The dynamical entropy of the quantum Arnold cat map

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Abstract

We present a rigorous computation of the dynamical entropy h of the quantum Arnold cat map. This map, which describes a flow on the non-commutative two-dimensional torus, is a simple example of a quantum dynamical system with optimal mixing properties, characterized by Lyapunov exponents $\pm \ln \lambda^+$, $\lambda^+ > 1$. We show that, for all values of the quantum deformation parameter, h coincides with the positive Lyapunov exponent of the dynamics.

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1 Introduction

The dynamical entropy is supposed to measure the asymptotic loss of information about the state of a dynamical system due to its time evolution. If a dynamical system is sufficiently differentiable, one expects that its entropy equals the sum of the positive Lyapunov exponents [Le]. These exponents measure the exponential divergence in time of small perturbations in initial conditions. It is, even for easily formulated dynamical systems such as the Hénon map, very difficult to obtain rigorous results. The Arnold cat map however, is quite tractable because it is, up to "cutting and pasting", linear. Still, because of the combined actions of stretching and folding, it is a paradigm of a non-linear system.

For classical dynamical systems, there is a well-established notion of dynamical entropy also known as the Kolmogorov-Sinai invariant [W]. An elementary construction of a dynamical entropy for discrete quantum systems was presented in [AF]. This construction is fundamentally connected to considering half-chain spin models of the system, with the unilateral shift as time evolution. The central notion is that of operational partition of unity as considered in [Li]. It is the aim of this note to compute the dynamical entropy h of the q-deformation of the Arnold cat map. This dynamical system lives on a non-commutative torus and it occurs naturally in the context of the quantum Hall effect [B]. It turns out that h is precisely equal to the positive Lyapunov exponent of the system and that it is therefore a good measure of the strong mixing properties of the q-Arnold cat map [ENTS]. In particular, h is independent of the deformation parameter q.

We should at this point comment on the relation between the dynamical entropy in the sense of [AF] and that in the sense of [CNT, ST]. For standard examples of dynamical systems, such as classical systems, shifts on a quantum spin chain and gauge-invariant quasi-free Fermion automorphisms, both schemes produce, up to structural constants, the same entropy, though it must be said that the computations in the [CNT] setup are much more elaborate and need additional regularity conditions on the invariant state. For less "physical" systems, such as the free shift, the difference becomes extreme [AN], but one could argue that such systems are irrelevant anyway.

The difference between the [CNT] and [AF] entropy becomes striking in the case of the q-Arnold cat map. Indeed, it is shown in [NT], that the [CNT] entropy vanishes almost everywhere as a function of q, while the excellent ergodic behaviour is independent of q [ENTS].

We briefly remind the construction of the entropy. Consider a discrete dynamical system i.e. a unital C*-algebra $\mathcal{A}$, an automorphism Θ of $\mathcal{A}$ and a state ω on $\mathcal{A}$ which is invariant under Θ. We will also need to specify a unital $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, globally invariant under Θ. A partition of size k is a k-tuple $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ of elements of $\mathcal{A}$ satisfying:

$$\sum_{i=0}^{k-1} x_i^* x_i = \mathbb{1}.$$

Two partitions $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ and $\mathcal{Y} = (y_0, y_1, \ldots, y_{\ell-1})$ can be composed to yield a new partition $\mathcal{X} \circ \mathcal{Y} = (x_0\, y_0, x_0\, y_1, \ldots, x_{k-1}\, y_{\ell-1})$ of size $k\,\ell$. Also, if $\mathcal{X}$ is a partition, then $\Theta(\mathcal{X}) = (\Theta(x_0), \Theta(x_1), \ldots, \Theta(x_{k-1})$ is again a partition.

