Geometric characterization of linearisable second–order differential equations

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Abstract

Given an Ehresmann connection on the tangent bundle $\tau : TM \to M$ we define a linear connection on the pull–back bundle $\tau^*(TM)$. With the aid of this tool, necessary and sufficient conditions for the existence of local coordinates in which a system of second–order differential equations is linear are derived.

1 Introduction

In previous papers [8, 9] we developed a theory of derivations of scalar and vector-valued forms along the tangent bundle projection $\tau : TM \to M$ of a manifold M. This theory was used in order to characterize in a geometric way when a system of second–order differential equations is separable, i. e. there exists a coordinate system on the base manifold in which the system decouples as a sum of one–dimensional independent differential equations (see [10]). Two fundamental operators associated with every vector field X along τ did appear in these papers. They were denoted D^V_X and D^H_X and called the horizontal and vertical covariant derivative operators. Their usefulness is partially based on the fact that they are derivations (of degree zero) and that the maps $X \mapsto D^H_X$ and $X \mapsto D^V_X$ are $C^\infty(TM)$–linear, so that there is a clear resemblance with a covariant derivative operator.

In this paper we show that this similarity is not a coincidence but a consequence of the existence of a covariant derivative on the pull–back bundle by τ of the tangent bundle. With the aid of this tool, we will geometrically characterize those second–order differential equations which are linearisable under a point transformation of coordinates.

In order to the paper be selfcontained, we give in Section 1 a summary of the results of the theory of derivations which are relevant for subsequent sections. In Section 2 we will introduce the linear connection associated to a non–linear connection on the tangent bundle and we will show the relation between the curvature of the linear connection and the tensors Rie and θ, that were previously defined in terms of the horizontal and vertical covariant derivatives. Some relations between Rie and θ which are just the expression of the Bianchi identities for the linear connection are also given. Finally, in Section 3 we study the problem of the linearisation of a second–order differential equations. Our results will be related to previous ones [1, 13]

2 Preliminaries

Let $\pi\colon E \to M$ be a fibre bundle and $\phi\colon N \to M$ a smooth map. A section of E along ϕ is a map $\sigma\colon N \to E$ such that $\pi \circ \sigma = \phi$. Equivalently, a section along ϕ can be considered as a section $\bar{\sigma}$ of the pull–back bundle $\phi^*\pi\colon \phi^*E \to N$ (see e. g. [11]). The relation between σ and $\bar{\sigma}$ is $\sigma = \phi[\pi] \circ \bar{\sigma}$, and we have the following commutative diagram

$$\begin{tikzcd} \phi^*E \arrow[r, "\phi[\pi]"] \arrow[d, shift right, "\phi^*\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[u, shift right, "\bar{\sigma}"'] \arrow[ur, "\sigma"] \arrow[r, "\phi"'] & M \end{tikzcd}$$

If E is a vector bundle then the set of sections along ϕ is a $C^\infty(N)$–module. The most interesting cases are $E = TM$, $(T^*M)^{\wedge p}$ or any other tensor bundle, and then a section of E along ϕ is called a vector field along ϕ, a p-form along ϕ, or a tensor field along ϕ (respectively). The set of vector fields along ϕ is denoted by $\mathcal{X}(\phi)$, and the set of p-forms along ϕ by $\bigwedge^p(\phi)$. In particular, we are interested in the case in which ϕ is the projection τ of the tangent bundle $\tau: TM \to M$. The easiest example of tensor field along τ is the composition $W \circ \tau$ for W a tensor field on the base manifold M. We will then say that W is a basic tensor field and we will simplify the notation writing just W instead of $W \circ \tau$. Another example is the canonical vector field along τ denoted by $\mathbf{T}$ and defined by the identity map on TM thought of as a section of TM along τ.

