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Nonunique Stationary States in Driven Systems; The Weakly Ionized Plasma

by

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Abstract

We study the electron velocity distribution $f(v,t)$ in a weakly ionized spatially homogeneous plasma in an external electric field E. f satisfies a Boltzmann type equation in which there are linear terms representing collisions with ions and neutral particles having a specified Maxwellian distribution as well as a nonlinear electron-electron (e-e) collision term. Using simplified collision terms we prove that, when e-e collisions dominate, the long time asymptotic behavior of f is non-unique for certain ranges of E. This provides a rigorous proof of a dynamical phase transition for a realistic (model) system.

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Introduction

The understanding of equilibrium phenomena, including their most interesting aspect, that of phase transitions, has enormously advanced in recent years. By contrast, our understanding of nonequilibrium phenomena is still very incomplete at the present time. In particular there is no general microscopic theory of nonequilibrium phase transitions. Theories of such phenomena rely mostly on the qualitative or numerical solutions of model hydrodynamical equations. These show how the great variety and complexity of observed nonequilibrium behavior in fluids, lasers, electron devices, chemical reactions, etc. arises from the nonlinearities in the macroscopic equations [1]. Despite the sucess of such phenomenological approaches, there remains the need for reliable microscopic theories to provide deeper insight into this important area of far from local equilibrium phenomena, encompassing enormous ranges of spatio-temporal scales, in plasmas, living systems and galaxies. Attempts so far in this direction make use of a variety of approaches. These include molecular dynamic simulations of Hamiltonian systems of particles ($\leq 10^6$) interacting via simple pair potentials (hard spheres or disks, cut-off Lenard-Jones, potential, etc.), and the study via analytic means and/or computer simulations of highly simplified (lattice) model systems [3]. Lacking thus far are proofs of the existence of a nonequilibrium phase transition due to cooperative effects for any physical system with realistic modeling of the dynamics.

In this note we prove such a result for a simplified, but still recognizable, model of a weakly ionized plasma in the presence of an external electric field. A more realistic version of this system has been studied in [4]. It was found there, using various approximations, that this system may undergo a phase transition, including hysteresis, as the magnitude of the field E is varied. Here we supply a rigorous mathematical proof of this phenomena. In particular we prove that there really exist stationary distributions $f(v)$ in the vicinity of any one of the stable fixed points on the hysteresis loop obtained in [4] from the solution of a pair of nonlinear equations for the hydrodynamical variables; current and temperature. We show furthermore that when the system is started near such a stable hydrodynamic state it will remain there forever. If, on the contrary, it is started near the unstable fixed point, it will leave the neighborhood after some finite time.

Our formal set up is as follows: We consider a weakly ionized gas in $\mathbb{R}^3$ in the presence of an externally imposed constant electric field E. The density of the gas, the degree of ionization and the strength of the field are assumed to be such that: (i) the interactions between the electrons can be described by some nonlinear, Boltzmann type collision operator, and (ii) collisions between the electrons and the heavy components of the plasma, ions and neutrals, are adequately described by assuming the latter ones to have a spatially homogeneous time independent Maxwellian distribution with an a priori given temperature. Under these conditions the time evolution of the spatially homogeneous electron velocity distribution function $f(v,t)$ will satisfy a Boltzmann type equation

$$\frac{\partial f(v,t)}{\partial t} = -E \cdot \nabla f + Lf + \epsilon^{-1} Q(f), \tag{1}$$

where ∇ is the gradient with respect to v and the mass and charge of the electron have been set equal to unity. The linear term Lf represents collisions with the ions and neutrals and we have introduced a coupling parameter ϵ^{-1} in front of the nonlinear term $Q(f)$ representing e-e interactions.

We refer the reader to refs. [4] for a detailed description of the different terms in (1). In the present work we use simplified collision terms: We replace the linear term by a Fokker-Planck type operator with a velocity dependent diffusion coefficient, chosen to have a maximum at $|v| = 0$ and approach a constant value when $|v| \to \infty$. This mimics the fact that the cross section of collisions of the electrons with the ions, is strongly peaked at low speeds (going to zero as $|v| \to \infty$) while collisions with neutrals has an approximately constant cross-section. In accordance with assumption (ii) L tries to bring f to equilibrium with the "thermal bath" represented by the massive neutrals, i.e. to a Maxwellian distribution with zero mean velocity and the a priori specified temperature of the neutrals (and ions). Choosing units in which this temperature is unity this equilibrium distribution of the electron velocities is, $M_n(v) \equiv (2\pi)^{-3/2} \exp[\frac{-v^2}{2}]$. In the absence of the external field E, $f(v,t)$ would approach $M_n(v)$, as $t \to \infty$, independent of the nature of the collisions between the electrons. For $E \neq 0$ we have to model these collisions. We simplify the collision term here keeping, however, its essential physical features: it conserves momentum and energy and vanishes when f is *any* Maxwellian. Thus while the electrons are being driven away from equilibrium with the ions and neutrals by the electric field their internal collisions try to bring the distribution to a general Maxwellian

$M(v; u, T) = (2\pi)^{-3/2} \exp[-(v-u)^2/2T]$ with the instantaneous value of momentum u and energy $e = \frac{1}{2}u^2 + \frac{3}{2}T$.