To any size k partition $\mathcal{X}$ we associate a $k \times k$ density matrix $\rho[\mathcal{X}]$ with (i,j) matrix-element $\omega(x_j^*\, x_i)$. We now define the dynamical entropy in terms of the von Neumann entropy of the density matrices generated by the partition $\mathcal{X}$ and Θ. More precisely, let $\mathcal{X}$ be any finite partition. $h_{(\Theta,\omega)}(\mathcal{X})$ is defined as:

$$\begin{aligned} h_{(\Theta,\omega)}(\mathcal{X}) &= \limsup_m \frac{1}{m} S(\rho[\Theta^{m-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]) \\ &= \limsup_m \frac{1}{m} \operatorname{Tr} \eta(\rho[\Theta^{m-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]). \end{aligned}$$

η is the standard entropy function $x \mapsto -x \ln x$ on $[0,1]$. The dynamical entropy $h_{(\mathcal{A}_0,\Theta,\omega)}$ is obtained by taking the supremum of $h_{(\Theta,\omega)}(\mathcal{X})$ over all finite partitions in $\mathcal{A}_0$:

$$h_{(\mathcal{A}_0,\Theta,\omega)} = \sup_{\mathcal{X} \subset \mathcal{A}_0} h_{(\Theta,\omega)}(\mathcal{X}).$$

2 The entropy of the q-Arnold cat map

We parametrize the two-dimensional torus $\mathbf{T}^2$ by a coordinate $z = (x, y) \in \mathbf{R}^2/\mathbf{Z}^2$, i.e. each point on $\mathbf{T}^2$ corresponds to a point $z \in [0, 1[\times[0, 1[$. The sides of the unit square with x or $y = 1$ are identified with those with x or $y = 0$. $\mathbf{T}^2$, equipped with the composition law

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 \bmod 1, y_1 + y_2 \bmod 1)$$

and with the usual convergence is a compact, commutative topological group. The normalized Haar measure $d\mathbf{x}$ on $\mathbf{T}^2$ assigns the value $(\delta_2 - \delta_1)(\epsilon_2 - \epsilon_1)$ to the open set $]\delta_1, \delta_2[\times]\epsilon_1, \epsilon_2[$, $0 \le \delta_1 < \delta_2 < 1$, $0 \le \epsilon_1 < \epsilon_2 < 1$.

Let $\mathrm{SL}(2, \mathbf{Z})$ be the group of all 2×2 matrices with integer coefficients and with determinant equal to 1. Due to these conditions, the natural action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$ preserves $d\mathbf{x}$. Each $T \in \mathrm{SL}(2, \mathbf{Z})$ defines a discrete dynamical system $(\mathbf{T}^2, T, d\mathbf{x})$ in the sense of [AA]. Such systems are called Arnold cat maps. In order to avoid trivialities we will assume that $\mathrm{Tr}\, T > 2$. The two Lyapunov exponents of T are $\pm \ln \lambda^+$, where λ^+ is the largest eigenvalue of T.

We will use a description of the cat map on the level of the algebra $\mathcal{C}$ of complex continuous functions on $\mathbf{T}^2$. Let u and v denote the functions $z \mapsto \exp(i2\pi x)$ and $z \mapsto \exp(i2\pi y)$. By the Stone-Weierstraß theorem, $\mathcal{C}$ is the abelian C*-algebra generated by the commuting unitaries u and v. It is the closure, in the uniform norm, of the algebra $\mathcal{D}$ generated by u and v. It will be useful to introduce for $\chi = (m, n) \in \mathbf{Z}^2$, the CCR-like notation $W(\chi) = u^m\, v^n$. The Haar measure $d\mathbf{x}$ translates in the state τ with $\tau(W(\chi)) = \delta_{\chi,0}$ and the dynamics determined by $T \in \mathrm{SL}(2, \mathbf{Z})$ becomes the $*$-automorphism $\Theta_T(W(\chi)) = W(\hat{T}\, \chi)$, $\hat{T}$ being the transpose of T. $(\mathcal{C}, \Theta_T, \tau)$ is now an algebraic dynamical system in the sense of [HK].