In two recent papers [8, 9] we have studied the algebra of derivations of scalar and vector valued forms along τ. A complete classification of the set of derivations of forms along τ was given with the aid of an Ehresmann connection in the tangent bundle, i. e. a horizontal subbundle of $T(TM)$ (see [11, 2, 3]). In presence of a connection, any vector field $Z \in \mathcal{X}(TM)$ can be decomposed as a sum $Z = X_1^H + X_2^V$ for X_1, $X_2 \in \mathcal{X}(\tau)$, where the indices H and V mean horizontal and vertical lift, respectively. In coordinates, if Z has the expression $Z = X^i\,\partial/\partial x^i + Y^i\,\partial/\partial v^i$, then X_1 and X_2 are given by $X_1 = X^i\,\partial/\partial x^i$ and $X_2 = (Y^i + \Gamma^i_j X^j)\,\partial/\partial x^i$, respectively. Here, the functions Γ^i_j are the coefficients of the connection, which are defined by the horizontal lift $H_i = \partial/\partial x^i - \Gamma^j_i\,\partial/\partial v^j$ of the coordinate vector field $\partial/\partial x^i$. If W is a vertical vector field on TM, the vector field X along τ such that $W = X^V$ will be denoted $W^{\downarrow}$. Note that if W is a horizontal vector field on TM then the vector field X along τ such that $X^H = W$ is just the projection of W, i. e. $X = T\tau \circ W$.

With the above decomposition in mind, we define two fundamental derivations D^H_X and D^V_X for every $X \in \mathcal{X}(\tau)$, called respectively the horizontal and vertical covariant derivatives, by mean of the equation

$$[X^H, Y^V] = \{D^H_X Y\}^V - \{D^V_Y X\}^H.$$

They are called covariant because they are $C^\infty(TM)$-linear in the subscript argument. We extend the action of these derivations to 1-forms along τ by imposing the duality rule

$$D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle,$$

and to general tensor fields along τ by imposing the Leibnitz rule for the tensor product. Derivations satisfying these properties are called self-dual. If X, Y are vector fields along τ with local expressions $X = X^i\,\partial/\partial x^i$ and $Y = Y^i\,\partial/\partial x^i$, then the coordinate expressions of $D^H_X Y$ and $D^V_X Y$ are

$$D^H_X Y = X^i(H_i Y^j + Y^k \Gamma^j_{ik})\,\frac{\partial}{\partial x^j}$$

$$D^V_X Y \;=\; X^i (V_i Y^j)\,\frac{\partial}{\partial x^j},$$

where $V_i = \partial/\partial v^i$ and $\Gamma^i_{jk} = V_k(\Gamma^i_j)$. It can be easily shown that a tensor field W along τ is basic if and only if $D^V W = 0$.

The commutators of two horizontal lifts and two vertical lifts are respectively given by

$$\begin{aligned}
[X^V, Y^V] &= [X,Y]_V^V \\
[X^H, Y^H] &= [X,Y]_H^H + \{R(X,Y)\}^V
\end{aligned}$$

where $[X,Y]_V = D^V_X Y - D^V_Y X$, $[X,Y]_H = D^H_X Y - D^H_Y X - T(X,Y)$ and where R and T are, respectively, the curvature and the torsion of the non–linear connection. It is easy to see (e. g. in coordinates) that if X and Y are vector fields on the base manifold M, then $[X,Y] = [X,Y]_H$.

The commutators of the covariant derivatives define two important 2-covariant tensor fields along τ, θ and Rie, taking values in the set of (1,1) tensor fields, as follows:

$$\begin{aligned}
[D^V_X, D^V_Y]Z &= D^V_{[X,Y]_V} Z \\
[D^V_X, D^H_Y]Z &= D^H_{D^V_X Y} - D^V_{D^H_Y X} Z + \theta(X,Y)Z \\
[D^H_X, D^H_Y]Z &= D^H_{[X,Y]_H} Z + D^V_{R(X,Y)} Z + \mathrm{Rie}(X,Y)Z.
\end{aligned}$$

Their coordinate expressions are

$$\begin{aligned}
\theta &= \Gamma^k_{jml}\, \mathrm{d}x^l \otimes \mathrm{d}x^j \otimes \left(\mathrm{d}x^m \otimes \frac{\partial}{\partial x^k}\right) \\
\mathrm{Rie} &= \frac{1}{2}[H_k(\Gamma^i_{lj}) - H_l(\Gamma^i_{kj}) + \Gamma^i_{kr}\Gamma^r_{lj} - \Gamma^i_{lr}\Gamma^r_{kj}]\mathrm{d}x^k \wedge \mathrm{d}x^l \otimes \left(\mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}\right),
\end{aligned}$$

where $\Gamma^i_{jkl} = V_l\,\Gamma^i_{jk}$. It follows from the coordinate expression of Rie that

$$\mathrm{Rie} = -D^V R,$$

i. e. if X, Y, Z are vector fields along τ, then $\mathrm{Rie}(X,Y)Z = -[D^V_Z R](X,Y)$.

