Inverse spectral theory for Jacobi matrices and their almost periodicity

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Abstract

In this paper we consider the inverse problem for bounded Jacobi matrices with nonempty absolutely continuous spectrum and as an application show the almost periodicity of some random Jacobi matrices. We do the inversion in two different ways. In the general case we use a direct method of reconstructing the Green functions. In the special case where we show the almost periodicity, we use an alternative method using the trace formula for points in the orbit of the matrices under translations. This method of reconstruction involves analyzing the Abel-Jacobi map and solving of the Jacobi inversion problem associated with an infinite genus Riemann surface constructed from the spectrum.

Contents

1 Introduction

In this paper we address the question of recovering a Jacobi matrix

$$Hu(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1), u \in l^2(\mathbb{Z}) \tag{1}$$

with $a_n \geq 0$ and b_n real, from its spectrum Σ, assuming it to be a compact set, and also the the question of the almost periodicity of the sequences a_n and b_n constructed from the spectrum. We consider the case, when the real part of the boundary values of the Green function for the vector δ_0 vanish almost everywhere on the spectrum. In this case we show that there is no uniqueness even when the Dirichlet eigen values of the half space problems are specified.

We use this theory to prove the almost periodicity of some random jacobi matrices with the spectrum having a band structure. The motivation for this work comes from the work on periodic Jacobi matrices and the inverse theory for Schrödinger operators. There is extensive work on inverse spectral theory in the sense of recovering the operators from given spectral quantities, for periodic Schrödinger operators in the literature, the work of Mckean-Moerbecke [31], McKean-Trubowitz [32], Trubowitz [38] being the some of these. There is also the work of Dubrovin-Matveev-Novikov [9], Levitan [27],[28],[29] for almost periodic potentials. In a general framework of ergodic potentials, the inverse spectral theory was initiated by Kotani, who showed the existence of ergodic Schrödinger operators associated with a class of spectral functions, and also discussed classical integrable systems in this framework in a series of papers ([18] - [21]). These inversion results of Kotani produced classes of potentials, while in the pathwise infromation, and the nature of the isospectral class obtained were discussed in Kotani-Krishna [23] for ergodic potentials and Craig [8] for a very general class of reflectionless potentials. In the discrete case there are several analogues of the above, for periodic examples, in the works of Kac-Moerbecke [14], [15], Dubrovin-Matveev-Novikov [9], Toda [37]. In the case of random Jacobi matrices, Carmona-Kotani [5] obtain an invariant probability measure associated with some spectral functions. With these examples in mind we wanted to address two questions, one is to invert a Jacobi matrix from its spectrum and the other is to identify the situations when the resulting matrix is almost periodic. For the case of finite band spectra we [2] proved almost periodicity of the Jacobi matrices.

More recently there has been some interest in the inverse spectral theory, and a very general set up is proposed by Gestezy-Simon, see the announcement [12], who identified the xi function as the central object for inverse theories. They also give a new proof for existence of absolutely continuous spectrum for almost mathieu operators with small coupling. There is also the work of Knill [17] on isospectral deformation for random Jacobi matrices. As this paper was nearing completion we received the beautiful lecture notes of Simon [36] on the applications of rank one perturbations to inverse theory, with some constructions similar to what we do in section 1.

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1.1 Ideas, strategies and limitations

In this subsection we discuss the questions we addressed in this paper and provide some clarifications of the assumptions we make use of and the results we obtain using our method and also discuss some of the limitations of our proofs.

In the section on Inverse spectral theory, we show how to reconstruct a Jacobi matrix from the spectrum when the Green function for the vector δ_0 of the Jacobi matrix has vanishing real part almost everywhere on the spectrum. The method of this section obtains the Jacobi matrix from some spectral data. However the reconstruction does not give a unique answer. In the Schròdinger case the source of non-uniqueness is expected to be the Dirichlet eigen values of the half space problems. However in the discrete case there are two additional sources of non-uniqueness even when the Dirichlet eigen values of the half space problems are specified. One is that even if the Jacobi matrix H has purely absolutely continuous spectrum Σ, the half space operators $H^{\pm}$ may have some singular spectrum in Σ. Alternately, even if $H^{\pm}$ have no singular spectrum in Σ, the spectral measures of $H^{\pm}$ associated with the vectors $\delta_{\pm 1}$ restricted to Σ may be different. In the last section we will see that for an ergodic Jacobi matrix with purely absolutely continuous spectrum given by a band structure both these sources of non-uniqueness will be absent.

The idea of showing almost periodicity is through using the Trace formula equation (60) and solving the analogue of the Dubrovin equation for the spectral parameters in this case to obtain the given operator. However there are a few differences. In the Schrödinger case (see Kotani-Krishna [23] or Craig [8]) the Dubrovin equation, $d\xi_i(t)/dt = W_i(\xi_i(t))$ for the spectral parameters $\xi(t)$ came for free, from the differential equation

$$(g''_\lambda(x,x) - 2(q-\lambda)g_\lambda(x,x))\, g_\lambda(x,x) - {g'_\lambda}^2 + 1 = 0$$

satisfied by the Green functions in the resolvent set. This equation was proved by Moser [30] and was also used in McKean-Moerbeke [31] and McKean-Trubowitz [32]. To solve for $\{\xi_i(t), t \in \mathbb{R}\}$, it is enough to get good estimates for $W_i(\xi)$ in the gaps (allowing for example Cantor like spectra) as done by Craig [8]. In the Jacobi case information in the gaps alone is not sufficient, we need to know even the behaviour of some spectral functions in the bands also, even in the case of reflectionless potentials, where all the Green functions $g_\lambda(n,n)$ have vanishing real parts in the spectrum.

Starting with section 3, we concentrate on a special class of spectra and set up the machinery required for proving almost periodicity of some Jacobi matrices. In section 3, we discuss an interpolation theorem for a class of analytic functions associated with the spectrum of a Jacobi matrix.

In the subsequent section, we discuss the Riemann surface and the Abel-Jacobi map on a class of divisors. We compute the image of the divisors in this class and for subsequent applications, extend the map to an infinite dimensional Banach space. The proposition (4.3) reminds one of the classical theorem of Abel, for compact surfaces, that the principal divisors are precisely those whose image in the Jacobi variety under the Abel-Jacobi map is 0. It is too tempting from our proofs to conclude a similar theorem for the surface $\mathcal{R}$. But unfortunately it is not entirely clear even when the set $\mathcal{E}$ is finite. The reason is that if we consider an arbitrary meromorphic function, on the Riemann surface $\mathcal{R}$ its behaviour on the bounding curves of the approximants is unclear, even under the simplifying assumptions used on the spectrum. Our proof of the computation of the image of the Abel-Jacobi map is essentially proving the bilinear relations of Riemann (though we do not state the relations in this paper) for a class of meromorphic differentials and these relations are valid only in a very special sense, and hence only for a special class of meromorphic differentials, in our context.

We would like to add a few more words about the Abel-Jacobi map whose properties might appear mysterious if not clarified. The set up and the ideas we use are from the beautiful papers of Levitan [27, 28, 29] who was working with Schrödinger operators. We believe that almost all the points we describe below are essentially contained in his work though they might not have been stated explicitly.

Classically (that is in the case of a compact Riemann surface) it is a map from the divisors on the surface to the Jacobi variety (a compact object). Even in our case the Jacobi variety is compact and we consider only some divisors in "real position" (see McKean-Trubowitz [32] for more detail on these) and hence the real part of the Jacobi variety. However the Jacobi variety is not a linear space. It is convenient to work on a linear space to be able to use the inverse function theorem , for Jacobi inversion, in this infinite dimensional setting.

The Abel-Jacobi map is set up on an infinite dimensional real Banach space (with its norm chosen using the geometrical conditions on the spectrum). In this framework, the Abel-Jacobi map may be **non-differentiable**. (The problems is that the family of functions f_{ii}s, introduced in the equation (45) are not necessarily equicontinuous family, so that the differentiability of $\mathcal{A}_1$ is not at all clear). It is however Lipshitz continuous and its inverse is also Lipshitz continuous and this is enough for the proofs of almost periodicity of some spectral parameters. To achieve this we split the Abel-Jacobi map into its "diagonal" and "off-diagonal" parts. The "diagonal" part is a Lipshitz continuous bijection and the "off-diagonal" part is a differentiable, periodic, (with its periods coming from the lattice of (π) integral points of the Banach space) and a compact map on the Banach space. This is the reason for introducing the auxiliary map $\mathcal{B}$.

We use throughout this paper the notation $\mathbb{Z}^{\pm}$ for $\{\pm 1, \pm 2, \pm 3, \cdots\}$, unless stated otherwise.

2 Inverse Spectral Theory

In this section we start with a compact subset Σ of $\mathbb{R}$ of positive Lebesgue measure and obtain a Jacobi matrix which has Σ as the essential closure of its absolutely continuous spectrum. The procedure involves constructing a likely candidate for the Green function at 0, by specifying its argument of the Herglotz function on the set and its complement. This allows us to determine the logarithm of the function from which the function is constructed by requiring a specific asymptotic behaviour. We then look at the inverse of the function so constructed, determine its poles outside the set Σ and construct two Herglotz functions which will be the Green functions of Jacobi matrices on square integrable sequences on the left and right half lattices. These will have simple spectrum and from these the full Jacobi matrix is recovered.

There is a long history for the inverse problem for a Jacobi matrix. There are the classical works of Akhiezer [1], Kac-Moerbecke [14], [15], Moerbeke [39] Dubrovin-Matveev-Novikov [9] done for the half lattice problem and for periodic sequences. The latest such inversion on the half lattice is in the work of Rajaram Bhat-Parthasarathy [34], in a different context. Some of the ideas used here in the construction of the Herglotz functions associated with a given set can be found in the works of Kotani [18],[19], [20], [21] , Kotani-Krishna [23], Craig [8], Levitan [27] [28], [29] , Gezstezy-Holden-Zhao-Simon [12] and Gezstezy-Simon [13]. In addition, the book of of Carmona-Lacroix [6] provides very good back ground for the inverse problems for random potentials.

We start by making an assumption on the set we would like to consider.

Assumption 1 *Let Σ be a compact subset of $\mathbb{R}$ of positive Lebesgue measure, which we write as the complement of the disjoint union of the gaps,*

$$\Sigma = \mathbb{R} \setminus [I(-\infty) \cup I(+\infty) \cup_{i=1}^{\infty} I_i] \tag{2}$$

where, we take $\tau_0 = inf\, \Sigma$ and $\tau_\infty = sup\ \Sigma$,

$$I(-\infty) = (-\infty\ ,\ \tau_0)\ ,\ I(\infty) = (\tau_\infty, \infty)\ ,\ I_i = (\tau_{2i-1}, \tau_{2i})$$

We consider the following set of sequences that will serve as the zeros of the function to be constructed. Henceforth we use the following notation

$$\begin{aligned}
\Pi &= \mathbb{C} \setminus \Sigma \quad \Psi = \prod_{i=1}^{\infty} \bar{I}_i \\
S_\xi &= \{\xi_1, \xi_2, \cdots \xi_n, \cdots\} \quad \xi \in \Psi \\
\mathcal{O} &= \cup_{i=1}^{\infty} I_i \ , \ \mathcal{O}_\xi = \mathcal{O} \cap S_\xi .
\end{aligned}$$

where the line above the set I_i indicates the closure of the set. For ξ in Ψ, consider a partition $S_\xi^+ \cup S_\xi^-$ of S_ξ into disjoint subsets $S_\xi^\pm$, we write

$$\mathcal{O}_\xi^\pm = \mathcal{O} \cap S_\xi^\pm . \tag{3}$$

Given this information we can proceed to construct a class of functions as follows. In the following for simplicity of notation we set

$$k(\lambda, x) = \frac{1}{(x-\lambda)} \quad \textit{and} \quad K(\lambda, x) = k(\lambda, x) - \frac{x}{1+x^2} \tag{4}$$

and choose the square root occuring in the log in the following lemma so that it is positive on (-∞,0).

Lemma 2.1 *Consider Σ as in assumption 1 and let $\xi \in \Psi$. Then there is a Herglotz function $F_\xi(\lambda)$ such that*

$$\begin{aligned}
F_\xi(\lambda) &= \int k(\lambda, x) d\sigma(x) + \frac{1}{2} \ \ log \frac{1}{(\tau_0 - \lambda)(\tau_\infty - \lambda)} \quad \lambda \in \mathbb{C}^+ \\
\int k(\lambda, x) d\sigma(x) &= \sum_{i=1}^{\infty} \ log \frac{(\xi_i - \lambda)}{(\tau_{2i} - \lambda)} + \ \ log \frac{(\xi_i - \lambda)}{(\tau_{2i-1} - \lambda)}
\end{aligned} \tag{5}$$

with σ a signed absolutely continuous measure of finite total variation. Fur-

ther $F_\xi(\lambda) \to log(-1/\lambda)$ *as* $\lambda \to \infty$ *and*

$$Im \;\; F_\xi(x) = \begin{cases} \frac{\pi}{2} & a.e. \quad x \in \Sigma \\ 0 & on \quad (-\infty, \xi_i) \cap I_i \;\; \forall i = 1, 2, \cdots \\ \pi & on \quad (\xi_i, \infty) \cap I_i \;\; \forall i = 1, 2, \cdots . \end{cases}$$

The sum in (5) converges compact uniformly in Π*.*

Proof: Consider a non negative bounded function, specified almost everywhere by,

$$\xi(x) = \begin{cases} \frac{\pi}{2} & a.e. \quad x \in \Sigma \\ 0 & on \quad (-\infty, \xi_i) \cup I_i \;\; \forall i = 1, 2, \cdots \\ \pi & on \quad (\xi_i, \infty) \cup I_i \;\; \forall i = 1, 2, \cdots . \end{cases}$$

Then $\int \xi(x)dx/(1 + x^2) < \infty$. Therefore for each c $\in \mathbb{R}$, the function

$$F_\xi(\lambda) = c + \frac{1}{\pi} \int K(\lambda, x)\xi(x)dx \tag{6}$$

is Herglotz. It is convenient sometimes to write F , as done by Craig [8], in terms of a signed measure. Therefore we choose another function,

$$\phi(x) = \frac{\pi}{2} \forall x \in [\tau_0, \tau_\infty] \;\; and \;\; \pi \;\; \forall x \in (\tau_\infty, \infty) \tag{7}$$

taken to be 0 otherwise, and define

$$G_\xi(\lambda) = \frac{1}{\pi} \int K(\lambda, x)\phi(x)dx. \tag{8}$$

We find that G is also Herglotz, since $\int \phi(x)dx/1 + x^2 < \infty$. We then write

$$F_\xi = F_\xi - G_\xi + G_\xi$$

and choose the number c so that F_ξ has the representation stated in the Lemma. The asymptotic behaviour for F_ξ comes from the second term in the expression for F_ξ given in the statement of the lemma, since the measure σ is finite. For clarity we write the measure σ below.

$$d\sigma(x) = \sum_{i=1}^{\infty} \frac{1}{2}\left(\chi_{(\tau_{2i-1},\xi_i)}(x) - \chi_{(\xi_i,\tau_{2i})}(x)\right) dx$$

We get the series expansion for F_ξ by integration by parts. Let S be a compact subset of Π. Then there is an $\epsilon > 0$, such that $\mathrm{dist}(\Sigma, S) > \epsilon$. We have the following estimates, for any i = 1,2,$\cdots$ and $\xi_i \neq \tau_{2i-1}$ or τ_{2i}. Clearly the estimate is trivial for $\xi_i = \tau_{2i-1}$ *or* τ_{2i}.

$$|\frac{(\xi_i - \lambda)}{(\tau_{2i-1} - \lambda)}| \leq 1 + |\frac{(\xi_i - \tau_{2i-1})}{(\tau_{2i-1} - \lambda)}| \leq 1 + |\frac{\ell_i}{\epsilon}| \tag{9}$$

and

$$|\frac{(\xi_i - \lambda)}{(\tau_{2i} - \lambda)}| \leq 1 + |\frac{(\xi_i - \tau_{2i})}{(\tau_{2i} - \lambda)}| \leq 1 + |\frac{\ell_i}{\epsilon}|. \tag{10}$$

where ℓ_i's are the gap lengths $|I_i|$. Their sum converges since, all I_i 's are a disjoint union of intervals contained in $[\tau_0, \tau_\infty]$. This shows the convergence, uniformly on compacts of Π, of the sum in equation (5). □

By exponentiating F_ξ of the above Lemma, we get the following proposition, where we define the formal products

$$P_\xi(\lambda) = \prod(\lambda - \xi_i) \quad and \quad R(\lambda) = (\lambda - \tau_0)(\lambda - \tau_\infty)\prod_i(\lambda - \tau_i).$$

The ratio of the formal products $P_\xi(\lambda)$ and $\sqrt{R(\lambda)}$ converge compact uniformly in Π, by the convergence of the sum in equation (5). It follows by lemma A.2 of the appendix, that the measure μ in the following representation has an absolutely continuous component supported on Σ.

