Renormalizability Proof for QED based on Flow Equations

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Abstract

We prove the perturbative renormalizability of Euclidean QED_4 using flow equations, i.e. with the aid of the Wilson renormalization group adapted to perturbation theory. As compared to Φ_4^4 the additional difficulty to overcome is that the regularization violates gauge invariance. We prove that there exists a class of renormalization conditions such that the renormalized Green functions satisfy the QED Ward identities and such that they are infrared finite at nonexceptional momenta. We give bounds on the singular behaviour at exceptional momenta (due to the massless photon) and comment on the adaptation to the case when the fermions are also massless.

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1 Introduction

About twenty years ago Wilson and his collaborators published their ideas on the renormalization group and effective Lagrangians [1], which have stimulated the progress of quantum field theory and statistical mechanics ever since. In 1984 Polchinski [2] showed that these ideas are suited for a treatment of the renormalization problem of perturbative field theory which does not make any use of Feynman diagrams and in particular sidesteps the complicated analysis of the divergence/convergence properties of the general bare or renormalized Feynman diagram. Instead he showed that the problem can be solved by bounding the solutions of a system of first order differential equations, the flow equations, which are a reduction of the Wilson flow equations to their perturbative content.
The present paper is part of a programme of the authors with the aim to show that the Polchinski method is suited to prove (in the sense of mathematical physics) the perturbative renormalizability of any by naive power counting renormalizable theory of physical interest. Polchinski's original proof for Euclidean massive Φ_4^4 was restricted to unphysical renormalization conditions (because they were imposed on the Green functions with an additional (large) infrared cutoff), and it was achieved by estimating the solutions of three types of flow equations for different quantities successively. In our first paper we redid Polchinski's proof with two essential modifications: By showing the effective Lagrangian to be the generating functional of the perturbative connected amputated Green functions (CAG) we could include any renormalization conditions (r.c.). Recently the construction of the analytical minimal subtraction scheme was performed explicitly [8]. By improving Polchinski's induction hypothesis for bounding the solutions of the flow equations we could reduce the proof to one type of flow equations (FE) only [3]. The method was then applied to prove the renormalization of composite operators, the Zimmermann identities, and the existence of the short distance expansion [4,5]. It turned out particularly suited for studying questions of convergence of the regularized theory to the renormalized one which go under the name of Symanzik's improvement programme [6]; see also [7], where the same question is analyzed in Polchinski's original framework. A recent proof by one of the authors also established a de Calan-Rivasseau bound for the large orders of perturbation theory - i.e. local Borel summability - for massive Φ_4^4, which shows that the FE method works beyond questions of perturbative finiteness [9]. In recent years there has also been increasing interest in the FE method from a more phenomenological point of view, i.e. with the aim to find new approximation schemes for the system of FE which differ from standard perturbation theory. In this case the FE are mostly presented and analysed in different form, namely for one particle

irreducible Green functions. For example critical exponents for Φ_4^4-type theories have been calculated in [10]. It has also been applied to the problem of bound states and vacuum condensates [11], see also [12].
If the FE are supposed to be suited for a renormalizability proof of, say, the standard model, it is necessary to cope with gauge theories. Gauge symmetries constitute a particular problem, since our framework crucially makes use of momentum space cutoffs, which necessarily violate gauge invariance, or - on the level of Green functions - the Ward identities (WIs). The problem is less severe for an Abelian gauge theory as QED due to the absence of photon self-interactions. Nevertheless it necessitates the introduction of new counter terms to render the Green functions finite. The theory including these new counter terms will be called a fermion photon theory in the following. It contains more free parameters than QED. We studied the renormalizability of QED in a recent letter [13]. There it was shown that there is a unique choice for the r.c. corresponding to the new counter terms such that the WIs are restored in the renormalized theory. This proves the renormalizability of perturbative Euclidean QED. In this paper we want to give a complete and fully rigorous proof of the renormalizability of perturbative QED. In particular we shall not make use of the nonexistent path integral measures to derive the WIs and their violation. And we want to go beyond the previous letter in that we do not restrict any more to a theory with a massive photon. The method of dealing with theories with massless particles has been developed previously for massless Φ_4^4 [14] and shall be applied to QED now, where we still restrict to the Euclidean framework, however. The renormalization of QED using noninvariant regularizations has also been studied by Feldman, Hurd, Rosen and Wright [15], Hurd [16] (this paper is closest in spirit to ours), Rosen and Wright [17]. These papers are based on the Gallavotti-Nicolò tree- formalism and they also include de Calan-Rivasseau type bounds on the large orders of perturbation theory and certain statements on Minkowski-space theory. Their method is still closer to Feynman diagram based proofs than Polchinski's method. They work in position space and they do not make explicit statements on the IR singularities for exceptional momentum configurations. Our method permits to analyze these singularities (see Proposition 6).
We proceed as follows. In section 2 we introduce the FE framework and the Lagrangian which for a special choice of counter terms will be proven later on to define perturbative QED in a general covariant gauge. In section 3 we prove the renormalizability of our $O(4)$- and charge conjugation invariant fermion photon theory, in which however for general counter terms or r.c. the WIs are violated. The reader unfamiliar with the subject is recommended to first read the papers [3] or [4], and [14] where we describe the same procedure more extensively and where the line of thought is not burdened by the heavy notation required due to the

QED symmetry structure. In section 4 we explicitly derive the violated WIs (vWIs) for the regularized theory as relations between CAGs. We show that there is a unique choice for the r.c. corresponding to those counter terms which manifestly vanish in invariantly regularized QED such that - for cutoff to infinity - the QED WIs are restored. In the last section we comment on the modifications necessitated when one regards massless fermions or does not renormalize at zero momentum.

2 The Fermion-Photon Theory - Definition and Flow Equations

As usual in the FE framework we start by defining the regularized propagators, here for photon and fermion, in Euclidean space. We set for $m > 0$

$$(D_\Lambda^{\Lambda_0}(k))_{\alpha\beta} := \frac{1}{k^2}[(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}) + \frac{1}{\lambda}\frac{k_\alpha k_\beta}{k^2}]\,(R(\Lambda_0,k) \;-\; R(\Lambda,k)) \tag{1}$$

$$S_\Lambda^{\Lambda_0}(p) \;:=\; \frac{1}{\not p + m}(R_m(\Lambda_0,p) \;-\; R_m(\Lambda,p))$$

with $\not p = p_\mu\gamma_\mu$, $\{\gamma_\mu,\gamma_\nu\} \;=\; -2\delta_{\mu\nu}$ for the Euclidean Dirac matrices. The functions R_a for $a \geq 0$ are characterized as follows:

$$R_a(\Lambda,k) \;=\; K(\frac{k^2+a^2}{\Lambda^2})\,, \quad R(\Lambda,k) \;:=\; R_0(\Lambda,k)\,,\ (\Lambda,k) \neq (0,0) \ \text{for}\ a=0\,. \tag{2}$$

Here $0 \leq \Lambda \leq \Lambda_0 \leq \infty$, and K satisfies

$$K \in C^\infty[0,\infty), \quad 0 \leq K \leq 1, \quad K(x) = 1 \quad \text{for} \quad x \leq 1, \quad K(x) = 0 \quad \text{for} \quad x \geq 4\,. \tag{3}$$

¿From (2),(3) we find $R \in C^\infty(\mathbf{R}_+ \times \mathbf{R}^4)$ and $R_m \in C^\infty([0,\infty) \times \mathbf{R}^4)$. We also have for $\Lambda > 0$ and for $0 \leq |k^2+a^2| \leq \Lambda^2$ or $4\Lambda^2 \leq |k^2+a^2|$

$$\partial^w R_a(\Lambda,k) = 0\,,\ w \neq 0, \quad \partial^w\partial_\Lambda R_a(\Lambda,k) = 0\,, \tag{4}$$

where the multiindex w indicates momentum derivatives

$$\partial^w = \partial^{w_1}\cdots\partial^{w_4} = \frac{\partial^{w_1}}{\partial k_1^{w_1}}\cdots\frac{\partial^{w_4}}{\partial k_4^{w_4}} \quad \text{for}\ k = (k_1,\ldots,k_4),\ w_i \in \mathbf{N}_0\,.$$

Replacing R by R_m (which is an improved version of the R used in [5]) for the massive fermions allows to obtain better statements on the IR behaviour later on, but R_m does not serve as an IR regulator and therefore it should not be used for the massless photon.

As can be seen in (1) we restrict to a general covariant gauge. The regularization breaks gauge invariance and consequently the WI's, but not $O(4)$-invariance. Due to this breaking of the WI's our interaction Lagrangian will also have to contain terms of dimension ≤ 4 which for invariant regularization need not be introduced- due to the WI's. We define $L^{\Lambda_0}(A,\overline{\psi},\psi)$ as

$$L^{\Lambda_0} := \int dx\{\frac{z_3}{4}F_{\mu\nu}^2 + \frac{\delta\lambda}{2}(\partial A)^2 + \frac{\delta\mu^2}{2}A^2 + z_4(A^2)^2 - z_2\overline{\psi}i\partial\!\!\!/\psi + \delta m\overline{\psi}\psi + e(1+z_1)\overline{\psi}A\!\!\!/\psi\}\,. \quad (5)$$

The notation is rather standard (including the summation convention), but we set

$$z_i := Z_i - 1, \quad 1 \leq i \leq 3 \quad (6)$$

as compared to standard textbooks. The WI's for invariantly regularized QED would then imply $z_1 = z_2$, $\delta\lambda = 0$, $z_4 = 0$, $\delta\mu^2 = 0$. The parameters z_i, $\delta\lambda$, $\delta\mu^2$ are formal power series in the coupling e. Apart from z_1 they have to be assumed to be of at least first order in e. For standard r.c. all constants are even of second order in e (see below (33)). The perturbative Green functions are obtained from (5) by the standard rules which imply that $\overline{\psi},\psi$ are viewed as independent elements of an infinite-dimensional (formal) Grassmann-algebra; $A_\mu(x)$ may be viewed as an element of $\mathcal{S}(\mathbf{R}^4)$.
As regards their transformation properties under $O(4)$ and charge conjugation C, we impose

$$O(4): \;\; \psi'(x') = S(\Lambda)\psi(x),\; \overline{\psi}'(x') = \overline{\psi}(x)S(\Lambda),\; A'_\mu(x') = \Lambda_{\mu\nu}A_\nu(x) \text{ with } x'_\mu = \Lambda_{\mu\nu}x_\nu\,, \quad (7)$$

$$C: \;\; \psi'(x) = -C^{-1}\overline{\psi}^T(x),\; \overline{\psi}'(x) = \psi^T(x)C\,,\; A'_\mu(x) = -A_\mu(x)\,, \quad (8)$$

where the charge conjugation matrix C fulfills

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \quad (\text{e.g. } C = \gamma_0\gamma_2). \quad (9)$$

Λ and $S(\Lambda)$ are the vector and spinor representations of $O(4)$ respectively. Using (8),(9) and $|\det\Lambda| = 1$ as well as the canonical assignments $dimA = 1$, $dim\psi = dim\overline{\psi} = 3/2$ we find

Lemma 1: L^{Λ_0} is the integral of the most general local polynomial of dimension ≤ 4 in the fields $A,\psi,\overline{\psi}$ and their derivatives which is $O(4)$ and charge conjugation invariant and fully symmetric under permutations of the A-fields.

The procedure to derive the FE's is analogous to that employed for Φ_4^4 [3,4]. We introduce the source functions

$$J_\mu(x),\; \eta(x),\; \overline{\eta}(x) \quad (10)$$

for the $A, \overline{\psi}, \psi$-fields and find for the generating functional of the perturbative regularized Green functions formally given by

$$\int DA\, D\overline{\psi} D\psi\; e^{-\frac{1}{2}<A,D_\Lambda^{\Lambda_0}A>}\; e^{<\overline{\psi},S_\Lambda^{\Lambda_0}\psi>}\; e^{-L^{\Lambda_0}+\int J\cdot A+\overline{\psi}\cdot\eta+\psi\cdot\overline{\eta}}$$

the following rigorous formula

$$Z_\Lambda^{\Lambda_0}(J,\overline{\eta},\eta) := e^{-L^{\Lambda_0}(\delta_J,-\delta_\eta,\delta_{\overline{\eta}})}\, e^{1/2<J,D_\Lambda^{\Lambda_0}J>}\, e^{<\overline{\eta},S_\Lambda^{\Lambda_0}\eta>}\,. \tag{11}$$

We assume $J_\mu \in \mathcal{S}(\mathbf{R}^4)$ and $\eta, \overline{\eta}$ to be Grassmann variables and we demand that all sources have the same $O(4)$ and C transformation properties as the respective fields. We employed the usual notation

$$< J, D_\Lambda^{\Lambda_0} J >= \int d^4x\, d^4y\; J_\mu(x)(D_\Lambda^{\Lambda_0})_{\mu\nu}(x-y)J_\nu(y) \;=\int \frac{d^4k}{(2\pi)^4} J_\mu(-k)(D_\Lambda^{\Lambda_0})_{\mu\nu}(k)J_\nu(k) \tag{12}$$

and similarly for $< \overline{\eta}, S_\Lambda^{\Lambda_0}\eta >$. We set $J_\mu(x) \;= \int \frac{d^4k}{(2\pi)^4} e^{ikx}\, J_\mu(k)$.
Then we introduce the functional Laplace operator

$$\triangle(\Lambda,\Lambda_0) \;=\; \triangle'(\Lambda,\Lambda_0) \;+\; \triangle''(\Lambda,\Lambda_0) \tag{13}$$

$$\triangle'(\Lambda,\Lambda_0) \;=< \delta_A, D_\Lambda^{\Lambda_0}\delta_A >,\; \triangle''(\Lambda,\Lambda_0) \;=< \delta_\psi, S_\Lambda^{\Lambda_0}\delta_{\overline{\psi}} >\,,$$

and find
Proposition 2:

$$Z_\Lambda^{\Lambda_0}(J,\overline{\eta},\eta) = e^{1/2<J,D_\Lambda^{\Lambda_0}J>}\, e^{<\overline{\eta},S_\Lambda^{\Lambda_0}\eta>}\, (e^{\triangle(\Lambda,\Lambda_0)}\; e^{-L^{\Lambda_0}(A,\overline{\psi},\psi)})|_{A=D_\Lambda^{\Lambda_0}J,\psi=S_\Lambda^{\Lambda_0}\eta,\overline{\psi}=\overline{\eta}S_\Lambda^{\Lambda_0}}\,. \tag{14}$$

Proof: In the proof we omit Λ, Λ_0 in $D, S, \triangle$. It may be performed in two steps. In the first step one shows

$$e^{-L^{\Lambda_0}(\delta_J,-\delta_\eta,\delta_{\overline{\eta}})}\, e^{1/2<J,D_\Lambda^{\Lambda_0}J>} \;=\; e^{1/2<J,D_\Lambda^{\Lambda_0}J>}\, (e^{\triangle'}\; e^{-L^{\Lambda_0}(A,-\delta_\eta,\delta_{\overline{\eta}})})|_{A=DJ}\,. \tag{15}$$

We omit the proof of (15) here since it is analogous to the proof of the corresponding statement for Φ_4^4 [3,4]. The way of proceeding may also be inferred from the treatment of the fermionic part, which is performed explicitly now. (14) follows immediately using (15), if we can show that

$$(e^{\triangle''}\; e^{-L^{\Lambda_0}(A,\overline{\psi},\psi)})|_{\psi=S\eta,\overline{\psi}=\overline{\eta}S} \;=\; e^{-<\overline{\eta},S\eta>}\, e^{-L^{\Lambda_0}(A,-\delta_\eta,\delta_{\overline{\eta}})}\, e^{<\overline{\eta},S\eta>}\,. \tag{16}$$

So we want to prove (16). We write in the proof for the r-th order perturbative contribution to L^{Λ_0}

$$L_r^{\Lambda_0}(\overline{\psi},\psi) \;=< \overline{\psi}, M_r\psi > \;+ B_r$$

with suitable local $A-$dependent $M_r(x)$. B_r contains the $\overline{\psi},\psi$-independent terms and is not of importance here. The first step is to derive a commutation relation for the functional differential operator $\triangle''$

$$[\triangle''\,,\,L_r^{\Lambda_0}\,]\;=\;a_r\,,\quad [(\triangle'')^2\,,\,L_r^{\Lambda_0}\,]\;=\;2a_r\,\triangle''\;+\;b_r$$

and by induction

$$[(\triangle'')^n\,,\,L_r^{\Lambda_0}\,]\;=\;na_r\,(\triangle'')^{n-1}\;+\;n(n-1)b_r(\triangle'')^{n-2}\,,\;n\geq 2\,, \tag{17}$$

where

$$a_r\;=\;-<\delta_\psi,SM_r\psi>\;+\;<\overline{\psi},M_rS\delta_{\overline{\psi}}>\,,\quad b_r\;=\;-<\delta_\psi,SM_rS\delta_{\overline{\psi}}>\;.$$