Choose $0 \le q < 2\pi$. A simple quantum deformation of the Arnold cat map is obtained by modifying the relation between the unitaries u and v to $u\, v = e^{iq}\, v\, u$. Putting for $\chi = (m, n) \in \mathbf{Z}^2$,

$$W(\chi) = e^{-iqmn/2}\, u^m\, v^n,$$

we obtain the relations

$$W(\chi_1)\,W(\chi_2) = e^{iq\sigma(\chi_1,\,\chi_2)/2}\,W(\chi_1+\chi_2), \quad \chi_1,\chi_2 \in \mathbf{Z}^2,$$

where $\sigma(\chi_1,\chi_2) = m_1\,n_2 - n_1\,m_2$. We will denote by $\mathcal{D}_q$ the algebra generated by the $W(\chi)$, $\chi \in \mathbf{Z}^2$ and by $\mathcal{C}_q$ the corresponding C*-algebra. The relation $\tau(W(\chi)) = \delta_{\chi,0}$ still defines a tracial state on $\mathcal{C}_q$ and, as $\sigma(\hat{T}\,\chi_1,\hat{T}\,\chi_2) = \sigma(\chi_1,\chi_2)$, $\Theta_T(W(\chi)) = W(\hat{T}\,\chi)$ extends to a $*$-automorphism of $\mathcal{C}_q$.

Theorem *Let λ^+ be the largest eigenvalue of the matrix $T \in \mathrm{SL}(2,\mathbf{Z})$ and assume $\mathrm{Tr}\,T > 2$. For any $0 \le q < 2\pi$ we have*

$$h_{(\mathcal{D}_q,\Theta_T,\tau)} = \ln\lambda^+.$$

Proof: The statement is a direct consequence of propositions 1 and 2. ■

3 The proofs

Let us introduce the following notation. For $f,g \in \ell^1(\mathbf{Z}^2)$ we put

$$W(f) = \sum_{\chi\in\mathbf{Z}^2} f(\chi)\,W(\chi).$$

Clearly $\|W(f)\| \le \|f\|_1$. Furthermore, $W(f)^* = W(f^\dagger)$ with $f^\dagger(\chi) = \overline{f(-\chi)}$, $W(f)\,W(g) = W(f*g)$ with

$$f*g(\chi) = \sum_{\xi\in\mathbf{Z}^2} \exp(iq\sigma(\chi,\xi))\,f(\chi-\xi)\,g(\xi),$$

$\tau(W(f)) = f(0)$ and $\tau(W^*(f)\,W(g)) = \langle f,g\rangle$. If we denote by $\ell^0(\mathbf{Z}^2)$ the space of complex sequences on $\mathbf{Z}^2$ with only a finite number of non-zero elements, then $\mathcal{D}_q$ is precisely $\{W(f) \mid f \in \ell^0(\mathbf{Z}^2)\}$. For any partition $\mathcal{X} = (x_0,\ldots,x_{k-1}) = (W(f_0),\ldots,W(f_{k-1}))$ in $\mathcal{D}_q$, we will denote

$$\mathrm{Support}(\mathcal{X}) = \bigcup_{j=0}^{k-1} \mathrm{Support}(f_j)$$

with $\mathrm{Support}(f) = \{\chi \in \mathbf{Z}^2 \mid f(\chi) \neq 0\}$.

The density matrix $\rho[\mathcal{X}]$ corresponding to the partition

$$\mathcal{X} = (W(f_0), \ldots, W(f_{k-1}))$$

in $\mathcal{D}_q$ has matrix elements

$$\rho[\mathcal{X}]_{ij} = \langle f_j \mid f_i \rangle$$

and therefore

$$S(\rho[\mathcal{X}]) = S([\langle f_j \mid f_i \rangle]_{ij}) = \mathrm{Tr}\, \eta([\langle f_j \mid f_i \rangle]_{ij}). \tag{1}$$