Proposition 2.2 *Consider Σ as in Assumption (1) and let $\xi \in \Psi$. Then there is a unique Herglotz function h_ξ analytic in Π satisfying the following properties.*

1. *h_ξ has simple zeros on $\mathcal{O}_\xi$.*

2. *$h_\xi = \frac{-1}{\lambda} + O(\frac{1}{\lambda^2})$ as $\lambda \to \infty$.*

3. *$h_\xi(x + i0)$ has the following values on $\mathbb{R}$.*

$$\begin{aligned} Reh_\xi(x+i0) &= 0 \ \ a.e. \ \ on \ \ \Sigma \ \ Imh_\xi(x+i0) = 0 \ , \ \forall x \in \mathbb{R} \notin \Sigma \\ Reh_\xi(x+i0) &> 0 \ , \ x \in I(-\infty) \cup_{i=1}^{\infty} (\xi_i, \infty) \cap I_i \\ Reh_\xi(x+i0) &< 0 \ , \ x \in I(\infty) \cup_{i=1}^{\infty} (-\infty, \xi_i) \cap I_i \end{aligned}$$

h_ξ also has the following Herglotz representation,

$$h_\xi(\lambda) = \int k(\lambda, x) d\mu(x) \tag{11}$$

with μ supported on Σ with the absolutely continuous component of μ having essential support Σ. The product representation

$$h_\xi(\lambda) = \frac{P_\xi(\lambda)}{\sqrt{R(\lambda)}} \tag{12}$$

is also valid with the products converging uniformly on compacts of Π.

Proof: We consider the function

$$h_\xi(\lambda) = exp(F_\xi(\lambda))$$

where F_ξ is as in lemma (2.1). Then the stated properties of h_ξ follow from those of F_ξ, the equation (5) and the estimates (9) and (10) used in concluding the convergence of the sum representing F_ξ. $\square$

The following lemma will be cruicial for solving the inverse problem. The lemma gives a Herglotz representation for the inverse of h_ξ given above.

Lemma 2.3 *The function $g_\xi = -{h_\xi}^{-1}$ is Herglotz and has the representation*

$$g_\xi(\lambda) = \lambda + d_\xi + \int k(\lambda, x) d\nu(x) \tag{13}$$

where ν is a finite positive measure with support $\Sigma \cup \mathcal{O}_\xi$ with the absolutely continuous part having support in Σ. Its singular part in $\mathcal{O}_\xi$ is pure point.

Proof: It is clear that g_ξ defined above is Herglotz, since h_ξ is Herglotz and has the stated representation , from proposition (A.2)(4) , since the support of the representing measure is compact and from the definition of g_ξ its behaviour at infinity is like λ. We let ν be the finite positive measure given by the Herglotz representation theorem. We note that equation (12) gives the product representation

$$g_\xi = -\frac{\sqrt{R(\lambda)}}{P_\xi(\lambda)} \tag{14}$$

with the product converging uniformly on compacts of Π. It is also clear that from the estimates of equation (5) , g_ξ has a simple pole in I_i, whenever $\xi_i \in I_i$. As for the set Σ, the limits Im $g_\xi(x + i0)$ exist finitely and are positive, at every point where $h_\xi(x + i0)$ has a positive finite imaginary part.

Since this happens a.e. on Σ, the same is true for g_ξ. This shows that the absolutely continuous part of ν is non zero and has essential support Σ. $\Box$

We write ν on Σ and $\mathcal{O}_\xi$ as ν_1 and ν_2 respectively, consider the partition of equation (3) and define $\nu_2^\pm \equiv \nu_2|_{\mathcal{O}_\xi^\pm}$. Given these we consider some positive measures, $\nu_{1\pm}$ so that $\nu_{1+} + \nu_{1-} = \nu_1$ as measures and define

$$\nu^\pm = \nu_{1\pm} + \nu_2^\pm \;\; and \;\; c^\pm = \int d\nu^\pm \tag{15}$$

We then have the following theorem, where we denote by $(\;)^{-ess}$ the closure up to sets of Lebesgue measure zero.

Theorem 2.4 *Consider the set $\Sigma \cup \mathcal{O}_\xi$, Σ as in assumption (1) and the measures $\nu^\pm$. Then there exist unique Jacobi matrices $H^\pm$ on $l^2(\mathbb{Z}^\pm)$ with simple spectra such that*

$$c^\pm (H^\pm - \lambda)^{-1}(\pm 1, \pm 1) = \int k(\lambda, x) d\nu^\pm(x).$$

In particular the absolutely continuous spectrum of $H^\pm$ is Σ^{-ess} and they have eigen values on $\mathcal{O}_\xi^\pm$ respectively.

Proof: We shall consider the + case the other one is similar. Consider $L^2(\Sigma \cup \mathcal{O}_\xi, \nu^+)$ and let M be the operator of multiplication by x on this space. Thus M is a bounded self-adjoint operator. The constant function 1 on $\Sigma \cup \mathcal{O}_\xi^+$ together with the monomials x^m , m = 1,2,$\cdots$ form a basis for the above Hilbert space, therefore by the Grahm-Schmidt procedure we can get an orthonormal basis $\{e_n\}$, n = 1,2,$\cdots$, out of these. We write the matrix elements of M as $M_{ij} =< e_i, Me_j >$ in this basis and see that for i > j +1 ,

M_{ij} is zero, since Me_j is at most a polynomial of degree j+1. On the other hand since $< e_i, Me_j > = \overline{< e_j, Me_i >}$, by the self-adjointness of M, it follows that for $i < j-1$ also M_{ij} is zero showing that M is tridiagonal in this basis. Clearly by the self-adjointness of M the diagonal entries are real and the off diagonal entries can be chosen to be positive, by choosing the phases of the vectors e_n appropriately.

We take the unitary isomorphism U from $L^2(\Sigma \cup \mathcal{O}_\xi, \nu^+)$ to $l^2(\mathbb{Z}^+)$ taking the basis $\{e_n\}$ to the canonical basis δ_n, We define $H^+ = UMU^{-1}$. Then $\langle e_n, Me_m \rangle = \langle \delta_n, H^+ \delta_m \rangle$. This gives us the uniqueness. We write the matrix elements of H^+ as

$$H^+_{i,i+1} = H^+_{i+1,i} = a_i \quad and \quad H^+_{ii} = b_i \quad i \in \mathbb{Z}^+.$$

We construct H^- on $l^2(\mathbb{Z}^-)$ similarly using ν^-. With these definitions the operators $H^\pm$ act on $l^2(\mathbb{Z}^\pm)$ as

$$(H^\pm)u(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1), |n| \neq 1$$

and

$$(H^+)u(1) = a_1 u(2) + b_1 u(1) \ , \ and \ \ (H^-)u(-1) = b_{-1} u(-1) + a_{-2} u(-2).$$

which proves the lemma. □

We also have by construction that, $M^\pm$ defined by

$$M^\pm(\lambda) \equiv \int k(\lambda, x) d\nu^\pm \ \text{ satisfies } \ M^\pm(\lambda) = c^\pm (H^\pm - \lambda)^{-1}(\pm 1, \pm 1) \tag{16}$$

for $\lambda \in \mathbb{C}^+$. Clearly the above expression can also be rewritten in the special case when $\nu_{1\pm} = \frac{1}{2}\nu_1$, as

$$M^\pm(\lambda) = \frac{1}{2} \int k(\lambda, x) d\nu \pm \int k(\lambda, x) d(\nu_2^+ - \nu_2^-). \tag{17}$$

For reconstructing the Jacobi matrix we set, using the function g_ξ of lemma (2.3) and positive squre roots of $c^\pm$,

$$a_0 = \sqrt{c^+}\ , \quad a_{-1} = \sqrt{c^-}\ , \quad \textit{and} \ \ b_0 = d_\xi .$$

and note that b_0 is real since g_ξ is Herglotz. We prefer to state the theorem by fixing the set of Dirichlet eigen values for the half space problems to emphasize the non-uniqueness present in the problem.

Theorem 2.5 *Consider the set Σ as in assumption (1) and the sets $\mathcal{O}_\xi^\pm$ associated with a point $\xi \in \Psi$ and a partition $S_\xi^\pm$. Then there exists a Jacobi matrix H with*

$$(H - \lambda)^{-1}(0,0) = h_\xi(\lambda)$$

The absolutely continuous spectrum of H is Σ^{-ess}, has multiplicity 2 and the half space operators $H^\pm$ have eigen values on $\mathcal{O}_\xi^\pm$. Such a H is unique if the spectral measures $\nu^\pm$ of $H^\pm$ for the vectors $\delta_{\pm 1}$ are also specified in which case we have

$$c^\pm (H^\pm - \lambda)^{-1} = \int k(\lambda, x) d\nu^\pm$$

Proof: Given Σ we construct h_ξ as in Proposition (2.2), g_ξ as in Proposition (2.3) and consider ν_1. Decompose ν_1 into its absolutely continuous ν_{1ac} and its singular ν_{1s} parts. Consider any partition $\rho_1 + \rho_2$ of unity and consider, the measures

$$\nu_{1+} = \rho_1 \nu_{1ac} \quad \nu_{1-} = \rho_2 \nu_{1ac} + \nu_{1s}$$

for example and let $\nu^\pm = \nu_{1\pm} + \nu_{2\pm}$. Then we consider the half lattice operators $H^\pm$ constructed in theorem (2.4) and define the operator H on

$l^2(\mathbb{Z})$ using $\nu^{\pm}$ by

$$(Hu)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1).$$

It is clearly a bounded self-adjoint operator and has Σ^{-ess} as its absolutely continuous spectrum of multiplicity 2, since H and $H^+ \oplus H^-$ differ by a (finite rank and hence) trace class operator and $H^+ \oplus H^-$ has Σ^{-ess} as its absolutely continuous spectrum with multiplicity 2. The remaining statement is clear by construction. Its uniqueness follows from the unique reconstruction of the operators $H^{\pm}$ from $\nu^{\pm}$ together with the determination of the numbers a_0, a_{-1} and b_0 are uniquely obtained from $\nu^{\pm}$ and h_{ξ}. The other properties are clear as in the previous proposition. $\square$

Remark 2.6 *The above theorem is a reformulation of the traditional way of stating the inverse theorem in terms of associating the operators $H^{\pm}$ to spectral parameters $\{\tau_i, \xi_i, \sigma_i\}$, with σ_i given values ± 1 whenever ξ_i is the eigen value of $H^{\pm}$ in the gaps. The formulation we give here is seems better. We should note here that there is a great deal of non-uniqueness in the above construction, even when the measures $\nu_{2\pm}$ are fixed. The special case $\nu_{1+} = \nu_{1-} = \frac{1}{2}\nu_1$ is valid for a class of ergodic Jacobi matrices, as we shall see in the last section. The asymmetry $a_0 = \int \nu^+$ while $a_{-1} = \int \nu^-$ comes from the definition of m-functions of the Jacobi matrix, see Simon [35] for example.*

3 Interpolation theorem

In this section we consider a class of analytic and meromorphic functions that would be candidates for the Green functions of Jacobi matrices. We impose further restrictions on the set Σ considered in the last section, and take the other quantities associated with Σ as before. We recall the the definition of the gaps I_i. We denote $I_k < I_i$ to mean I_k is to the left of I_i. This gives an ordering of the gaps which is convenient for deducing the properties of some of the functions to be introduced in this section and which will be used later. For a given gap I_i we denote the infimum of its distance from other gaps as $4s_i$, that is $4s_i = \inf_{k \neq i} \; dist(I_i, I_k)$. We call s_0, s_∞ the distances of I$(-\infty)$ and I(∞) from the nearest gaps. We also set $q_i = \ell_i / s_i$ and we take without loss of generality that $q_i < 1$. We also note that assumption (1) implies that ℓ_i and s_i are summable. With these notations we make the following assumptions.

Assumption 2 *Let Σ be a compact set satisfying assumptions (1) and in addition suppose,*

1. $s_i > 0$ *for all i =1,2,$\cdots$ and for i=0,∞.*
2. $\sum_{i=1}^{\infty} q_i < \infty$ *and* $\sum_{i=1}^{\infty} q_i / s_i < \infty$.
3. *Let $\mathcal{E}$ denote the set of accumulation points of the boundary points of Σ and $\mathcal{E}_1$ the set of accumulation points of $\mathcal{E}$ and $\mathcal{E}_1$ is a finite set.*

We would like to recover a class of functions from their values at a given set of points as in the case of a polynomial of degree n which can be recovered from its values on a set of n+1 points. This is the main idea of the interpolation theorem. Towards stating the interpolation, we consider a collection of points one each from the closure of each gap, or the set of points coming from all gaps except one. The notation $\infty - 1$ is strange, but as in the work of McKean-Trubowitz [32], it is natural. Associated with these points we define classes of analytic functions below. To start with we define a set , where the union is the disjoint union, given by

$$\Psi_{\infty-1} = \bigcup_{i=1}^{\infty} \Psi^i \quad , \ \Psi^i = \prod_{k \neq i} \bar{I}_k. \tag{18}$$

Recall proposition (2.2) where we associated a unique function h_ξ with Σ and $\xi \in \Psi$. These collection of functions will be denoted as,

$$\mathcal{H} = \{ h_\xi : \ \xi \in \Psi \quad \}. \tag{19}$$

Given a point $\xi \in \Psi$, we can write it as (ξ_i, ξ^i) with $\xi_i \in \bar{I}_i$ and $\xi^i \in \Psi^i$. We then consider the analytic function in Π given by,

$$\omega_{\xi^i} = \frac{1}{\lambda - \xi_i} h_\xi. \tag{20}$$

It is not hard to see from the product representation for h_ξ in Π that ω_{ξ^i} is independent of $\xi_i \in \bar{I}_i$. This class of functions has the following properties in the gaps.

Lemma 3.1 *Suppose Σ satisfies assumptions (2). Then the following bounds are valid for any λ. We set dist(λ, I_i) to be d_i, i = 1,2,$\cdots$, dist(λ, τ_0) = d_0 and dist(λ, τ_∞) = d_∞.*

$$|\frac{(\xi_i - \lambda)}{(\tau_{2i-1} - \lambda)}| \leq 1 + |\frac{(\xi_i - \tau_{2i-1})}{(\tau_{2i-1} - \lambda)}| \leq 1 + |\frac{\ell_i}{d_i}| \tag{21}$$

and

$$|\frac{(\xi_i-\lambda)}{(\tau_{2i}-\lambda)}| \leq 1+|\frac{(\xi_i-\tau_{2i})}{(\tau_{2i}-\lambda)}| \leq 1+|\frac{\ell_i}{d_i}|. \tag{22}$$

$$|\frac{1}{\sqrt{(\tau_0-\lambda)(\tau_\infty-\lambda)}}| \leq |\frac{1}{\sqrt{d_0 d_\infty}}|. \tag{23}$$

Proof: The proof is trivial once we allow both sides to be infinite. □

For the following proposition, we set the distances of τ_0 and τ_∞ from I_k to be d_{0k} and $d_{\infty k}$.

Lemma 3.2 *Consider the ω_η for $\eta \in \Psi_{\infty-1}$. Then it has the following properties. There is an i$\in \mathbb{Z}^+$ such that:*

1. $\omega_\eta(x) > 0 \; x \in I_i$.
2. $\omega_\eta(x) > 0 \; x \in I(-\infty) \cup \left[\bigcup_{I_k<I_i}(\tau_{2k-1},\eta_k)\right] \cup \left[\bigcup_{I_k>I_i}(\eta_k,\tau_{2k})\right]$.
3. $\omega_\eta(x) < 0 \; x \in I(\infty) \cup \left[\bigcup_{I_k>I_i}(\tau_{2k-1},\eta_k)\right] \cup \left[\bigcup_{I_k<I_i}(\eta_k,\tau_{2k})\right]$.
4. $\omega_\eta(\lambda) = O(\frac{1}{\lambda^2})$, $\lambda \to \infty$
5. *The following bounds are valid in $\bar{I}_k$ for each $k \neq i$,*

$$\omega_\eta(\lambda) \leq |\frac{q_k}{\sqrt{d_{0k}d_{\infty k}}}\frac{\prod_{j\neq k}(1+\frac{\ell_j}{s_j})}{\sqrt{(\lambda-\tau_{2k-1})(\lambda-\tau_{2k})}}|$$

and for $k = i$,

$$|\frac{\prod_{j\neq i}(1-\frac{\ell_j}{s_j})}{(\tau_\infty-\tau_0)}| \leq |\sqrt{(\lambda-\tau_{2i-1})(\lambda-\tau_{2i})}\omega_\eta(\lambda)| \leq |\frac{\prod_{j\neq i}(1+\frac{\ell_j}{s_j})}{\sqrt{d_{0i}d_{\infty i}}}|. \tag{24}$$

6. $\omega_\eta(\lambda + i0)$ *is integrable on* $\mathbb{R}$.