(17) implies

$$[e^{\triangle''}\,,\,L_r^{\Lambda_0}\,]\;=\;(a_r\;+\;b_r)e^{\triangle''}\,. \tag{18}$$

Now we may get rid of a_r,b_r using essentially the same mechanism by which they were produced, namely we find

$$L_r^{\Lambda_0}(-\delta_\eta,\delta_{\overline{\eta}})\;e^{<\overline{\eta},S\eta>}\;=\;e^{<\overline{\eta},S\eta>}\,(L_r^{\Lambda_0}\;+\;a_r\;+\;b_r)|_{\psi=S\eta,\overline{\psi}=\overline{\eta}S}\,, \tag{19}$$

so that (18),(19) together give

$$(e^{\triangle''}\,L_{r_1}^{\Lambda_0}\ldots\,L_{r_k}^{\Lambda_0})|_{\psi=S\eta,\overline{\psi}=\overline{\eta}S}\;=\;e^{-<\overline{\eta},S\eta>}\,e^{<\overline{\eta},S\eta>}\,\prod_{i=1}^{k}(L_{r_i}^{\Lambda_0}\;+\;a_{r_i}\;+\;b_{r_i})|_{\psi=S\eta,\overline{\psi}=\overline{\eta}S}\;=$$

$$=\;e^{-<\overline{\eta},S\eta>}\,L_{r_1}^{\Lambda_0}(-\delta_\eta,\delta_{\overline{\eta}})\ldots L_{r_k}^{\Lambda_0}(-\delta_\eta,\delta_{\overline{\eta}})\;e^{<\overline{\eta},S\eta>}\,,$$

from which (16) immediately follows on expanding $e^{-L^{\Lambda_0}}$. (Note that a factor $e^{\triangle''}$ on the r.h.s. of the second equation is replaced by 1, when we regard the equations as equations for functionals, not operators.) ∎

We may then introduce the generating functionals $W^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta)$ of the (nontrivial, regularized) connected Green functions, and $L^{\Lambda,\Lambda_0}(A,\overline{\psi},\psi)$ of the (nontrivial, regularized) connected amputated Green functions (CAG), given by

$$e^{-(W^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta)+f.i.)}\;=\;Z_\Lambda^{\Lambda_0}(J,\overline{\eta},\eta)\,,\;e^{-(L^{\Lambda,\Lambda_0}(A,\overline{\psi},\psi)+f.i.)}\;=\;e^{\triangle(\Lambda,\Lambda_0)}\,e^{-L^{\Lambda_0}(A,\overline{\psi},\psi)}\,, \tag{20}$$

where $f.i.$ (for field-independent) is defined such that

$$W^{\Lambda,\Lambda_0}(0,0,0)\;=\;L^{\Lambda,\Lambda_0}(0,0,0)\;=\;0\,.$$

Thus $f.i.$ also depends on Λ, Λ_0, and the volume has to be kept finite as long as we deal with $f.i.$. Since we are not interested in $f.i.$ we spare ourselves being precise here and refer to [3,4] instead. Note that

$$L^{\Lambda_0,\Lambda_0} \;\equiv\; L^{\Lambda_0}\,. \tag{21}$$

The FE is then obtained by taking derivatives w.r.t. Λ on both sides of (20),2^{nd} equ. We obtain

$$\begin{aligned}\partial_\Lambda L^{\Lambda,\Lambda_0} \;&=\; \partial_\Lambda \triangle(\Lambda,\Lambda_0) \;-\; 1/2 < \delta_A L^{\Lambda,\Lambda_0}, (\partial_\Lambda D_\Lambda^{\Lambda_0})\delta_A L^{\Lambda,\Lambda_0} > \\ &+ < \delta_\psi L^{\Lambda,\Lambda_0}, (\partial_\Lambda S_\Lambda^{\Lambda_0})\delta_{\overline{\psi}} L^{\Lambda,\Lambda_0} > \;-\; \partial_\Lambda f.i.\end{aligned} \tag{22}$$

To proceed further we expand L in terms of powers of external fields and orders of perturbation theory in momentum space

$$L^{\Lambda,\Lambda_0} \;=\; \sum_{r\geq 1} e^r\, L_r^{\Lambda,\Lambda_0} \tag{23}$$

and

$$\begin{aligned}L_r^{\Lambda,\Lambda_0} \;=\; \sum_{m+n>0} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4p_{2n-1}}{(2\pi)^4}\, (\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n})_{\mu_1\ldots\mu_m i_1\ldots i_n j_1\ldots j_n}(k_1,\ldots k_m, p_1,\ldots p_{2n-1}) \times \\ \times A_{\mu_1}(k_1)\ldots A_{\mu_m}(k_m)\overline{\psi}_{i_1}(p_1)\ldots\overline{\psi}_{i_n}(p_n)\psi_{j_1}(p_{n+1})\ldots\psi_{j_n}(p_{2n})\;,\end{aligned} \tag{24}$$

where for $n \neq 0$ we set $p_{2n} := -k_1 - \ldots k_m - p_1 - \ldots - p_{2n-1}$. We did not write explicitly the case $n=0$ where by momentum conservation $k_m = -k_1 - \ldots - k_{m-1}$.
The following symmetry properties of the $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ follow from the properties of L and $\triangle$:

(i) $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n} = 0$, if $m + 2n > 4r$ (connectedness)

(ii) $\mathcal{L}^{\Lambda,\Lambda_0,r}_{2m+1,0} = 0$ (charge conjugation symmetry, Furry's theorem)

(iii) $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ may (and will) be chosen fully symmetric under permutations of $(k_1,\mu_1),\ldots,$ (k_m,μ_m) and fully antisymmetric under permutations of $(p_1,i_1),\ldots,(p_n,i_n)$ and $(p_{n+1},j_1),\ldots,(p_{2n},j_n)$.

(iv) The $O(4)$-transformation properties of $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ are indicated through the vector and spinor indices $\mu_1\ldots i_1\ldots j_1\ldots$

(v) $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ is in $C^\infty((0,\Lambda_0]\times\mathbf{R}^{4(m+2n-1)})$ as a function of $\Lambda, k_1\ldots p_{2n-1}$, due to the smoothness of the regularized propagators.

The FE for the coefficient functions $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ is then obtained from (22),(23) by identifying the coefficients of m photon and n fermion and antifermion fields in (22). We obtain:

$$
\begin{aligned}
&(\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n})_{\mu_1\ldots\mu_m i_1\ldots i_n j_1\ldots j_n}(k_1,\ldots k_n,p_1,\ldots p_{2n-1}) \;= \\
&= -\binom{m+2}{2}\int \frac{d^4k}{(2\pi)^4}\partial_\Lambda R(\Lambda,k)\,D_{\mu\nu}(k)\times \\
&\times(\mathcal{L}^{\Lambda,\Lambda_0,r}_{m+2,2n})_{\mu\nu\mu_1\ldots\mu_m i_1\ldots i_n j_1\ldots j_n}(k,-k,k_1,\ldots k_n,p_1,\ldots p_{2n-1})\,+ \\
&+(n+1)^2(-1)^n\int\frac{d^4p}{(2\pi)^4}\partial_\Lambda R_m(\Lambda,p)S_{ji}(p)\times \\
&(\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n+2})_{\mu_1\ldots\mu_m i i_1\ldots i_n j j_1\ldots j_n}(k_1,\ldots k_n,-p,p_1,\ldots p_n,p,\ldots,p_{2n-1})\,+ \\
&+\sum_{m'+m''=m+2,n'+n''=n,r'+r''=r}\frac{m'\,m''}{2}(-1)^{n'n''}[\partial_\Lambda R(\Lambda,k')\,D_{\mu\nu}(k')\times \\
&(\mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'})_{\mu\mu_1\ldots\mu_{m'-1}i_1\ldots i_{n'}j_1\ldots j_{n'}}(k',k_1,\ldots k_{m'-1},p_1,\ldots,p_{n'},p_{n+1},\ldots p_{n+n'-1})\times \\
&(\mathcal{L}^{\Lambda,\Lambda_0,r''}_{m'',2n''})_{\nu\mu_{m'}\ldots\mu_m i_{n'+1}\ldots i_n j_{n'+1}\ldots j_n}(-k',k_{m'},\ldots k_m,p_{n'+1},\ldots,p_n,p_{n+n'},\ldots p_{2n-1})]_{SAS}\,+ \\
&+\sum_{m'+m''=m,n'+n''=n+1,r'+r''=r} n'\,n''(-1)^{(n'+1)n''}[\partial_\Lambda R_m(\Lambda,p')\,S_{ji}(p')\times \\
&(\mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'})_{\mu_1\ldots\mu_{m'}i_1\ldots i_{n'}jj_1\ldots j_{n'}}(k_1,\ldots k_{m'},p_1,\ldots,p_{n'},p,'\,p_{n+1},\ldots p_{n+n'-1})\times \\
&(\mathcal{L}^{\Lambda,\Lambda_0,r''}_{m'',2n''})_{\mu_{m'+1}\ldots\mu_m i i_{n'+1}\ldots i_n j_{n'+1}\ldots j_n}(k_{m'+1},\ldots k_m,-p',p_{n'+1},\ldots,p_n,p_{n+n'},\ldots p_{2n-1})]_{SAS}\,.
\end{aligned}
\tag{25}
$$

The momenta k',p' are determined by momentum conservation. SAS indicates symmetrization w.r.t. photon and antisymmetrization w.r.t. i) fermion and ii) antifermion momenta and indices. Many of the details of (25) are not important for us. The important points are the following:
(i) The r.h.s. contains only $\mathcal{L}$ terms for which either r is of smaller value than that of the l.h.s. or, if not, $m+n$ is of larger value than that of the l.h.s. This together with (i) after (24) fixes the induction scheme through which we will estimate the solutions of (25).
(ii) The induction Ansatz will be determined by the power counting w.r.t. Λ for the differentiated regularized propagators. For a complete estimate of the solutions we will (as always) need the equations generated from (25) by taking $|w|\in\mathbf{N}$ momentum derivatives. As regards notation, we set

$$
w\in\mathbf{N}_0^{4(m+2n-1)}\qquad w=(w_1,\ldots,w_{m+2n-1})\qquad w_i=(w_{i,1},\ldots,w_{i,4}) \tag{26}
$$
$$
|w|=\sum|w_{i_\nu}|\qquad \partial^w=\partial^{w_{1,1}}\cdots\partial^{w_{m+2n-1,4}}=\frac{\partial^{w_{1,1}}}{\partial k_{1,1}^{w_{1,1}}}\cdots\frac{\partial^{w_{m+2n-1,4}}}{\partial p_{2n-1,4}^{w_{m+2n-1,4}}}
$$

3 UV- and IR-finiteness of the fermion-photon theory

3.1 UV-finiteness

The proof of UV- and IR-finiteness proceeds similarly as in Φ_4^4 [3,4,14]. We start with the UV-problem. That means we choose a scale $\Lambda_1 > 0$, for simplicity $\Lambda_1 = 1$, and want to show that $\lim_{\Lambda_0\to\infty} \mathcal{L}_{m,2n}^{1,\Lambda_0,r}$ exists for all $m+n>0,\ r \geq 1$ and arbitrary (bounded) momenta. The proof requires that we fix all terms of (mass) dimension ≤ 4 which are not automatically zero due to the symmetry structure of the theory, through renormalization conditions (r.c.) at Λ_1. The symmetry structure, i.e. invariance under the Euclidean group and charge conjugation- has been fixed through the structure of L^{Λ_0} (the particular values of the z_i, $\delta\lambda$, $\delta\mu^2$, δm are not yet fixed) and through $\triangle$. Since we are dealing with a partially massless theory, the CAG without IR regularization can generally be expected to exist in momentum space only, if certain restrictions on the r.c. are obeyed and if the momentum configuration is nonexceptional (see [14,18]). More precise statements will follow. As long as we keep $\Lambda \geq 1$ we need not care about these restrictions, but we will choose the renormalization points such that the notation is as simple as possible and such that we need not change them when we go down to $\Lambda = 0$, namely it turns out that all renormalizations for the photon Green functions $\mathcal{L}_{m,0}$, $m \leq 4$ should be performed at zero momentum in order to obtain reasonably simple IR bounds. The photon mass term has to vanish at 0 momentum and $\Lambda = 0$ for the theory to exist. In massless QED the renormalization conditions for the $\partial^w \mathcal{L}_{m,2n}$ with $m + 3n + |w| = 4$ have to be imposed at nonvanishing momenta, however (ch.5). We noted already that due to C-invariance

$$\mathcal{L}_{m,0}^{\Lambda,\Lambda_0,r} \equiv 0 \quad \text{for}\, m \,\text{odd, in particular for m=1,3.} \tag{27}$$

¿From $O(4), C$-invariance of L^{Λ,Λ_0} and permutation (anti)symmetry of $\mathcal{L}_{m,2n}^{\Lambda,\Lambda_0,r}$ we obtain for the remaining terms at zero momentum and for given Λ, Λ_0, r (which we suppress)

$$(\mathcal{L}_{2,0})_{\mu\nu}(0) \sim \delta_{\mu\nu}\,,\ \partial_\rho(\mathcal{L}_{2,0})_{\mu\nu}(0) = 0\,,\ \partial_\sigma\partial_\rho(\mathcal{L}_{2,0})_{\mu\nu}(0) = c\,\delta_{\mu\nu}\delta_{\rho\sigma} + c'\,(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\,, \tag{28}$$

$$(\mathcal{L}_{4,0})_{\mu\nu\rho\sigma}(0) \sim f_{\mu\nu\rho\sigma} := 1/3(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\,, \tag{29}$$

$$(\mathcal{L}_{0,2})_{ij}(0) \sim \delta_{ij}\,, \quad \partial_\mu(\mathcal{L}_{0,2})_{ij}(0) \sim (\gamma_\mu)_{ij}\,, \tag{30}$$

$$(\mathcal{L}_{1,2})_{\mu ij}(0) \sim (\gamma_\mu)_{ij}\,. \tag{31}$$

Thus 7 independent constants fix the terms of dimension ≤ 4. The structure of L^{Λ_0} determines the b.c. for $\Lambda = \Lambda_0$. (5) tells us that at
$\underline{\Lambda = \Lambda_0}$:

$$\partial^w(\mathcal{L}_{m,2n}^{\Lambda_0,\Lambda_0,r})(k_1,\ldots,p_{2n-1}) \equiv 0, \quad \text{if}\ m + 3n + |w| \geq 5. \tag{32}$$

We impose at
$\underline{\Lambda = 1}$, $r \geq 2$:

(i) $(\mathcal{L}^{1,\Lambda_0,r}_{2,0})_{\mu\nu}(0) \;=\; \frac{1}{2}\,\delta\mu^{2,1}_r \delta_{\mu\nu}\;,$

(ii) $\partial_\sigma\partial_\rho(\mathcal{L}^{1,\Lambda_0,r}_{2,0})_{\mu\nu}(0) \;=\; \frac{1}{2}\,\delta\lambda^1_r\,(\delta_{\mu\rho}\delta_{\nu\sigma} \,+\, \delta_{\mu\sigma}\delta_{\nu\rho}) \,+\, \frac{1}{4}\,z^1_{3r}(2\delta_{\mu\nu}\delta_{\sigma\rho} \,-\, \delta_{\mu\rho}\delta_{\nu\sigma} \,-\, \delta_{\mu\sigma}\delta_{\nu\rho})$
(so that $\frac{1}{2}\,z_3$ corresponds to c, and $\frac{1}{2}\,\delta\lambda$ corresponds to $c' + 1/2\,c$ in (28))

$$\text{(iii)}\ (\mathcal{L}^{1,\Lambda_0,r}_{4,0})_{\mu\nu\rho\sigma}(0) \;=\; z^1_{4,r}\,f_{\mu\nu\rho\sigma}\;, \tag{33}$$

(iv) $(\mathcal{L}^{1,\Lambda_0,r}_{0,2})_{ij}(0) \;=\; \delta m^1_r\,\delta_{ij}\;,$

(v) $\partial_\mu(\mathcal{L}^{1,\Lambda_0,r}_{0,2})_{ij}(0) \;=\; -z^1_{2r}\,(\gamma_\mu)_{ij}\;,$

(vi) $(\mathcal{L}^{1,\Lambda_0,r}_{1,2})_{\mu ij}(0) \;=\; z^1_{1,r-1}(\gamma_\mu)_{ij}\;.$

$\underline{\Lambda = 1}$, $r = 1$: $\qquad (\mathcal{L}^{1,\Lambda_0,1}_{1,2})_{\mu ij}(0) \;=\; (\gamma_\mu)_{ij}\;.$

We assume (and this is standard) that all renormalization constants apart from z^1_1 vanish for $r = 1$. This somewhat simplifies the notation. Leaving out this restriction is possible, but not of much interest. It is easy to see from the FE (and obvious to anyone acquainted with QED) that the $\mathcal{L}^{\Lambda,\Lambda_0,1}_{m,2n}$ then also vanish for $\Lambda \neq 1$ for $m \neq 1$ or $n \neq 1$. Apart from this restriction all constants are completely arbitrary numbers which later on will be uniquely fixed by the r.c. which we impose at $\Lambda = 0$. They are of course assumed to be independent of Λ_0.

