar y_1)\varphi^{n-1}_{F(\bar y_1)}(\Sigma_1) - A^{-1}_{22}(\bar y_2)\varphi^{n-1}_{F(\bar y_2)}(\Sigma_2)\|_q$$

$$\le \|A^{-1}_{22}(\bar y_1)\varphi^{n-1}_{F(\bar y_1)}(\Sigma_1) - A^{-1}_{22}(\bar y_1)\varphi^{n-1}_{F(\bar y_2)}(\Sigma_2)\|_q + \|[A^{-1}_{22}(\bar y_1) - A^{-1}_{22}(\bar y_2)]\varphi^{n-1}_{F(\bar y_2)}(\Sigma_2)\|_q$$

$$\le k_2\|\varphi^{n-1}_{F(\bar y_1)}(\Sigma_1) - \varphi^{n-1}_{F(\bar y_2)}(\Sigma_2)\|_q + K_2 L_3 \rho_q(\bar y_1, \bar y_2)$$

$$\le k_2\|\varphi^{n-1}_{F(\bar y_1)}(\Sigma_1) - \varphi^{n-1}_{F(\bar y_2)}(\Sigma_1)\|_q + k_2\|\varphi^{n-1}_{F(\bar y_2)}(\Sigma_1) - \varphi^{n-1}_{F(\bar y_2)}(\Sigma_2)\|_q + K_2 L_3 \rho_q(\bar y_1, \bar y_2)$$

$$\le k_2 C \rho^\alpha_q(\bar y_1, \bar y_2) + k_2 L_4 \|\Sigma_1 - \Sigma_2\|_q + K_2 L_3 \rho_q(\bar y_1, \bar y_2).$$

where L_3 is the bound of the norms of $\varphi^{n-1}_{F(\bar y_2)}(t)$ and L_4 is the Lipschitz constant of $\varphi^{n-1}_{F(\bar y_2)}(t)$ with respect to t in the metric ρ_q. Since we already know that $(\varphi^m_{\bar y})$ converges to $\varphi_{\bar y}$, these two constants L_3, and L_4 can be chosen independent of $\bar y$ and m.

On the other hand

$$\|\Sigma_1 - \Sigma_2\|_q = \|A_{11}(\bar y_1)t + A_{12}(\bar y_1)\varphi^{n-1}_{\bar y_1)}(t) + g_{1,\bar y_1}(t, \varphi^{n-1}i, \bar y_1(t))$$

$$-[A_{11}(\bar y_2)t + A_{12}(\bar y_2)\varphi^{n-1}_{\bar y_2}(t) + g_{1,\bar y_2}(t, \varphi^{n-1}_{\bar y_2}(t))]\|_q$$

$$\le \|A_{11}(\bar y_1)t - A_{11}(\bar y_2)t\|_q + \|A_{12}(\bar y_1)\varphi^{n-1}_{i,y_1}(t) - A_{12}(\bar y_2)\varphi^{n-1}_{\bar y_2}(t)\|_q$$

$$+\|g_{1,\bar y_1}(t, \varphi^{n-1}_{\bar y_1}(t)) - g_{1,\bar y_2}(t, \varphi^{n-1}_{\bar y_2}(t))\|_q$$

$$\le K_3 \rho_q(\bar y_1, \bar y_2) + [\epsilon_3 C \rho^\alpha_q(\bar y_1, \bar y_2) + K_3 L_3 \rho_q(\bar y_1, \bar y_2)] + \epsilon_2 C \rho^\alpha_q(\bar y_1, \bar y_2).$$

Combining all these estimations together we have

$$\|\varphi^n_{\bar y_1}(t) - \varphi^n_{\bar y_2}(t)\|_q \le [k_2 C + K_4 \rho^{(1-\alpha)}_q(\bar y_1, \bar y_2) + K_5\epsilon_4]\rho^\alpha_q(\bar y_1, \bar y_2),$$

where ϵ_4, depending on ϵ_1, ϵ_2, and ϵ_3, can be taken arbitrarily small and K_4, K_5 are two constants. Thus, if $\rho(\bar{y}_1, \bar{y}_2)$ is chosen to be sufficiently small, we have

$$k_2 C + K_4 \rho_q^{(1-\alpha)}(\bar{y}_1, \bar{y}_2) + K_5 \epsilon_4 \leq C.$$

This concludes the proof of Theorem 4.