Proof: Since η belongs to $\Psi_{\infty-1}$, which is a disjoint union of Ψ^k's, there is an i $\in \mathbb{Z}^+$ with $\eta \in \Psi^i$. We consider any point $\xi_i \in I_i$ and let $\xi = (\xi_i, \eta)$. Then $\xi \in \Psi$ and we have a h_ξ associated with this ξ. Using this we define the ω_η as in equation (20). Then the properties (1)-(3) stated for ω_η are clear from those of h_ξ , from proposition (2.2) and the fact that $1/(\lambda - \xi_i)$ is negative in (-∞, ξ_i) and positive in (ξ_i, ∞). The asymptotic behaviour of ω_η is clear from that of h_ξ. Given the asymptotic behaviour and the analyticity of ω_η in I(-∞) and I(∞), it is integrable outside the compact set [a,b] with $a < \tau_0$ and $b > \tau_\infty$. Therefore we consider a compact set [a,b] and show the integrability there. As before we associate a $\xi \in \Psi$ with η with ξ_i chosen in (for example the mid point of the gap) I_i. We write the product representation of equation (12) for h_ξ and write it as

$$h_\xi = \frac{(\lambda - \xi_k)}{\sqrt{(\lambda - \tau_{2k-1})(\lambda - \tau_{2k})}} g_k(\lambda) \tag{25}$$

with

$$g_k(\lambda) = \frac{\sqrt{(\lambda - \tau_{2k-1})(\lambda - \tau_{2k})}}{(\lambda - \xi_k)} h_\xi(\lambda).$$

Then in the expression for $g_k(\lambda)$, the numbers $\tau_{2k-1}, \tau_{2k}, \xi_k$ do not occur in the numerator or denominator. So for each $\lambda \in I_k, k \neq i$, the dist $(\lambda, \bar{I}_j) > s_j$, so that we use the bounds of lemma (3.1) with $d_j = s_j$, and $d_0 = d_{0k}$ and $d_\infty = d_{\infty k}$ both of which are bigger than s_k by assumption (2), to conclude that the bounds,

$$\omega_\eta(\lambda) \leq |\frac{1}{\sqrt{d_{0k} d_{\infty k}}} \frac{\ell_k}{s_k} \frac{\prod_{j \neq k}(1 + \frac{\ell_j}{s_j})}{\sqrt{(\lambda - \tau_{2k-1})(\lambda - \tau_{2k})}}|$$

are valid for $\lambda \in I_k, k \neq i.$ As for the case k = i, the bound

$$\omega_\eta(\lambda) \leq |\frac{1}{d_{0i}d_{\infty i}} \frac{\prod_{j\neq i}(1+\frac{\ell_j}{s_j})}{\sqrt{(\lambda-\tau_{2i-1})(\lambda-\tau_{2i})}}|$$

is clear, since the factors $(\lambda - \xi_i)$ occuring in the numerator of the product representation of h_ξ and that occuring in the denominator , in the definition of ω_η, cancel. The lower bounds of (5)are deduced similarly, by noting that the distance of a point in I_i to either of τ_0, τ_∞ is bounded above by $\tau_\infty - \tau_0$. The infinite products occuring in the numerators of the inequalities in (5) converge by the , summability of q_i of assumptions (2). The integrability in the closure of the gaps follows from (5). In the case of λ in Σ, under the assumptions (2) and theorem (A.2.3) the limits $h_\xi(\lambda + i0)$ exist everywhere in the interior of Σ and and are purely imaginary there by construction of h. Therefore the absolute value of $h_\xi(\lambda+i0)$ is just the density of the absolutely continuous measure (or a constant multiple of it) representing h_ξ hence it is integrable once we note that the factor $1/(\lambda - \xi_i)$ is uniformly bounded in $\lambda \in \Sigma$ for each fixed i, by the choice of ξ_i. $\square$

We consider the real Banach space , with q_i as in assumption (2) and δ_i denoting the point measure at i with unit mass,

$$\mathcal{K} = \quad \ell^\infty_{\mathbb{R}}(\mathbb{Z}^+, \sigma) \quad , \sigma = \sum_{i=1}^{\infty} \quad {q_i}^{-1}\delta_i$$

Using this Banach space and the set $\Psi_{\infty-1}$ we can define a class of functions $\mathcal{H}_{\infty-1}$ analytic in Π as,

$$\mathcal{H}_{\infty-1} = \left\{ \sum_{i=1}^{\infty} \kappa_i \omega_{\xi^i} : \xi^i \in \Psi^i \quad and \quad \kappa \in \mathcal{K} \right\} \tag{26}$$

We will use the functions in $\mathcal{H}_{\infty-1}$ to construct a class of differentials of the first kind on a Riemann surface to be considered in the next section. The properties of functions in $\mathcal{H}_{\infty-1}$ will also be important for the properties of the Abel-Jacobi map that will be constructed later. Therefore we start with some of the simple properties. To do this we need a technical lemma.

Lemma 3.3 *Given any N, we have finitely many, say M(N), non intersecting rectangular curves R_N^i, i = 1,..,M(N) with the following properties.*

1. *Each R_N^i has its sides parallel and perpendicular to the axes and each side parallel to the y-axis goes through the mid point of some band in Σ.*

2. *Every point of $\mathcal{E}$ is enclosed by some R_N^i.*

3. *There is a sequence of positive numbers m_N increasing to infinity such that the sum of the perimeters ,$Per(R_N^i)$, of the curves R_N^i satisfies,*

$$\sum_{i=1}^{M(N)} Per(R_N^i) < \frac{1}{m_N}.$$

Proof: Suppose the cardinality of the set $\mathcal{E}_1$ is ℓ and let the minimum of the distance between the points of $\mathcal{E}_1$ be d, which is positive since the points are distinct. We fix an N and consider the closed intervals J_N^i , i=1,..,ℓ of positive length smaller than $d/\ell 2^{N+1}$ with each point $p_i \in \mathcal{E}_1$ as its mid point and such that the end points of the intervals do not coincide with any point of $\mathcal{E}$. This choice is possible since the points in $\mathcal{E}_1$ are finite. Then it is clear that the number of points of $\mathcal{E}$ in $[\tau_0, \tau_\infty] \setminus \cup_i^\ell J_N^i$ is finite. Let this number be $M(N)$. Let the minimum of their distance be $d(M(N))$. We now pick intervals J_N^j, j = ℓ+1,..,M(N)+ℓ, of length equal to $d(M(N))/(M(N)2^{N+1})$ with these points as their mid points. By our choice, the end points of these intervals do not coincide with any points of $\mathcal{E}$. The choice of the rectangular curves is made as follows. Consider a point $p_j \in \mathcal{E}_1 \cup [\mathcal{E} \setminus \cup_i^\ell J_N^i]$, j =1,..,M(N)+$\ell$. Then by assumptions (2) it follows that there are bands b_j^1 and b_j^2, to the left and right of p_j respectively contained in J_N^j. It is also clear, since the endpoints of J_N^i i = 1,..,ℓ are not points of $\mathcal{E}$, we can choose b_j^1, b_j^2 such that the points of $\mathcal{E} \cap I_N^i$ are to the left of b_j^2 and to the right of b_j^1. For each j, now, we choose the rectangular curve R_N^j with sides parallel and perpendicular to the axes such that , the sides parallel to the y-axis passes through the mid points of b_j^1 and b_j^2. The sides parallel to the x-axis are at a distance equal to maximum of dist(p_j, b_j^k) , k = 1,2. Clearly the perimeters of R_N^j satisfy the bounds $Per(R_N^j) \leq const.1/(\ell 2^{N+1})$ for j=1,..,ℓ and $Per(R_N^j) \leq const.1/(M(N)2^{N+1})$ for j = ℓ+1,.,M(N)+1. The properties listed in the lemma are clear with the redefinition of M(N)+ℓ as M(N) and $m_N = const.\ 2^{N+1}$. □

For later use we shall consider, a different set r_N of rectangles for each N given by $\cup_{i=1}^{M(N)} r_N^i$. Where r_N^i is a rectangle with its sides parallel to the x-and y-axes, the distance of the sides parallel to the x-axis being s_0, one of the sides parallel to the y-axis passes through τ_0 and the other side contains a side of R_N^i given in the above lemma.

We first show an interpolation theorem for ω_η's, for which recall the definition of $\xi^i \in \Psi_{\infty-1}$ associated with $\xi \in \Psi$ and the definition of the set S_ξ

given in the last section.

Proposition 3.4 *Consider $\eta \in \Psi_{\infty-1}$, and consider ω_η. Then there is an $\imath \in \mathbb{Z}^+$ such that for any $\xi \in \Psi$, we have the following relation as analytic functions in Π.*

$$\omega_\eta(\lambda) = h_\xi(\lambda)\sum_{k=1}^{\infty}\frac{D_k^i}{(\lambda-\xi_k)} \tag{27}$$

where

$$D_k^i = \left(\frac{\xi_k-\eta_k}{\xi_k-\xi_i}\right)\prod_{j\neq k,i}\frac{(\xi_k-\eta_j)}{(\xi_k-\xi_j)} \quad D_i^i = \prod_{j\neq i}\frac{(\xi_i-\eta_j)}{(\xi_i-\xi_j)}. \tag{28}$$

with the sum converging compact uniformly in $\mathbb{C}\setminus\mathcal{E}\cup S_\xi$.

Proof: Since η is in $\Psi_{\infty-1}$, there is an $\imath$ such that $\eta \in \Psi^i$. This is the $\imath$ stated in the proposition. So we fix the i and consider the rational function $f(\lambda) = \omega_\eta(\lambda)/h_\xi(\lambda)$ in Π. By using the product representation for h_ξ and the definition of ω_η we see that formally,

$$f(\lambda) = \frac{1}{(\lambda-\xi_i)}\prod_{j\neq i}\frac{(\lambda-\eta_j)}{(\lambda-\xi_j)}.$$

By using the estimates

$$\left|\frac{(\lambda-\eta_j)}{(\lambda-\xi_j)}\right| \leq 1+\left|\frac{(\xi_j-\eta_j)}{(\lambda-\xi_j)}\right| \leq 1+\left|\frac{\ell_j}{(\lambda-\xi_j)}\right|. \tag{29}$$

and the convergence of the products $\prod(1+q_i)$ a computation shows that the product defining $f(\lambda)$ converges uniformly in compacts of $\mathbb{C}\setminus(\mathcal{E}\cup S_\xi)$ and it is also clear that the points ξ_k are simple poles of $f(\lambda)$ in $\mathbb{C}\setminus\mathcal{E}$. Thus showing that $f(\lambda)$ extends to a meromorphic function, with simple poles at ξ_k's, in $\mathbb{C}\setminus\mathcal{E}$. The residues of the function $f(\lambda)$ at the poles ξ_k when computed are precisely the quantities D_k^i stated in the proposition.

On the other hand the function

$$g(\lambda) = f(\lambda) - \sum_{j=1}^{\infty} \frac{D_k^i}{(\lambda - \xi_j)}$$

is analytic in $\mathbb{C}\backslash\mathcal{E}$ and the bounds of equation (29) together with the bounds $(\lambda - \xi_j) > s_j$, which is valid in view of assumption (2), for λ on the rectangles, shows that we have

$$\sup_{\lambda\in R_N^l} g(\lambda) \leq \prod(1+q_j) + \sum_{k=1}^{\infty} \prod_{j\neq k,i} (1+q_j)\frac{\ell_k}{s_k}$$

where we have used the estimates for D_k^i from the lemma (3.5) which will be shown below. The right hand side in the above inequality is independent of N,l etc., so that from the definition of the rectangles and the proposition (3.3), the above uniform bound on them shows that the points in $\mathcal{E} \setminus \mathcal{E}_1$ are not essential singularities of g(λ). The same bound also shows that

$$\lim_{N\to\infty} \int_{\lambda\in\cup_{i=l}^{M(N)} R_N^l} (\lambda - x)^m g(\lambda) = 0$$

for each positive integer m, and hence in the Laurent series expansion of g at the points $x \in (\mathcal{E} \setminus \mathcal{E}_1)$ all the negative coefficients vanish, showing that g can be extend as an analytic function to $\mathbb{C} \setminus \mathcal{E}_1$. A similar argument shows that g can be extended to an analytic function in $\mathbb{C}$. But g has zeros at ξ_i and these accumulate to some point in $\mathcal{E}$, its region of analyticity, showing that g has to be identically zero. □

Lemma 3.5 *Consider $\eta \in \Psi^i$ and $\xi \in \Psi$. Let D_k^i be as in proposition (3.4). Then the following bounds are valid for k≠i,*

$$|D_k^i| \leq |\frac{\xi_k - \eta_k}{\xi_k - \xi_i} \prod_{j\neq k,i} (1+q_j)| \leq C \ \ min(q_k, \frac{\ell_k}{s_i}) \ , \ C_1 \leq |D_i^i| \leq C_2$$

with C_1, C_2 independent of i.

Proof: From the definition of D_k^i it is clear that we can write

$$D_k^i = \frac{\lambda - \eta_k}{\lambda - \xi_i} \prod_{j \neq k,i} \frac{\lambda - \eta_j}{\lambda - \xi_j} |_{\lambda = \xi_k}$$

which shows using the bounds of equation (29) with $\lambda = \xi_k$, and the lower bounds $\xi_k - \xi_i \geq min(s_i, s_k)$ implied by assumption (2). The uniform upper and lower bounds, in the case of k=i are obvious from the expressions. □

Theorem 3.6 (Interpolation theorem) *Consider f in $\mathcal{H}_{\infty-1}$ and let $\xi \in \Psi$. Then we have the following relation as analytic functions in Π.*

$$f(\lambda) = h_\xi(\lambda) \sum_{i=1}^{\infty} \frac{C_i f(\xi_i)}{(\lambda - \xi_i)} \ , \quad C_i = \frac{\sqrt{R(\xi_i)}}{\prod_{j \neq i}(\xi_i - \xi_j)}. \tag{30}$$

In particular f can be written as

$$f(\lambda) = \sum_{i=1}^{\infty} \kappa_i(f) \omega_{\xi^i} \ , \ \kappa(f) \in \mathcal{K} \tag{31}$$

Proof: We consider a function f $\in \mathcal{H}_{\infty-1}$ given by

$$f(\lambda) = \sum_{i=1}^{\infty} \kappa_i \omega_{\zeta^i} \ , \ \kappa \in \mathcal{K}$$

with $\zeta^i \in \Psi^i$. Then using the proposition (3.4) we see that each of the ω_{ζ^i} 's can be written as

$$\omega_{\zeta^i}(\lambda) = h_\xi(\lambda) \sum_{k=1}^{\infty} \frac{D_k^i}{(\lambda - \xi_k)}.$$

¿From this the theorem follows after an interchange of sum and the relation $C_k f(\xi_k) = \sum_{i=1}^{\infty} D_k^i \kappa_i$. The necessary convergences can be checked using the bounds

$$|\kappa_k(f)| = |C_k f(\xi_k)| = \sum_{i=1}^{\infty} |D_k^i \kappa_i| \leq |D_k^k \kappa_k| + |\sum_{i \neq k}^{\infty} |\kappa_i D_k^i| \leq C q_k$$

which follow from the bounds in the lemma (3.5) and from the definition of $\mathcal{K}$. □

Since any two elements of $\mathcal{H}_{\infty-1}$ can be written interms of a single $h_\xi, \xi \in \Psi$, addition of two elements, via the above theorem gives again an element of $\mathcal{H}_{\infty-1}$. Therefore $\mathcal{H}_{\infty-1}$ is a linear space, in fact a vector space over the reals. This fact will be useful for the following corollary.