To prove the UV finiteness of the fermion-photon theory we introduce the (by now standard) (semi-)norms $\|\;\|_{(a,b)}$ defined as

$$\|\partial^z\,f\|_{(a,b)} \;=\; \sup_{x_1,\ldots,x_n,w,i_1,\ldots,i_l,|x_i|\leq max(a,b)} |\partial^w f_{i_1,\ldots,i_l}(x_1,\ldots,x_n)|\;, \tag{34}$$

where $z = |w|$, $z \in \mathbf{N}_0$, $a, b \geq 0$ and for any system of sufficiently smooth functions $f_{i_1,\ldots,i_l}$: $\mathbf{R}^n \to \mathbf{C}$ with i_j running through some finite set. We find for $1 \leq \Lambda \leq \Lambda_0$ and any fixed $B > 0$ and $a \geq 0$:

$$\|\partial^z R_a(\Lambda,\cdot)\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-z}, \quad \|\partial^z\partial_\Lambda R_a(\Lambda,\cdot)\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-z-1} \tag{35}$$

$$\|\partial^z R_a(\Lambda,\cdot)\,f_1\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-2-z},\ z > 0\,,\ \ \|\partial^z\partial_\Lambda R_a(\Lambda,\cdot)\,f_1\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-3-z}, \tag{36}$$

$$\|\partial^z R_a(\Lambda,\cdot)\,f_2\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-1-z},\ z > 0\,,\ \ \|\partial^z\partial_\Lambda R_a(\Lambda,\cdot)\,f_2\|_{(2\Lambda,B)} \;\leq\; c(z)\Lambda^{-2-z}, \tag{37}$$

$$\text{where } f_1(k) = k^{-2},\ f_{2,ij}(p) = (\frac{1}{\not p + m})_{ij},\ c(z) \text{ is some suitable constant.} \tag{38}$$

For $0 < \Lambda \le 1$ we also find

$$|\partial^w R(\Lambda,k)| \le c(z)(\sup(\Lambda,|k|))^{-z}, \quad |\partial^w \partial_\Lambda R(\Lambda,k)| \le c(z)(\sup(\Lambda,|k|))^{-z-1}. \tag{39}$$

$$|\partial^w R_m(\Lambda,p)| \le c(z), \; |\partial^w \partial_\Lambda R_m(\Lambda,p)| \le c(z)\,.$$

($c(z)$ also depends on the mass m, which we do not indicate since m is fixed).
Now we may state the UV-renormalizability of the fermion-photon theory through

Proposition 3: For $1 \le \Lambda \le \Lambda_0 < \infty$ we have the following estimates

(i) $\|\partial^z \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}\|_{(2\Lambda,B)} \le \Lambda^{4-m-3n-z} Plog\Lambda$ (UV-boundedness)

(ii) $\|\partial_{\Lambda_0}\partial^z \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}\|_{(2\Lambda,B)} \le (\frac{\Lambda}{\Lambda_0})^2 \Lambda^{3-m-3n-z} Plog\Lambda_0$ (UV-renormalizability),

where we denote (as usually) by $Plog\Lambda$ a polynomial in $log\Lambda$ with nonnegative coefficients independent of Λ, Λ_0, but depending on m,n,r,z,B.
Proof: We proceed in the standard way [3,4] by induction on r. For given r we descend in the values of $m+2n$ (remember (i) after (24)) and then in the values of z for fixed $m+2n$ starting from some arbitrary $z_{\max}$.
<u>$r=1$</u>:
(a) $m+3n+|w| \ge 5$: $\partial^w \mathcal{L}^{\Lambda,\Lambda_0,1}_{m,2n} \equiv 0$ from the b.c. (32) and the FE.
(b) $m+3n+|w| \le 4$: The r.c. (33) (plus subsequent comments) and the FE tell us that

$$(\mathcal{L}^{\Lambda,\Lambda_0,1}_{1,2})_{\mu ij}(0) = (\gamma_\mu)_{ij}$$

Using (a) and Taylor's theorem we find $(\mathcal{L}^{\Lambda,\Lambda_0,1}_{1,2})_{\mu ij}(k,p) = (\gamma_\mu)_{ij}$. (33) and the FE also tell us that all other $\mathcal{L}^{\Lambda,\Lambda_0,1}_{m,2n}$ vanish. Thus (i),(ii) are true for $r=1$.
<u>$r-1 \to r$</u>:
We assume to have verified the bound (i) for any m,n,z and $r' \le r-1$ for $r \ge 2$ and for r and all m',n',z' with $m'+2n' > m+2n$ or with $m'+2n' = m+2n$ and $z' > z$. We prove it now for r and (m,n,z) and start with
(a) $m+3n+z \ge 5$: As for Φ^4_4 we may write an estimated FE which is in shorthand notation (leaving out indices and collecting all Λ,Λ_0-independent constants into one c)

$$\|\partial^z\partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}\|_{(2\Lambda,B)} \le c\,\{\int_\Lambda^{2\Lambda} dt(\|\partial^z \mathcal{L}^{\Lambda,\Lambda_0,r}_{m+2,2n}\|_{(2\Lambda,B)} + t\,\|\partial^z \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n+2}\|_{(2\Lambda,B)})$$

$$+ \sum (\Lambda^{-3-z'''}\|\partial^{z'} \mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'}\|_{(2\Lambda,B)}\; \|\partial^{z''} \mathcal{L}^{\Lambda,\Lambda_0,r''}_{m'',2n''}\|_{(2\Lambda,B)}) + \tag{40}$$

$$+ \sum (\Lambda^{-2-z'''} \|\partial^{z'} \mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'} \|_{(2\Lambda,B)} \ \|\partial^{z''} \mathcal{L}^{\Lambda,\Lambda_0,r''}_{m'',2n''} \|_{(2\Lambda,B)}) \}$$

The sums are over the same values as in (25) and additionally over all $z', z'', z''' \geq 0$ with $z' + z'' + z''' = z$. We used (35)-(37). The bound (i) then follows from (32) and on integration of (40) from Λ_0 to Λ, since the r.h.s. of (40) is bounded using (i), by induction.
(b) $m + 3n + z \leq 4$:
(b1) $m + 3n + z = 4$: Use the r.c. (33), the FE and (a) to verify

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(0)| \ \leq \ \Lambda^{4-m-3n-|w|} Plog\Lambda\,, \quad 1 \leq \Lambda \leq \Lambda_0$$

for any choice of indices.
Once this has been achieved we may pass on to arbitrary momenta using the Schlömilch formula as in Φ^4_4:

$$f(p) \ = \ f(0) \ + \ p_\mu \int_0^1 d\lambda\, \partial_{k_\mu} f(k)\,, \ \ k = \lambda p. \tag{41}$$

The integrated derivative has $m + 3n + z = 5$ and is thus already bounded by induction for bounded momenta. So (i) can again be verified.
(b2) $m + 3n + z = 3$, $m + 3n + z = 2$ are then subsequently verified in the same manner. Note that we have to proceed in this order to be able to estimate the integrated derivative on the r.h.s. of (41) by induction.
For (ii) we do not give an explicit proof, but refer to Φ^4_4. The essential points are the following:
1. Differentiate both sides of the FE w.r.t. Λ_0 and write again an estimated form of this equation corresponding to (40).
2. Use the same induction scheme as before to estimate the r.h.s. of this estimated FE.
3. Use the Λ_0-independence of the r.c. to realize that the boundary terms $\partial_{\Lambda_0} \partial^w \mathcal{L}^{1,\Lambda_0,r}_{m,2n}$ vanish for $m + 3n + |w| \leq 4$ at zero momentum. This is the important change as compared to the proof of (i). (At $\Lambda = \Lambda_0$ we use as before the b.c. (32)). Use again (41) to go away from zero momentum. From this it is then straightforward to verify the bound (ii). ■

Referring to earlier papers [3,6] we note in passing that a statement like (ii) also holds if we soften the requirements of Λ_0-independent r.c. and/or $\partial^w \mathcal{L}^{\Lambda_0,\Lambda_0,r}_{m,2n} \ = \ 0$ for $m + 3n + |w| \geq 5$ to only requiring that these terms are suppressed by powers of Λ_0 according to their power counting dimension. This freedom may also be used to improve on the rate of convergence in (ii), see [6].

3.2 IR-Finiteness

Now we turn to the IR part of the problem. Proposition 3 tells us that for $\Lambda \geq 1$ the $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ exist for $\Lambda_0 \to \infty$ and for arbitrary indices and momenta bounded in modulus by B.

Looking at $0 < \Lambda \leq 1$ we want to show that for suitable b.c. the $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ exist for $\Lambda \to 0$, if the external momenta are chosen nonexceptional, i.e. no partial sums vanish. We again proceed in analogy to Φ^4_4 [14]. I.e. we first define an IR index g for any configuration of m photon and $2n$ fermion momenta, taylored such that we can prove inductively with the help of the FE

$$|\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(k,p)| \leq \Lambda^{-2g} Plog \Lambda^{-1}\,, \text{ if } \Lambda \to 0 \tag{42}$$

for any exceptional momentum configuration. Afterwards we can prove finiteness for nonexceptional momenta. All momenta are from now on supposed to be bounded by B.

The proof of a formula as (42) with the use of the FE can work only if the IR indices of the momentum sets on the r.h.s. of the FE obey sufficiently strong bounds in terms of the index of the momentum set appearing on the l.h.s. We need the following definitions to proceed in this direction:

Definition 1: A set of photon and fermion-antifermion momenta $\{k_1, \ldots, k_m, p_1, \ldots p_{2n}\}$ denoted also as $\{q_1, \ldots, q_{m+2n}\}$ [1] is called admissible w.r.t. QED, if

(i) $m + 3n > 2$ $\quad (ii)$ m even, if $n = 0$ $\quad (iii)$ $\sum_i k_i + \sum_j p_j = 0$.

Definition 2: An admissible momentum set (a.m.s.) Q is called exceptional, if there exists $Q_1 \subset Q$, $\emptyset \neq Q_1 \neq Q$, such that $\sum_{Q_1} q_i = 0$. Otherwise it is called nonexceptional. $\sum_{Q_1} := \sum_{q_i \in Q_1}$.

Definition 3: A partition $Z(Q)$ of an a.m.s. Q is a system of nonempty subsets $E_\nu \subset_{\neq} Q$, $\nu = 1, \ldots, N$ with

$$(i) \quad Q = \bigcup_{\nu=1}^{N} E_\nu \qquad (ii) \quad E_\nu \cap E_\mu = \emptyset, \text{ if } \nu \neq \mu \qquad (iii) \quad \sum_{E_\nu} q_i = 0$$

(iv) $\quad E_\nu$ contains the same number of fermion and antifermion (fe-afe) momenta.

For any partition $Z(Q)$ we define the subsets and numbers

$$\begin{aligned} A(Z) &= \{E_\nu \in Z |\ E_\nu \text{ consists of a single photon momentum }\},\ a := |A|\,, \\ B(Z) &= \{E_\nu \in Z |\ E_\nu \text{ consists of only} \geq 2 \text{ photon momenta}\}\,,\ b := |B|\,, \\ D(Z) &= Z \setminus (A(Z) \cup B(Z))\,,\ d = |D|\,. \end{aligned} \tag{43}$$

Definition 4: The IR index $g_Z(Q)$ of a partition $Z(Q)$ is defined as follows:

$$g_Z(Q) \;=\; \sup(0, \frac{a}{2} + b + \frac{3}{2}d - 2)$$

[1] We regard q_i and $q_j (i \neq j)$ as different entities, even if $q_i = q_j$ as elements of $\mathbf{R}^4$, since they belong to different fields or external lines. q_i may be thought of as a mapping $i \mapsto q_i$, we do not develop this point explicitly, however.

The IR-index g of an a.m.s. Q is defined to be

$$g(Q) \;=\; 0\,,\ \text{if no } Z(Q) \text{ exists and } \ g(Q) \;=\; \max_{Z(Q)}\, g_Z(Q) \text{ otherwise.}$$

So in particular $g(Q) = 0$, if Q is nonexceptional. As a motivation for Definition 4 note that by naive power counting one photon contributes one power in the IR cutoff to the IR singularity. This explains $\frac{a}{2}$, b. The momenta in D contribute more since they may flow across a subdiagram into one one-particle-reducible photon line and then contribute again via this line. On inspecting examples one finds that a constant $(= 2)$ may be subtracted. Using Definition 5 below we will obtain better IR bounds than with Definition 4 which is not optimal in this respect, but this requires additional effort. For an a.m.s. Q and pairs $\{k, -k\}$, $\{p, -p\}$ of photon and fermion-antifermion momenta we finally define the sets

$$Q_A = \{k, -k\} \cup Q, \quad Q_F = \{p, -p\} \cup Q.$$

And for $\emptyset \neq Q_1 \subset_{\neq} Q, \quad Q_2 := Q \setminus Q_1$ we set

$$Q'_A = Q_1 \cup \{k'\}, \quad Q'_F = Q_1 \cup \{p'\}, \quad Q''_A = Q_2 \cup \{k''\}, \quad Q''_F = Q_2 \cup \{p''\} \tag{44}$$

where the new momenta k', k'', p', p'' have values

$$k' = -\sum_{Q_1} q_i\,,\ p' = -\sum_{Q_1} q_i\,,\ k'' = -k'\,,\ p'' = -p'.$$

After these definitions we can now prove

Lemma 4: Let Q be an a.m.s. Suppose $Q_A, Q'_A, Q''_A, Q_F, Q'_F, Q''_F$ (44) are also a.m.s. Then we have

(a1) $g(Q_A) \leq g(Q) + 1$, if all q_i vanish or -for $\sup |q_i| > 0$- if $|k| \leq \eta$, where $\eta > 0$ is defined as

$$\eta(Q) := \frac{1}{2} \inf_J \eta_J, \tag{45}$$

and the inf is over all sets J with $J \subset_{\neq} \{1, \ldots, m+2n\}$ such that $|\sum_{i\in J} q_i| =: \eta_J > 0$.

(a2) $g(Q_A) \leq g(Q) + \frac{3}{2}$,

(b1) $g(Q'_A) + g(Q''_A) + 1 \leq g(Q)$, if $k' = 0$,

(b2) $g(Q'_A) + g(Q''_A) \leq g(Q)$,

(c) $g(Q_F) \leq g(Q) + \frac{3}{2}$,

(d) $g(Q'_F) + g(Q''_F) \leq g(Q)$.

Remarks:
In the proof we will denote the IR indices of $Q_A, Q_F, Q'_A, \dots$ by $g_A, g_F, g'_A, \dots$ and suitable partitions of these sets maximizing g_Z will be denoted by $Z_A, Z_F, Z'_A, \dots$.
Proof:
(a1) $g_A \leq 1$: trivial. $g_A \geq 1$:
(1) Assume there exists $E \in Z_A$ with $E \supset \{k, -k\}$, and set $Z := (Z_A \setminus \{E\}) \cup \{E \setminus \{k, -k\}\}$, and verify that $g_Z(Q) \geq g_A - 1$, whether $\{E \setminus \{k, -k\}\}$ is empty or not.
(2) If E as in (1) does not exist, then for suitable ν, ν', $k \in E_\nu$, $-k \in E_{\nu'}$, and we set $Z := (Z_A \setminus \{E_\nu, E_{\nu'}\} \cup \{(E_\nu \setminus \{k\}), (E_{\nu'} \setminus \{-k\})\}$. Again $g_Z(Q) \geq g_A - 1$, whatever $E_\nu, E_{\nu'}$ are. Note that the sum over the momenta in $E_\nu \setminus \{k\}$ still vanishes due to the supplementary condition on k which here implies $k = 0$.
(a2): The proof is as for (a1) except for the last case, where we have to set $Z := (Z_A \setminus \{E_\nu, E_{\nu'}\} \cup \{(E_\nu \setminus \{k\}) \cup (E_{\nu'} \setminus \{-k\})\}$ so that $g_Z(Q) \geq g_A - 3/2$.
(b1) If $g'_A, g''_A = 0$, set $Z = \{Q_1, Q_2\}$ (44), so that $g_Z(Q) = 1$. Now observe that for any a.m.s. a partition Z maximizing g_Z may always be chosen such that $a = |A(Z)|$ is maximal. For the rest of the proof we shall assume a to be maximal in any maximizing partition to appear. Now if $g'_A > 0$, $g''_A = 0$ set $Z = (Z'_A \setminus \{\{ k'\}\}) \cup \{Q_2\}$, which gives $g_Z = g'_A - 1/2 + 3/2$. Finally for $g'_A, g''_A > 0$ set $Z := (Z'_A \setminus \{\{ k'\}\}) \cup (Z''_A \setminus \{\{ k''\}\})$ so that $g_Z = g'_A - 1/2 + g''_A - 1/2 + 2$.
(b2) $g'_A, g''_A = 0$ is trivial. For $g'_A > 0$, $g''_A = 0$ take away from Z'_A the set E' containing k' and replace it by $(E' \setminus \{k'\}) \cup \{Q_2\}$ to verify (b2). For $g'_A, g''_A > 0$ (b2) is verified on defining $Z := (Z'_A \setminus \{E'\}) \cup (Z''_A \setminus \{E''\}) \cup \{(E' \setminus \{k'\}) \cup (E'' \setminus \{k''\})\}$. Whatever E', E'' are, we even find $g_Z \geq g'_A + g''_A + 1/2$.
(c) One easily convinces oneself that a maximizing partition Z_F of Q_F may be chosen such that for some ν: $E_\nu = \{p, -p\}$. Then $g_{Z_F}(Q_F) \leq g(Q) + 3/2$ is obvious.
(d) The proof is the same as for (b2) replacing $k \to p, A \to F$. ∎

The IR index of Definition 4 is slightly more crude than that of [14], which however facilitated the proof of Lemma 4 considerably. Since we want to show that all renormalizations may be performed at zero momentum (for massive fermions) we need a somewhat sharper version.
Definition 5: For an a.m.s. $Q = \{k_1, \dots, k_m, p_1, \dots, p_{2n}\}$ set

$$g_1(Q) = g(Q) - \frac{1}{2}, \quad \text{if} \tag{46}$$

(i) Q contains at most one fe-afe pair, and
(ii) Q is such that $g(Q) > 0$ and such that $g(Q)$ takes the maximal value possible for the given number of photon and fe-afe momenta in Q.