Proof of Theorem 5 We first observe that the $\mathbb{Z}^2$ action τ induced by (F, S) is expansive on (Δ, ρ_q) for any $0 < q < 1$ and the conjugacy maps h and h^{-1} are both uniformly continuous in the metric ρ_q, $0 < q < 1$. Thus, the $\mathbb{Z}^2$-action τ induced by (Φ, S) is also expansive in metric ρ_q on Δ_Φ. Then, Theorem 5 becomes a direct corollary of Proposition 1. It is easy to see that for any $0 < q_1 < q_2 < 1$, the metrics ρ_{q_1} and ρ_{q_2} are Hölder equivalent, by which we mean that the following inequality holds for some constants $C > 0$ and $0 < b < 1$:

$$\rho_{q_2}(\bar{x}, \bar{y}) \leq C \rho_{q_1}^b(\bar{x}, \bar{y}).$$

Therefore, the equilibrium states for φ are independent of the choices of the metric ρ_q, $0 < q < 1$.

Proof of Lemma 6.1 Properties 1 and 2 are obvious.

Let us denote the boundary of the Markov Partition $\cup_{i=1}^m R_i$ of Λ by ∂R, then the boundary of the Markov partition of Δ is $\mathcal{B} = \cup_{i \in \mathbb{Z}} \mathcal{B}(i)$, where

$$\mathcal{B}(i) = \{\bar{x} = (x_j)_{j \in \mathbb{Z}} : x_i \in \partial R\}.$$

Property 3 follows directly from the fact that $\pi : \Sigma_A \backslash \pi^{-1}(\partial R) \to \Lambda$ is injective.

Proof of Lemma 6.2 1) Let $\mathcal{U} = (U_i)$ be an open finite covering of Δ. The the collection of sets $\bar{\pi}^{-1}(\mathcal{U}) = (\bar{\pi}^{-1}(U_i))$ is an open cover of $\Sigma_A^{\mathbb{Z}}$. Note that

$$P_\tau(\varphi) = \sup_{\mathcal{U}} P_\tau(\varphi, \mathcal{U}) = \sup_{\mathcal{U}}(\lim_{n,m \to \infty} \frac{1}{nm} \log Z_{X_{n,m}}(\varphi, \mathcal{U}))$$

and that

$$Z_X(\varphi, \mathcal{U}) = \min_{(B_j)} \{\sum_j \exp[\inf_{\bar{y} \in B_j} \sum_{x \in X} \varphi(\tau^x \bar{y})]\}$$

$$= \min_{(\bar{\pi}^{-1} B_j)} \{\sum_j \exp[\inf_{\bar{\xi} \in \bar{\pi}^{-1} B_j} \sum_{x \in X} \varphi^*(\tau \bar{\xi})\} = Z_X(\varphi^*, \bar{\pi}^{-1}(\mathcal{U})).$$

We have $P_\tau(\varphi) \leq P_{\tau^*}(\varphi^*)$.

2) We need only to observe that $\bar{\pi} : \Sigma_A^{\mathbb{Z}} \to \Delta$ is one-to-one except for those points in $\bar{\pi}^{-1}(\mathcal{B})$ and $\mu(\mathcal{B}) = \mu^*(\bar{\pi}^{-1}(\mathcal{B})) = 0$.

Proof of Theorem 6 1) By Lemma 6.2 $h_\tau(\mu)+\int\varphi d\mu = h_\tau(\mu^*)+\int\varphi d\mu^* = P_\tau(\varphi^*) \geq P\tau(\varphi)$. This means μ is an equilibrium state for φ and $P_\tau(\varphi^*) = P\tau(\varphi)$.

2) We first use Hahn-Banach theorem to show that for any probability measure μ τ-invariant on Δ, there exists a τ-invariant probability measure ν on $\Sigma_A^{\mathbb{Z}}$ with $\mu(E) = \mu^*(\bar{\pi}^{-1}(E))$ for any Borel set $E \subset \Delta$.