Corollary 3.7 *Suppose f is in $\mathcal{H}_{\infty-1}$ such that it vanishes at a point each in each of the gaps. Then $f \equiv 0$.*

Proof: Consider the point of $\xi \in \Psi$ such that S_ξ is the set of zeros of f , in the gaps. Then, by the interpolation theorem, we can write f in terms of h_ξ, and the numbers $f(\xi_i)$, as in equation (30), so that f is identically zero. □

In the following we choose a basis for $\mathcal{H}_{\infty-1}$ which will come in useful for the analysis on the Riemann surface and in terms of which the Abel-Jacobi map will be defined later.

Proposition 3.8 *There exists a $\zeta \in \Psi$ and a collection $\{\sigma_i\}$ of positive numbers $C \leq \sigma_i \leq D$ such that $f(\lambda) = \sigma_i \omega_{\zeta^i}$ have the following properties.*

1. $\int_{I_j} |f_i(x)| \;\; dx < \infty \;\; \forall j$.

2. $\int_{I_j} f_i(x) \;\; dx \;\; = 0 \;\; \forall j \neq \; i$.

3. $\int_{I_i} f_i(x) \;\; dx \;\; = \frac{\pi}{2}$.

Proof: The first property mentioned in the definition is automatic for f_i defined with any $\zeta \in \Psi$, from the estimates (5) of lemma (3.2). For the second we consider an i and fix it. Then we consider a positive integer n and consider the following subset of Ψ^i

$$\Lambda(n) = \prod_{\substack{j \leq n+1 \\ j \neq i}} \bar{I}_j \prod_{\substack{j > n+1 \\ j \neq i}} \{\tau_{2j-1}\}$$

and consider ξ^i from $\Lambda(n)$. It is clear that the functions ω_{ξ^i} with ξ^i coming from $\Lambda(n)$ are strictly positive in the gaps $I_j \;, j = i \;\; and \;\; j > n+1$ and they are real in I_j for the remaining values of j. Further it is also clear ,from lemma (3.2) that whenever $1 \leq j \leq n+1, j \neq i$, we have $\omega_{\xi^i}(\lambda) > 0 \;\; or \;\; < 0$ according as $\lambda > \xi^i_j \;\; or \;\; < \xi^i_j$ in I_j. In the proof below we take i=1, however the proof works for any i. (For i not equal to 1 it may be necessary to multiply F^ns by ± 1). We consider the following function F^n of n real variables.

$$F^n_j(x_2, \cdots, x_{n+1}) = \int_{I_j} \omega_{\xi^1}(\lambda) d\lambda \;\;\; j = 2, \cdots, n+1$$

with $x_j = \xi^1_j \;\;\; j = 2, \cdots, n+1$. Then from the properties of ω_{ξ^1}, when the first n coordinates of ξ^1 vary, we find that

$$F^n_j(x_2, \cdots, \tau_{2k-1}, \cdots, x_{n+1}) < 0 \;\;\; k = 2, \cdots, n+1$$

and

$$F_j^n(x_2, \cdots, \tau_{2k}, \cdots, x_{n+1}) > 0 \quad , k = 2, \cdots, n+1.$$

It is also clear that F^n is a continuous function from $\times_{j=2}^{n+1} \bar{I}_j$ to $\mathbb{R}^n$ satisfying the assumptions of theorem (A.3), so that the existence of a point $\xi^1(n)$ in Ψ^i at which F^n vanishes is guaranteed. When Ψ^1 is equipped with the topology of pointwise convergence, the subsequence of the points $\xi^1(n)$ converges, to say ζ^1, in Ψ^1. Under this convergence, we have the point wise convergence of $\omega_{\xi^1(n)}$ as an integrable function on each $I_j \;\; j = 1, 2, \cdots$. It is also clear from the estimates (5) of lemma (3.2) , valid for any $\eta \in \Psi_{\infty-1}$, that $\omega_{\xi^1(n)}$ are integrable uniformly in each $I_k, k = 1, 2, \cdots$. Thus by Lebesgue dominated convergence theorem we have that

$$\int_{I_k} \omega_{\zeta^1}(\lambda) d\lambda = 0 \forall k \neq i.$$

It is also clear that for k = 1, the above integral is nonzero and in fact positive hence the normalization is a matter of choosing an overall real multiplicative constant σ_1. For later purposes, we shall write the functions f_i as

$$f_i(\lambda) = \sigma_i \omega_{\zeta^i}(\lambda), \quad \textit{with} \;\; C \leq \sigma_i \leq D. \tag{32}$$

The upper and lower bounds for σ_i follow from the estimates (5) of lemma (3.2) and assumption (2.1), which also show that the constants C and D are independent of i. $\square$

Proposition 3.9 *Assume that Σ satisfies assumptions (2). Assume further that the measure μ in the Herglotz representation of proposition (2.2) is absolutely continuous for each $\xi \in \Psi$. Then there exists a ζ in Ψ such that h_ζ has the following properties.*

1. $\int_{I_j} h_\zeta(x) \quad dx \ = 0 \ \ \forall j$.

2. *Re* $\int_{\Sigma\cap(-\infty,\tau_{2i})} h_\zeta(x) \quad dx \ = 0 \ \ \forall \ \ i$

Proof: Given any $\xi \in \Psi$ we construct h_ξ as in proposition (2.2). By the assumption on Σ, the limits $h_\xi(\lambda+i0)$ of the functions h_ξ of proposition (2.2) exist everywhere in the interior of Σ and by assumption there is no singular part for μ so that $d\mu(x) = Imh_\xi(x+i0)dx$. The product representation for h_ξ also shows, using estimates of lemma (3.1), that $h_\xi(\lambda)$ is integrable in each I_i. The proof of the first item follows the proof of the previous proposition, by which we make a choice of a fixed point $\zeta \in \Psi$, so we omit it. The second item follows since Re h_ζ is zero almost everywhere in the spectrum. $\square$

There is an interesting corollary of proposition (3.8) which will be useful in showing that the Abel-Jacobi map to be introduced later is onto. In the following we use the usual convention that if x < y , then $\int_y^x f(\lambda) = -\int_x^y f(\lambda)$.

Corollary 3.10 *Consider two points ζ and η in Ψ. If for all $f \in \mathcal{H}_{\infty-1}$,*

$$\sum_{j=1}^{\infty} \int_{\eta_j}^{\zeta_j} f(\lambda)d\lambda = 0$$

then $\zeta \equiv \eta$.

Proof: Consider a point ξ^1 in Ψ^1 obtained as follows. For $j \neq 1$, we take $\xi^1_j = \zeta_j$ whenever $\eta_j < \zeta_j$, and $\xi^1_j = \zeta_j$ otherwise. With this choice, the convention mentioned before the corollary and from the properties (1,2,3) of, ω_{ξ^1} of lemma (3.2), it follows that for each j = 1,2,$\cdots$

$$\int_{\eta_j}^{\zeta_j} \omega_{\xi^1} d\lambda \geq 0.$$

Since $\omega_{\xi^1} \in \mathcal{H}_{\infty-1}$ the corollary is immediate. $\square$

4 Analysis on a Riemann surface

We consider a Riemann surface associated with the set Σ of the last section and consider the Abel-Jacobi map on the surface. We formulate the Abel-Jacobi map on a Banach space.

4.1 The Riemann Surface

In this section we construct a Riemann surface $\mathcal{R}$ whose branch points are the τ_i's and the τ_∞, τ_0's. We take two copies of the Riemann sphere and delete the points of $\mathcal{E}$ from each. The rest of the procedure to construct $\mathcal{R}$ is similar to the construction of a hyperelliptic Riemann surface with finite number of branch points. On each sphere we cut the real line along the spectral bands and glue the resulting spheres together by joining the upper lips of the cuts of the first one to the corresponding lower lips of the cuts of the second one and vice versa. Thus we get a non-compact hyperelliptic surface of infinite genus. This surface will be a two sheeted branched cover of the Riemann sphere with the points of $\mathcal{E}$ removed from it.

We denote by λ the projection map from $\mathcal{R}$ into the Riemann sphere. On $\mathcal{R}$ we give a coordinate system as follows. We denote the points of $\lambda^{-1}(\infty)$ in the upper and lower sheets as ∞_1 and ∞_2. At ∞_1 we give local coordinates by mapping a neighbourhood of it to a neighbourhood of the origin in $\mathbb{C}$ by the map $z(\infty_1) = 0$ and $z(p) = \frac{1}{\lambda(p)}$ for other points p in the neighbourhood. Similarly we give coordinates around ∞_2. At point p_n corresponding to a branch point τ_n we give the coordinate system given by $z^2 = \lambda(p) - \tau_n$. For the rest of the points we give coordinate systems via the projection map λ. Thus the Riemann surface is a two sheeted branched cover of $\{\tau_0, \tau_\infty, \tau_{2i}$ *and* $\tau_{2i+1}\}$ i = 1,2,.... as the branch points.

We now give some definitions, see figure 1.

Definition 4.1 *The closed loop which starts from τ_{2i-1} , goes to τ_{2i} through the embedded spectral gap on the lower sheet and comes back to τ_{2i-1} through the similar spectral gap on the upper sheet is called the i'th β- cycle and is denoted by β_i.*

Let P_0 be any point on the embedded real line on the lower sheet to the left of τ_0. The closed loop, lying on the lower sheet, that starts from some point on β_i going to P_0 and coming back to the same point through the other side of the lower sheet is called the i'th alpha cycle and is denoted by α_i.

We recall that a differential of the first kind on $\mathcal{R}$ is one which can be written in local coordinates z as $g(z)dz$ with g holomorphic .

We shall consider a class of differentials on $\mathcal{R}$ which will be normalized on using the α and β cycles considered above.

Lemma 4.2 *Each of the functions of class $\mathcal{H}_{\infty-1}$ considered in the last section gives rise to a differential of the first kind on $\mathcal{R}$.*

Proof: Consider the differentials $f(\lambda)d\lambda$ on $\mathcal{R}$ with f $\in \mathcal{H}_{\infty-1}$. Apriori this is a differential defined on $\lambda^{-1}(\Pi)$ and a computation in each of the local coordinate charts on $\mathcal{R}$ shows that it extends to a differential of the first kind on the whole of $\mathcal{R}$. $\square$

We fix a basis of the differentials of the first kind on $\mathcal{R}$. Recall the functions f_i of proposition (3.8). Using these we form the following differentials of the first kind on $\mathcal{R}$ given by

$$d\omega_i(p) = f_i(\lambda(p))d\lambda(p)$$

where the right hand side is written in local coordinates at p in $\mathcal{R}$.

Then the normalizations of proposition (3.8) immediately imply on computation using the product representation for f_i's , the the choice of the α and β cycles, that

$$\int_{\beta_j} d\omega_i = 2\int_{I_j} f_i(\lambda)d\lambda = \delta_{ij}\pi \tag{33}$$
$$\int_{\alpha_j} d\omega_i = 2\int_{(-\infty,\tau_{2i-1})\cap\Sigma} f_i(\lambda)d\lambda = \pi_{ij}$$

with π_{ij} is purely imaginary.

Connected with the basis given above and a fixed point p_0 in $\mathcal{R}$ we define the Abel map

$$\omega_i(p) = \int_{p_0}^{p} d\omega_i \tag{34}$$

which is a function from $\mathcal{R}$ to $\mathbb{C}$ for each i. It is not well defined since the right hand side depends on the path of integration. It is however well defined as a map into the Jacobi variety of $\mathcal{R}$. We shall not go into this further. We shall use the Abel map for defining the Abel-Jacobi map later.

The above collection of differentials of the first kind form indeed a basis for a real Hilbert space of holomorphic differentials of the first kind. It is possible to define and analyze the period matrix, prove the bilinear relations of Riemann and define the Theta functions on it. However, we shall not discuss these points in this work. For the present purposes it is enough to quickly get a consequence of an analogue of the bilinear relations of Riemann. More precisely we are interested in the image of the Abel-Jacobi map on a class of divisors. To present this we consider the subset of $\mathcal{R}$ coming from

$$\Phi = \lambda^{-1}(\Psi) \cup \lambda^{-1}(\infty)$$

where Ψ is as in the section 2.

We recall that a divisor is a formal sum

$$D = \sum_{P \in \mathcal{R}} D(P)P$$

with D(P) an integer valued function on $\mathcal{R}$. We shall denote by the support of a divisor as the set of points P with D(P) non zero. We recall that the divisor D_f of a meromorphic function is $D_f(P) = ord_P(f)$. A divisor D is said to be principal if it is the divisor of a meromorphic function. We consider below a special subclass of principal divisors D given by

$$D = K[P_\infty - Q_\infty] + \sum_{i=1}^{\infty} P_i - Q_i \tag{35}$$

for $P, Q \in \Phi$ for an integer K.

The divisors of the above type will occur as the divisors of some m-functions of random Jacobi matrices which will be considered in the last section.

We shall denote the two sheets of $\mathcal{R}$ by $\mathcal{R}^{\pm}$ in the analysis below. We consider the rectangular curves introduced in the last section in lemma (3.3) , and let

$$S_N^j = \lambda^{-1}(R_N^j) = \cup_{k=1}^2 S_N^j(k) \ , \ 1 \leq j \leq M(N) \ \ \forall N.$$

Since each point of R_N^j is away from the branch points, R_N^j has two preimages in $\mathcal{R}$ and these will be disjoint,(see figure 2 to get an idea of the preimages, where dotted lines indicate change of sheets) so that the above union is a disjoint union. We denote the closed region in $\mathbb{C}$ bounded and enclosed by the rectangles R_N^j of lemma (3.3) by $\bar{R_N^j}$ and let

$$\bar{S_N^j} = \lambda^{-1}(\bar{R_N^j}) \ , \ 1 \leq j \leq M(N) \ \ \forall N.$$

We first note that since the Riemann surface we have is of infinite genus, we need to do an approximation to obtain the relation (36). To this end we consider the open region $\mathcal{R}_N$ of $\mathcal{R}$ obtained by taking the sets $\bar{S}_N^k$, defined above (if necessary with smoothed out to make the boundary smooth)

$$\mathcal{R}_N = \mathcal{R} \setminus \ \cup_{i=1}^{M(N)} \bar{S}_N^k.$$

Then $\mathcal{R}_N$ is increasing with N in the set theoretic sense. Further, $\mathcal{R}_N$ is a Riemann surface by its own right. It can be compactified by adding 2M(N) disks. Therefore if we fix a collection of α and β cycles in $\mathcal{R}_N$, there is a normal polygon D_N, (we refer to [10] for a construction of this domain) with the number of sides equal to 4 times the number of beta cycles in $\mathcal{R}_N$ and having 2M(N) holes corresponding to the removal of M(N) disjoint closed sets $\bar{S_N^i}$. (We should remember here that each of $\bar{S_N^i}$ is a closed subset of

$\mathcal{R}$ having two preimages for each point of $\bar{R^i_N}$ except the branch points.) Suppose that the number of beta cycles in $\mathcal{R}_N$ is k(N). Then the 4k(N) sides of the polygon are indexed as a_i or $b_i$according to whether they correspond to traversing an alpha or a beta cycle in the anti-clockwise or as a_i^{-1} or b_i^{-1} corresponding to traversing them in the clockwise direction. Further the sides are such that $a_i, b_i, a_i^{-1} and b_i^{-1}$ occur as consecutive sides of the polygon for each i. There are 2M(N) holes H_j in D_N we shall index them so that ∂H_{2j-1} and ∂H_{2j},j = 1,2, $\cdots$,M(N), are $S^j_N(1)$ and $S^j_N(2)$ respectively.

We find that the Abel map ω_i in D_N is not single valued, though it has analytic continuation along any path in D_N, the values along two different paths differing by an integer multiple of its integrals on the boundaries of the holes. Therefore to obtain analytic functions corresponding to the the Abel map, which essentially means that we ensure that the paths of integration do not wind around the holes, we make cuts in D_N along paths $\gamma_{2j-1}, \gamma_{2j}$ connecting a fixed vertex to the boundary of the holes $\partial H_{2j-1}, \partial H_{2j}$, so that we obtain a simply connected domain which we call E_N. Recall the set r_N of rectangles defined after lemma (3.3). For example the $\gamma_{2j-1}, \gamma_{2j}$ can be chosen to be the arcs $r^j_N(1)$ and $r^j_N(2)$ respectively given by the portions of $\lambda^{-1}(r^i_N)$ in $\lambda^{-1}(\mathbb{C}^{\pm}) \cap \mathcal{R}^+$ for the purpose of estimates needed later.