We will first concentrate on partitions generated by $f_j \in \ell^0(\mathbf{Z}^2)$ that live on a single point of $\mathbf{Z}^2$. Let for $k \in \{1, 2, \ldots\}$, $\mathcal{X}$ be such a partition

$$\mathcal{X} = \Big(\frac{e^{i\alpha_0}\, W(\chi_0)}{\sqrt{k}}, \cdots, \frac{e^{i\alpha_{k-1}}\, W(\chi_{k-1})}{\sqrt{k}}\Big), \tag{2}$$

with $0 \leq \alpha_j < 2\pi$ and $\chi_j \in \mathbf{Z}^2$. We define an equivalence relation between the indices $\{0, \ldots, k-1\}$ by putting $a \equiv b$ if $\chi_a = \chi_b$. The $\#[a] \times \#[a]$ matrix $P_{[a]}$, with r, s matrix-element $(\exp i(\alpha_r - \alpha_s))/\#[a]$ projects onto a one-dimensional subspace of $\mathbf{C}^{\#[a]}$. The density matrix $\rho[\mathcal{X}]$ can be written as

$$\rho[\mathcal{X}] = \bigoplus_{[a]} \frac{\#[a]}{k}\, P_{[a]}.$$

Therefore

$$S(\rho[\mathcal{X}]) = \sum_{[a]} \eta(\frac{\#[a]}{k}).$$

Lemma 1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be partitions of the form*

$$\mathcal{X} = \Big(\frac{e^{i\alpha_0}\, W(\chi_0)}{\sqrt{k}}, \cdots, \frac{e^{i\alpha_{k-1}}\, W(\chi_{k-1})}{\sqrt{k}}\Big),$$

and

$$\mathcal{Y} = \Big(\frac{e^{i\beta_0}\, W(\psi_0)}{\sqrt{\ell}}, \cdots, \frac{e^{i\beta_{\ell-1}}\, W(\psi_{\ell-1})}{\sqrt{\ell}}\Big).$$

Then $S(\rho[\mathcal{X} \circ \mathcal{Y}]) \geq S(\rho[\mathcal{X}])$.

Proof: Putting $(i,j) \equiv (a,b)$ if $\chi_i + \psi_j = \chi_a + \psi_b$, we can again express the entropy $S(\rho[\mathcal{X} \circ \mathcal{Y}])$ as

$$S(\rho[\mathcal{X} \circ \mathcal{Y}]) = \sum_{[(i,j)]} \eta\Big(\frac{\#[(i,j)]}{k\,\ell}\Big).$$

Moreover

$$\#[(i,j)] = \sum_{\substack{[a],[b] \\ [(a,b)]=[(i,j)]}} \#[a]\,\#[b].$$

Define now $c([(i,j)])$ by

$$c([(i,j)]) = \sum_{\substack{[a] \\ \exists b\,\text{s.t.}\,[(a,b)]=[(i,j)]}} \frac{\#[a]}{k}.$$

Using $\eta(x\,y) = x\,\eta(y) + y\,\eta(x)$ and the positivity and concavity of η we obtain

$$\begin{aligned} S(\rho[\mathcal{X} \circ \mathcal{Y}]) &= \sum_{[(i,j)]} \eta\Big(\frac{\#[(i,j)]}{k\,\ell}\Big) \\ &= \sum_{[(i,j)]} \eta\Big(\sum_{\substack{[a],[b] \\ [(a,b)]=[(i,j)]}} \frac{\#[a]}{k}\,\frac{\#[b]}{\ell}\Big). \end{aligned}$$