Otherwise set

$$g_1(Q) = g(Q). \tag{47}$$

Now we can prove

Lemma 5: With the assumptions of Lemma 4 and the additional requirement that none of the a.m.s. Q_A, Q'_A, Q''_A, Q appearing below consist of four photon momenta only we have:

(a1) $g_1(Q_A) \le g_1(Q) + 1$, if all q_i vanish or -for $\sup|q_i| > 0$- if $|k| \le \eta$,

(a2) $g_1(Q_A) \le g_1(Q) + \frac{3}{2}$,

(b1) $g_1(Q'_A) + g_1(Q''_A) + 1 \le g_1(Q)$, if $k' = 0$,

(b2) $g_1(Q'_A) + g_1(Q''_A) \le g_1(Q)$,

(c) $g_1(Q_F) \le g_1(Q) + 2$,

(d1) $g_1(Q'_F) + g_1(Q''_F) \le g_1(Q)$, if $p' = 0$.

(d2) $g_1(Q'_F) + g_1(Q''_F) \le g_1(Q) + 1/2$.

Proof: The notation is as in the proof of Lemma 4. We have only to look at the cases where $g_1(Q) < g(Q)$ for the a.m.s. Q appearing on the r.h.s.:

(a1),(a2): If Q is such as in (46) (i),(ii), then Q_A also fulfills these conditions and (a) follows from Lemma 4.

(b1),(b2): Due to our restrictions on the sets Q'_A, Q''_A, Q we find $g_1(Q) \ge 3/2$ ($= 3/2$ for the case of one fe-afe pair and 5 photons), if $g_1(Q) < g(Q)$: Thus $g'_{1A}, g''_{1A} = 0$ is trivial. If $g'_{1A} + g''_{1A} > 0$ we only have to verify the case where $g_1(Q) < g(Q)$, but $g'_{1A} + g''_{1A} = g'_A + g''_A$. Going through the cases as in Lemma 4 and using the fact that $g(Q)$ is maximal whereas g'_A and/or g''_A are not, produces the required inequalities.

(c),(d2) follow trivially from Lemma 4. (They are however not sufficient to prove IR finiteness, if massless fermions are present.)

(d1) follows by the same line of arguments as (b2). ∎

With the aid of Lemmas 4 and 5 we can prove the existence of the IR limit, provided we use the following class of r.c. (see also (33)) at

$\underline{\Lambda = 0}$, $r \ge 2$: $(\mathcal{L}^{0,\Lambda_0,r}_{2,0})_{\mu\nu}(0) = 0$, $\partial_\sigma\partial_\rho(\mathcal{L}^{0,\Lambda_0,r}_{2,0})_{\mu\nu}(0) = 0$, $(\mathcal{L}^{0,\Lambda_0,r}_{4,0})_{\mu\nu\rho\sigma}(0) = 0$,

$$(\mathcal{L}^{0,\Lambda_0,r}_{0,2})_{ij}(0) = \delta m^R_r\,\delta_{ij}\,,\ \partial_\mu(\mathcal{L}^{0,\Lambda_0,r}_{0,2})_{ij}(0) = -z^R_{2r}\,(\gamma_\mu)_{ij}\,,\ (\mathcal{L}^{0,\Lambda_0,r}_{1,2})_{\mu ij}(0) = z^R_{1,r-1}(\gamma_\mu)_{ij}\,. \tag{48}$$

$\underline{\Lambda = 0}$, $r=1$: $(\mathcal{L}^{0,\Lambda_0,1}_{1,2})_{\mu ij}(0) \,=\, (\gamma_\mu)_{ij}$.
A special case of (48) are the following r.c.: Take the r.c. (48) with

$$\delta m^R_r \,=\, 0,\; z^R_{2r} \,=\, 0,\; z^R_{1r} \,=\, 0. \tag{49}$$

It is also possible to impose r.c. at nonvanishing or nonexceptional momenta. In the presence of massless fermions this is even necessary (see ch.5). Note however that $\mathcal{L}^{0,\Lambda_0,r}_{2,0}$ always has to vanish at 0 momentum. We cannot prove IR convergence if we impose r.c. with nonvanishing values of $\partial_\sigma\partial_\rho(\mathcal{L}^{0,\Lambda_0,r}_{2,0})_{\mu\nu}(0)$, $(\mathcal{L}^{0,\Lambda_0,r}_{4,0})_{\mu\nu\rho\sigma}(0)$. We could perform the proof on renormalizing all $\mathcal{L}^{0,\Lambda_0,r}$ at nonexceptional momenta (where however $\mathcal{L}^{0,\Lambda_0,r}_{2,0}$ always has to be tuned such that it vanishes at 0 momentum). This introduces new technicalities and the statements obtained on the IR behaviour are in general not stronger than in massless QED (ch.5).
What is crucial for us however is that the r.c.(48) will turn out such that the QED WI's are restored for the respective theory in the limits $\Lambda \to 0$, $\Lambda_0 \to \infty$ (ch.4). But they are not the most general r.c. with this property since the WI's leave free the r.c. for the transverse photon propagator.
Before we prove our statement on the existence of the IR limit we want to remind the reader of two facts about exceptional momentum sets stated already in [14]. The first is: For any a.m.s. $Q = \{q_1, \ldots, q_{m+2n}\}$ there exists $\varepsilon(Q) > 0$ and a neighbourhood

$$U_\varepsilon(Q) = \{\{\hat q_1, \ldots, \hat q_{m+2n}\} \,|\, (q_i - \hat q_i)^2 \le \varepsilon^2,\, 1 \le i \le m+2n,\, \sum_{i=1}^{m+2n} \hat q_i = 0\}, \tag{50}$$

such that for any $\hat Q = \{\hat q_1, \ldots, \hat q_{m+2n}\} \,\in U_\varepsilon(Q):\; g(\hat Q) \le g(Q)$.
This holds since for all partitions of Q all subsets $S \subset_{\neq} Q$ which are not an element of any $Z(Q)$ have $\sum_{q_i\in S} q_i \neq 0$. Take ε so small that all these inequalities still hold in $U_\varepsilon(Q)$.

The second is on the sets of nonexceptional momenta M_{m+2n}, as subsets of $\mathbf{R}^{4(m+2n-1)}$:

$$M_{m+2n} := \{(q_1, \ldots, q_{m+2n-1}) \in \mathbf{R}^{4(m+2n-1)} |\, \sum_{i\in J} q_j \neq 0 \text{ for all } J \subset_{\neq} \{1, \ldots, m+2n\}\} \tag{51}$$

(as usual $q_{m+2n} = -q_1 - \ldots - q_{m+2n-1}$). The sets M_{m+2n} are obviously open in $\mathbf{R}^{4(m+2n-1)}$.
Now we prove

Proposition 6: Let $\Lambda \le 1 \le \Lambda_0 \le \infty$ and $r \ge 1$. All (independent) momenta are assumed to be bounded by $B > 0$ (arbitrarily fixed).

(a) The (connected amputated) renormalized Green functions of the perturbative fermion photon theory, defined through (32) and the renormalization conditions (48), which are given as

$$\mathcal{L}^{\Lambda_0,r}_{m,2n}(q_1,\ldots,q_{m+2n-1}) := \lim_{\Lambda\to 0}\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(q_1,\ldots,q_{m+2n-1}),$$

in particular $\mathcal{L}^{r}_{m,2n}(q_1,\ldots,q_{m+2n-1}) := \mathcal{L}^{\infty,r}_{m,2n}(q_1,\ldots,q_{m+2n-1})$, exist in $C^\infty(M_{m+2n})$ (see (51)), and in $C^\infty(M_{m+2n})$ we may interchange the limits:

$$\lim_{\Lambda\to 0}\lim_{\Lambda_0\to\infty}\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n} = \lim_{\Lambda_0\to\infty}\lim_{\Lambda\to 0}\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n} = \mathcal{L}^{r}_{m,2n}\,.$$

Furthermore $\mathcal{L}^{\Lambda,\Lambda_0,r}_{2,0} \in C^2([0,\infty)\times\mathbf{R}^4)$, $\mathcal{L}^{\Lambda,\Lambda_0,r}_{4,0} \in C^0([0,\infty)\times\mathbf{R}^{12})$ and $\mathcal{L}^{\Lambda,\Lambda_0,r}_{0,2} \in C^1([0,\infty)\times\mathbf{R}^4)$, $\mathcal{L}^{\Lambda,\Lambda_0,r}_{1,2} \in C^0([0,\infty)\times\mathbf{R}^8)$ as functions of Λ and the (independent) momenta.

(b) Let $Q=\{q_1,\ldots,q_{m+2n}\}$ be an a.m.s. (Def.1).

(b1) If Q is nonexceptional or- for $m=2, n=0$, if $q_1\neq 0$ - we have

$$\partial^w\mathcal{L}^{\Lambda_0,r}_{m,2n}(q_1,\ldots,q_{m+2n-1}) = \lim_{\Lambda\to 0}\partial^w\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(q_1,\ldots,q_{m+2n-1})$$

uniformly in $U_\varepsilon(Q)$.

(b2) Assume the a.m.s. Q is such that $Q=Q_a\cup Q_b$, where $Q_a\neq\emptyset$ and Q_b has the form $Q_b=\{q^{(b)}_1,-q^{(b)}_1,\ldots,q^{(b)}_l,-q^{(b)}_l\}$, and such that for any $E\subset Q$ with $\sum_E q_i=0$ either $Q_a\subset E$ or $Q_a\cap E=\emptyset$. Let $Q_a=\{q_1,\ldots,q_s\}$; we denote by Q_{a_1} some (arbitrary) subset of $s-1$ momenta of Q_a and by $\partial^w_{a_1}$ any sequence of $|w|$ derivatives w.r.t. to momenta in Q_{a_1}. Finally we denote by $N_\varepsilon(Q_b)$ the set $\{\hat{q}^{(b)}_1,-\hat{q}^{(b)}_1,\ldots,\hat{q}^{(b)}_l,-\hat{q}^{(b)}_l\}$ with $|\hat{q}^{(b)}_i-q^{(b)}_i|<\varepsilon$. Then we claim

$$|\partial^w_{a_1}\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{Q}_b,\hat{Q}_{a_1})| \leq \Lambda^{-2g_1(Q)}Plog\Lambda^{-1}\,. \tag{52}$$

The statement is uniform in $U_\varepsilon(Q_a)\cup N_\varepsilon(Q_b)$. The constants in $Plog$ depend on $\varepsilon,Q,r,m,n,|w|$. The notation in (52) is slightly abusive in that it requires that we parametrize $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ possibly in terms of momenta differing from the standard choice (24), and in different order.

(b3) For a general a.m.s. Q we have (for $\Lambda>0$)

$$|\partial^w\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{q}_1,\ldots,\hat{q}_{m+2n-1})| \leq \Lambda^{-2g_1(Q)-|w|}Plog\Lambda^{-1} \tag{53}$$

uniformly for $\hat{Q}\in U_\varepsilon(Q)$, the constants in $Plog$ depend on $\varepsilon,Q,r,m,n,|w|$.

(c) For $m+3n+|w| \le 4$ we obtain the bounds (for given r and $\Lambda > 0$)

$$|\partial^w \mathcal{L}_{2,0}^{\Lambda,\Lambda_0,r}(k)| \le \Lambda^{3-|w|} Plog \Lambda^{-1}\,, \quad |\partial^w \mathcal{L}_{4,0}^{\Lambda,\Lambda_0,r}(k_1,k_2,k_3)| \le \Lambda^{1-|w|} Plog \Lambda^{-1}\,, \tag{54}$$

and for the r.c.(49) and $r \ge 2$ (second inequ.) also

$$|\partial^w \mathcal{L}_{0,2}^{\Lambda,\Lambda_0,r}(p)| \le \Lambda^{2-|w|} Plog \Lambda^{-1}\,, \quad |\partial^w \mathcal{L}_{1,2}^{\Lambda,\Lambda_0,r}(k,p)| \le \Lambda^{1-|w|} Plog \Lambda^{-1}\,. \tag{55}$$

Those statements in (54),(55) for which the r.h.s. vanishes for $\Lambda \to 0$ hold uniformly only for $|k|, |k_i|, |p| \le \Lambda$ (or $O(1)\Lambda$), the others hold uniformly for $|k|, |k_i|, |p| \le B$. In case of the r.c.(48) the statement (55) holds only for $|w| \ge 2$ (first inequ.) resp. $|w| \ge 1$ (second inequ.).

All statements in (b),(c) are uniform in Λ_0.
Remarks : We left and leave out indices on the $\mathcal{L}_{m,2n}^{\Lambda,\Lambda_0}(q_1,\ldots,q_{m+2n-1})$ whenever possible, and we abbreviate (in slightly abusive shorthand notation) $\mathcal{L}_{m,2n}^{\Lambda,\Lambda_0}(Q) = \mathcal{L}_{m,2n}^{\Lambda,\Lambda_0}(q_1,\ldots,q_{m+2n-1})$ etc. For a given momentum set $Q = \{q_1,\ldots,q_{m+2n}\}$ we denote by $\hat{Q} = \{\hat{q}_1,\ldots,\hat{q}_{m+2n}\}$ a momentum set such that $\{\hat{q}_1,\ldots,\hat{q}_{m+2n}\}$ is in $U_\varepsilon(Q)$ and by $Q_A(k)$ or shortly Q_A the set $\{k,-k,q_1,\ldots,q_{m+2n}\}$ etc. (cf. Lemma 4). The symbol ε will always denote a positive number, chosen sufficiently small case per case (we do not introduce $\varepsilon', \varepsilon'', \ldots$) such that the respective estimate holds uniformly in $U_\varepsilon(\ldots)$. ε depends in particular on the respective $\eta(Q)$(45). $c, c_1, \ldots$ denote positive Λ, Λ_0-independent constants. The proof of Proposition 6 is in many aspects analogous to that of Theorem 1 in [14]. Here we are slightly shorter.
Proof: We use the standard FE induction scheme which proceeds upwards in r and for given r downwards in $l = m+3n$ using (i) after (24) (see also Prop.3).
(A) $r=1$: The b.c. (32) and the r.c. (48) give vanishing $\mathcal{L}_{m,2n}^{\Lambda,\Lambda_0,1}$ apart from $\mathcal{L}_{1,2}^{\Lambda,\Lambda_0,1}$ (see also Proposition 3). The r.h.s. of the FE for $\partial_\Lambda \mathcal{L}_{1,2}^{\Lambda,\Lambda_0,1}$ vanishes identically in $\Lambda, \Lambda_0, k\,, p$. Thus

$$(\mathcal{L}_{1,2}^{\Lambda,\Lambda_0,1})_{\mu ij}(k,p) = (\gamma_\mu)_{ij}\,,\ \mathcal{L}_{m,2n}^{\Lambda,\Lambda_0,1} \equiv 0,\ m \neq 1 \,\mathrm{or}\, n \neq 1\,.$$