Proposition 4.3 *Consider two points $P, Q \in \Phi$ and suppose ϕ is a meromorphic function on $\mathcal{R}$ such that it has simple poles at Q_i, simple zeros at P_i i=1,2,$\cdots$, and has a zero and a pole of order K at P_∞ and Q_∞ respectively.*

Suppose further ϕ satisfies the conditions that,

$$\lim_{N\to\infty}\sum_{j=1}^{M(N)}|\int_{\partial H_j}\frac{d\phi}{\phi}|+|\int_{\partial H_j}\omega_i|=0 \quad \sup_{1\le j\le 2M(N)}|\int_{\gamma_j}\frac{d\phi}{\phi}|<\infty.$$

Then for each $i\in\mathbb{Z}^+$, we have

$$\sum_{k=1}^{\infty}\int_{Q_k}^{P_k}d\omega_i+K\int_{Q_\infty}^{P_\infty}d\omega_i=0 \quad mod \quad \pi \tag{36}$$

Here the paths of integrations from Q_k to P_k are taken taken to lie in $\lambda^{-1}(\bar{I}_k)$. For the index ∞ the path lies in $\lambda^{-1}((-\infty,\tau_0))$.

Proof: We note that since ϕ is a meromorphic function on $\mathcal{R}$ it gives rise to an abelian differential $d\phi/\phi$ and $\phi^{'}/\phi$ in local coordinates has poles precisely at the points P_i, Q_i, P_∞ and Q_∞, hence its divisor is of the form in equation (35). Since only finitely many of the P_i , Q_i fall in $\mathcal{R}_N$, $d\phi/\phi$ has finitely many poles in $\mathcal{R}_N$.

We pass to the simply connected region E_N obtained from the normal polygon of $\mathcal{R}_N$. We note that in this region the differentials $d\omega_i$ are exact. Therefore the integrals $\omega_i(z)=\int_{z_0}^{z}d\omega_i$ are analytic functions in E_N as long as the paths of integration lie in the interior of E_N. Then $\omega_i d\phi/\phi$ is also a meromorphic differential and has poles precisely at the poles of $d\phi/\phi$. We consider the integral of $\omega_i d\phi/\phi$ along the boundary C of E_N, slightly deformed if necessary, such that none of the poles of the differential fall on it. Then we have, by applying the residue theorem, in the simply connected region E_N,

$$\frac{1}{2\pi i}\int_C\omega_i\frac{d\phi}{\phi} = \sum_{j=1}^{k(N)}[\omega_i(P_j)-\omega_i(Q_j)]+K[\omega_i(P_\infty)-\omega_i(Q_\infty)] \tag{37}$$

$$= \sum_{k=1}^{k(N)} \int_{Q_i}^{P_i} d\omega_i + \int_{Q_\infty}^{P_\infty} d\omega_i$$

We compute the quantities on the left directly. The integral over the curve C can be written as

$$\begin{aligned} \int_C \omega_i \frac{d\phi}{\phi} &= \sum_{i=1}^{k(N)} \int_{a_i \cup b_i \cup a_i^{-1} \cup b_i^{-1}} \omega_i \frac{d\phi}{\phi} \\ &+ \sum_{j=1}^{2k(N)} \int_{\gamma_j \cup \gamma_j^{-1} \cup \partial H_j} \omega_i \frac{d\phi}{\phi} \end{aligned} \tag{38}$$

by a slight deformation of the contour if necessary. To compute this integral explicitly we use the following properties of the values of ω_i on the corresponding points on the a,b ,γ's and $a^{-1}, b^{-1}, \gamma^{-1}$s. We denote the points on a_j^{-1} corresponding to the point p in a_j as p^* and similarly for the b's and the γ's.

$$\frac{\phi'}{\phi}(p^*) = \frac{\phi'}{\phi}(p), \tag{39}$$

$$\omega_i(p^*) = \begin{cases} \omega_i(p) + \pi\delta_{ij} & p \in a_j \\ \omega_i(p) + \pi_{ij}, & p \in b_j \\ \omega_i(p) + \int_{\partial H_j} \omega_i, & p \in \gamma_j. \end{cases}$$

Using these relations , and the fact that integrating $(\phi)'/\phi$ over an alpha or a beta cycle gives the change in the argument of ϕ, which is $2\pi i$ times an integer, we compute the integrals in equation (38) to obtain the value of the integral as

$$\int_{a_i \cup b_i \cup a_i^{-1} \cup b_i^{-1}} \omega_i \frac{d\phi}{\phi} = \sum_{j=1}^{k(N)} \pi\delta_{ij}(2\pi i m_j) + \pi_{ij}(2\pi i l_j) \tag{40}$$

for some integers m_j and l_j. Therefore collecting the above equations together we have,

$$\begin{aligned}\sum_{j=1}^{k(N)}[\omega_i(P_j)-\omega_i(Q_j)] &= -K[\omega_i(P_\infty)-\omega_i(Q_\infty)]+\pi\delta_{ij}m_i \\ &\quad +\sum_{j=1}^{k(N)}\pi_{ij}l_j+e_N \qquad (41)\end{aligned}$$

where we have taken

$$e_N=\frac{1}{2\pi i}\left\{\sum_{j=1}^{2M(N)}\int_{\partial H_j}\omega_i\frac{d\phi}{\phi}+\sum_{j=1}^{2M(N)}(\int_{\partial H_j}\omega_i)(\int_{\gamma_j}\frac{d\phi}{\phi})\right\}.$$

We can compute the left hand side of the above equation on $\mathcal{R}$ to get

$$\begin{aligned}\sum_{j=1}^{k(N)}\int_{Q_j}^{P_j}d\omega_i &= -K\int_{Q_\infty}^{P_\infty}d\omega_i+\pi\delta_{ij}m_i \\ &\quad +\sum_{j=1}^{k(N)}\pi_{ij}l_j+e_N \qquad (42)\end{aligned}$$

At this stage of the approximation , the sums are finite. We will need only real part of the above equation in the limiting case so that we can take the real parts at this stage. Then the π_{ij} terms drop out, since it is purely imaginary. By assumption, the term e_N goes to zero and the real part of $\omega_i(P_j)-\omega_i(Q_j)$ can be evaluated using $d\omega_i$ with the path of integration in $\lambda^{-1}(\bar{I}_j)$ and a path that lies in $\lambda^{-1}((-\infty,\tau_0))$ for the $j=\infty$, term, so that the quantities are real. □

In view of the choice of paths in the equation (36), we use a continuous parametrization for $\lambda^{-1}(\bar{I}_j)$ and rewrite the equation (36) as follows.

Denote the points x_j in $\lambda^{-1}(\bar{I}_j)$ by

$$x_i=\tau_{2i-1}+\ell_i\ sin^2(\theta_i) \qquad (43)$$

where $0 \leq \theta_i < \frac{\pi}{2}$, if $x_i \in \mathcal{R}^+$ and $\frac{\pi}{2} \leq \theta_i < \pi$, if $x_i \in \mathcal{R}^-$.

In this parametrization we denote the angles corresponding to P_j and Q_j as θ_j and ψ_j respectively. Then using the definition of $d\omega_i$, in terms of the functions f_i of the Definition (3.8),equations (34) and change variables to the angular variables θ, we have

$$\sum_{k=1}^{\infty} \int_{\psi_k}^{\theta_k} f_{ik}(\theta) d\theta = -Kc_i + \pi m_i. \tag{44}$$

We write the f_{ik} below for clarity, where we denote $s(\theta) = sin^2(\theta)$.

$$\begin{aligned} f_{ik}(\theta) \quad = \quad & \frac{\sigma_i}{\sqrt{\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_0)(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_\infty)}} \\ & \times \frac{(\ell_k\ (s(\theta) + \tau_{2k-1} - \zeta_k^i)}{\sqrt{(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2i-1})(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2i})}} \\ & \times \prod_{j \neq i,k} \frac{(\ell_k\ (s(\theta) + \tau_{2k-1} - \zeta_j^i)}{\sqrt{(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2j-1})(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2j})}}. \end{aligned} \tag{45}$$

We show that f_{ik} satisfy the following bounds as functions from $[0,\pi]$ to $\mathbb{R}$.

Lemma 4.4 *The f_{ik} defined above are periodic functions of θ and they satisfy the following bounds for each $k \neq i$.*

$$\sup_\theta |f_{ik}(\theta)| \leq C_1 q_k \quad and \quad \sup_\theta |f_{ik} f_{kk}^{-1}(\theta)| \leq C_2 q_k.$$

For $k = i$, we have the following upper and lower bounds,

$$C_3 \leq |f_{ii}(\theta)| \leq C_4$$

with the constants C_3, C_4 independent of i.

Proof: The quantities f_{ik} s are nothing but $\sqrt{(\lambda-\tau_{2k-1})(\lambda-\tau_{2k})}\sigma_i\omega_{\zeta^i}$ of the equation (32), written in θ variables, so that the estimates are clear from the estimates of lemma (3.2) and the estimates for σ_i coming from the equation (32). $\square$

At this stage we state a corollary of the interpolation theorem that will be useful in the next subsection in showing the invertibility of a linear map.

Corollary 4.5 *Consider a set of points θ_j one in each $[0,\pi/2]$ or $[\pi/2,\pi]$, j=1,2$\cdots$. Then for each j, there exists a sequence of numbers C_i^j, with $C_i^j \leq Cq_i$, i =1,2,.., such that, $\sum_{i=1}^{\infty} C_i^j f_{ik}(\theta_k) = 0$ for each $k \neq j$ and $\sum_{i=1}^{\infty} C_i^j f_{ij}(\theta_j) \neq 0$.*

Proof: Consider the point x $\in \Psi$ given by,

$$x_i = \tau_{2i-1} + \ell_i sin^2(\theta_i) \tag{46}$$

and consider $\omega_{x^j} \in \mathcal{H}_{\infty-1}$ for any given j fixed. Consider, C_i^j given by,

$$C_i^j = \int_{I_i} \omega_{x^j}(\lambda)d\lambda.$$

Then using the bounds (5) of lemma (3.2), we have the estimate

$$C_i^j = \int_{I_i} \omega_{x^j}(\lambda)d\lambda \leq Cq_i \quad C_1 \leq C_j^j = \int_{I_j} \omega_{x^j}(\lambda)d\lambda \leq C_2.$$

with the constants independent of i. This implies that for fixed j, the vector C^j is in $\mathcal{K}$. Therefore by the interpolation theorem, for each k, we have the relation,

$$\sqrt{(\lambda-\tau_{2k-1})(\lambda-\tau_{2k})}\omega_{x^j}(\lambda) = \sum_{i=1}^{\infty} C_i^j f_i(\lambda)\sqrt{(\lambda-\tau_{2k-1})(\lambda-\tau_{2k})}.$$

Evaluating the left hand side at the points x_k^j and writing the right hand side of the above equation using the equations (45, 46), we have the required statements at the points θ_k, since the left hand side vanishes at the points x_k^j for each k $\neq j$. For k = j , the left hand side is non-zero. Clearly this argument is valid for any j. $\square$

4.2 The Abel-Jacobi map

We consider here the Abel-Jacobi map of equation (44) and prove its properties. To start with we consider the following real Banach space. We consider the numbers s_i of assumption (2.1), consider the finite measure $\mu = \sum_{i=1}^{\infty} s_i \delta_i$ and consider

$$X = l^2_{\mathbb{R}}(\mathbb{Z}^+, \mu) \quad and \quad X_{\mathbb{Z}} = \{x \in X : x_i \in \pi\mathbb{Z} \ \ \forall i\}\,. \tag{47}$$

We note that the topology on X is such that the l^∞ unit ball is compact in X. Therefore the set $\mathcal{B}_X = X/X_{\mathbb{Z}}$ is compact in X.

We define the Abel-Jacobi map $\mathcal{A}$ from X to itself as

$$\mathcal{A}_i(\vec{\theta}) = \sum_{j=1}^{\infty} \int_0^{\theta_j} f_{ij}(\theta) d\theta$$

and split it into two parts, the "diagonal" $\mathcal{A}_1$ and the "off diagonal" $\mathcal{A}_2$, as

$$\mathcal{A}_{1i}(\vec{\theta}) = \int_0^{\theta_i} f_{ii}(\theta) d\theta \quad , \ \mathcal{A}_{2i}(\vec{\theta}) = \sum_{j \neq i}^{\infty} \int_0^{\theta_j} f_{ij}(\theta) d\theta.$$

Henceforth we use a vector notation for points of X, $X_{\mathbb{Z}}$ etc., Then $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ have the following properties.

Proposition 4.6 *The maps $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ map X to itself and satisfy*

1. $\mathcal{A}(\vec{\theta} + \vec{m}) = \mathcal{A}(\vec{\theta}) + \vec{m}$,

2. $\mathcal{A}_1(\vec{\theta} + \vec{m}) = \mathcal{A}_1(\vec{\theta}) + \vec{m}$,

3. $\mathcal{A}_2(\vec{\theta} + \vec{m}) = \mathcal{A}_2(\vec{\theta})$,

for each $\vec{\theta}$ inX and $\vec{m} \in X_{\mathbb{Z}}$.

Proof: We note that the functions $f_{ik}(\theta)$ are periodic of period π. Therefore the nomalizations of equation (34), imply that for k $\neq$ i,

$$\int_0^{\theta+\pi l} f_{ik}(\theta) = \int_0^{\theta} f_{ik}(\theta) \quad \forall l \in \mathbb{Z}$$

and

$$\int_0^{\theta+\pi l} f_{ii}(\theta) = \int_0^{\theta} f_{ii}(\theta) + \pi l \quad \forall l \in \mathbb{Z}.$$

These periodicity relations together with the uniform boundedness of the functions f_{ik} when θ varies over $[0,\pi]$, shows that $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ map X to itself. The stated periodicity relations of the proposition are again easy consequences of the above periodicity of f_{ik} s and the normalizations of proposition (3.8). $\square$

We note that the periodicity of $\mathcal{A}_2$ shows that it is a bounded map from X to itself. This will be useful in the following.

We would like to show that the map $\mathcal{A}$ is a Lipschitz continuous bijection from X to itself , with its inverse also being Lipschitz. To do this we consider an auxiliary map

$$\mathcal{B} = (I + \mathcal{A}_2 \circ {\mathcal{A}_1}^{-1}).$$

We will show below that $\mathcal{B}$ is a differentiable , invertible map of X to itself and its derivative is bounded on X. Similar properties are true for its inverse. To do this we first consider the matrices A_1, A_2 and B of partial derivatives of $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_2 \circ \mathcal{A}_1{}^{-1}$. For a point $\vec{\theta} \in$ X,

$$A_1(\vec{\theta})_{ij} = f_{ii}(\theta_i)\delta_{ij} \quad A_2(\vec{\theta})_{ij} = f_{ij}(\theta_j)(1-\delta_{ij}) \quad B(\vec{\theta}) = A_2 \circ A_1^{-1}(\vec{\theta}). \tag{48}$$

Let $\mathcal{L}_X$ denote the space of bounded linear operators on X. In the proposition below the continuity of A_2 from X to $\mathcal{L}_X$ is not clear since the equicontinuity of the family $\{f_{ii}\}$ of maps is not clear.