Let $\tilde{b}$ be such that $[(a,\tilde{b})] = [(i,j)]$, then

$$\begin{aligned} S(\rho[\mathcal{X} \circ \mathcal{Y}]) &= \sum_{[(i,j)]} \eta\Big(c([(i,j)])\,\Big(\sum_{\substack{[a] \\ \exists b\,\text{s.t.}\,[(a,b)]=[(i,j)]}} \frac{1}{c([(i,j)])}\,\frac{\#[a]}{k}\,\frac{\#[\tilde{b}]}{\ell}\Big)\Big) \\ &\geq \sum_{[(i,j)]} c([(i,j)])\,\eta\Big(\sum_{\substack{[a] \\ \exists b\,\text{s.t.}\,[(a,b)]=[(i,j)]}} \frac{1}{c([(i,j)])}\,\frac{\#[a]}{k}\,\frac{\#[\tilde{b}]}{\ell}\Big) \\ &\geq \sum_{[(i,j)]} \sum_{\substack{[a] \\ \exists b\,\text{s.t.}\,[(a,b)]=[(i,j)]}} \frac{\#[a]}{k}\,\eta\Big(\frac{\#[\tilde{b}]}{\ell}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{[b]} \eta(\frac{\#[b]}{\ell}) \\
&= S(\rho[\mathcal{Y}])
\end{aligned}
$$

Proposition 1 *Let λ^+ denote the largest eigenvalue of T, then*

$$h_{(\mathcal{D}_q,\Theta_T,\tau)} \geq \ln \lambda^+.$$

Proof: Assume first that we can find, for each $m \in \mathbf{N}_0$, a partition $\mathcal{X}$ of the form

$$\mathcal{X} = (\frac{W(\chi_0)}{\sqrt{k}}, \cdots, \frac{W(\chi_{k-1})}{\sqrt{k}}),$$

such that $h_{(\mathcal{D}_q,\Theta_T^m,\tau)}(\mathcal{X}) \geq \ln[(\lambda^+)^m]$, where $[\nu]$, $\nu \in \mathbf{R}$ is the integer part of ν. We enlarge, for $n \in \mathbf{N}_0$, the partition

$$\mathcal{X}_1(m,n) = \Theta_T^{m(n-1)}(\mathcal{X}) \circ \cdots \circ \Theta_T^m(\mathcal{X}) \circ \mathcal{X}$$

to

$$\mathcal{X}_2(m,n) = \Theta_T^{m(n-1)}(\mathcal{X}) \circ \Theta_T^{m(n-1)-1}(\mathcal{X}) \cdots \circ \Theta_T(\mathcal{X}) \circ \mathcal{X}.$$

Both partitions are of the form (2). Remark that for two such partitions $\mathcal{Y}_1$ and $\mathcal{Y}_2$, $S(\rho[\mathcal{Y}_1 \circ \mathcal{Y}_2]) = S(\rho[\mathcal{Y}_2 \circ \mathcal{Y}_1])$. Repeated applications of this remark and of lemma 1 yield

$$
\begin{aligned}
h_{(\mathcal{D}_q,\Theta_T,\tau)} &\geq \limsup_n \frac{1}{m(n-1)+1} S\Big(\rho[\mathcal{X}_2(m,n)]\Big) \\
&\geq \frac{1}{m} \limsup_n \frac{1}{n} S\Big(\rho[\mathcal{X}_1(m,n)]\Big) \\
&\geq \frac{1}{m} \ln[(\lambda^+)^m].
\end{aligned}
$$

The statement of the proposition now follows by taking the limit $m \to \infty$.

It remains to prove the assumption at the beginning of the proof. Choose any $0 \neq \chi \in \mathbf{Z}^2$ and consider the partition

$$\mathcal{X} = \Big(\frac{W(0)}{\sqrt{[(\lambda^+)^m]}}, \frac{W(\chi)}{\sqrt{[(\lambda^+)^m]}}, \cdots, \frac{W(([(\lambda^+)^m]-1)\chi)}{\sqrt{[(\lambda^+)^m]}}\Big).$$