So the proposition is true for $r=1$.
(B) $r>1$: We assume the proposition to be true for r', l' with $r' < r$, $l' \in \mathbf{N}$ and for r, l' with $l' > l$. We prove it now for r, l. We start with proving (b),(a) for
(B1) $m+3n+|w| \ge 5$, $m+3n > 2$: First we prove
(b3): Bounding the $\mathcal{L}'$s on the r.h.s. of the FE for $\partial_\Lambda \mathcal{L}_{m,2n}^{\Lambda,\Lambda_0,r}$ with the aid of the induction hypothesis and using Lemma 5 c),a) we obtain

$$|\partial^w \mathcal{L}_{m,2n+2}^{\Lambda,\Lambda_0,r}(p,-p,\hat{Q})| \;\le\; \Lambda^{-2g_1(Q)-4-|w|}\, Plog \Lambda^{-1}\,,$$

$$|\partial^w \mathcal{L}_{m+2,2n}^{\Lambda,\Lambda_0,r}(k,-k,\hat{Q})| \;\le\; \Lambda^{-2g_1(Q)-2-|w|}\, Plog \Lambda^{-1}\,,$$

if $|k| \leq \eta(Q)$ or if all momenta vanish, and generally

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0,r}_{m+2,2n}(k,-k,\hat{Q})| \leq \Lambda^{-2g_1(Q)-3-|w|}\, Plog\Lambda^{-1}\,.$$

The sums on the r.h.s. of (25) can be bounded - using Lemma 5 b),d), (4),(39) and the induction assumption - by $\Lambda^{-2g_1(Q)-1-|w|}\, Plog\Lambda^{-1}$. Here we note that in the cases where Lemma 5 b1) cannot be applied the corresponding contributions vanish for $\Lambda < 1/2\,\eta(Q)$ due to (4), since $|k'|, |p'| \geq 2\eta - O(\varepsilon)$ in $\hat{Q}$. For $\Lambda \geq 1/2\,\eta(Q)$ they can be absorbed in the constants of $Plog$. All previous bounds are by induction assumption uniform in the respective U_ε's. By a standard compactness argument the first two thus hold uniformly in $\{(p,-p) \mid |p| \leq \eta\} \times U_\varepsilon(Q)$ respectively in $\{(k,-k) \mid |k| \leq \eta\} \times U_\varepsilon(Q)$. ¿From (1)-(4),(25) we thus obtain in $U_\varepsilon(Q)$

$$|\partial^w \partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{Q})| \leq \int_\Lambda^{2\Lambda} dt\; t^3 [t^{-2g_1-4-|w|} Plogt^{-1} + t^{-3-2g_1-2-|w|} Plogt^{-1} +$$

$$+\,\theta(t-\eta)\,\eta^{-3-2g_1-3-|w|}\; Plog\eta^{-1}] + \Lambda^{-2g_1-1-|w|} Plog\Lambda^{-1} \;\leq\; \Lambda^{-2g_1-1-|w|} Plog\Lambda^{-1}.$$

Here $g_1 = g_1(Q)$, $\eta = \eta(Q)$. Integrating now from 1 to $\Lambda < 1$ shows

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{Q})| \leq \Lambda^{-2g_1-|w|} Plog\Lambda^{-1} + |\partial^w \mathcal{L}^{1,\Lambda_0,r}_{m,2n}(\hat{Q})|. \tag{56}$$

The last term is independent of Λ and uniformly bounded in Λ_0 by Proposition 3. So it may be absorbed in the first. Proposition 3 was proven for r.c. imposed at $\Lambda = 1$. Now we impose them at $\Lambda = 0$. So we have to show that both classes of r.c. are in one-to-one relation. That this is true indeed can be seen from the FE when integrating from $\Lambda = 1$ to $\Lambda = 0$. Imposing the r.c. at $\Lambda = 0$ one then finds that $\mathcal{L}^{1,\Lambda_0,r}_{m,2n}$ fulfill r.c. at $\Lambda = 1$ of the form (33). The way of proceeding can also be inferred from (B2) below. Strictly speaking this uniqueness statement is also part of the induction hypothesis.

(b2) If Q is as in (b2), then $Q_F = (Q_b \cup \{p,-p\}) \cup Q_a$, $Q_A = (Q_b \cup \{k,-k\}) \cup Q_a$ also fulfill the assumptions of (b2), if $0 < |p|, |k| \leq \eta(Q)$. The important point to note is that if the intermediate momenta p', k' appearing on the r.h.s. of the FE fulfill $|p'|, |k'| \leq \eta(Q)$ then the derivatives $\partial^w_{a_1}$ ($w \neq 0$) applied to k' (and p') give zero by our assumptions. (Note that k', p' need not vanish in this case if the external momenta are taken in $U_\varepsilon(Q_a) \cup N_\varepsilon(Q_b)$.) Using these facts the verification of (b2) proceeds as that of (b3). We again have to use Lemma 5 and the induction assumption to bound separately the regions where $\Lambda > \eta/2$ and $\Lambda \leq \eta/2$, and we also have to use again the compactness argument from the proof of (b3).

(b1) For Q nonexceptional, the sets $\hat{Q}_F := \{p,-p\} \cup \hat{Q}$, $\hat{Q}_A := \{k,-k\} \cup \hat{Q}$, $0 < |p|, |k| \leq \eta(Q)$, fulfill the assumptions of (b2), furthermore the momenta k', p' appearing on the r.h.s. of (25) fulfill $|k'|, |p'| \geq \eta(Q)$ in $\hat{Q}$. This implies as in (b2) (since $g_1(Q_F)$, $g_1(Q_A) = 0$)

$$|\partial^w \partial_\Lambda \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{Q})| \leq Plog\Lambda^{-1}.$$

Integrating from 1 to Λ proves the existence of $\lim_{\Lambda\to 0}\partial^w\partial_\Lambda\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}(\hat{Q})$ uniformly in $U_\varepsilon(Q)$ and therefore (b1).

(a) now follows (for $m+3n+|w|\ge 5,\ m+3n>2$) from the proof of (b1), since $|w|$ may take any (finite) value. In particular we may interchange the limits using a standard $\varepsilon/4$-argument for:

$$\mathcal{L}^{\delta\to 0,\infty}-\mathcal{L}^{0,\Lambda_0\to\infty}=\mathcal{L}^{\delta\to 0,\infty}-\mathcal{L}^{\delta,\infty}+\mathcal{L}^{\delta,\infty}-\mathcal{L}^{\delta,\Lambda_0}+\mathcal{L}^{\delta,\Lambda_0}-\mathcal{L}^{0,\Lambda_0}+\mathcal{L}^{0,\Lambda_0}-\mathcal{L}^{0,\infty}$$

and our knowledge on the IR and UV-limits.

(B2): Now we prove (b3),(b2),(b1),(a) for $m+3n+|w|\le 4$ or $m+3n=2$ as well as (54),(55). We start from $m+3n=4$,$m=4$: The r.c. fix the value of $\mathcal{L}^{0,\Lambda_0,r}_{4,0}(0)$, which is in one-to-one relation to $\mathcal{L}^{1,\Lambda_0,r}_{4,0}(0)$ through

$$\mathcal{L}^{1,\Lambda_0,r}_{4,0}(0)=\mathcal{L}^{0,\Lambda_0,r}_{4,0}(0)\,+\int_0^1 dt\,\partial_t\mathcal{L}^{t,\Lambda_0,r}_{4,0}(0)$$

because the integrand is independent of the r.c. for $\mathcal{L}^{0,\Lambda_0,r}_{4,0}(0)$ to order r: It is given by the r.h.s. of the FE. Noting that for $m+3n=4$ and any Q we have $g_1(Q)=0$, $g_1(Q_A)\le 1/2$, $g_1(Q_F)\le 1$ we find by induction, also using (c) to lower order

$$\partial^w\partial_\Lambda\mathcal{L}^{\Lambda,\Lambda_0,r}_{4,0}(\hat{Q})\ \le\ \Lambda^{-|w|}Plog\Lambda^{-1}$$

uniformly in $U_\varepsilon(Q)$, including the case where all q_i vanish.

Using a compactness argument and integrating over Λ we then deduce

$$\mathcal{L}^{0,\Lambda_0,r}_{4,0}\in C^0(\mathbf{R}^{12})\,,\quad\text{and}\quad \mathcal{L}^{\Lambda,\Lambda_0,r}_{4,0}(Q)=\mathcal{L}^{0,\Lambda_0,r}_{4,0}(Q)\,+\,O(\Lambda\,Plog\Lambda^{-1})$$

uniformly in $Q=\{\{q_1,q_2,q_3\}\,|\,|q_i|\le B\}$. The statements in (a),(b1),(c) follow from the previous estimates and (B1) (where $|w|\ge 1$ is included), the uniformity of the limit $\Lambda\to 0$ and the Schlömilch formula together with the r.c. The treatment of $\mathcal{L}^{\Lambda,\Lambda_0,r}_{1,2}$ in case of the r.c. (49) is analogous to that of $\mathcal{L}^{0,\Lambda_0,r}_{4,0}$ and we do not repeat the argument. In case of the r.c.(48) $\mathcal{L}^{\Lambda,\Lambda_0,r}_{1,2}(0,0)$ may be nonzero for $\Lambda\to 0$. But the regularity properties in (a),(c) are verified as before using in particular the fact that R_m vanishes for $\Lambda<m/2$ to bound the $\sum$-terms in the FE. Note that the statements on $\mathcal{L}^{\Lambda,\Lambda_0,r}_{1,2}$ for the r.c. (48) could not be verified with our method if we regularized the fermions in the same way as the photons using R instead of R_m. Now we come to $m=2,n=0$. The statements on $\mathcal{L}_{2,0}$ in (a),(b1),(c) are again proven by bounding the r.h.s. of the FE by induction for any k. We obtain

$$|\partial^w\partial_\Lambda\mathcal{L}^{\Lambda,\Lambda_0,r}_{2,0}(k)|\ \le\ \Lambda^{2-|w|}\,Plog\Lambda^{-1}\,. \tag{57}$$

The bounds are as usually uniform in the respective U_ε. (57) for $|w| \le 1$ only holds for $|k| \le c\,\Lambda$. Integration over Λ and the r.c. - or for $m + |w| \ge 5$ the b.c. at $\Lambda = 1$ (or at $\Lambda = \Lambda_0$, cf. the remark in (B1))- the usual uniformity and compactness arguments and Taylor expansion around zero momentum then provide the estimates in (c) and the statements of (a). The last case to treat is $m = 0, n = 1$. We again have to distinguish between the r.c. (49) and (48). But the way of proceeding is as previously for $\mathcal{L}_{1,2}$. To verify the statements we again need the regularity of R_m around 0. ■

We have seen in the end of the previous proof that our techniques really require different regularizations R, R_m to prove the Proposition. Using R throughout we can only prove results as sketched in sect.5 which also hold in massless QED. It is a straightforward exercise to show that replacing R_m by different smoothed versions of R (see e.g.[5]) produces the same results on taking the limits (one estimates the difference of the two regularized versions).

4 Violation and restoration of the Ward Identities

We start with a few introductory remarks forgetting about regularization, Λ, Λ_0 etc. The standard QED Ward Identity (WI) may be expressed in terms of the generating functional Z (11),(14) as

$$\{\lambda\Box\partial_\mu\delta_{J_\mu(x)} \;-\; ie\eta\delta_{\eta(x)} \;+\; ie\overline{\eta}\delta_{\overline{\eta}(x)} \;+\; \partial_\mu J_\mu(x)\}Z(J,\overline{\eta},\eta) \;=\; 0 \tag{58}$$

In terms of W with $Z = e^{-(W+f.i.)}$ we obtain

$$\{-\lambda\Box\partial_\mu\delta_{J_\mu(x)} \;+\; ie\eta\delta_{\eta(x)} \;-\; ie\overline{\eta}\delta_{\overline{\eta}(x)}\}W(J,\overline{\eta},\eta) \;=\; -\partial_\mu J_\mu(x) \tag{59}$$

or

$$(\delta_{\chi(x)}W(J_\mu - \partial_\nu(D^{-1})_{\nu\mu}\chi\,,\,\overline{\eta}e^{-ie\chi}\,,\,e^{ie\chi}\eta))|_{\chi=0} \;=\; -\partial_\mu J_\mu(x) \tag{60}$$

D^{-1} is the inverse photon propagator and χ decribes the gauge transformations, we assume $\chi \in \mathcal{S}(\mathbf{R}^4)$:

$$A_\mu \to A_\mu + \partial_\mu\chi,\ \psi \to e^{-ie\chi}\psi,\ \overline{\psi} \to e^{ie\chi}\overline{\psi}\,. \tag{61}$$

Now we look at the regularized theory (see (11),(20)). For safeness we keep $0 < \Lambda \le \Lambda_0 < \infty$. Due to the violation of gauge invariance implied by the momentum cutoff the WI's will also be violated. (60) leads us to define

$$J_\mu(\chi) \;=\; J_\mu - \partial_\nu(D^{-1})_{\nu\mu}\chi,\ \overline{\eta}(\chi) \;=\; \overline{\eta}\,e^{-ie\chi},\ \eta(\chi) \;=\; e^{ie\chi}\eta \tag{62}$$

(position space arguments are suppressed), and we set

$$Z_\Lambda^{\Lambda_0}(J,\overline{\eta},\eta;\chi) \;=\; Z_\Lambda^{\Lambda_0}(J(\chi),\overline{\eta}(\chi),\eta(\chi))\,,\ W_\chi^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta) \;=\; W^{\Lambda,\Lambda_0}(J(\chi),\overline{\eta}(\chi),\eta(\chi)) \tag{63}$$

$$L_\chi^{\Lambda_0}(\delta_J, -\delta_\eta, \delta_{\overline{\eta}}) \;=\; L^{\Lambda_0}(\delta_{J(\chi)}, -\delta_{\eta(\chi)}, \delta_{\overline{\eta}(\chi)}) \;=\; L^{\Lambda_0}(\delta_J, -e^{-ie\chi}\delta_\eta, e^{ie\chi}\delta_{\overline{\eta}}) \tag{64}$$

so that

$$Z_\Lambda^{\Lambda_0}(J, \overline{\eta}, \eta; \chi) \;=\; e^{-(W_\chi^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta) + f.i.(\chi))} \;=\; e^{-L_\chi^{\Lambda_0}(\delta_J, -\delta_\eta, \delta_{\overline{\eta}})}\; e^{\frac{1}{2}<J(\chi), D_\Lambda^{\Lambda_0}\, J(\chi)> + <\overline{\eta}(\chi), S_\Lambda^{\Lambda_0}\, \eta(\chi)>}\,. \tag{65}$$

In deriving the violated WI's (vWI's) we are only interested in contributions of first order in χ. We find

$$L_\chi^{\Lambda_0}(\delta_J, -\delta_\eta, \delta_{\overline{\eta}}) \;=\; L^{\Lambda_0} \,+\, e\, z_2 < \chi, \partial_\mu \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}}) > \;\text{(exactly)} \tag{66}$$

$$1/2\, < J(\chi), D_\Lambda^{\Lambda_0}\, J(\chi) > = \,1/2\, < J, D_\Lambda^{\Lambda_0}\, J > \,+\, < \chi, \partial D^{-1}\, D_\Lambda^{\Lambda_0}\, J > \,+ O(\chi^2)\,, \tag{67}$$

$$< \overline{\eta}(\chi), S_\Lambda^{\Lambda_0}\, \eta(\chi) > = < \overline{\eta}, S_\Lambda^{\Lambda_0}\, \eta > \,+ ie(< \overline{\eta}, S_\Lambda^{\Lambda_0}\, \chi\, \eta > \,-\, < \overline{\eta}, \chi\, S_\Lambda^{\Lambda_0}\, \eta >) \,+\, O(\chi^2)\,. \tag{68}$$

We thus may rewrite (65) to first order in χ

$$\begin{aligned} & e^{-(W_0^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta) + f.i.(0))}\, (- < \chi, \delta_\chi (W_\chi^{\Lambda,\Lambda_0} \,+\, f.i.)|_{\chi=0} >) \;= \\ & e^{-L_0^{\Lambda_0}}[\, -e\, z_2\; < \chi, \partial_\mu \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}}) > \,+\, < \chi, \partial D^{-1}\, D_\Lambda^{\Lambda_0}\, J > \,- \\ & -ie\, \{ < \overline{\eta}, \chi\, \delta_{\overline{\eta}} > \,+\, < \delta_\eta, \chi\, \eta >' \}]\; e^{1/2\, <J, D_\Lambda^{\Lambda_0}\, J> + <\overline{\eta}, S_\Lambda^{\Lambda_0}\, \eta>}\,, \end{aligned} \tag{69}$$

where $< \delta_\eta, \chi\, \eta >'$ means that we subtract the contribution where δ_η applies to the η in $< \ldots >$. Now we find

$$\begin{aligned} & [\, L_0^{\Lambda_0}(\delta_J, -\delta_\eta, \delta_{\overline{\eta}}),\, < \chi, \partial D^{-1}\, D_\Lambda^{\Lambda_0}\, J >] \;= \\ & < \chi, (\partial D^{-1}\, D_\Lambda^{\Lambda_0})_\mu\, (\, \delta_{\mu\nu}(-z_3 \Box \,+\, \delta\mu^2) \,+\, (z_3 \,-\, \delta\lambda) \partial_\mu \partial_\nu\,)\, \delta_{J_\nu} \, > \,- \\ & -\, 4 z_4\, f_{\mu\nu\rho\sigma}\; < \chi, (\partial D^{-1}\, D_\Lambda^{\Lambda_0})_\mu\, \delta_{J_\nu}\, \delta_{J_\rho}\, \delta_{J_\sigma} \, > \,+ e\, (1 + z_1) \; < \chi, (\partial D^{-1}\, D_\Lambda^{\Lambda_0})_\mu\, \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}} \, > \,. \end{aligned} \tag{70}$$