Lemma 4.7 *Consider X and $\mathcal{L}_X$ as metric spaces equipped with the respective norm topologies. Then the maps $A_1(\vec{\theta})$ are bounded from X to $\mathcal{L}_X$. The maps $A_2(\vec{\theta})$ and $B(\vec{\theta})$ are compact operator valued and are continuous from X to $\mathcal{L}_X$.*

Proof: The boundedness of $A_1(\vec{\theta})$ as a linear operator from X to X, for any $\vec{\theta}$ is clear from the estimates of lemma (4.4). The estimates of lemma (4.4), and the uniform boundedness of q_i/s_i coming from assumption (2.2) show that for any $\vec{\psi} \in X$,

$$|(A_2(\vec{\theta})\vec{\psi})_i| \leq \sum_{k=1}^{\infty} |f_{ik}(\theta_k)\psi_k| \leq C \sum_{k=1}^{\infty} s_k |\psi_k| = C\|\vec{\psi}\|.$$

This estimate shows that $A_2(\vec{\theta})$ maps a bounded subset of X into a precompact subset of X for each fixed $\vec{\theta}$, showing the compactness of $A_2(\vec{\theta})$. The estimate also shows that the norms $\|A_2(\vec{\theta})\|$ are uniformly bounded on X. The proof for $B(\vec{\theta})$ follows from compactness of A_2 and the boundedness of

A_1^{-1} coming from the bounds of lemma (4.4). We will show the continuity of $A_2(\vec{\theta})$, the proof for B is similar using the estimates of lemma (4.4). Consider any $\vec{\psi}$ in X and consider $\theta, \phi \in$ X and an ϵ fixed. Then we have

$$\begin{aligned}
&\|(A_2(\vec{\theta}) - A_2(\vec{\phi}))\vec{\psi}\| \\
&\qquad = \sum_{i=1}^{\infty} s_i \Big|\sum_{k\neq i}^{\infty} (f_{ik}(\theta_k) - f_{ik}(\phi_k))\psi_k\Big| \\
&\qquad = \sum_{i=1}^{M(\epsilon)} s_i \Big|\sum_{k(\neq i)=1}^{l(\epsilon)} (f_{ik}(\theta_k) - f_{ik}(\phi_k))\psi_k\Big| \\
&\qquad\quad + \sum_{i=1}^{M(\epsilon)} s_i \Big|\sum_{k(\neq i)=l(\epsilon)+1}^{\infty} (f_{ik}(\theta_k) - f_{ik}(\phi_k))\psi_k\Big| \\
&\qquad\quad + \sum_{i=M(\epsilon)+1}^{\infty} s_i \Big|\sum_{k\neq i}^{\infty} (f_{ik}(\theta_k) - f_{ik}(\phi_k))\psi_k\Big| \qquad (49)
\end{aligned}$$

Given ϵ we can choose $M(\epsilon)$ and $l(\epsilon)$ so that the last two terms in the above inequality are bounded by C$\epsilon\|\vec{\psi}\|$, using the summability of s_i, the bounds of lemma (4.4) for f_{ik}s and the estimate $q_k \leq Cs_k$ coming from the summability of q_k/s_k. Therefore we concentrate on the first term in the inequalities. We can find, for each i = $1,\cdots,M(\epsilon)$ and k = $1, \cdots, l(\epsilon)$, numbers δ_{ik} so that, by continuity of f_{ik}s,

$$|f_{ik}(\theta_k) - f_{ik}(\phi_k)| \leq s_k\epsilon \quad \textit{whenever} \quad |\theta_k - \phi_k| < \delta_{ik}$$

Now choose δ so that

$$\delta = \inf_{\substack{i=1,\cdots,M(\epsilon)\\ k(\neq i)=1,\cdots,l(\epsilon)}} \{\delta_{ik}, s_k\delta_{ik}\}\,.$$

Then it follows that if $\|\vec{\theta} - \vec{\phi}\| < \delta$, we have for each i = $1,\cdots,M(\epsilon)$, k$\neq i, =$ $1,\cdots,l(\epsilon)$,

$$s_k|\theta_k - \phi_k| < \delta \leq s_k\delta_{ik} \Rightarrow |\theta_k - \phi_k| < \delta_{ik} \Rightarrow |f_{ik}(\theta_k) - f_{ik}(\phi_k)| \leq s_k\epsilon.$$

These estimates show that in first term in the inequalities (49) are bounded by $C\epsilon||\psi||$ again by the summability of s_is. These inequalities together show that given ϵ positive we have a δ such that,

$$||(A_2(\vec{\theta}) - A_2(\vec{\phi}))\vec{\psi}|| \leq C\epsilon||\vec{\psi}|| \quad whenever \quad ||\vec{\theta} - \vec{\phi}|| < \delta.$$

$\square$

Theorem 4.8 *Consider the maps $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{B}$. Then we have the following properties.*

1. *$\mathcal{A}_1$, ${\mathcal{A}_1}^{-1}$ are Lipshitz continuous bijections of X.*
2. *$\mathcal{B}$ is differentiable from X to itself with the derivative invertible and uniformly bounded on X.*
3. *$\mathcal{B}$ and $\mathcal{A}$ are Lipschitz continuous bijections of X.*
4. *$\mathcal{A}^{-1}$ is a Lipschitz continuous map of X to itself.*

Proof:

1. The property (1) of proposition (4.6), shows that it is enough to consider the map $\mathcal{A}_1$ on $\mathbb{B}_X$. For each i, the continuous map $g_i(\theta_i) = (\mathcal{A}_1(\vec{\theta}))_i$, satisfies, $g_i(0) = 0, g_i(\pi) = \pi$ is strictly monotone on [0,π], hence it is a bijection of [0,π] to itself. This shows that $\mathcal{A}_1$ is a bijection of $\mathbb{B}_X$ to itself. On the other hand , the functions $f_{ii}(\theta_i)$ and $f_{ii}^{-1}(\theta_i)$ are uniformly bounded above and below,by lemma (4.4), showing that both $\mathcal{A}_1$ and ${\mathcal{A}_1}^{-1}$ are Lipschitz on $\mathbb{B}_X$.

2. We prove here that $\mathcal{A}_2$ is differentiable, the proof for $\mathcal{A}_2 \circ \mathcal{A}_1{}^{-1}$ is similar. Let $A_2(\vec{\theta})$ denote the matrix of partial derivatives of $\mathcal{A}_2$ at the point $\vec{\theta}$ in X. Then we have for any $\vec{h} \in$ X (indeed we can take h in $\mathcal{B}_X$ without loss of generality by the periodicity of $\mathcal{A}_2$),

$$\|\mathcal{A}_2(\vec{\theta}+\vec{h}) - \mathcal{A}_2(\vec{\theta}) - A_2(\vec{\theta})\vec{h}\| \tag{50}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} s_i \mid \mathcal{A}_2(\vec{\theta}+\vec{h})(i) - \mathcal{A}_2(\vec{\theta})(i) - (A_2(\vec{\theta})\vec{h})(i) \mid \\
&= \sum_{i=1}^{\infty} s_i \mid \sum_{k\neq i}[\int_{\theta_k}^{\theta_k+h_k} f_{ik}(\phi_k)d\phi_k - f_{ik}(\theta_k)h_k] \mid \\
&= \sum_{i=1}^{\infty} s_i \mid \sum_{k\neq i}[f_{ik}(\tilde{\theta}_k) - f_{ik}(\theta_k)]h_k \mid \\
&\leq \|A_2(\vec{\tilde{\theta}}) - A_2(\vec{\theta})\|\|h\|
\end{aligned}$$

We have used the mean value theorem for the functions $\int f_{ik}$ in the second step above and chose the point $\tilde{\theta_k}$ from $[\theta_k, \theta_k + h_k]$. Now the continuity of A_2, mentioned in lemma shows that as $\|h\|$ goes to zero, $\vec{\tilde{\theta}} \to \vec{\theta}$ in X, showing the differentiability, by lemma (4.7). Similarly the differentiability of $\mathcal{A}_2 \circ \mathcal{A}_1{}^{-1}$ and hence that of $\mathcal{B}$ is shown. Therefore, by lemma (4.7) the derivative $I + B(\vec{\theta})$ of $\mathcal{B}$ at $\vec{\theta}$ is uniformly bounded and continuous from X to $\mathcal{L}_X$. To show the invertibility of (I + $B(\vec{\theta})$), it is enough to show that zero is not an eigenvalue, by the Fredholm alternative, since B is a compact operator. Suppose there is a ϕ in X such that

$$(I + B(\vec{\theta}))\vec{\phi} = 0.$$

Then, since A_1 is an invertible operator, it follows that,

$$(A_1(\vec{\theta}) + A_2(\vec{\theta}))\vec{\psi} = 0$$

for some $\vec{\psi}$ in X. Writing the above equation explicitly interms of the matrix elements, we have that

$$\sum_{j=1}^{\infty} f_{ij}(\theta_j)\psi_j = 0 \quad \forall i.$$

Then by corollary (4.5), we can choose numbers C_i^k such that after an interchange of sums we get,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_i^k f_{ij}(\theta_j) = \left(\sum_{i=1}^{\infty} C_i^k f_{ik}(\theta_k)\right)\psi_k = 0 \quad \forall k.$$

Since the sum in the parantheses is non-zero by lemma (4.5), we have that $\psi_k = 0$ for each k. The continuity of (I + $B(\vec{\theta})$) as a map from X to $\mathcal{L}_X$ implies, by an application of the resolvent equation, the inverse is also continuous and hence uniformly bounded on compacts of X. Further, the periodicity of the operator valued function $B(\vec{\theta})$, with its periods in $X_{\mathbb{Z}}$, coming from the periodicity of f_{ij}s, shows that $\|(I + B(\vec{\theta}))^{-1}\|$ is uniformly bounded on X.

3. To show that $\mathcal{B}$ and $\mathcal{A}$ are bijections of X to itself, we note that by (1) , it is enough to show that $\mathcal{A}$ is injective and $\mathcal{B}$ is surjective. Suppose there are two points $\vec{\theta}$ and $\vec{\psi}$ in X such that $\mathcal{A}(\vec{\theta}) = \mathcal{A}(\vec{\psi})$. Let η and ζ be in Ψ such that $\eta_j = \tau_{2j-1} + \ell_j sin^2(\theta_j)$ and $\zeta_j = \tau_{2j-1} + \ell_j sin^2(\psi_j)$. Then $\mathcal{A}(\vec{\theta}) = \mathcal{A}(\vec{\psi})$ implies,

$$\sum_{j=1}^{\infty} \int_{\eta_j}^{\zeta_j} f_i(\lambda) d\lambda = 0.$$

for each i. The interpolation theorem implies that

$$\sum_{j=1}^{\infty} \int_{\eta_j}^{\zeta_j} f(\lambda) d\lambda = 0.$$

for each f$\in \mathcal{H}_{\infty-1}$. Then corollary (3.10) implies that $\eta_j = \zeta_j$ for each j. This implies in turn that $\vec{\theta}$ and $\vec{\psi}$ differ at most by an element of $X_{\mathbb{Z}}$ so that $\vec{\theta} = \vec{\psi} + \vec{m}$. Using proposition (4.6(1)) we see that,

$$\mathcal{A}(\vec{\theta}) = \mathcal{A}(\vec{\psi} + \vec{m}) = \mathcal{A}(\vec{\theta}) + \vec{m}$$

which implies that $\vec{m}$ is zero.

The differentiability of $\mathcal{B}$ allows us to use the inverse function theorem on X, to show that the range of $\mathcal{B}$ is open. We will show that the range of $\mathcal{B}$ is closed showing, by connectedness of X, that $\mathcal{B}$ is onto. The map $\mathcal{B}$ satisfies $\mathcal{B}(\vec{\theta} + \vec{m}) = \mathcal{B}(\vec{\theta}) + \vec{m}$ for all $\vec{m} \in X_{\mathbb{Z}}$. We also have that if $x \in Ran\mathcal{B}$ and $\vec{m} \in X_{\mathbb{Z}}$, then $x + \vec{m} \in Ran\mathcal{B}$. Consider y^n in $Ran\mathcal{B}$, converging to some y in X. Then, decompose y^n and y as $(y^n) + [y^n]$, (y)+[y] into their parts in $\mathbb{B}_X$ and $X_{\mathbb{Z}}$. Notice that $[y^n] - [y] = y^n - y - (y^n) + (y)$. Since $(y^n) - (y)$ belongs to a compact set, some subsequence of this converges, say to x and it is clear that x $\in X_{\mathbb{Z}}$. It is also clear that $[y^n] - [y] - x$ goes to zero for this subsequence. Therefore y is in Ran $\mathcal{B}$ follows from showing that $Ran(\mathcal{B}) \cap \mathbb{B}_X$ is closed. Let $\mathcal{C}$ denote $\mathcal{A}_2 \circ {\mathcal{A}_1}^{-1}$, then by lemma (4.6), $\mathcal{C}$ is also a periodic bounded map of X to itself with the coordinate maps $\mathcal{C}_i$ uniformly bounded on X. Consider a sequence $\vec{\theta^n} \in Ran(\mathcal{B}) \cap \mathcal{B}_X$ converging to $\vec{\theta}$ in $\mathcal{B}_X$. Then the sequence $\vec{\psi^n}$ of vectors in X, with $\mathcal{B}(\vec{\psi^n}) = \vec{\theta^n}$, are in a compact set, since,

$$\theta_i^n = \mathcal{B}(\vec{\psi^n})_i = \psi_i^n + \mathcal{C}(\vec{\psi^n})_i \Rightarrow \psi_i^n = \theta_i^n - \mathcal{C}(\vec{\psi^n})_i$$

and $\mathcal{C}(\vec{\psi^n})_i$ are bounded uniformly the bound independent of $\vec{\psi^n}$ and i. Hence there is a subsequence $\vec{\psi^{n_k}}$ of $\vec{\psi^n}$ that converges to some point $\vec{\psi}$ in X. Therefore by the continuity of $\mathcal{B}$, it is clear that $\vec{\theta}$ is precisely the image of $\vec{\psi}$ under $\mathcal{B}$.

4. The inverse $\mathcal{B}^{-1}$ of $\mathcal{B}$ exists by (4) and it is also differentiable with the derivative $(I + B(\vec{\theta}))^{-1}$ uniformly bounded on X by (2) so it is Lipschitz. The Lipschitz continuity of $\mathcal{A}$ now follows from those of $\mathcal{B}$ and $\mathcal{A}_1$. □

Finally we end this section showing the almost periodicity of the preimages of the orbits of nc, for a given vector c in X.

Theorem 4.9 *Consider any two vectors $\vec{c}$ and $\vec{d}$ in X. Consider the sequence of points $\xi(n) \in \Psi$ with $\xi_i(n) = \tau_{2i-1} + \ell_i sin^2(\mathcal{A}_i(n\vec{c}+\vec{d}))$, $n \in \mathbb{Z}$. Then $\xi_i(n)$ is an almost periodic sequence , in n , for each i.*

Proof: Let $y_i(n) = nc_i + d_i \bmod \pi$, then $sin^2(\mathcal{A}^{-1}(nc+d)) = sin^2(\mathcal{A}^{-1}(y(n)))$. Therefore for the stated almost periodicity, it is enough to consider y(n). Given an ϵ we truncate y(n) to $y^K(n)$ where $y^K(n)_i$ is zero for $i > K$ and equals $y_i(n)$ for $i \leq K$, so that $\|y(n) - y^K(n)\| < \epsilon$. Then $y^K(n)$ is an element of $\mathbb{R}^K/\pi\mathbb{Z}^K$, on which it satisfies $\sup_n \|y^K(n) - y^K(n+N)\| < \epsilon$ for an N depending upon ϵ, by an application of Poincare recurrence theorem.

Therefore by the Lipschitz continuity of $sin^2 \circ \mathcal{A}_i$, coming from the Lipschitz continuity of sin^2 and $\mathcal{A}$ we have that

$$\begin{aligned}
\sup_n |\xi_i(n) - \xi_i(n+N)| &\leq \frac{\ell_i}{s_i} s_i |\mathcal{A}_i^{-1}(y(n)) - \mathcal{A}_i^{-1}(y(n+N))| \\
&\leq Cq_i \|y(n) - y(n+N)\| \\
&< Cq_i \|y^K(n) - y^K(n+N)\| + \epsilon < D\epsilon
\end{aligned}$$

which is the almost periodicity claimed in the theorem. □

5 Random Jacobi Matrices

Consider $(\Omega, \mathcal{B}, \mathbb{P})$, where $\Omega =$ the space of $\mathbb{R}^+ \times \mathbb{R}$ valued bounded sequences (a_n, b_n), $\mathcal{B} =$ the Borel σ -algebra on Ω generated by the topology of point wise convergence on it and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{B})$. Let T denote the translation T on Ω given by $(T(a,b))(n) = (a_{n-1}, b_{n-1})$. We assume that,

Assumption 3 *$\mathbb{P}$ is invariant and ergodic with respect to T and*

$$\mathbb{E}_{\mathbb{P}}\{|log\ a_0^\omega|\} < \infty.$$

Then corresponding to each $\omega \in \Omega$ we have a self-adjoint operator acting on $\ell^2(\mathbb{Z})$ given by, [5]

$$(H^\omega u)(n) = a_{n-1}^\omega u(n-1) + a_n^\omega u(n+1) + b_n^\omega u(n) \quad u \in \ell^2(\mathbb{Z}) \tag{51}$$

It was proved by Pastur [33] under the assumptions on $\mathbb{P}$, that the spectrum Σ of the operators H^ω is a non random set. It was shown by Kunz-Souillard [25] that the spectral type of the operators is also non random, that is there are non random sets Σ_{ac} and Σ_{sc} and Σ_{pp} which are respectively the absolutely continuous, singular continuous and pure point spectra of the operators H^ω for almost all ω.