The vertical covariant derivative of the canonical vector field along τ is the identity tensor $D^V\mathbf{T} = I$, while the horizontal covariant derivative defines an important (1,1) tensor field associated to the connection which is called the tension $\mathbf{t} = -D^{H}\mathbf{T}$. Its coordinate expression is

$$\mathbf{t} = (\Gamma^i_j - \Gamma^i_{jk} v^k) \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}.$$

Notice that $\mathbf{t} = 0$ iff the connection is linear, i. e. the functions Γ^i_{jk} do not depend on v^l. In this case the tensor Rie coincides with the Riemann curvature tensor of the linear connection.

3 Induced linear connection

We remind that a covariant derivative on a vector bundle $\pi\colon E \to N$ is a $C^\infty(N)$–linear operator mapping every vector field V of N onto a derivation D_V of the $C^\infty(N)$–module $\mathrm{Sec}\,(\pi)$ of sections of the bundle E, such that it coincides precisely with V on $C^\infty(N)$. That is, D satisfies the following properties:

1. $\mathrm{D}_{V+W}\sigma = \mathrm{D}_V\sigma + \mathrm{D}_W\sigma$
2. $\mathrm{D}_{fV}\sigma = f\mathrm{D}_V\sigma$
3. $\mathrm{D}_V(\sigma + \rho) = \mathrm{D}_V\sigma + \mathrm{D}_V\rho$
4. $\mathrm{D}_V(f\sigma) = (Vf)\sigma + f\mathrm{D}_V\sigma$,

for all $V, W \in \mathcal{X}(N)$, $f \in C^\infty(M)$ and $\sigma, \rho \in \mathrm{Sec}\,(\pi)$. A covariant derivative on a bundle is equivalent to a linear connection on such bundle (see [11, 6]).

As we mentioned in the last section, the operators D^{H} and D^{V} are similar to actual covariant derivatives. Taking into account that sections of the bundle $\tau^*\tau\colon \tau^*(TM) \to TM$ are (canonically identified with) vector fields along τ, it is clear that the horizontal and vertical covariant derivatives must be related with a true covariant derivative on such bundle. Indeed, for every vector field W on TM we define the operator $\widetilde{\mathrm{D}}_W$ by

$$\widetilde{\mathrm{D}}_W Z = \{P_V[P_H(W), Z^{V}]\}^{\downarrow} + T\tau \circ [P_V(W), Z^{H}], \qquad (*)$$

where Z is a vector field along τ and P_H, P_V are the horizontal and vertical projectors defined by the connection. The $\mathbb{R}$–linearity of $[\cdot\,,\cdot]$ implies that $\widetilde{\mathrm{D}}$

satisfies properties (1) and (3). We now prove that $\widetilde{\mathrm{D}}$ satisfies (2): if F is a function on TM, then using $[FP_H(W), Z^V] = -Z^V F\, P_H(W) + F[P_H(W), Z^V]$ and $[FP_V(W), Z^H] = -Z^H F\, P_V(W) + F[P_V(W), Z^H]$ we have

$$\begin{aligned}\widetilde{\mathrm{D}}_{FW} Z &= -Z^V F\, P_V P_H(W) + F\{P_V[P_H W, Z^V]\}^{\downarrow} + FT\tau \circ [P_V(W), Z^H] \\ &= F\, \widetilde{\mathrm{D}}_W Z,\end{aligned}$$

because $P_V P_H = 0$. Furthermore, $[P_H(W), FZ^V] = P_H(W)F\, Z^V + F[P_H(W), Z^V]$ and $[P_V(W), FZ^H] = P_V(W)F\, Z^H + F[P_V(W), Z^H]$, so that

$$\begin{aligned}\widetilde{\mathrm{D}}_W(FZ) &= \{P_H(W)F\, Z^V + FP_V[P_H W, Z^V]\}^{\downarrow} \\ &\quad + T\tau \circ ((P_V(W)F)Z^H + F[P_V(W), Z^H]) \\ &= (P_H(W)F + P_V(W)F)Z + F(P_V[P_H(W), Z^V])^{\downarrow} \\ &\quad + FT\tau + ([P_V(W), Z^H]),\end{aligned}$$

and taking into account that $P_H + P_V = I$, we find that $\widetilde{\mathrm{D}}$ satisfies (4). Thus, $\widetilde{\mathrm{D}}$ is a covariant derivative in the pull–back bundle by τ.