Note that the commutator commutes with $L_0^{\Lambda_0}$. For the last term we write

$$< \chi, \partial_\mu R_\Lambda^{\Lambda_0}\, \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}} > = < \chi, \partial_\mu\, \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}} > + < \chi, (\partial D^{-1}\, D_\Lambda^{\Lambda_0} - \partial)_\mu\, \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}} >\,, \tag{71}$$

where $R_{a,\Lambda}^{\Lambda_0}(p) \,:=\, R_a(\Lambda_0, p) - R_a(\Lambda, p), R_\Lambda^{\Lambda_0} := R_{0,\Lambda}^{\Lambda_0}$. Taking the first term on the r.h.s.of (71) together with the two terms in curly brackets in (69) we find on application of derivatives

$$\begin{aligned} & [\, e\, < \chi, \partial_\mu\, \delta_\eta\, \gamma^\mu\, \delta_{\overline{\eta}} \, > \, - ie\{ < \overline{\eta}, \chi\, \delta_{\overline{\eta}} > \,+\, < \delta_\eta, \chi\, \eta >' \}]\; e^{<\overline{\eta}, S_\Lambda^{\Lambda_0}\, \eta>} \;= \\ & =\; ie\, \{ < \overline{\eta}, (R_{m,\Lambda}^{\Lambda_0} \,-\, 1)\, \chi\, \delta_{\overline{\eta}} \, > \,+\, < \delta_\eta, \chi\, (R_{m,\Lambda}^{\Lambda_0} \,-\, 1)\, \eta >' \}\; e^{<\overline{\eta}, S_\Lambda^{\Lambda_0}\, \eta>}\,, \end{aligned} \tag{72}$$

where we used $S\, \overleftarrow{\partial\!\!\!/} \,=\, -i1 \,+\, imS,\; \partial\!\!\!/ S \,=\, i1 \,-\, imS$. Using (70)-(72) in (69) and commuting $e^{-L_0^{\Lambda_0}}$ through (70) (trivial), and the curly brackets in (72), finally gives

$$e^{-(W_0^{\Lambda,\Lambda_0}(J,\overline{\eta},\eta) + f.i.(0))}\, [- < \chi, \delta_\chi (W_\chi \,+\, f.i.)|_{\chi=0} > \,-\, < \chi, \partial R_\Lambda^{\Lambda_0}\, J > \,+$$

$$+\, ie\{ <\overline{\eta}, r^{\Lambda_0}_{m,\Lambda}\, \chi\, \delta_{\overline{\eta}}> \,+\, <\delta_\eta, \chi\, r^{\Lambda_0}_{m,\Lambda}\, \eta>' \} W_0^{\Lambda,\Lambda_0}] \,= \tag{73}$$
$$= \{ e^{-L_0^{\Lambda_0} + <\chi, O_\Lambda^{\Lambda_0}>}\; e^{(1/2 <J, D_\Lambda^{\Lambda_0} J> + <\overline{\eta}, S_\Lambda^{\Lambda_0} \eta>)} \}|_{1^{st}\ \text{order in}\ \chi} ,$$

with the following explanations:

1. $r^{\Lambda_0}_{a,\Lambda}(p) = R^{\Lambda_0}_{a,\Lambda}(p) - 1$, $r^{\Lambda_0}_{a,\Lambda}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\, r^{\Lambda_0}_{a,\Lambda}(p)$.
2. $O_\Lambda^{\Lambda_0}$ collects the outcome of the commutators. In (73) it carries the arguments $O_\Lambda^{\Lambda_0}(\delta_J, -\delta_\eta, \delta_{\overline{\eta}})$ (as $L_0^{\Lambda_0}$). It has the form

$$O_\Lambda^{\Lambda_0}(A, \overline{\psi}, \psi)(x) \,=\, e\, z_2\, (\partial_\mu \overline{\psi} \gamma^\mu \psi)(x) \,-\, e\, z_1\, [\, \partial_\mu R_\Lambda^{\Lambda_0}\, \overline{\psi} \gamma^\mu \psi](x)$$
$$- e\, (\, \partial_\mu r_\Lambda^{\Lambda_0}\, \overline{\psi} \gamma^\mu \psi)(x) \,-\, [\, \partial_\mu R_\Lambda^{\Lambda_0}\, (\, \delta^{\mu\nu}(-z_3 \Box \,+\, \delta\mu^2) \,+\, (z_3 \,-\, \delta\lambda) \partial^\mu \partial^\nu\,)\, A_\nu\,](x)$$
$$- 4 z_4\, f_{\mu\nu\rho\sigma}\, [\partial_\mu R_\Lambda^{\Lambda_0}\, A_\nu\, A_\rho\, A_\sigma\,](x) \tag{74}$$
$$+\, ie\, \{ i z_2 \int d^4 z\, [\overline{\psi}(z) \overleftarrow{\partial\!\!\!/}_z\, r^{\Lambda_0}_{m,\Lambda}(z - x)\, \psi(x) \,-\, \overline{\psi}(x)\, r^{\Lambda_0}_{m,\Lambda}(x - z)\, \partial\!\!\!/_z\, \psi(z)]$$
$$- \delta m \int d^4 z\, [\overline{\psi}(z)\, r^{\Lambda_0}_{m,\Lambda}(z - x)\, \psi(x) \,-\, \overline{\psi}(x)\, r^{\Lambda_0}_{m,\Lambda}(x - z)\, \psi(z)]$$
$$- e(1 + z_1) \int d^4 z\, [\overline{\psi}(z) A\!\!\!/(z)\, r^{\Lambda_0}_{m,\Lambda}(z - x)\, \psi(x) \,-\, \overline{\psi}(x) r^{\Lambda_0}_{m,\Lambda}(x - z) A\!\!\!/(z)\, \psi(z)] \}$$

   The curly brackets correspond to the contribution of those in (72), on commuting through as in (70).
3. The term $< \chi, O >$ has been raised to the exponent. This is allowed since we regard only the first order in χ.

As can be seen from (74), all terms in O vanish, if we formally let $\Lambda \to 0, \Lambda_0 \to \infty$ and require $z_1 = z_2,\ z_4 = 0, \delta\lambda = 0,\ \delta\mu^2 = 0$. Thus $O_\Lambda^{\Lambda_0}$ collects the gauge symmetry violating contributions, and (73) has been derived to control them, again using flow equations. (73) is to be understood as usually in the sense of perturbation theory, i.e. as a relation between r-th order terms of the perturbative expansion. Furthermore we are dealing only with the coefficient functions for a given number of external $J, \eta, \overline{\eta}$ or χ-fields. The coefficient functions $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$ exist for any positive value of Λ at arbitrary momenta. The same is then obviously true for the $\mathcal{W}^{\Lambda,\Lambda_0,r}_{m,2n}$ (see below) which appear on the l.h.s. of (73). Eliminating the exponential in (73) we may write this equation as

$$- < \chi, (\delta_\chi W_\chi^{\Lambda,\Lambda_0})|_{\chi=0} > \,-\, < \chi, \partial R_\Lambda^{\Lambda_0}\, J > \,+$$
$$+\, ie\{ <\overline{\eta}, r^{\Lambda_0}_{m,\Lambda}\, \chi\, \delta_{\overline{\eta}}> \,+\, <\delta_\eta, \chi\, r^{\Lambda_0}_{m,\Lambda}\, \eta>' \} W_{(0)}^{\Lambda,\Lambda_0} \,=\, -W_{(1)}^{\Lambda,\Lambda_0} . \tag{75}$$

Here $W_{(1)}^{\Lambda,\Lambda_0}$ is defined as follows. We set

$$e^{-(W_O^{\Lambda,\Lambda_0}+f.i.(\chi))} := e^{-L_0^{\Lambda_0}+<\chi,O_\Lambda^{\Lambda_0}>}\, e^{(1/2<J,D_\Lambda^{\Lambda_0}J>+<\overline{\eta},S_\Lambda^{\Lambda_0}\eta>)}\,. \tag{76}$$

Thus W_O is the generating functional of the connected unamputated Green functions with insertions of O and we expand

$$W_O^{\Lambda,\Lambda_0} = W_{(0)}^{\Lambda,\Lambda_0} + W_{(1)}^{\Lambda,\Lambda_0} + O(\chi^2)\,. \tag{77}$$

Therefore $W_{(1)}$ generates those with one O-insertion. In the same way as for the L^{Λ,Λ_0} we expand

$$W_{(0)}^{\Lambda,\Lambda_0} = \sum_{r\geq 0} e^r\, W^{\Lambda,\Lambda_0,r}\,,\; W_{(1)}^{\Lambda,\Lambda_0} = \sum_{r\geq 0} e^r\, W_{(1)}^{\Lambda,\Lambda_0,r}\,, \tag{78}$$

$$W^{\Lambda,\Lambda_0,r} = \sum_{m+2n>0} W_{m,2n}^{\Lambda,\Lambda_0,r}\,,\; W_{(1)}^{\Lambda,\Lambda_0,r} = \sum_{m+2n>0} W_{m,2n,1}^{\Lambda,\Lambda_0,r} \tag{79}$$

and finally

$$W_{m,2n,1}^{\Lambda,\Lambda_0,r} = \int \frac{d^4q}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4p_{2n-1}}{(2\pi)^4}\,(\mathcal{W}_{m,2n,1}^{\Lambda,\Lambda_0,r})(q,k_1,\ldots k_m,p_1,\ldots p_{2n-1})\times \tag{80}$$
$$\times\,\chi(q)\,J(k_1)\ldots J(k_m)\overline{\eta}(p_1)\ldots\ldots\eta(p_{2n})\,,$$

and similarly for $\mathcal{W}_{m,2n}^{\Lambda,\Lambda_0,r}$. For shortness we left out indices. The $\mathcal{W}'s$ are assumed to have the same symmetry and antisymmetry properties as the $\mathcal{L}'s$. We know already from ch.3 that the $\mathcal{W}_{m,2n}^{\Lambda,\Lambda_0,r}$ are finite in the limit $\Lambda\to 0$ for those momentum configurations for which the $\mathcal{L}_{m,2n}^{\Lambda,\Lambda_0,r}$ are finite in the same limit *and* for which no external photon momentum vanishes, since the external lines are no more amputated in W.
With these definitions we get from (75) (remembering (62), (63)) for $r\geq 1$:

$$(m+1)\,iq_\rho D_{\rho\mu}^{-1}(q)\mathcal{W}_{m+1,2n}^{\Lambda,\Lambda_0,r}(q,k_1,\ldots k_m,p_1,\ldots p_{2n-1})_{\mu\mu_1\ldots j_n}+$$
$$+\,i\sum_{a=1}^{n}\{\,R_{m,\Lambda}^{\Lambda_0}(p_a)\mathcal{W}_{m,2n}^{\Lambda,\Lambda_0,r-1}(k_1,\ldots k_m,p_1,\ldots,p_a+q,\ldots,p_{2n-1})_{\mu_1\ldots j_n}-$$
$$-\,R_{m,\Lambda}^{\Lambda_0}(p_{n+a})\mathcal{W}_{m,2n}^{\Lambda,\Lambda_0,r-1}(k_1,\ldots k_m,p_1,\ldots,p_{n+a}+q,\ldots,p_{2n-1})_{\mu_1\ldots j_n}\,\}+$$
$$+\,\mathcal{W}_{m,2n,1}^{\Lambda,\Lambda_0,r}(q,k_1,\ldots k_m,p_1,\ldots,p_{2n-1})_{\mu_1\ldots j_n} = 0 \tag{81}$$

In the derivation of (81) we used $1+r_{m,\Lambda}^{\Lambda_0}=R_{m,\Lambda}^{\Lambda_0}$. (81) for $r=0$ is realized to be trivially fulfilled. Since $R_{m,\Lambda}^{\Lambda_0}$ has a well-defined limit for $\Lambda\to 0$, IR-finiteness of $\mathcal{W}_{m,2n,1}^{\Lambda,\Lambda_0,r}$ for $\Lambda\to 0$ may be inferred from that of $\mathcal{W}_{m,2n}^{\Lambda,\Lambda_0,r}$ (see also Proposition 7). We can pass from unamputated to amputated quantities. We define the generating functional of the UV- and IR- regularized CAG with $O_\Lambda^{\Lambda_0}$-insertions in analogy with (11), (14), (20), (76) as

$$e^{1/2\,<J,D_\Lambda^{\Lambda_0}\,J>+<\overline{\eta},S_\Lambda^{\Lambda_0}\,\eta>}\ \exp\{-(L^{\Lambda,\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(DJ,S\eta,\overline{\eta}S)\ +\ f.i.)\} \tag{82}$$

$$:=\ \exp\{-(L^{\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(\delta_J,-\delta_\eta,\delta_{\overline{\eta}})\}\ e^{1/2\,<J,D_\Lambda^{\Lambda_0}\,J>+<\overline{\eta},S_\Lambda^{\Lambda_0}\,\eta>}$$

with

$$L^{\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}\ :=\ L^{\Lambda_0}-\ <\chi,O_\Lambda^{\Lambda_0}>\,. \tag{83}$$

¿From these definitions we will be able to derive a FE for the $L_{m,2n,1}$ defined as in (78)-(80) with $W\to L$, $\mathcal{W}\to\mathcal{L}$. The aim to arrive at such a FE was the reason for raising $<\chi,O_\Lambda^{\Lambda_0}>$ to the exponent in (73), (76). The vWI's (81) in terms of the $\mathcal{L}_{m,2n}$, $\mathcal{L}_{m,2n,1}$ take the form for $r\geq 2$:

$$i(m+1)\,q_\mu\,R_\Lambda^{\Lambda_0}(q)\,\mathcal{L}^{\Lambda,\Lambda_0,r}_{m+1,2n}(q,k_1,\dots k_m,p_1,\dots p_{2n-1})_{\mu\mu_1\dots j_n}\,+$$

$$+\,i\sum_{a=1}^{n}\{\,((S(-p_a))^{-1}\,S_\Lambda^{\Lambda_0}(-p_a-q))_{i_a i'_a}\,\mathcal{L}^{\Lambda,\Lambda_0,r-1}_{m,2n}(k_1,\dots k_m,p_1,\dots,p_a+q,\dots,p_{2n-1})_{\mu_1\dots i'_a\dots j_n}$$
$$-\,\mathcal{L}^{\Lambda,\Lambda_0,r-1}_{m,2n}(k_1,\dots k_m,p_1,\dots,p_{n+a}+q,\dots,p_{2n-1})_{\mu_1\dots j'_a\dots j_n}(S_\Lambda^{\Lambda_0}(p_{n+a}+q)\,(S(p_{n+a}))^{-1})_{j'_a j_a}\,\}$$
$$+\,\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n,1}(q,k_1,\dots k_m,p_1,\dots,p_{2n-1})_{\mu_1\dots j_n}\ =\ 0\,. \tag{84}$$