Consider the operator H^ω for ω in support of $\mathbb{P}$. The following facts are contained essentially in Carmona-Kotani [5] and in the proofs of Simon [35]. These are infact worked out the in the book of Carmona-Lacroix [6]. H^ω is a bounded self-adjoint operator and hence $\lambda \in \mathbb{C}^+ \cup \mathbb{C}^-$ belongs to the resolvent of H^ω. The Wronskian of two solutions f, g of the eigenvalue equation $(H^\omega - \lambda)u = 0$ is given by

$$W(f, g) = a_n[f(n+1)g(n) - f(n)g(n+1)]$$

and is a constant. By Weyl theory,there are unique solutions, for the eigenvalue equation $(H^\omega - \lambda)\ u = 0$ which are integrable at $\pm\infty$ for λ in the resolvent set of H^ω. Denote them by $u^\omega_{\pm,\lambda}$. Then we have the expression for Green function given by

$$g^\omega_\lambda(n, n) = \frac{u^\omega_{+,\lambda}(n)u^\omega_{-,\lambda}(n)}{W^\omega} \tag{52}$$

where W^ω is the Wronskian of the solutions $u^\omega_{\pm,\lambda}$ given by $a^\omega_0(u^\omega_{+,\lambda}(1)u^\omega_{-,\lambda}(0) - u^\omega_{+,\lambda}(0)u^\omega_{-,\lambda}(1))$. The Weyl functions given by

$$m^\omega_+(\lambda)(n) = -\frac{u^\omega_{+,\lambda}(n+1)}{a^\omega_n u^\omega_{+,\lambda}(n)} \quad and \quad m^\omega_-(\lambda)(n) = -\frac{u^\omega_{-,\lambda}(n-1)}{a^\omega_{n-1}u^\omega_{-,\lambda}(n)} \tag{53}$$

are related to the operators $H^\omega_{\pm,n}$ of H^ω restricted to the subspaces $\ell^2[n+1, \infty)$ and $\ell^2(-\infty, n-1]$ via

$$m^\omega_{\pm,n}(\lambda) = (H^\omega_{\pm,n} - \lambda)^{-1}(n \pm 1, n \pm 1).$$

We define the scaled m-functions $M_\pm$ by

$$M^\omega_{+,n}(\lambda) = a_n^2 m^\omega_{+,n}(\lambda) \ \ and \ \ M^\omega_{-,n}(\lambda) = a_{n-1}^2 m^\omega_{-,n}(\lambda). \tag{54}$$

Then the $M_\pm$ satisfy the following equations.

$$\begin{aligned} M^\omega_{+,n}(\lambda) &= b^\omega_n - \lambda - (a^\omega_{n-1})^2 (M^\omega_{+,n-1})^{-1} \ \ and \\ M^\omega_{-,n}(\lambda) &= b^\omega_n - \lambda - (a^\omega_n)^2 (M^\omega_{-,n+1})^{-1} \end{aligned} \tag{55}$$

The M-functions and the Green function are related by,

$$g^\omega_\lambda(n,n) = \frac{-1}{M^\omega_{+,n}(\lambda) + M^\omega_{-,n}(\lambda) + \lambda - b^\omega_n}. \tag{56}$$

Under the Assumptions on the probability measure $\mathbb{P}$, the Lyapunov exponent $\gamma(\lambda)$ exists for all $\lambda \in \mathbb{C}^+$ and is related to the m-functions by,

$$\mathbb{E}_\mathbb{P}\left\{ \ log\ |M_{+,0}(\lambda)| \right\} = -\gamma(\lambda) + \mathbb{E}_\mathbb{P} \ \{log\ a_0\}. \tag{57}$$

Then by a combination of theorems of Ishii-Pastur, Kotani and Simon it follows that

$$\Sigma_{ac}(H^\omega) = \{E : \gamma(E + i0) = 0\}^{-ess}$$

$^-$ess denoting closure upto sets of Lebesgue measure zero. There is also a probability measure on $\mathbb{R}$ supported on the spectrum called the density of states d$\mathbf{n}$ and the following Thouless formula relating the Weyl functions and the density of states is valid [5] and [7].

$$\mathbb{E}_\mathbb{P}\left\{ \ log\ M_{+,0}(\lambda) \right\} = \frac{1}{\pi} \int_\mathbb{R} log \ \ \frac{1}{\xi - \lambda} \ \ d\mathbf{n}(\xi). \tag{58}$$

For a sequence ω_m converging to ω in the topology of Ω , the operators H^{ω_m} converge strongly, since H^ω are bounded operators with a uniform bound on their norms, by assumption, for $\omega \in \Omega$. Therefore the Green functions converge compact uniformly in $\mathbb{C}^+$. Using this fact one can get the following theorem of Kotani, on the lines of Kotani [18] Simon [35] or Craig [8].

Theorem 5.1 *Let $\mathbb{P}$ satisfy assumption (3) and let the spectrum of H^ω satisfy assumptions (1, 2). Then everywhere on the interior of the spectrum Σ of H^ω, the following relation is valid*

$$Im\ [M_{+,n} - M_{-,n}](\lambda + i0) = 0\ ,\ Re\ [M_{+,n}(\lambda + i0) + M_{-,n}(\lambda + i0) + \lambda - b_n] = 0 \tag{59}$$

for all ω in the support of $\mathbb{P}$.

We consider next the trace formulae, well known in the periodic examples and recently constructed for the general bounded Jacobi matrices by Gesztezy-Holden-Simon-Zhao [12]. In the following we set

$$g_\lambda(n,n) \equiv (H - \lambda)^{-1}(n,n)\ ,\ n \in \mathbb{Z}.$$

We however state the trace formula for a special case we are interested in.

Theorem 5.2 (Trace Formula) *Consider a Jacobi matrix given in equation (1) with its spectrum Σ satisfying the assumptions (1). Suppose the Green function $g_\lambda(n,n)$ has vanishing real part almost everywhere on Σ. Then there is a unique point $\xi(n) \in \Psi$ such that*

$$\begin{aligned} b_n &= \frac{1}{2}(\tau_0 + \tau_\infty) + \frac{1}{2}\sum_{i=1}^{\infty}(\tau_{2i-1} + \tau_{2i} - 2\xi_i(n)) \\ a_n^2 + a_{n-1}^2 &= \frac{1}{2}(b_n)^2 + \frac{1}{4}(\tau_0^2 + \tau_\infty^2) + \frac{1}{4}\sum_{i=1}^{\infty}\tau_{2i-1}^2 + \tau_{2i}^2 - 2\xi_i(n)^2 \end{aligned} \tag{60}$$

Proof: We note that the Green function is real and increasing in the gaps I_i so that it has at most one zero in each gap. When there is a zero in I_i we call the zero to be $\xi_i(n)$, otherwise we take $\xi_i(n)$ to be τ_{2i-1} or τ_{2i} according as $g_\lambda(n,n)$ is positive or negative in I_i. With this choice we have a point $\xi(n) \in \Psi$ and we also have by assumption that the Green function has vanishing real part a.e. on the spectrum Σ. Therefore $h_{\xi(n)}$ constructed, from Σ and $\xi(n)$, in proposition (2.2) agrees with $g_\lambda(n,n)$ everywhere in Π. Therefore the product representation of equation (12) is valid for $g_\lambda(n,n)$ in Π. The coefficients of $1/\lambda^2$ and $1/\lambda^3$ in the asymptotic expansion of $g_\lambda(n,n)$ are respectively the left hand sides of the first two relations of the Trace formula stated in the theorem. The right hand sides are the respective coefficients coming from the product representation. $\square$

5.1 Ergodic potentials with band spectrum

In the section we show the existence of probability measures satisfying our Assumptions (3) such that their spectra satisfy the Assumptions (1,2). We use the inverse spectral theory of Carmona-Kotani [5] to explicitly construct such measures given the spectrum and also present the theorem of Kotani on the ergodic selection of such a measure. Since the proofs of these theorems are essentially contained in the works cited above we do not give proof for theorem (5.4). This theory appears also in the book of Carmona-Lacroix [6].

Theorem 5.3 *Consider the set Σ satisfying Assumptions (2). Then there exists a Herglotz function w satisfying the following properties.*

1. *w and w' and are Herglotz with w having the representation,*

$$w(\lambda) = \frac{1}{\pi} \int log \, \frac{1}{\xi - \lambda} dn(\xi)$$

   for an absolutely continuous measure dn supported on Σ.

2. *w(λ)$\sim$ log $-1/\lambda$ as $\lambda \to \infty$*

3. *$w(\mathbb{C}^+) \subseteq (-\infty, c] \times i[0, \pi]$, and on Σ, $-\gamma$ = Re(w) - c = 0 for some finite c and $\int_{\mathbb{R}} \gamma(\xi) dn(\xi) = 0$.*

Proof: Consider the set Σ as in Assumption 2.1 and consider the point $\zeta \in \Psi$ chosen in proposition (3.9) and the corresponding h_ζ of proposition (3.9). Then, the estimates of proposition (5.6), which are also valid for h_ζ will show that the measure dn in the representation

$$h_\zeta(\lambda) = \frac{1}{\pi} \int k(\xi, \lambda) dn(\xi)$$

is absolutely continuous, supported on Σ and the distribution function n(ξ) = dn$(-\infty, \xi]$ satisfies n(∞) = π. Therefore $dn(\xi) = Im \; h_\zeta(\xi + i0) d\xi$. Using dn we consider another Herglotz function w such that

$$w(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} log \, \frac{1}{(\xi - \lambda)} dn(\xi) \tag{61}$$

and show that this is the function stated in the theorem. The above function is well defined for λ in $\mathbb{C}^+$. A computation shows that $w'(\lambda) = h_\zeta(\lambda)$ for

$\lambda \in \mathbb{C}^+$. Since $Im\ h_\zeta(\lambda + i0)$ has at most at most inverse square root singularity at the boundary points of Σ, as can be seen from its product representation, $\log|\xi - Re\lambda|$ is integrable with respect to dn. Therefore by Lebesgue dominated convergence theorem the limits $w(\xi + i0)$ exist for all $\xi \in \mathbb{R}$. We consider the function $w(\lambda) - w(\tau_0)$ for $\lambda \in \mathbb{C}^+$, then the integral representation,

$$g(\lambda) = w(\lambda) - w(\tau_0) = \int_{\tau_0}^{\lambda} w'(\xi) d\xi = \int_{\tau_0}^{\lambda} h_\zeta(\xi) d\xi \tag{62}$$

is valid for $\lambda \in \mathbb{R}$ with the path of integration in $\mathbb{C}^+$. The limits $h_\zeta(\lambda + i0)$ exist a.e. so we can take the path of integration along the real axis. Consider the real part of g. For $\lambda \in I(-\infty)$ it is obvious that real part of g is negative. As for λ in a given gap I_i, we have by proposition (3.9), together with the fact that the real part of h_ζ in the spectrum is zero, that

$$Re\ \ g(\lambda) = \int_{\tau_{2i-1}}^{\lambda} h_\zeta(x) dx$$

is negative, since h_ζ is an increasing function having at most one zero in the each gap I_i. Finally the real part of g in $I(\infty)$ is negative, since h_ζ is negative there and the value of Re g at any point λ in $I(\infty)$ is given by $\int_{\tau_\infty}^{\lambda} h_\zeta$, by using proposition (3.9). The analysis to show that the imaginary part of g(λ) is increasing from 0 to π is similar. Once we obtain the values of w(λ) on the boundary, by equations (61), (62) and by the fact that Im $w'(\lambda) > 0$ for $\lambda \in \mathbb{C}^+$ it is clear that

$$Re\ \ w \in (-\infty, c]\ , \quad Im\ \ w \in [0, \pi]\ , \quad c = \omega(\tau_0)\ , \quad \forall\ \ \lambda \in \bar{\mathbb{C}}^+ .$$

The statements of (3) are now clear. $\square$

Given the w function satisfying the conditions of Theorem 5.3, we can find a probability measure $\mathbb{P}$ on $(\mathbb{R}^+ \times \mathbb{R})^{\mathbb{Z}}$ which is invariant under translations, by the Theorem 4.8 of [5]. This shows from their proofs that $\mathbb{E}_{\mathbb{P}} \{log\ a_0\} = c < \infty$. The ergodicity of such a measure follows if the density of states and the Lyapunov exponent satisfy the condition

$$\int \gamma(\xi) dn(\xi) = 0$$

which is true in our case. The proof that this implies the existence of an ergodic measure is almost identical to the proof of Kotani [18], Lemmas 7.11 and Theorem 6.3 which can be proved for the w function satisfying the properties of Theorem 5.3. Since the measure dn above has compact support (Σ), the sequences (a_n, b_n) in the support of the measure so constructed will be bounded. Hence we have,

Theorem 5.4 *There exists an invariant and ergodic probability measure* $\mathbb{P}$ *on* Ω *satisfying assumption (3) having spectrum as in assumptions (1,2).*

5.2 Almost Periodicity

We prove the almost periodicity of the random jacobi matrices, in this section. We start with getting expressions for the M-functions and show a technical lemma on their boundedness which verifies some conditions required for the application of proposition (4.3). We recall the rectangles R_N^i defined in lemma (3.3) and r_N defined after it, we set $R_N = \cup_{i=1}^{M(N)} R_N^i$ for each N. We consider $M_\pm$ given in equation (54) and drop the superscript ω.

Proposition 5.5 *Consider $\mathbb{P}$ satisfying assumptions (3),with the spectrum Σ of H^ω purely absolutely continuous and satisfy assumptions (1,2). Then there exist unique $\xi(n) \in \Psi$, a unique partition $\mathcal{O}^{\pm}_{\xi(n)}$ of $\mathcal{O}_{\xi(n)}$ into disjoint subsets and unique measures $\nu_{2,n}$ supported on $\mathcal{O}_{\xi(n)}$ such that the M-functions have the following representation,*

$$M^{\omega}_{\pm,n}(\lambda) = -\frac{1}{2}g^{\omega}_{\lambda}(n,n) \pm \int k(\lambda,x)d(\nu^{+}_{2,n} - \nu^{-}_{2,n}) \tag{63}$$

with $\nu^{\pm}_{2,n} = \nu_{2,n}|_{\mathcal{O}^{\pm}_{\xi(n)}}$ given by

$$\nu^{\pm}_{2,n}(\xi_i(n)) = \frac{\sqrt{R(\xi_i(n))}}{\prod_{j\neq i}(\xi_i(n) - \xi_j(n))} \quad for \quad \xi_i(n) \in \mathcal{O}_{\xi(n)}.$$

In the above equation, the square root taken so that , the right hand side is positive.

Proof: We fix a ω and work with it in the following, so that the superscripts are dropped. We note that for each n the real part of the Green functions $g_\lambda(n,n)$ are zero in interior of the spectrum and it is real and increasing in the gaps I_i. Therefore for each n we get a point $\xi(n) \in \Psi$ as in the proposition (2.2) and conclude that $g_\lambda(n,n) = h_{\xi(n)}(\lambda)$, h as in proposition (2.2). By lemma (5.1) we have that $0 < ImM_{+,n} = ImM_{-.n} < \infty$ in the interior of Σ. Therefore in the Herglotz representation for the M-functions, the singular part can only be supported in the set $\partial\bar{\Sigma} \cup S_{\xi(n)}$. This set being closed and countable, the singular part can only be pure point. We now use the relations, coming from $(a+b)^2 - (a-b)^2 = 4ab$, (F is defined so that we need not write long expressions later)

$$F(\lambda) \equiv (M_{+,n} - M_{-,n} - \lambda + b_n)^2 = g_\lambda(n,n)^{-2} + 4a_n^2 \frac{g_\lambda(n+1,n+1)}{g_\lambda(n,n)}$$

and the product representations for $h_{\xi(n)}, h_{\xi(n+1)}$ to conclude that the right hand side is an analytic function in $\mathbb{C} \setminus (\mathcal{E} \cup S_{\xi(n)})$. Therefore the left hand side is a meromorphic function on $\mathbb{C}$, with possibly essential singularities at the points of $\mathcal{E}$. We first rule out the set of points of $S_{\xi(n)} \cap \Sigma$ from being singularities. If there is a singularity at any of the points in this set, then the product representations, will show that the right hand side in the above equation has a simple pole at this ponit, while the left hand side has a double pole giving a contradiction. This leaves us with only the set $\mathcal{E} \cup \mathcal{O}_{\xi(n)}$. We rule out the possibility of essential singularities at the points of $\mathcal{E}$ by estimating the right hand side on the sequence R_N of rectangles of lemma (3.3), using the product representation for $h_{\xi(n)}, h_{\xi(n+1)}$ and the bounds of lemma (3.1), to show that the limits

$$\lim_{N \to \infty} \int_{R_N} F(\lambda) d\lambda = 0.$$

For a given point p in $\mathcal{E} \setminus \mathcal{E}_1$, we again do the estimates for $(\lambda - p)^m F(\lambda)$, for each positive integer m to show that there the negative terms in the Laurent series expansion for F at these points vanish. Similar method shows that none of the points of $\mathcal{E}_1$ is a pole of any order.