As we said above, every vector field W on TM can be decomposed as a sum of a vertical plus a horizontal lift of a vector field along τ, $W = X^H + Y^V$. The following relation holds

$$\widetilde{\mathrm{D}}_{X^H + Y^V} = D^H_X + D^V_Y.$$

Indeed, taking into account the equation defining D^H and D^V, we obtain

$$\begin{aligned}\widetilde{\mathrm{D}}_{X^H + Y^V} Z &= (P_V[X^H, Z^V])^{\downarrow} + T\tau + ([Y^V, Z^H]) \\ &= ((D^H_X Z)^V)^{\downarrow} + T\tau + ((D^V_Y Z)^H) \\ &= D^H_X Z + D^V_Y Z.\end{aligned}$$

We have proved the following result:

Theorem 1 *Given a connection on the tangent bundle TM to a manifold M, there exists one linear connection $\widetilde{\mathrm{D}}$ on the pull–back bundle $\tau^*\tau\colon \tau^*(TM) \to TM$ such that*

$$\widetilde{\mathrm{D}}_{X^H + Y^V} = D^H_X + D^V_Y.$$

for all X, $Y \in \mathcal{X}(\tau)$. Such connection is explicitly given by equation (*) and will be said to be the linear connection induced by the non–linear connection on TM.

We remind that if $\phi: N \to M$ is a smooth map and $H(E)$ is an Ehresmann connection on the bundle $\pi: E \to M$, then $(T\phi[\pi])^{-1}(H(E))$ is an Ehresmann connection on the bundle $\phi^*\pi: \phi^*E \to N$. This connection is known as the induced conection by ϕ. We remark that in the case $\phi = \tau$ the induced connection is different from the induced linear connection here defined. In particular, if the original connection is non linear, then the induced connection is non linear too.

If we choose local coordinates (x^i, v^i, w^i) on $\tau^*(TM)$ such that

$$\tau^*\tau(x^i, v^i, w^i) = (x^i, v^i) \quad \text{and} \quad \tau[\tau](x^i, v^i, w^i) = (x^i, w^i),$$

then the expressions of the horizontal lifts of the coordinate vector fields with respect to the induced linear connection are

$$\left(\frac{\partial}{\partial x^i}\right)^{\widetilde{H}} = \frac{\partial}{\partial x^i} - w^k\,\Gamma^j_{ik}\frac{\partial}{\partial w^j} \qquad \left(\frac{\partial}{\partial v^i}\right)^{\widetilde{H}} = \frac{\partial}{\partial v^i}.$$

Consider now the Riemann curvature tensor

$$\widetilde{\mathrm{Rie}}(V, W)Z = [\widetilde{\mathrm{D}}_V, \widetilde{\mathrm{D}}_W]Z - \widetilde{\mathrm{D}}_{[V,W]}Z.$$

of the connection $\widetilde{\mathrm{D}}$. We will show that the tensors Rie and θ are but components of $\widetilde{\mathrm{Rie}}$.

Theorem 2 *The curvature of the connection $\widetilde{\mathrm{D}}$ is given by*

$$\begin{aligned}
\widetilde{\mathrm{Rie}}(X^H, Y^H)Z &= \mathrm{Rie}(X, Y)Z \\
\widetilde{\mathrm{Rie}}(X^V, Y^H)Z &= \theta(X, Y)Z \\
\widetilde{\mathrm{Rie}}(X^V, Y^V)Z &= 0
\end{aligned}$$

for $X, Y, Z \in \mathcal{X}(\tau)$.