We label each element of

$$\mathcal{X}_1(m,n) = \Theta_T^{m(n-1)}(\mathcal{X}) \circ \cdots \circ \Theta_T^m(\mathcal{X}) \circ \mathcal{X}$$

by a n-tuple $(a_{n-1}, \ldots, a_1, a_0)$, $a_j \in \{0, 1, \ldots, [(\lambda^+)^m] - 1\}$. The mappings

$$(a_{n-1}, \ldots, a_1, a_0) \mapsto \sum_{j=0}^{n-1} a_j \, [\lambda^+]^{\pm j}$$

are injective. As the eigenvalues of $\hat{T}^m$ are $(\lambda^+)^{\pm m}$, also

$$(a_{n-1}, \ldots, a_1, a_0) \mapsto \sum_{j=0}^{n-1} a_j \, (\lambda^+)^{\pm j}$$

is injective. Therefore $\rho[\mathcal{X}_1(m,n)]$ is diagonal with equal entries $[(\lambda^+)^m]^{-n}$. It follows that $h_{(\mathcal{D}_q, \Theta_T^m, \tau)}(\mathcal{X}) \geq \ln[(\lambda^+)^m]$.

■

In order to obtain an upper bound for h, we have now to consider general partitions in $\mathcal{D}_q$.

Lemma 2 *Consider a partition* $\mathcal{X} = (W(f_0), \ldots, W(f_{k-1}))$ *in* $\mathcal{D}_q$, *then*

$$S(\rho[\mathcal{X}]) \leq \ln \# \Big(\mathrm{Support}(\mathcal{X})\Big).$$

Proof: As $\mathcal{X}$ is a partition we have

$$\sum_{j=0}^{k-1} W(f_j)^* \, W(f_j) = \mathbb{1}.$$

In particular it follows that

$$\sum_{j=0}^{k-1} \|f_j\|^2 = 1.$$

Consider an orthonormal basis $\{e_i \mid i = 0, \ldots, k-1\}$ of $\mathbf{C}^k$, then $\rho[\mathcal{X}]$ is the restriction of the pure state determined by the normalized vector

$$\sum_j e_j \otimes f_j \in \mathbf{C}^k \otimes \ell^2(\mathbf{Z}^2)$$

to the factor $\mathcal{M}_k$ in $\mathcal{M}_k \otimes \mathcal{B}(\ell^2(\mathbf{Z}^2))$, where $\mathcal{M}_k$ are the $k \times k$ matrices with complex entries. As the restrictions of a pure state on a tensor product are given by density matrices that have the same eigenvalues taking multiplicities into account, except possibly for the eigenvalue zero [AL], we can conclude from (1) that:

$$S(\rho[\mathcal{X}]) = S([\langle f_j, f_i\rangle]_{ij}) = S(\sum_{j=0}^{k-1} |f_j\rangle\langle f_j|) \leq \ln \#\big(\mathrm{Support}(\mathcal{X})\big).$$

$\blacksquare$

Proposition 2 *Let λ^+ denote the largest eigenvalue of T, then*

$$h_{(\mathcal{D}_q, \Theta_T, \tau)} \leq \ln \lambda^+.$$

Proof: Let e_+ and e_- be the eigenvectors of T which belong to the eigenvalues λ^+ and $\lambda^- = 1/\lambda^+$. For any $\mathcal{X} \subset \mathcal{D}_q$, there exist $a, b > 0$ with $\mathrm{Support}(\mathcal{X}) \subset R(a, b)$, where

$$R(a, b) = \{\chi \in \mathbf{Z}^2 \mid \chi = \alpha e_+ + \beta e_-, |\alpha| < a, \, |\beta| < b\}.$$

Then

$$\mathrm{Support}(\Theta_T^{n-1}(\mathcal{X}) \circ \ldots \circ \Theta_T(\mathcal{X}) \circ \mathcal{X}) \subset R(a\frac{\lambda^{+n} - 1}{\lambda^+ - 1}, b\frac{1}{1 - \lambda^-}).$$

Due to 2

$$S(\rho[\Theta_T^{n-1}(\mathcal{X}) \circ \ldots \circ \mathcal{X}]) < \mathrm{constant} + n \ln \lambda^+.$$

$\blacksquare$

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