Consider the linear functional on a subspace of $C(\Sigma_A^{\mathbb{Z}})$ defined by $g\bar{\pi} \to \int g d\mu$, where $g \in C(\Delta, \rho_q)$. This functional is positive and its norm is 1. Thus it corresponds to a probability measure β on $\Sigma_A^{\mathbb{Z}}$. The relation $\int g\bar{\pi}d\beta = \int g d\mu$ implies $\beta(\bar{\pi}^{-1}(E)) = \mu(E)$. However, β may be not τ-invariant on $\Sigma_A^{\mathbb{Z}}$. We define

$$\nu := weak^* - \lim_{(n,m)\to\infty} \frac{1}{nm}\left(\sum_{x\in X_{n,m}} \tau^x \beta\right).$$

Then, ν is τ-invariant and we still have $\mu(E) = \nu(\bar{\pi}^{-1}(E))$. Thus

$$\int g\bar{\pi}d\nu = \int g d\mu. \tag{6.1}$$

Now by Lemma 6.2 $h_\tau(\nu) + \int \varphi^* d\nu \geq h_\tau(\mu) + \int\varphi d\mu = P_\tau(\varphi) = P_\tau(\varphi^*)$. Thus ν is an equilibrium state. So, we set $\mu^* = \nu$.

Proof of Theorem 7 1) We follow the proof of a similar property in the finite dimensional case by Bowen [B]. Let us consider the boundary points of the Markov Partition of Λ: $\partial R = \partial^s R \cup \partial^u R$, where

$$\partial^s R = \cup_{i=1}^m \{x \in R_i : x \notin \text{interior of } V^u(x)\cap R_i\};$$

$$\partial^u R = \cup_{i=1}^m \{x \in R_i : x \notin \text{interior of } V^s(x)\cap R_i\}.$$

Then the boundary of the Markov partition of Δ $\mathcal{B} = \cup_{i\in\mathbb{Z}}\mathcal{B}(i)$, where

$$\mathcal{B}(i) = \{\bar{x} = (x_j)_{j\in\mathbb{Z}} : x_i \in \partial R\}.$$

Let $\mathcal{B}^s(i) = \{\bar{x} = (x_j)_{j\in\mathbb{Z}} : x_i \in \partial^s R\}$ and $\mathcal{B}^u(i) = \{\bar{x} = (x_j)_{j\in\mathbb{Z}} : x_i \in \partial^u R\}$. In order to show $\mu^*(\bar{\pi}^{-1}(\mathcal{B})) = 0$, we will show that for each i, $\mu^*(\bar{\pi}^{-1}(\mathcal{B}^s(i))) = 0$ and $\mu^*(\bar{\pi}^{-1}(\mathcal{B}^u(i))) = 0$.

Consider $D_s(i) = \bar{\pi}^{-1}(\mathcal{B}^s(i))$. $D_s(i)$ is a closed subset of $\Sigma_A^{\mathbb{Z}}$ and $\sigma^t D_s(i) \subset D_s(i)$. As μ^* is σ^t invariant, $\mu^*(\sigma_t)^n D_s(i) = \mu^*(D_s(i))$ for any $n \in I\!N$. Since $(\sigma^t)^{n+1}D_s(i) \subset (\sigma^t)^n D_s(i)$, one has

$$\mu^*(\cap_{n\geq 0}(\sigma^t)^n D_s(i)) = \mu^*(D_s(i)).$$

Now μ^* is σ^t ergodic and positive on open set and the the set $\cap_{n\geq 0}(\sigma^t)^n D_s(i)$ is σ^t-invariant and closed . Then, $\cap_{n\geq 0}(\sigma^t)^n D_s(i)$ has measure zero $\mu^*(D_s(i)) = 0$. Similarly one can show that $\mu^*(\bar{\pi}^{-1}(\mathcal{B}^u(i))) = 0$.

2) Let μ^* be a Gibbs state for a Hölder continuous function ψ on $\Sigma_A^{\mathbb{Z}}$. Without loss of generality, we will prove $\mu^*(\bar{\pi}^{-1}(\mathcal{B}^s(0))) = 0$.