The nonequilibrium "phase transition" in this system, which we show that f and thus also the current and energy of the electrons undergo, can be understood as a transition, when the speed of the electrons increases with the field from a low energy regime in which there is a strong coupling to the ions to a high energy regime where this coupling essentially vanishes. In the absence of electron-neutral collisions this leads to a runaway situation [5]. With neutrals present the stationary distribution would be smooth as E changes in the absence of the nonlinear coupling induced by the electron-electron collisions, the frequently used "electron swarm" approximation [6]. These collisions produce a cooperative effect which prevents the distribution from deviating too much from a Maxwellian. The strength of this effect is controlled by the parameter ϵ, which measures the effective mean time between electron-electron collisions. Our proofs apply to the situation when ϵ is "small enough".

The precise form of the collision operator L we shall use is

$$Lf(v) = \nabla \cdot \left(D(v) M_n(v) \nabla \left(\frac{f(v)}{M_n(v)} \right) \right) \tag{2}$$

where

$$D(v) = a \exp(-b|v|^2/2) + c \tag{3}$$

for some strictly positive constants a, b and c. For $Q(f)$ we take the BGK model of the Boltzmann collision kernel [7],

$$Q(f) = M_f - f \tag{4}$$

where for any velocity distribution f, M_f denotes the Maxwellian distribution with the same first and second moments as f. Since $Q(f)$ drives f close towards the Maxwellian manifold M_f we should have, formally at least, that in the limit $\epsilon \to 0$, f will equal M_f for all time $t > 0$. To keep track of the evolution of f we would then need only keep track of the moments

$$u(t) \equiv \int_{\mathbb{R}^3} v f(v,t) \mathrm{d}^3 v, \tag{5}$$

and

$$e(t) \equiv \frac{1}{2} \int_{\mathbb{R}^3} v^2 f(v,t) \mathrm{d}^3 v \quad . \tag{6}$$

Using the prescription $f = M_f$, one easily obtains

$$\frac{du}{dt} = F(u,e) = E - u[c + a\exp(-w)\frac{1+b}{(1+b/\beta)^{5/2}}], \tag{7a}$$

$$\frac{de}{dt} = G(u,e) = Eu - c[2e(1-\beta) + \beta u^2] - \frac{a\exp(-w)}{(1+b/\beta)^{\frac{5}{2}}}[2e(1-\beta) + u^2(\beta - b\frac{1-\beta}{b+\beta})], \tag{7b}$$

where $\beta^{-1} = T = \frac{2}{3}(e - \frac{1}{2}u^2)$, and $w = bu^2/2(1+bT)$. Solving (7) for the stationary values of u and e gives, for certain ranges of $|E|/c$, depending on b and c/a, three such pairs, two stable, and one unstable, see Fig. 1.

Our primary goal here is to show that this situation actually does hold for small, but positive, values of ϵ; i.e. that the interaction between the hydrodynamic and the non–hydrodynamic modes does not destroy this picture involving only the hydrodynamic modes. The following theorems enable us to do this.

Theorem 1. *Let f be a solution of (1) with $f(v,0) = M_0(v; u_0, T_0)$ some arbitrary Maxwellian. Then there is a constant K depending only on a, b, c, $|E|$ and e_0 such that*

$$\|f(v,t) - M_{f(\cdot,t)}(v)\| \le K\epsilon^{1/2}, \tag{8}$$

$\|\cdot\|$ denoting the $L^1(\mathbb{R}^3)$–norm.

Theorem 2. *Let (u^*, e^*) be a stationary point of (7) and T^* the corresponding temperature. Then for ϵ small enough, there exists a unique stationary point in an ϵ-neighborhood of $M(v; u^*, T^*)$, and there are no other stationary solutions of (1) that do not correspond to stationary solutions of (7).*

Moreover, if (u^*, e^*) is a stable stationary solution of (7), then there is a neighborhood of $M(v; u^*, T^*)$ that is stable for (1). More precisely, given any $\delta > 0$, there is an ϵ greater than zero such that if

$$\|f(v,0) - M(v, u^*, T^*)\| \le \epsilon \tag{9}$$

then the $f_\epsilon(v,t)$ which solves (1) with the value of ϵ satisfies

$$\|f_\epsilon(v,t) - M(v; u^*, T^*)\| \le \delta \tag{10}$$

for all $t \geq 0$.

Likewise, the unstable stationary solutions of (7) can be shown to correspond to unstable stationary solutions of (1).

The detailed description of the stationary solutions follows by their almost explicit construction. In fact one can show that the moments (u_ϵ, e_ϵ) of the stationary solution f_ϵ satisfy a closed equation of the form

$$F(u_\epsilon, e_\epsilon) + \epsilon \tilde{F}_\epsilon(u_\epsilon, e_\epsilon) = 0, \tag{11a}$$

$$G(u_\epsilon, e_\epsilon) + \epsilon \tilde{G}_\epsilon(u_\epsilon, e_\epsilon) = 0, \tag{11b}$$

with suitable $\tilde{F}_\epsilon$ and $\tilde{G}_\epsilon$.