Proof: For two horizontal vector fields X^H and Y^H, we have

$$\begin{aligned}
\widetilde{\mathrm{Rie}}(X^H, Y^H)Z &= [D^H_X, D^H_Y]Z - \widetilde{\mathrm{D}}_{\{[X,Y]_H\}^H + R(X,Y)^V}Z \\
&= [D^H_X, D^H_Y]Z - D^H_{[X,Y]_H}Z - D^V_{R(X,Y)}Z \\
&= \mathrm{Rie}(X, Y)Z.
\end{aligned}$$

Similarly, for X^V and Y^H, we obtain

$$\begin{aligned}
\widetilde{\mathrm{Rie}}(X^V, Y^H)Z &= [D^V_X, D^H_Y]Z - \widetilde{\mathrm{D}}_{\{D^V_X Y\}^H - \{D^H_Y X\}^V} Z \\
&= [D^V_X, D^H_Y]Z - D^H_{D^V_X Y} Z + D^V_{D^H_Y X} Z \\
&= \theta(X, Y)Z.
\end{aligned}$$

Finally, for two vertical vector fields we find

$$\begin{aligned}
\widetilde{\mathrm{Rie}}(X^V, Y^V)Z &= [D^V_X, D^V_Y]Z - \widetilde{\mathrm{D}}_{\{[X,Y]_V\}^V} Z \\
&= [D^V_X, D^V_Y]Z - D^V_{[X,Y]_V} Z \\
&= 0.
\end{aligned}$$

Having a linear connection, we have at our disposal a menagerie of results. In particular, the Bianchi identities

$$\sum_{\text{cyclic}} \widetilde{\mathrm{D}}_U \widetilde{\mathrm{Rie}}(V, W) = 0,$$

implies some relations that Rie and θ must satisfy. They are:

$$\begin{gathered}
\theta(X, Y)Z - \theta(Z, Y)X = 0, \\
D^H_Z \theta(X, Y) - D^H_Y \theta(X, Z) - \theta(X, T(Y, Z)) + D^V_X \mathrm{Rie}(Y, Z) = 0, \\
\textstyle\sum_{\text{cyclic}} \{D^H_X \mathrm{Rie}(Y, Z) + \mathrm{Rie}(X, T(Y, Z)) - \theta(R(X, Y), Z)\} = 0.
\end{gathered}$$

The proof is a matter of straightforward calculation and will be omited. This Bianchi identities were first found in [7] by calculating the Jacobi identities for D^H and D^V. We also mention that in [8] we found another set of relations between the torsion and the curvature of the non–linear connection, called the Bianchi identities for the non–linear connection.

4 Characterization of linearisable SODE

In this section we will consider the following problem: Given a system of second–order differential equations

$$\ddot{x}^i = f^i(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n), \qquad i = 1, \dots, n,$$

we ask under what circumstances there exists a coordinate transformation

$$\bar{x}^i = \xi^i(x^1, \ldots, x^n), \qquad i = 1, \ldots, n,$$

such that in the coordinates $\bar{x}^i$ the system of second–order equations turns out to be linear:

$$\ddot{\bar{x}}^i = A^i_j \dot{\bar{x}}^j + B^i_j \bar{x}^j + C^i, \qquad i = 1, \ldots, n,$$

for some contants A^i_j, B^i_j and C^i.

In geometric terms, a system of second–order equations (SODE) on a manifold M is interpreted as a vector field Γ in the tangent bundle to M, such that $T_v\tau(\Gamma(v)) = v$ for every $v \in TM$. Every SODE defines a connection on the tangent bundle (see [3, 2]). In coordinates, if f^i are the forces defined by a SODE Γ, $f^i = \Gamma v^i$, then the coefficients of the connection are $\Gamma^i_j = -\frac{1}{2}\partial f^i/\partial v^j$. It follows that the functions $\Gamma^i_{jk} = \partial \Gamma^i_j/\partial v^k$ are symmetric in the subscripts j, k, that is, the torsion of the connection vanishes. In fact, the vanishing of the torsion is a necessary and sufficient condition for the connection to be defined by a SODE.

More directly related to the SODE, there are a derivation ∇ called the dynamical covariant derivative and a (1,1) tensor field Φ called the Jacobi endomorphism. They can simultaneously be defined by the equation

$$[\Gamma, X^H] = \{\nabla X\}^H + \{\Phi(X)\}^V,$$

and contain the whole information about the SODE. In fact, on functions ∇ coincides with $\mathcal{L}_\Gamma$. The action of ∇ over coordinate vector fields is determined by the coefficients of the connection in such coordinates

$$\nabla\left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_j \frac{\partial}{\partial x^i},$$

so that the dynamical covariant derivative of $X = X^i\,\partial/\partial x^i$ has the expression

$$\nabla X = (\Gamma X^i + \Gamma^i_j X^j)\frac{\partial}{\partial x^i}.$$

In terms of the induced linear connection we have $\nabla X = \widetilde{\mathrm{D}}_\Gamma X + \mathbf{t}(X)$. The coordinate expression of the Jacobi endomorphism is