Note that $\partial^s R$ is a finite union of local stable manifolds intersecting with Λ. Let us define the following subsets of $\Sigma_A^{\mathbb{Z}}$. For each fixed element $\xi^* \in \Sigma_A$ and $k \in \mathbb{Z}$

$$E_k(\xi^*) = \{\bar{\xi} = (\xi_i) : \xi_i \in \Sigma, i \in \mathbb{Z}, \xi_0(j) = \xi^*(j), j \leq k\}.$$

One can see that the set $\bar{\pi}^{-1}(\mathcal{B}^s(0))$ is a subset of a finite union of the sets defined above. So it suffices to show that $\mu^*(E_k(\xi^*)) = 0$. Without loss of generality we will further assume $k = 0$ and ξ^* is fixed. We consider the following sequence of sets that converges to $E_0(\xi^*)$.

$$E^m = \{\bar{\xi} = (\xi_i) : \xi_i \in \Sigma, i \in \mathbb{Z}, \xi_0(j) = \xi^*(j), -m \leq j \leq 0\}.$$

One can easily see that $\lim_{m\to\infty} \mu^*(E^{(m)}) = \mu^*(E_0(\xi^*))$. Next we estimate $\mu^*(E^{(m)})$.

Since the set of all Gibbs states for ψ is the closed convex hull of the thermodynamic limits of the Gibbs distributions, we may assume that μ^* is such a limit, i.e., $\lim_{n\to\infty} \mu_n^* = \mu^*$, where μ_n^* are the measures on $\Sigma_A^{\mathbb{Z}}$ determined by the Gibbs distributions

$$p_{X_n}(\bar{\xi}_{X_n}, \bar{\eta}^*_{X_n^C}) = \frac{1}{\sum_{\bar{\eta}:\bar{\eta}|_{X_n^C}=\bar{\eta}^*|_{X_n^C}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}) - \psi(\tau^x(\bar{\xi}_{X_n}, \bar{\eta}^*_{X_n^C})))}$$

$$= \frac{\exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x(\bar{\xi}_{X_n}, \bar{\eta}^*_{X_n^C})))}{\sum_{\bar{\eta}:\bar{\eta}|_{X_n^C}=\bar{\eta}^*|_{X_n^C}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}))}$$

and $X_n \to \mathbb{Z}^2$. In the above expressions, $\bar{\eta}^*$ is any fixed configuration in $\Sigma_A^{\mathbb{Z}}$, $\bar{\xi}_{X_n}$ is the restriction of the configuration $\bar{\xi}$ to the subset X_n and $\bar{\eta}^*_{X_n^C}$ is the restriction of $\bar{\eta}^*$ to X_n^C.

For convenience we introduce the following notation. Let N be the smallest positive integer such that every entry of the matrix A^N is positive. $V_0 = \{(0, j) \in \mathbb{Z}^2, j \in \mathbb{Z}\}$. $S_{m+N} = \{(0, j) \in \mathbb{Z}^2, -m - N \leq j \leq N\}$. $W_n = X_n \cap V_0$. We assume $X_n \supset S_{m+N}$. Thus

$$\mu_n^*(E^{(m)}) = \frac{\sum_{\bar{\eta}:\ \bar{\eta}|_{X_n^C}=\bar{\eta}*|_{X_n^C},\ \bar{\eta}|_{S_m}=\xi^*|_{S_m}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}))}{\sum_{\bar{\eta}:\ \bar{\eta}|_{X_n^C}=\bar{\eta}*|_{X_n^C}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}))}$$

$$= \frac{\sum_{\bar{\eta}_{X_n\setminus V_0}} \sum_{\bar{\eta}_{W_n\setminus S_m}:\bar{\eta}|_{S_m}=\xi^*|_{S_m}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}))}{\sum_{\bar{\eta}_{X_n\setminus V_0}} \sum_{\bar{\eta}_{W_n}} \exp(\sum_{x\in\mathbb{Z}^2} \psi(\tau^x\bar{\eta}))},$$

where the sum $\sum_{\bar\eta_{X_n\setminus V_0}}$ is taken over all configurations $\bar\eta$ on $X_n\setminus V_0$ with $\bar\eta|_{X_n^C}=\bar\eta^*_{X_n^C}$. The restriction of any configuration in the above expressions to X_n^C is always $\bar\eta^*_{X_n^C}$. Other sums are defined in a similar way.