The unviolated WI's, which we want to recover for $\Lambda\to 0$, $\Lambda_0\to\infty$ are obtained from (84) on replacing $R_\Lambda^{\Lambda_0}$ by 1, $S_\Lambda^{\Lambda_0}$ by S and $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n,1}$ by 0. The first two replacements are true for $\Lambda\to 0$, $\Lambda_0\to\infty$ and finite nonvanishing momenta. That the last is also true on taking limits is a consequence of the FE for $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n,1}$ and the boundary conditions at $\Lambda=\Lambda_0$ (see (83),(74)) and at $\Lambda=0$. The latter will follow from (84) and the r.c. for the $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n}$. FE's for Green functions with operator insertions have been studied extensively in [4] in the Φ_4^4-context. For the present case they have already been presented in [13]. In the same way as in Proposition 2 we find

$$\exp\{-(L^{\Lambda,\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(A,\overline{\psi},\psi)\ +\ f.i.)\}\ =\ e^{\triangle(\Lambda,\Lambda_0)}\ \exp\{-(L^{\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(A,\overline{\psi},\psi))\} \tag{85}$$

and

$$\exp\{-(L^{\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(\delta_J,-\delta_\eta,\delta_{\overline{\eta}})\}\ e^{1/2\,<J,D_\Lambda^{\Lambda_0}\,J>}\ e^{<\overline{\eta},S_\Lambda^{\Lambda_0}\,\eta>}\ = \tag{86}$$

$$e^{1/2\,<J,D_\Lambda^{\Lambda_0}\,J>}\ e^{<\overline{\eta},S_\Lambda^{\Lambda_0}\,\eta>}\exp\{-(L^{\Lambda,\Lambda_0}_{<\chi,O_\Lambda^{\Lambda_0}>}(A,\overline{\psi},\psi)\ +\ f.i.)\}|_{A=D_\Lambda^{\Lambda_0}J,\,\psi=S_\Lambda^{\Lambda_0}\eta,\,\overline{\psi}=\overline{\eta}S_\Lambda^{\Lambda_0}}$$

for the generating functional of the regularized CAG with $O_\Lambda^{\Lambda_0}$-insertions. The FE is obtained as (22) by taking a Λ-derivative on both sides (replace L by$L_{<\chi,O_\Lambda^{\Lambda_0}>}$ in (22)). If we were to apply the Λ-derivative also to the Λ-dependent term $O_\Lambda^{\Lambda_0}$ we would obtain a much more complicated equation than (22), however. We therefore fix the Λ-parameter in $O_\Lambda^{\Lambda_0}$ to be equal to δ : $O_\Lambda^{\Lambda_0} \to O_\delta^{\Lambda_0}$. We choose $0 < \delta << m$. The respective δ-dependent $L^{\Lambda,\Lambda_0,r}_{m,2n,1}$ or $\mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n,1}$ are then denoted as $L^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1}$, $\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1}$ and we define $L^{\delta,\Lambda,\Lambda_0,r}_{m,2n} := L^{\Lambda,\Lambda_0,r}_{m,2n}$. For $\delta = \Lambda$ obviously $\mathcal{L}^{\Lambda,\Lambda,\Lambda_0,r}_{m,2n,1} = \mathcal{L}^{\Lambda,\Lambda_0,r}_{m,2n,1}$. We are interested in the limit $\delta = \Lambda \to 0$. Following these remarks we thus take a Λ-derivative of the following equation (cf. (85)):

$$\exp\{-(L^{\Lambda,\Lambda_0}_{<\chi,O_\delta^{\Lambda_0}>}(A,\overline{\psi},\psi) + f.i.)\} = e^{\triangle(\Lambda,\Lambda_0)} \exp\{-(L^{\Lambda_0}_{<\chi,O_\delta^{\Lambda_0}>}(A,\overline{\psi},\psi))\}$$

Expanding in powers of e and of the external fields $A,\overline{\psi},\psi$ we obtain to zeroth order in χ the FE (25) and for the first order terms we find

$$\partial_\Lambda L^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1} = (\partial_\Lambda \triangle')\, L^{\delta,\Lambda,\Lambda_0,r}_{m,2n+2,1} + (\partial_\Lambda \triangle'')\, L^{\delta,\Lambda,\Lambda_0,r}_{m+2,2n,1} +$$

$$\sum_1 \int \frac{d^4k_0}{(2\pi)^4}\frac{1}{2} D_{\mu\nu}(k_0)\,(\partial_\Lambda R(\Lambda,k_0))\,(\delta_{A_\mu(k_0)} L^{\delta,\Lambda,\Lambda_0,r'}_{m',2n',s'})\,(\delta_{A_\nu(-k_0)} L^{\delta,\Lambda,\Lambda_0,r''}_{m'',2n'',s''}) - \tag{87}$$

$$\sum_2 \int \frac{d^4p_0}{(2\pi)^4} S_{ij}(p_0)\,(\partial_\Lambda R_m(\Lambda,p_0))\,(\delta_{\psi_i(p_0)} L^{\delta,\Lambda,\Lambda_0,r'}_{m',2n',s'})\,(\delta_{\overline{\psi}_j(-p_0)} L^{\delta,\Lambda,\Lambda_0,r''}_{m'',2n'',s''})\ ,$$

where $\sum_1$ is over $r'+r''=r, m'+m''=m+2, n'+n''=n, s'+s''=1$ and $\sum_2$ is over $r'+r''=r, m'+m''=m, n'+n''=n+1, s'+s''=1$ and we denote $L_{m,2n,0}=L_{m,2n}$. The equation analogous to (25) is then

$$\partial_\Lambda\,(\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1})_{\mu_1,\ldots j_n}(q,k_1,\ldots,p_{2n-1}) =$$

$$= -\binom{m+1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}D_{\mu\nu}(k)\,(\partial_\Lambda R(\Lambda,k))\,(\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m+2,2n,1})_{\mu\nu\mu_1,\ldots j_n}(q,k,-k,k_1,\ldots,p_{2n-1}) +$$

$$+(-1)^n(n+1)^2\int\frac{d^4p}{(2\pi)^4}S_{ji}(p)\times$$

$$\times(\partial_\Lambda R_m(\Lambda,p))\,(\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m,2n+2,1})_{\mu_1\ldots ii_1\ldots i_nj\ldots j_n}(q,k_1,\ldots,k_m,-p,p_1,\ldots,p_n,p,\ldots,p_{2n-1}) +$$

$$\sum_1 m'm''(-1)^{n'n''}[\,(\partial_\Lambda R(\Lambda,k'))\,D_{\mu\nu}(k')\,(\mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'})_{\mu\mu_1\ldots j_{n'}}(k',k_1,\ldots,k_{m'-1},p_1,\ldots,p_{n+n'-1})\times$$

$$(\mathcal{L}^{\delta,\Lambda,\Lambda_0,r''}_{m'',2n'',1})_{\nu\mu_{m'}\ldots j_n}(q,-k',k_{m'},\ldots,p_{2n-1})\,]_{SAS} + \sum_2(-1)^{n'n''+n''}n'n''[\,(\partial_\Lambda R_m(\Lambda,p'))\,S_{ji}(p')\times \tag{88}$$

$$\times(\mathcal{L}^{\Lambda,\Lambda_0,r'}_{m',2n'})_{\mu_1\ldots\mu_{m'}i_1\ldots i_{n'}jj_1\ldots j_{n'}}(k_1,\ldots,k_{m'},p_1,\ldots,p_{n'},p',\ldots,p_{n+n'-1})\times$$

$$\times(\mathcal{L}^{\delta,\Lambda,\Lambda_0,r''}_{m'',2n'',1})_{\mu_{m'+1}\ldots\mu_m ii_{n'+1}\ldots i_nj_{n'}\ldots j_n}(q,k_{m'+1},\ldots,k_m,-p',p_{n'+1},\ldots,p_n,p_{n+n'},\ldots,p_{2n-1})\,]_{SAS}$$

$$+\sum_2(-1)^{n'n''+n''}n'n''[\,(\partial_\Lambda R_m(\Lambda,p'))\,S_{ji}(p')\times$$

$$\times\, (\mathcal{L}^{\delta,\Lambda,\Lambda_0,r'}_{m',2n',1})_{\mu_1\ldots\mu_{m'} i_1\ldots i_{n'} j j_1\ldots j_{n'}}(q,k_1,\ldots,k_{m'},p_1,\ldots,p_{n'},p',\ldots,p_{n+n'-1})\,\times$$

$$\times\, (\mathcal{L}^{\Lambda,\Lambda_0,r''}_{m'',2n''})_{\mu_{m'+1}\ldots\mu_m i i_{n'+1}\ldots i_n j_{n'}\ldots j_n}(k_{m'+1},\ldots,k_m,-p',p_{n'+1},\ldots,p_n,p_{n+n'},\ldots,p_{2n-1})\,]_{SAS}\,.$$

$\sum_1, \sum_2$ are defined as in (87) with the exception that we do not sum over s', s''. Otherwise the notation corresponds to that of (25).

To bound the solutions of (88) we have again to look at the b.c. For $\Lambda = \Lambda_0$ they follow from (83), (74) (with $\Lambda = \delta$ in (83),(74)). We find at

$\underline{\Lambda = \Lambda_0}$ for $r \geq 1$:

$$\begin{aligned}
(\mathcal{L}^{\delta,\Lambda_0,\Lambda_0,r}_{1,0,1})_\mu(q) \;&=\; -iq_\mu\, R^{\Lambda_0}_\delta(q)\,(q^2\delta\lambda_r \,+\, \delta\mu_r^2)\\
(\mathcal{L}^{\delta,\Lambda_0,\Lambda_0,r}_{3,0,1})_{\nu\rho\sigma}(q,k_1,k_2) \;&=\; -4iq_\mu\, R^{\Lambda_0}_\delta(q)\, f_{\mu\nu\rho\sigma}\, z_{4r}\,, \quad (q=-k_1-k_2-k_3)\\
(\mathcal{L}^{\delta,\Lambda_0,\Lambda_0,r}_{0,2,1})_{ij}(q,p) \;&=\; i\not{q}\,[z_{2,r-1} \,-\, R^{\Lambda_0}_\delta(q)\, z_{1,r-1} \,+\, \delta_{1r}\, r^{\Lambda_0}_\delta(q)\,] \,+ \qquad (89)\\
+\, iz_{2,r-1}\,(-\not{p}\, r^{\Lambda_0}_{m,\delta}(p) \,&+\, (\not{q} \,+\, \not{p})\, r^{\Lambda_0}_{m,\delta}(q+p)) \,+\, i\delta m_{r-1}\,(r^{\Lambda_0}_{m,\delta}(p) \,-\, r^{\Lambda_0}_{m,\delta}(q+p))\\
(\mathcal{L}^{\delta,\Lambda_0,\Lambda_0,r}_{1,2,1})_{\mu ij}(q,k,p) \;&=\; i(\delta_{r,2} \,+\, z_{1,r-2})\,(r^{\Lambda_0}_{m,\delta}(p) \,-\, r^{\Lambda_0}_{m,\delta}(q+p))\,.
\end{aligned}$$

Remember $r^{\Lambda_0}_{a,\delta}(p) \;=\; r^{\Lambda_0}_{a,\delta}(-p)$, $\;z_{is} := 0$ for $s \leq 1$ and note that $r^{\Lambda_0}_{m,\delta}(p) = R_m(\Lambda_0,p) - 1$ since $\delta < m/2$ (see (4),(74)). All other $\mathcal{L}^{\delta,\Lambda_0,\Lambda_0,r}_{m,2n,1}$ vanish. The coefficients $z_{ir}, \delta m_r, \delta\mu_r^2, \delta\lambda_r$ obey the bounds

$$|z_{ir}|,\; |\delta\lambda_r| \,\leq\, Plog\Lambda_0\,, \tag{90}$$

$$|\delta m_r| \,\leq\, \Lambda_0\, Plog\Lambda_0\,,\;\; |\delta\mu_r^2| \,\leq\, \Lambda_0^2\, Plog\Lambda_0 \tag{91}$$

due to Proposition 3. We also have to look at the boundary conditions at $\Lambda = \delta$. Since the $\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{m,2n,1}$ are given in terms of the vWI's (84) we can calculate the boundary values using (84). Looking first at zero momenta we realize that these boundary values for the terms with $m + 3n + |w| \leq 4$ often vanish due to the $O(4)$ and C symmetries of the theory which simplifies our task considerably. As can be seen from (61) and (8), χ has to transform as a scalar (trivially) under $O(4)$, whereas $\chi \to -\chi$ under charge conjugation. From this and (74) we then deduce $O(4)$ and charge conjugation invariance of $L^{\Lambda,\Lambda_0}_{<\chi,O^{\Lambda_0}_\delta>}$. Using also the invariance of the theory without insertions this implies

(i) $\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{m,0,1} \;\equiv\; 0$ for m even (C)

(ii) $(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_\mu(0) \;=\; 0$ ($O(4)$)

(iii) $\partial_\mu(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_\nu(0) \;=\; c^\delta_{1r}\delta_{\mu\nu}$ ($O(4)$)

(iv) $\partial_\mu\partial_\nu(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_\rho(0) \;=\; 0$ ($O(4)$)

(v) $\partial_\mu\partial_\nu\partial_\rho(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_\sigma(0) \;=\; c^\delta_{2r} f_{\mu\nu\rho\sigma}$ $\quad$ ($O(4)$ and permutation symmetry for μ,ν,ρ) $\quad$ (92)

(vi) $(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(0) \;=\; 0$ ($O(4)$ and C)

(vii) $\partial_{q_\mu}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(0) \;=\; c^{\delta}_{3r}\,(\gamma_\mu)_{ij} \;=\; -\,\partial_{p_\mu}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(0)$ ($O(4)$ and C)

(viii) $(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\mu\nu\rho}(0) \;=\; 0$ ($O(4)$ and Bose symmetry)

(ix) $\partial_{q_\mu}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\nu\rho\sigma}(0) \;=\; c^{\delta}_{4r}\,f_{\mu\nu\rho\sigma} \;=\; -\,\partial_{k_{j,\mu}}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\nu\rho\sigma}(0)$

(x) $(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,2,1})_{\mu ij}(0) \;=\; 0$ (C).

In the derivation of (ix) it was not sufficient to use the symmetries, which also allow for a term $c'^{\delta}_{4r}(\delta_{\mu\nu}\,\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\,\delta_{\nu\rho})$. This term is excluded however using the vWI (84) which gives

$$(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\nu\rho\sigma}(q,k_1,k_2) \;=\; -4i\,q_\mu R^{\Lambda_0}_{\delta}(q)(\mathcal{L}^{\delta,\Lambda_0,r}_{4,0})_{\mu\nu\rho\sigma}(q,k_1,k_2),\quad |q|,|k_i| < \Lambda_0\ . \tag{93}$$

We remark that the existence of the derivatives in (92) for $\delta = 0$ can be inferred from (84). For $|q|$ between δ and 2δ the functions $\partial^w R^{\Lambda_0}_{\delta}(q)$ change rapidly and have large derivatives $\sim O(\delta^{-|w|})$. The task of finding suitable bounds for the $\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1}$ is therefore considerably simplified by restricting to the region $|q| > 2\delta$ (see below). This is compatible with our way of inductively estimating the solutions of the FE since q appears as a fixed parameter on both sides of (88). Such restricted bounds are sufficient for our purposes since we let $\delta \to 0$ in the end. In this case we also have to use boundary conditions for the FE in which the value of q fulfills $|q| > 2\delta$. To be definite we choose some q with $|q| = 3\delta$, and the second momentum argument appearing in $\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{m,2n,1}$ is then chosen as $-q$, the others being 0. Using (93) and (48) as well as (54) we find

$$|(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\mu\nu\rho}(q,-q,0)| \le \delta^2\, P\log\delta^{-1}\,, |\partial_{q_\mu}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{3,0,1})_{\nu\rho\sigma}(q,-q,0) - (-4i\,z^R_{4r}\,f_{\mu\nu\rho\sigma})| \le \delta\, P\log\delta^{-1}\,. \tag{94}$$

Using again the vWI (84) we also find from (48) and (54)

$$|(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_{\mu}(q)| \le\; \delta^4\, P\log\delta^{-1}\,,\; |\partial_\mu(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_{\nu}(q) - (-i\delta\mu^{2R}_r\,\delta_{\mu\nu})| \le \delta^3\, P\log\delta^{-1}\,, \tag{95}$$

$$|\partial_\mu\partial_\nu(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_{\rho}(q)| \le\; \delta^2\, P\log\delta^{-1}\,,\; |\partial_\mu\partial_\nu\partial_\rho(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,0,1})_{\sigma}(q) \;-\; (-6i\delta\lambda^R_r\,f_{\mu\nu\rho\sigma})| \le\; \delta\, P\log\delta^{-1}\,.$$

For $m = 0, n = 1, r \ge 2$ (84) takes the form

$$\begin{aligned}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(q,-q) \;&=\; -iq_\mu(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,2})_{\mu ij}(q,-q) + \\ (-i)[(\frac{\not{q}+m)}{m}(\mathcal{L}^{\delta,\Lambda_0,r-1}_{0,2}(0))_{ij} \;&-\; (\mathcal{L}^{\delta,\Lambda_0,r-1}_{0,2})(-q)\,\frac{m}{\not{q}+m})_{ij}]\,,\end{aligned} \tag{96}$$

where we used the fact that all $R^{\Lambda_0}_{a,\delta}$'s equal 1 for the chosen momentum arguments. Using the continuity of $\mathcal{L}_{1,2}$ and the continuous differentiability of $\mathcal{L}_{0,2}$ we obtain from (96)

$$|(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(q,-q)| \leq \delta^2 Plog\delta^{-1}\,, \tag{97}$$

$$|\partial_{q_\mu}(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{0,2,1})_{ij}(q,-q) - \{-iz^R_{1,r-1} - i[-z^R_{2,r-1} + \frac{2}{m}\delta m^R_{r-1}]\}(\gamma_\mu)_{ij}| \leq \delta Plog\delta^{-1}\,.$$

For $r=1$ one realizes on going back to (81) that $\mathcal{W}^{\delta,\delta,\Lambda_0,1}_{0,2,1}(q,p)=0$ for $|q|,|p|\leq \Lambda_0/2$ which implies $\mathcal{L}^{\delta,\delta,\Lambda_0,1}_{0,2,1}(q,p)=0$ for $|q|,|p|\leq\Lambda_0/2$. All other $\mathcal{L}^{\delta,\delta,\Lambda_0,1}_{m,2n,1}$ also vanish by inspection of the vWI's (trivially). Similar considerations finally show

$$|(\mathcal{L}^{\delta,\delta,\Lambda_0,r}_{1,2,1})_{\mu ij}(q,-q,0)| \leq \delta\, Plog\delta^{-1}\,. \tag{98}$$