An application of the implicit function theorem provides the moments (u_ϵ, e_ϵ) of the stationary solution in an ϵ-neighborhood of (u^*, e^*). These moments determine M_{f_ϵ} and hence f_ϵ itself, using the stationary equation. This construction is peculiar of the choice of the BGK collision kernel, but the results should extend to more general collisions.

The stability statements in theorem (2) are consequence of a more general result: if we compute $u(t)$ and $e(t)$ for a solution $f(v,t)$ of (1) satisfying $f(v,0) = M_0(v; u_0, T_0)$ then $u(t)$ and $e(t)$ satisfy

$$\begin{aligned} \frac{d}{dt} u(t) &= F(u,e) + \epsilon^{1/4}\gamma(t) \\ \frac{d}{dt} e(t) &= G(u,e) + \epsilon^{1/4}\eta(t) \end{aligned} \tag{12}$$

where F and G are defined in (7) and $\gamma(t), \eta(t)$ are bounded uniformly in t with a bound independent of ϵ.

Our proof of Theorems 1 and 2, whose details will be presented elsewhere, has several ingredients. First we prove by direct calculations and "interpolation inequalities" some a priori bounds on the moments and smoothness of $f(v,t)$. We then prove an entropy production inequality to get strong bounds on the tendency of the BGK operator to keep f nearly Maxwellian.

To get an idea of how the latter works consider the time evolution of the system's "free energy" $A(t) = [e - T_n s]$

$$A(t) = \int [\frac{1}{2}v^2 + \log f] f(v,t) d^3v = \int \log(f/M_n) f d^3v - \frac{3}{2}\log 2\pi \tag{13}$$

where T_n is the ion and neutral temperature which we have set equal to 1. Using the properties of L and Q we find after some manipulations

$$\frac{d}{dt}A(t) = E \cdot u - 4\int D(v)M_n(v)[\nabla\sqrt{f/M_n}]^2 d^3v - \frac{1}{\epsilon}\int \log(f/M_f)[f - M_f]d^3v. \quad (14)$$

Combining (14) with other inequalities and bounds leads to the inequality

$$\int \log(f/M_f)f(v,t)d^3v \le \epsilon K, \quad (15)$$

where K is a constant. This shows that f has to stay close to M_f when ϵ is small.

Our methods allow us also to take into account an initial layer, i.e. we need not start with a Maxwellian $f(v,0)$. We can also extend our results to more realistic collision kernels both linear and nonlinear. We have not tried to formulate this here since these required additional conditions and our main interest is to show rigorously how nonuniqueness of the electron distribution in the limit $t \to \infty$ can come about from the dynamics of the interactions between the electrons in a strongly driven nonequilibrium system. Just how strong the field needs to be depends of course on the model. It is easy to see that for $E = 0$, $f(v,t) \to M_n(v)$ independent of the initial $f(v,0)$ exponentially fast in time. We can expect the same to be true for sufficiently small fields, with the current u linear in E. For our particular model this will also occur for very large $|E|$, since for large velocities $D(v) \to c$ and for a constant $D(v) = c$ the stationary solution is $M(v;u,T)$ with $u = E/c$ and $T = T_n = 1$ in our units. With more realistic electron-neutral collisions the large $|E|$ behavior will presumably be that of a Druyvestan distribution [6].

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References

[1] M.Cross and P.C. Hohenberg, Rev. Mod. Phys., **65**, 881 (1993); P. Bergé, Y. Pomeau, C. Vidal, *Order within Chaos*, J. Wiley (1984); A.V. Gaponov-Grekhov, M.I. Rabinovich, *Nonlinearities in Action*, Sprinegr (1988).

[2] J. Koplik, J.R. Banavar and J.F. Willemson, Phys. Rev. Lett. **60**. 1282 (1988); Phys. Fluids A, **1**, 781 (1989); D. Rappaport, Phys. Rev. Lett. **60**, 2480 (1988).

[3] B. Schmittmann, Intl. J. Mod. Physics, **B4**, 2269 (1990); B. Schmittmann and R.K.P. Zia, preprint; P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); T.M. Liggett, *Interacting Particle Systems*, Springer Verlag (1985).

[4] A. Rokhlenko, Phys. Rev.A, **43**, 4438 (1991); A. Rokhlenko and J.L. Lebowitz, Phys. Fluids B: Plasma Physics, **5**, 1766 (1993).

[5] c.f. R. Balescu, *Transport Processes in Plasmas*, North-Holland (1988), Chapter 19.

[6] R.N. Franklin, *Plasma Phenomena in Gas Discharges*, Clarendon Press, Oxford, 1976.

[7] c.f. C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer Verlag, 1988.

$$b = \frac{a}{c} = 0.1$$

u is the drift velocity

$e = \frac{u^2}{2} + \frac{3}{2}T$ is the mean energy