Now we estimate the following quotient

$$\frac{\sum_{\bar\eta_{W_n\setminus S_m}:\bar\eta|_{S_m}=\xi^*|_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}{\sum_{\bar\eta_{W_n}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}$$

$$=\frac{\sum_{\bar\eta_{W_n\setminus S_{m+N}}}\sum_{\bar\eta_{S_{m+N}}:\bar\eta|_{S_m}=\xi^*|_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}{\sum_{\bar\eta_{W_n\setminus S_{m+N}}}\sum_{\bar\eta_{S_{m+N}}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}.$$

Let

$$Q_1=\frac{\sum_{\bar\eta_{S_{m+N}}:\bar\eta|_{S_m}=\xi^*|_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}{\sum_{\bar\eta_{S_{m+N}}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x\bar\eta))}.$$

Let P denote the size of the matrix A. Then,

$$Q_1\le P^N\max_{\bar\eta_{X_n\setminus S_m}}\frac{\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x(\xi^*_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}{\sum_{\bar\eta_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x(\bar\eta_{S_m},\bar\eta|_{X_n^C\setminus S_m}))},$$

where the maximum is taken over all $\bar\eta$ on $X_n\setminus S_m$.

We see that

$$Q_2=\frac{\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x(\xi^*|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}{\sum_{\bar\eta_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x(\bar\eta|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}$$

$$=\frac{1}{\sum_{\bar\eta_{S_m}}\exp(\sum_{x\in\mathbb{Z}^2}\psi(\tau^x(\bar\eta|_{S_m},\bar\eta|_{X_n^C\setminus S_m})-\psi(\tau^x(\xi^*|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}.$$

One can see that there exists an ϵ_1 independent of m and X_n such that

$$\sum_{x\in\mathbb{Z}^2\setminus S_m}|\psi(\tau^x(\bar\eta|_{S_m},\bar\eta|_{X_n^C\setminus S_m})-\psi(\tau^x(\xi^*|_{S_m},\bar\eta|_{X_n^C\setminus S_m})|\le\epsilon_1 m$$

and ϵ_1 can be taken small if c is small. Thus $Q_2\le Q_3\exp\epsilon_1 m$, where

$$Q_3=\frac{\exp(\sum_{x\in S_m}\psi(\tau^x(\xi^*|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}{\sum_{\bar\eta_{S_m}}\exp(\sum_{x\in S_m}\psi(\tau^x(\bar\eta|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))}$$

is a Gibbs distribution on Σ_A. Let us arbitrarily fix a configuration $\bar\eta_1$ over $\mathbb{Z}^2\setminus V_0$. Let ν be the unique Gibbs state for $\psi_{\bar\eta_1}=\psi(\bar\eta_{V_0},\bar\eta_1)$. Then, when m is sufficiently large, there exists $\epsilon_2>0$ such that $\frac{1}{m}\log Q_3$

$$=\frac{1}{m}\sum_{x\in S_m}\psi(\tau^x(\xi^*|_{S_m},\bar\eta|_{X_n^C\setminus S_m})-\frac{1}{m}\log\sum_{\bar\eta_{S_m}}\exp(\sum_{x\in S_m}\psi(\tau^x(\bar\eta|_{S_m},\bar\eta|_{X_n^C\setminus S_m}))$$

$$\leq \int \psi_{\bar{\eta}_1} d\nu - P(\psi_{\bar{\eta}_1}) + \epsilon_2 = -h_\nu + \epsilon_2,$$

where $P(\psi_{\bar{\eta}_1})$ is the topological presure over Σ_A , $h_\nu > 0$ is the metric entropy, and $\epsilon_2 \to 0$ when $c \to 0, m \to \infty$.

Thus, $Q_3 \leq \exp(-h_\nu + \epsilon_2)m$ and $Q_1 \leq P^N \exp(-h_\nu + \epsilon_1 + \epsilon_2)m$. Therefore, $\mu^*(E_0(\xi^*)) = \lim_{m\to\infty} \mu^*(E^m) = 0$.

The statements 1 and 2 of Theorem 8 are direct consequences of Proposition 3 and Theorems 6 and 7. The mixing property with respect to both F and S comes from the equality (6.1)

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