$$\Phi = \left(-\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j)\right)\,\mathrm{d}x^j \otimes \frac{\partial}{\partial x^i},$$

and it satisfies the following relation

$$\Phi(X) = R(\mathbf{T}, X) - D^{H}_{X}\nabla\mathbf{T}.$$

Moreover, Φ is related with the curvature by mean of

$$(D^{V}_{X}\Phi)(Y) - (D^{V}_{Y}\Phi)(X) = 3R(X, Y).$$

Furthermore, we have the following commutators:

$$\begin{aligned}
[\nabla, D^{V}_{X}]Y &= D^{V}_{\nabla X}Y - D^{H}_{X}Y, \\
[\nabla, D^{H}_{X}]Y &= D^{H}_{\nabla X}Y + D^{V}_{\Phi(X)}Y - [D^{V}_{Y}\Phi](X) - R(X, Y),
\end{aligned}$$

for X and Y vector fields along τ.

We now turn to the question of the linearisability of second–order ordinary differential equations. In geometric terms, a SODE Γ is said to be linearisable in velocities (v–linearisable) if for each point $m \in M$ there exists a set of local coordinates (x^i) in a neighborhood $\mathcal{U}$ of m such that the coordinate expression of the forces defined by Γ is of the form

$$f^i(x, v) = A^i_j(x)v^j + b^i(x)$$

for some functions A^i_j, $b^i \in C^\infty(\mathcal{U})$.

Theorem 3 *A* SODE Γ *is v–linearisable if and only if the connection $\widetilde{\mathrm{D}}$ is flat.*

Proof: Assume that Γ is v–linearisable. In the coordinates in which Γ is linear in the velocities the coefficients of the connection defined by Γ are $\Gamma^i_j = -\frac{1}{2}A^i_j(x)$. Thus $\Gamma^i_{jk} = 0$, from where it follows $\widetilde{\mathrm{Rie}} = 0$.

Conversely, if $\widetilde{\mathrm{D}}$ is flat, then in a neighbourhood of each point there exists a basis $\{X_i\}$ of $\mathcal{X}(\tau) = \mathrm{Sec}\,(\tau^*\tau)$ such that $\widetilde{\mathrm{D}}_W X_i = 0$ for every vector field W on TM. Since $\widetilde{\mathrm{D}}_{X^V} = D^{V}_{X}$ we have that X_i are basic vector fields. Moreover, taking into account that $\widetilde{\mathrm{D}}_{X^H} = D^{H}_{X}$ and that $[X, Y] = D^{H}_{X}Y - D^{H}_{Y}X$ we find that $[X_i, X_j] = 0$. Thus, there exist local coordinates (x_i) for M such that $X_i = \partial/\partial x^i$. In these coordinates the functions Γ^i_{jk} vanish, from where it follows that Γ^i_j are basic functions and that the expression of the forces is an affine function of the velocities.

It is worthy of note that the forces depend only on the positions iff, being linearisable, the tension of the connection vanishes. Also note that the existence of a v–linearisable SODE imposes severe restrictions to the topology of the manifold M. Indeed, since $\theta = 0$, we have that if X and Y are vector fields on M, then $D^{\scriptscriptstyle H}_X Y$ is also a basic vector field, so that it defines a linear connection D on M. Since the curvature tensor of this linear connection is just the Rie tensor, we have that D is flat. In particular, M is locally isomorphic to an ordinary affine space (endowed with the natural connection on it).

A SODE Γ is said to be linearisable if for each point $m \in M$ there exist local coordinates (x^i) in a neighbourhood $\mathcal{U}$ of m such that the coordinate expressions of the forces defined by Γ are of the form

$$f^i(x,v) = A^i_j v^j + B^i_j x^j + C^i,$$

for some constants A^i_j, B^i_j and C^i.

Theorem 4 *A SODE Γ is linearisable if and only if the connection $\widetilde{\mathrm{D}}$ is flat and the Jacobi endomorphism is parallel.*

Proof: If Γ is linearisable, then it is v–linearisable, and therefore $\widetilde{\mathrm{D}}$ is flat. In coordinates in which Γ is linear we have that the components of the Jacobi endomorphism are constant $\Phi^i_j = -B^i_j - \frac{1}{4}A^i_k A^k_j$, from where $\widetilde{\mathrm{D}}\Phi = 0$ readily follows.