(93),(94),(95),(98) now tell us that all terms of dimension ≤ 4 which do not vanish a priori by symmetry, are explicitly given in terms of the renormalization constants from (48) - up to corrections bounded by $\delta\, Plog\delta^{-1}$. The aforementioned restriction on the values of q is implemented by adapting the norms (34), with the aid of which we estimated the solutions of the FE, to the present situation:
For a system a functions $f_{\mu_1,\ldots,i_1,\ldots,j_1,\ldots}(q,k_1,\ldots,p_1,\ldots,p_{2n-1})$ depending on q and on photon momenta $k_1,\ldots,k_m$ and fermion and antifermion momenta $p_1,\ldots,p_{2n}$ we define

$$||\partial^{|w|}f||^{\delta}_{(a,b)}$$

as in (34) with the additional restriction that we only take the sup over the momenta fulfilling $|q|>2\delta$.
We also need an adaptation of Def.1-5 in sect.3.2. The following changes are necessary: Q is now the set $\{q,k_1,\ldots,p_1,\ldots,p_{2n}\}$. In Def.1 we replace (ii) by: (ii') m odd if $n=0$. In Def.3 one of the sets E_ν now contains q. This set is then counted as if it were $E_\nu\setminus\{q\}$ in all subsequent definitions, and we find immediately that Lemmas 4, 5 may be restated for the new situation without change. Now we prove:

Proposition 7: Let $B>0$ be any fixed constant, $|w|\leq 4$, $0<\delta<<m$, $|q|>2\delta$, $\delta\leq\Lambda\leq\Lambda_0<\infty$. All subsequent estimates are uniform in δ and Λ_0. For the r.c. (48) together with the following restriction

$$z^R_{1,r} = z^R_{2,r} - \frac{2}{m}\delta m^R_r\,,\ r\geq 1, \tag{99}$$

in particular for the r.c.(49), we obtain the following bounds for $\Lambda\geq 1$

$$||\partial^{|w|}\mathcal{L}^{\delta,\Lambda,\Lambda_0,r}_{m,2n,1}||^{\delta}_{(2\Lambda,B)} \leq (\frac{\Lambda}{\Lambda_0}\, Plog\Lambda_0\, +\, \delta\, Plog\delta^{-1})\,\Lambda^{4-m-3n-|w|}\,. \tag{100}$$

The constants in $Plog$ depend on $m, n, r, |w|, B$.
Now let $\delta \le \Lambda \le 1$ and all momenta $q, k_1, \ldots, p_{2n-1}$ be bounded by B. We find for nonexceptional momentum configurations Q (or for $\{q, -q\}$ if $m = 1, n = 0$)

$$|\partial^w \mathcal{L}_{m,2n,1}^{\delta,\Lambda,\Lambda_0,r}(q, k_1, \ldots, p_{2n-1})| \le \Lambda_0^{-1}\, Plog \Lambda_0 + \delta\, Plog \delta^{-1}\,. \tag{101}$$

The constants in $Plog$ now also depend on η (45), but (101) holds uniformly in $U_\varepsilon(\{q, \ldots, p_{2n}\})$ (see Proposition 6).
Furthermore we find in the same sense as in Proposition 6 for exceptional momentum sets $Q = \{q, k_1, \ldots, p_{2n}\}$

$$|\partial^w \mathcal{L}_{m,2n,1}^{\delta,\Lambda,\Lambda_0,r}(\hat{q}, \hat{k}_1, \ldots, \hat{p}_{2n-1})| \le \Lambda^{-2g_1(Q)-|w|}\, (\Lambda_0^{-1}\, Plog \Lambda_0 + \delta\, Plog \delta^{-1}) \tag{102}$$

uniformly in $U_\varepsilon(Q)$. The statement analogous to (b2) in Proposition 6 is the following: Let $Q = \{q, k_1, \ldots, p_{2n}\}$ be such that $Q = Q_a \cup Q_b$ and $q \in Q_a$, where Q_b is of the form $Q_b = \{q_1^{(b)}, -q_1^{(b)}, \ldots, q_l^{(b)}, -q_l^{(b)}\}$. And let Q_a be such that for any $E \subset Q$ with $\sum_E q_i = 0$ either $Q_a \subset E$ or $E \cap Q_a = \emptyset$. Write $Q_a = \{q, q_1, \ldots, q_s\}$; we denote by Q_{a_1} any strict subset of Q_a and by $\partial_{a_1}^w$ any sequence of w derivatives w.r.t. to $q_i \in Q_{a_1}$. Then we claim

$$|\partial_{a_1}^w \mathcal{L}_{m,2n,1}^{\delta,\Lambda,\Lambda_0,r}(Q_b, \hat{Q}_{a_1})| \le \Lambda^{-2g_1(Q)}\, (\Lambda_0^{-1}\, Plog \Lambda_0 + \delta\, Plog \delta^{-1})\,, \tag{103}$$

where the statement is uniform in $U_\varepsilon(Q_a)$ (see Proposition 6).
For $m + 3n + |w| \le 4$ we obtain the bounds

$$|\partial^w \mathcal{L}_{3,0,1}^{\delta,\Lambda,\Lambda_0,r}(q, k_1, k_2)| \le \Lambda^{2-|w|}\, Plog \Lambda^{-1}\,, \ |\partial^w \mathcal{L}_{1,0,1}^{\delta,\Lambda,\Lambda_0,r}(q)| \le \Lambda^{4-|w|}\, Plog \Lambda^{-1}\,, \tag{104}$$

$$|\partial^w \mathcal{L}_{1,2,1}^{\delta,\Lambda,\Lambda_0,r}(q, k, p)| \le \Lambda^{1-|w|}\, Plog \Lambda^{-1}\,, \ |\partial^w \mathcal{L}_{0,2,1}^{\delta,\Lambda,\Lambda_0,r}(q, p)| \le \Lambda^{2-|w|}\, Plog \Lambda^{-1}\,. \tag{105}$$

(104), (105) are uniform in δ, Λ_0. As in Prop.6,(c) those statements for which the r.h.s. vanishes for $\Lambda \to 0$ hold uniformly only for momenta bounded by $O(1)\Lambda$, otherwise they hold for momenta $\le B$.
Proof: The proof is in many aspects similar to that of Propositions 3 and 6. We will concentrate on those aspects which are new. The two contributions appearing on the r.h.s. of (100)-(103) enter through the boundary conditions at large and small Λ. We use the standard inductive scheme. The statements for $r = 1$ are immediately verified, since all $\mathcal{L}_{m,2n,1}^{\delta,\Lambda,\Lambda_0,1}$ vanish. For $r > 1$ we go down in $m + 2n$ and for given $m + 2n$ down in $|w|$. We start with $m + 3n + |w| \ge 5$. At $\Lambda = \Lambda_0$ the bound (100) is verified using (89)-(91). Using the induction assumption and the bounds from Prop.3 on the r.h.s. of the FE (88) we also verify (100) down to $\Lambda = 1$.
Now we may integrate further down to $\Lambda \ge \delta$. At $\Lambda = 1$ (101) to (105) for $m + 3n + |w| \ge 5$

are true since (100) has been verified for $\Lambda = 1$. We start verifying (102) by estimating the r.h.s. of the FE with the aid of the induction hypothesis and Prop.6. The proof proceeds then as that of (53), Prop.6. As there we also need the statements (104) to lower order in the respective estimates. Having verfied (102) we may also prove (103) and (101), (104) and (105).
Coming now to $m+3n+|w| \le 4$ we have as usually to integrate the FE from the lower end, here from $\Lambda = \delta$ upwards, with the momenta fixed at some renormalization point. It follows from (92)-(98), (99) that all $\partial^w \mathcal{L}^{\delta,\delta,\Lambda_0;r}_{m,2n,1}$ with $m+3n+|w| \le 4$ fulfill the estimates (101) to (105) for the momentum arguments imposed in (94)-(96). Integrating then the FE from δ to $\Lambda > \delta$ at these arguments and using the induction hypothesis on the r.h.s. we verify (104), (105) also for $\Lambda > \delta$. The next step is then to go from the renormalization points to arbitrary momenta q, p, k_1, k_2, k ($\le B$) via the Schlömilch formula (41), starting from $m+3n = 4$ and treating then $m+3n = 3, |w| = 1, 0$ and $m = 1, |w| = 3, 2, 1$. Using the induction hypothesis allows then to verify all statements (100) to (105) for abitrary momenta bounded by B. ■
An immediate consequence of Proposition 7 and (84) is now
Proposition 8: (Restoration of the Ward Identities)
Sending the UV-cutoff Λ_0 to ∞ and δ to 0 (in arbitrary order) the connected amputated Green functions $\mathcal{L}^r_{m,2n}$ fulfill the standard QED Ward identities. That means - for momenta for which they are well-defined - they satisfy the equations (for $r > 1$)

$$
\begin{aligned}
&(m+1) q_\mu (\mathcal{L}^r_{m+1,2n})_{\mu\mu_1\ldots j_n}(q, k_1, \ldots, p_{2n-1}) = \\
&-\sum_{a=1}^{n} [((S(-p_a))^{-1}\, S(-p_a - q)\, \mathcal{L}^{r-1}_{m,2n}(k_1, \ldots, p_a + q, \ldots, p_{2n-1}))_{\mu_1,\ldots,j_n} - \\
&(\mathcal{L}^{r-1}_{m,2n}(k_1, \ldots, p_{n+a} + q, \ldots, p_{2n-1}) S(p_{n+a} + q)\, (S(p_{n+a})^{-1})_{\mu_1,\ldots,j_n}].
\end{aligned}
\tag{106}
$$

5 On Massless QED

In this paper we have treated Euclidean QED with massive fermions. In view of physical reality one should also find a way to pass to the Minkowski metric which we intend to do. The case of massless fermions is less important from this point of view, still there are massless fermions in the standard model (and maybe in nature). So we shortly indicate the modifications necessary in this case without giving a proof. In massless QED we have to regularize the fermion propagator by $R(\Lambda, p)$ instead of $R_m(\Lambda, p)$, since R_m is not an infrared regulator. This change induces a deterioration in the IR estimates (see (39)).
The definition of the index g (Def.4) has to be changed as follows:

(i) Momentum sets Q consisting of two fermion momenta only are no more admissible.

(ii) Assume a momentum subset E_ν (43) or Q itself can be subdivided into two subsets $E_{\nu 1}$, $E_{\nu 2}$ such that the sum over the momenta in both vanishes and such that both contain an odd number of momenta from $\{p_1, \ldots, p_{2n}\}$. In this case Q is called exceptional and the set E_ν contributes $3/2$ to $g_Z(Q)$ (as before) if it consists of two single momenta, it contributes 2 if one subset $E_{\nu i}$ consists of more than one momentum, and it contributes $5/2$ otherwise.

These changes are then sufficient to prove $g(Q'_F) + g(Q''_F) + 1/2 \le g(Q)$, if $p' = 0$ instead of Lemma 4 (d). We need this sharpened inequality to prove Proposition 6 in the massless case. The improved index g_1 is of no use any more since Lemma 5 (c),(d1) are no more sufficient to bound the $\mathcal{L}^{\Lambda}_{m,2n}(Q)$ by $\Lambda^{-2g_1} Plog\Lambda^{-1}$. The new index g is then such that $\mathcal{L}^{\Lambda,\Lambda_0}_{4,0}$, $\mathcal{L}^{\Lambda,\Lambda_0}_{1,2}$ and $\partial_\mu \mathcal{L}^{\Lambda,\Lambda_0}_{0,2}$, $\partial_\mu\partial_\nu \mathcal{L}^{\Lambda,\Lambda_0}_{2,0}$ are allowed to be logarithmically divergent for $\Lambda \to 0$ at zero momentum, when estimated with the aid of the FE. All these terms therefore have to be renormalized at nonexceptional momenta whereas the undifferentiated two point functions have to be renormalized at zero momentum. Renormalization at nonexceptional momenta induces notational complications. Regard e.g. $\mathcal{L}_{1,2}$ (leaving out upper indices). We find from symmetry considerations

$$(\mathcal{L}_{1,2})_{\mu,i,j} = (\gamma_\mu)_{ij}\, l_1(k,p) \,+\, k_\mu \not{p}_{ij}\, l_2(k,p) + \, p_\mu \not{k}_{ij}\, l_3(k,p) \,+\, k_\mu \not{k}_{ij}\, l_4(k,p) \,+\, p_\mu \not{p}_{ij}\, l_5(k,p)$$

(to be compared with (31)), where the l_i depend on k, p only through $O(4)$-invariant combinations. The function l_1 is then fixed by a r.c. at some nonexceptional momentum configuration $\{k, p, -k-p\}$, whereas $l_{2,3,4,5}$ are to be calculated from terms with $m + 3n + |w| \ge 5$. To solve for l_2 choose e.g. k, p nonexceptional such that $p = (0, p_2, 0, 0)$, $k = (0, 0, k_3, 0)$. Then

$$-4p_2\, l_2(k,p) \;=\; tr(\gamma_2 \partial_{k_1} (\mathcal{L}_{1,2})_1(k,p))\,,$$

and similar expressions for $l_{3,4,5}$. Arbitrary nonexceptional momenta can now be reached on application of the Schlömilch formula.

Observing these changes and imposing r.c. as described above it is then straightforward to rewrite Proposition 6. In part (a) the degree of smoothness is generally reduced by 1 ($C^2(\mathbf{R}^4) \to C^1(\mathbf{R}^4)$, $C^1(\mathbf{R}^4) \to C^0(\mathbf{R}^4)$), whereas $\mathcal{L}^{\Lambda,\Lambda_0}_{4,0}$, $\mathcal{L}^{\Lambda,\Lambda_0}_{1,2}$ may diverge logarithmically at exceptional momenta for $\Lambda \to 0$. In (b3) we have to replace g_1 by the new g. In (c) we claim

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0,r}_{2,0}(k)| \le \Lambda^{2-|w|} Plog\Lambda^{-1},$$

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0,r}_{0,2}(p)| \le \Lambda^{1-|w|} Plog\Lambda^{-1},$$

in the same sense as in Proposition 6.

These changes in the IR behaviour and the corresponding modifications of the r.c. have to be taken into account in ch.IV, i.e. in the relations between the $\mathcal{L}^{\Lambda}_{m,2n}$ and$\mathcal{L}^{\Lambda}_{m,2n,1}$ for $\Lambda = 0$ (92), ... These then also have to be exploited at the new renormalization points. This does not change the statement on the restoration of the WI's at nonexceptional momenta. But the proof of the statement corresponding to Proposition 7 becomes more complicated. This is due to the fact that on the r.h.s. of (89) all $r^{\Lambda_0}_{m,\delta}$ are to be replaced by $r^{\Lambda_0}_{\delta}$ which have large derivatives for small δ. A restriction as $|q| > 2\delta$ is no more sufficient to exclude their appearance since the $r^{\Lambda_0}_{\delta}$ also carry arguments p which in turn appear as integration variables on the r.h.s. of the FE. Therefore we have to make a new induction hypothesis. The first step is again to adapt the norms (34) to the new situation by the following definition:

For a system a functions $f_{\mu_1,\ldots,i_1,\ldots,j_1,\ldots}(q, k_1, \ldots, p_1, \ldots, p_{2n-1})$ depending on q and on photon momenta $k_1, \ldots, k_m$ and fermion and antifermion momenta $p_1, \ldots, p_{2n}$ we define

$$||\partial^{|w|} f||^{\delta}_{(a,b)}$$

as in (34) with the additional restriction that we only take the sup over the momenta fulfilling

$$|q| > 2\delta \text{ and also: } |p_i|,\ |q + p_i + S_i|\ > 2\delta \ \text{ or } \ |p_i|,\ |q + p_i + S_i|\ < \delta\,,\ i = 1, \ldots, n\,, \quad (107)$$

where S_i denotes any (possibly empty) subsum over momenta from $\{q, \ldots, \ldots p_{2n}\} \setminus \{q, p_i\}$ which contains as many fermion as antifermion momenta (for $n = 0$ only q is restricted, as in the massive case).

The bounds of Prop.7 now hold again if the conditions (106) are satisfied. If they are not satisfied we claim weaker bounds to hold which are obtained from those of Prop.7 on multiplication by $\delta^{-|w|}$. Since the volume of those regions in p_i-space where (107) is violated is $O(\delta^4)$ the factor $\delta^{-|w|}$ is compensated by the integration volume on performing the momentum integral on the r.h.s. of the FE, as long as we restrict to $|w| \leq 4$ (which is sufficient for us). The restoration of the WI's is then obtained letting $\delta \to 0$, $\Lambda_0 \to \infty$ as before.

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