The sum and total of all this analysis is that the singular parts of $\nu^{\pm}$ can only be supported on $\mathcal{O}_{\xi(n)}$. The relation $M_+(M_- + \lambda - b_n)(\lambda) = -a_n^2 g_\lambda(n+1, n+1)/g_\lambda(n,n)$ shows that a point of $\xi_i(n) \in I_i$ can be an eigen value of only one of $H_{\pm . n}$, since the right hand side has a simple pole at $\xi_i(n)$. Therefore specifying them, which we can do in principle if we know H, will give us a partition, of $\mathcal{O}_{\xi(n)}$ stated in the proposition. Then the measures $\nu_{2,n}^{\pm}$ can be obtained uniquely as

$$\nu^{\pm}_{2,n} = \nu|_{\mathcal{O}^{\pm}_{\xi(n)}}$$

and the expression for $\nu_{2,n}$ is clear from the product representation in Π, of the meromorphic function $g_\lambda(n,n)$ coming from proposition (2.3), by a computation of the residues. $\square$

Lemma 5.6 *Assume that $\mathbb{P}$ satisfies assumptions (3) and the spectrum of H^ω satisfies assumptions (1,2). Then for each n the M functions $M_{\pm,n}(\lambda)$ and $(M_{\pm,n}(\lambda))^{-1}$ are uniformly bounded on the set of rectangles R_N defined in the lemma 3.3 Moreover the derivatives of $m^{\pm}_{\lambda}(n)$ are uniformly bounded on $R_N \cap (\mathbb{C} \setminus \Sigma)$. These functions are also bounded uniformly on the set r_N. All the bounds are uniform in N.*

Proof: We consider $\mathbb{P}$ satisfying assumptions (3) such that the corresponding random Jacobi matrix satisfies assumptions (1,2). Consider a ω in the support of $\mathbb{P}$ fixed. We use the product representation of (14) for $g_\lambda(n,n)$, and the expressions of equation (63) for the M-functions. Then the bounds of lemma (3.1), prove the lemma for $M_{\pm,n}(\lambda)$. On the other hand for the inverses of the m functions, we use the relations of equation (55) on the portion of R_N in the resolvent of $H^{\pm}$ and extend the bounds to all of R_N in the spectrum since the bounds are uniform in $R_N \setminus \mathbb{R}$. For fixed n these bounds will depend on $1/a_n^2$. As for the derivatives we consider the second term on the right hand side of equation (63). The summability of the derivatives follows from the expressions for $\nu_{2,n}$ coming from the last lemma, the lower bound s_i on the distance of R_N to I_i, bounds of lemma (3.1) and assumptions (2). Regarding the boundedness of the derivative of the inverse of the Green function, we note that it is a meromorphic function in $\mathbb{C} \setminus \Sigma$ and the derivative is given by

$$\begin{aligned} \frac{d}{d\lambda} g_\lambda(n,n)^{-1} &= \frac{1}{2} g_\lambda(n,n)^{-1} \times \left(\frac{1}{(\lambda-\tau_0)^2} + \frac{1}{(\lambda-\tau_\infty)^2} \right. \\ &\quad \left. + \sum_i \frac{\tau_{2i-1}-\xi_i(n)}{(\lambda-\tau_{2i-1})(\lambda-\xi_i(n))} + \frac{\tau_{2i}-\xi_i(n)}{(\lambda-\tau_{2i})(\lambda-\xi_i(n))} \right) \end{aligned}$$

The bounds of lemma (3.1) show that apriori the derivative is bounded on $R_N \cap [\mathbb{C} \setminus \Sigma]$, and with a uniform bound since the distance of R_N to I_i is bounded below by s_i and by assumption (2), q_i/s_i is summable. Hence even the lim sup of the derivative along any sequence on the rectangle approaching the real axis, both from $\mathbb{C}^+$ and from $\mathbb{C}^-$ is uniformly bounded. ¿From the construction of r_N it is clear that the uniform bounds are valid on r_N also. □

Now we are ready to prove the almost periodicity of the random Jacobi matrix.

Theorem 5.7 *Let $\mathbb{P}$ satisfy assumptions (3) and let the spectrum Σ satisfy the assumptions (1,2). Then every point ω in the support of $\mathbb{P}$ is an almost periodic sequence, in the sense that b_n^ω and a_n^ω are almost periodic sequences for each ω.*

Proof: We consider a ω fixed and drop the superscript in the following. We can check from the definitions of $M_{\pm,n}$ written in terms of the solutions

$u_{\pm}(n)$, and the product representations for the Green functions valid under our assumptions, that

$$\begin{aligned}\prod_{i=0}^{n-1}[-a_i^{-2}][M_{+,i}(\lambda)][-M_{-,i}(\lambda)-\lambda+b_i] &= \frac{u_-(n)u_+(n)}{u_-(0)u_+(0)} \\ &= \frac{g_\lambda(n,n)}{g_\lambda(0,0)} \\ &= \prod_{k=1}^{\infty}\frac{(\lambda-\xi_k(n))}{(\lambda-\xi_i(0))}. \qquad (64)\end{aligned}$$

This equation shows that the product on the left hand side has zeros and poles at the ponts of $S_{\xi(n)} \cup S_{\xi(0)}$ and exactly one of $\prod M_{+,i}, \prod(M_{-,i}+\lambda-b_i)$ will have a pole in the set $S_{\xi(0)} \cap \mathcal{O}$. On the other hand the representation for $M_{\pm,i}$ coming from equation (17) show that we have the expressions,

$$M_{+,i-1} = \frac{1}{2}[\int k(\lambda,x)d(\nu_{2,i}^+ - \nu_{2,i}^-) + \lambda - b_i] + g_\lambda(i,i)^{-1}$$

$$-M_{-,i-1} - \lambda + b_i = \frac{1}{2}[\int k(\lambda,x)d(\nu_{2,i}^+ - \nu_{2,i}^-) + \lambda - b_i] - g_\lambda(i,i)^{-1}.$$

¿From the above equations, it is clear that both M_+ and $-M_- - \lambda + b_i$ are the two branches of the same meromorphic function ϕ_i when lifted to the Riemann surface $\mathcal{R}$. Consider the meromorphic function, $\phi(n)$, on $\mathcal{R}$ given by

$$\phi(n) = \prod_{i=0}^{n-1}\phi_i.$$

This function has simple zeros in $\lambda^{-1}(S_{\xi(n)})$ and simple poles in $\lambda^{-1}(S_{\xi(0)})$ and by the previous arguments, there is only one zero and one pole in each of $\lambda^{-1}(\bar{I}_i)$, i = 1,2,$\cdots$. Let us denote these points as $\xi_i^*(n)$ and $\xi_i^*(0)$. As for the points in $\lambda^{-1}(\infty)$, the asymptotic behaviour of $M_{+,i} \approx -1/\lambda$ and $-M_{-,i} - \lambda + b_i \approx -\lambda$ shows that ϕ_i has a pole and a zero at each point of $\lambda^{-1}(\infty)$ of order 1. Therefore the divisor of the meromorphic differential $d\phi(n)/\phi(n)$ is given by $D = \sum_{i=1}^{\infty}[\xi_i^*(n) - \xi_i^*(0)] + n[\infty_1 - \infty_2]$. It follows from lemma (5.6) that the meromorphic functions ϕ_i and hence their product

$$\phi(n) = \prod_{i=0}^{n-1} \phi_i$$

satisfy the assumptions of proposition (4.3) since $\phi_i(p)$ agrees with $M_{\pm,i}(\lambda(p))$ for $p \in \mathcal{R}^{\pm} \cap \lambda^{-1}(\Pi)$. Therefore the Abel-Jacobi map $\mathcal{A}$ corresponding to the point $\xi^*(n)$ has the image satisfying the assumptions of theorem (4.9). This shows that each of the points $\xi_i^*(n)$, hence their projections $\xi_i(n)$ to $\bar{I}_i$, are almost periodic for each i. Now the Trace formulae , together with the uniform, in n, convergence of the sums in the Trace formulae give the almost periodicity of $a_n^2 + a_{n-1}^2$ and b_n. To show the almost periodicity of $a_n^2 - a_{n-1}^2$, we compute this difference to be $\int \nu_{2,n}^+ - \int \nu_{2,n}^-$, by looking at the $1/\lambda$ term in the asymptotic expansion of $M_+ - M_-$ using the proposition (5.5). We note that this difference is nothing but the sum of the residues of the meromorphic function ϕ_n (the suffix n is correct, we emphasize this) at its poles ξ_i^* in $\lambda^{-1}(I_i)$, (the preimages of the open gaps, not closed gaps). This computation gives us after changing variables to the angular coordinates,

$$\begin{aligned} \int \nu_{2,n}^+ - \int \nu_{2,n}^- &= \sum_{i:\xi_i(n)\in I_i} \sqrt{(\xi_i^*(n) - \tau_{2i-1})(\tau_{2i} - \xi_i^*(n))}\ \ g_i(n) \\ &= \sum_{i:\xi_i(n)\in I_i} \ell_i\ \ sin(\theta_i(n))\ \ cos(\theta_i(n))\ \ g_i(n) \end{aligned}$$

where we have taken,

$$\begin{aligned} \theta_j(n) &= \mathcal{A}_j^{-1}(n\vec{c} + \vec{d}) \\ \xi_j^*(n) &= \tau_{2j-1} + \ell_j sin^2(\theta_j(n)) \end{aligned}$$

$$g_i(n) \;=\; \sqrt{(\tau_0-\xi_i(n))(\tau_\infty-\xi_i(n))}\prod_{j\neq i}\frac{\sqrt{(\tau_{2j-1}-\xi_i(n))(\tau_{2j}-\xi_i(n))}}{(\xi_i(n)-\xi_j(n))}.$$

We note here that the signs ± 1 in front of $\nu^{\pm}_{2,n}$ have been absorbed into the angles, since we have that modulo π, $\theta_i(n)$ is in $[0,\,\pi/2)$ whenever $\xi^*_i(n)\in\mathcal{R}^+\cap\lambda^{-1}(I_i)$ and it is in $[\pi/2,\,\pi)$ when $\xi^*_i(n)$ is in $\mathcal{R}^-\cap\lambda^{-1}(I_i)$. We also note that the $g_i(n)$ are positive. The almost periodicity of $a_n^2-a_{n-1}^2$, follows from the above two equations, using the almost periodicity of $sin(\theta_i(n))$ and $cos(\theta_i(n))$ and those of $\xi_j(n)$, using the following facts, together with the uniform, in n, convergence of the products and sums. If two sequences c_n and d_n are almost periodic, then their sums, products are almost periodic. Their ratios are also almost periodic if the absolute values of the denominators have a strict positive lower bound. The positive square roots of an almost periodic sequence with positive entries is also almost periodic as can be seen from the proof below.

Since by above $a_n^2+a_{n-1}^2$ and $a_n^2-a_{n-1}^2$ are almost periodic, we see that a_n^2 is almost periodic. Now consider the positive square roots a_n. Then consider ϵ^4 and an N such that $|a_n^2-a_{n+N}^2|<\epsilon^4$. For this N, consider the collection of points n such that $a_{n+N}^2<\epsilon^2$. Then for these points we have

$$|a_n-a_{n+N}| = |\sqrt{(a_n^2-a_{n+N}^2)+a_{n+N}^2}-\sqrt{a_{n+N}^2}| \le \sqrt{\epsilon^4+\epsilon^2}+\epsilon < 3\epsilon.$$

As for the set of points where $a_{n+N}^2>\epsilon^2$, we use the relation

$$|a_n-a_{n+N}| = |\frac{(a_n^2-a_{n+N}^2)}{a_n+a_{n+N}}| \le \epsilon^2 < \epsilon$$

using the positivity of a_n. $\square$

Finally we remark that the equation of motion (55) for the m-functions shows that $\xi_i(n)$ does not pause in the gaps as n varies. If there is an n such that $\xi_i(n) = \xi_i(n+1)$, then it must be that these two points are eigen values of different half space problems. We also note that the Jacobi matrices with $a_n \equiv const$ satisfy the condition that $\int d(\nu^+_{2,n} - \nu^-_{2,n}) = 0$ for each n.

A Herglotz representation theorems

In this appendix we collect some of the standard theorems on Herglotz functions, and state most of them without proof. We refer to equations 4 for the definition of the kernels K and k. The following theorems are standard and can be found in Kotani [18] , Simon [35], Craig [8] or in the appendix of Figotin-Pastur [11].

Theorem A.1 (Herglotz representation) *Let F be Herglotz, then there are $a \geq 0, b \in \mathbb{R}, \mu$ such that the representation*

$$F(z) = az + b + \frac{1}{\pi} \int K(\lambda, x) d\mu(x) \tag{65}$$

is valid, with μ a positive borel measure satisfying $\int 1/(1+x^2) d\mu(x) < \infty$.

Theorem A.2 *Let F and G be a Herglotz functions, then the following are valid*

1. *The non tangential limits F(x+i0) exist finitely almost everywhere with respect to Lebesgue measure.*

2. *If F(x + i0) = G(x + i0) , on a set of positive Lebesgue measure then $F \equiv G$.*

3. If Im F(x+i0) = 0 a.e. on an open interval I, then F is analytic on I.

4. If the measure μ representing F, has compact support, then it is finite and the representation theorem for F becomes

$$F(\lambda) = a\lambda + b + \frac{1}{\pi}\int k(\lambda, x) d\mu(x).$$

5. The absolutely continuous part of the representing measure μ can be recovered from F by the following formula. Let B be any Borel set,

$$\mu(B) = \lim_{\epsilon \to 0} \int_B Im \quad F(x + i\epsilon)dx$$

in particular μ_{ac} has density $Im \quad F(x + i0)$.

6. The singular part of μ is supported on $\{x : \lim_{\epsilon \to 0} F(x + i\epsilon) = \infty\}$.

Proof: The statement (4) follows from the trivial estimate that when the support of μ is bounded say , sup $\{|x| : x \in supp\ \mu\}$ = a, then $1/1 + x^2 \leq 1/1 + a^2$. Then we can absorb the quantity $\int (x/1 + x^2) d\mu(x)$ into b, hence the stated representation. The proof for the rest can be found in the works quoted above. □

The next theorem, on the existence of zeros of a class of continuous functions, is from Deimlig [16]. Consider a bounded open set I of $\mathbb{R}^n$ with ∂I denoting its boundary. The topological degree of a C^1 map f at a point y not in the image of the boundary is defined as

$$d(f, I, y) = \sum_{y \in f^{-1}(y)} sgn J_f(x) \ , \quad y \in \mathbb{R}^n \setminus f(\partial I \cup S_f).$$

where S_f is the set of singular values of f and J_f the Jacobian. This definition is extended for C^2 maps and for continuous maps it is defined in terms of C^2 maps close to it, we refer the reader to the definition 2.2 of [16] for an exact definition. It follows from the definition of the degree that the identity map has degree one. The degree of a continuous map is useful in determining if it can take some values. As an application of the theory of degrees we have the following theorem.

Theorem A.3 *Let $I_k = [c_k, d_k]$; $k = 1, 2, .., n$ be intervals in the real line. Define $I = \prod_{k=1}^{n} I_k$. Let $F : I \rightarrow \mathbb{R}^n$ be a continuous function such that $F_k(x_1, .., c_k, .., x_n) < 0$ and $F_k(x_1, .., d_k, .., x_n) > 0$ $\forall k = 1, .., n$, where $F_k(.)$ denotes the k'th coordinate of F. Then F has a zero in the interior of I.*

Proof: Without loss of generality we can assume that $c_k < 0$ and $d_k > 0$ $\forall k$. The straight line homotopy $x \rightarrow tF(x) + (1-t)x; 0 \leq t \leq 1$ gives a homotopy between F and the identity map. The conditions $F(x_1, .., c_k, .., x_n) < 0$ and $F(x_1, .., d_k, .., x_n) > 0$ ensure that throughout the homotopic deformation no point in the the boundary of I goes to zero. Hence the conditions of (d3) of Theorem 3.1 in [16] are valid showing that the topological degree of F for the point 0 in $\mathbb{R}^n$ is the same as that of the identity map which is 1. This implies by (d4) of Theorem 3.1 in [16] that F has a zero in the interior of I. $\square$

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