Conversely, assume that $\widetilde{\mathrm{D}}$ is flat and $\widetilde{\mathrm{D}}\Phi = 0$. Since $\theta = 0$, we have $D^{\scriptscriptstyle H} \circ D^{\scriptscriptstyle V} = D^{\scriptscriptstyle V} \circ D^{\scriptscriptstyle H}$ and thus $D^{\scriptscriptstyle V}\mathbf{t} = -D^{\scriptscriptstyle V} D^{\scriptscriptstyle H}\mathbf{T} = -D^{\scriptscriptstyle H} D^{\scriptscriptstyle V}\mathbf{T} = -D^{\scriptscriptstyle H} I = 0$. Moreover, since Φ is basic the curvature vanishes, from where we have

$$0 = D^{\scriptscriptstyle V}\Phi = -D^{\scriptscriptstyle V} D^{\scriptscriptstyle H}\nabla\mathbf{T} = -D^{\scriptscriptstyle H} D^{\scriptscriptstyle V}\nabla\mathbf{T} = -D^{\scriptscriptstyle H} D^{\scriptscriptstyle H}\mathbf{T} = D^{\scriptscriptstyle H}\mathbf{t}.$$

Thus the tension is a parallel tensor field. In coordinates in which the coefficients Γ^i_{jk} vanish we have that $\mathbf{t}^i_j = \Gamma^i_j$ and Φ^i_j are constant. It follows that $\partial f^i/\partial v^j$ and $\partial f^i/\partial x^j$ are constant too, and thus Γ is linear in these coordinates.

As a consequence of the relations between the Jacobi endomorphism, the curvature and the Rie tensor we have the following result:

Theorem 5 *A SODE Γ is linearisable if and only if $\theta = 0$, $D^{\scriptscriptstyle H}\Phi = 0$ and $D^{\scriptscriptstyle V}\Phi = 0$.*

Proof: ¿From $D^V\Phi = 0$ we have $R = 0$, and thus $\mathrm{Rie} = -D^V R = 0$.

To finish the paper we consider some previous results in the literature. In [13] the following characterization was given: A SODE Γ is linearisable iff:

1. $\Gamma = \Lambda + A_0^V + A_1(\mathbf{T})^V$, where Λ is a spray, A_0 is a vector field on M and A_1 is a (1,1) tensor field on M.

2. The curvature of the connection D defined by Λ is flat.

3. $\mathrm{D}A_1$ and D^2A_0 vanishes.

In order to relate it with our results, we note that the vanishing of θ is equivalent to (1) and that the connection defined by the spray Λ is given by $\mathrm{D}_X Y = D^H_X Y$ for $X, Y \in \mathcal{X}(M)$. Thus D is flat iff Rie vanishes. Finally, since $A_1 = -2\mathbf{t}$ we have $\mathrm{D}A_1 = 0$ iff the tension is parallel. In such a case, $\mathrm{D}A_0 = -\Phi - \mathbf{t}^2$, from where we deduce that $\mathrm{D}^2A_0 = 0$ if and only if the Jacobi endomorphism is parallel.

Kamran [5] has also studied the problem of linearisation by using Cartan's method of equivalence. Kamran dealt with time–dependent coordinate transformations and therefore their results are not applicable to our case. On the other hand, Chern [1] has solved the equivalence problem under spatial diffeomorphism. The invariants founded there correspond to $\mathbf{T}$, $\nabla\mathbf{T}$, Φ, R and θ, and the covariant derivatives of $\nabla\mathbf{T}$, one of them being the tension, $\mathbf{t} = -D^H\mathbf{T} = D^V\nabla\mathbf{T}$. If follows that if R and θ vanish and moreover Φ and $\mathbf{t}$ are constant (i. e. parallel), then the SODE is equivalent to a linear one. From our results it follows that some of this conditions are redundant.

Finally it is worthy to mention that a diferent kind of linearisability problem has been treated in [4, 12]. There, the problem is to linearise a non–autonomous SODE by coordinate transformations of the form $\bar{t} = \tau(t, x)$, $\bar{x} = \xi(t, x)$. These coordinate changes induce projective transformations on $\mathbb{R} \times TM$ so that the formalism used here is not directly applicable. We hope that further research will allow us to solve that problem.

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