Tree expansion and multiscale analysis for KAM tori

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Abstract: *We prove that the perturbative expansion for the KAM invariant tori of the Thirring model (with interaction depending also on the action variables) is convergent by using techniques usual in quantum field theory like the multiscale decomposition and the tree expansion. The proof follows the ideas of Eliasson, [3], and extends the results found in the case of an action-independent interaction potential, [4].*

Keywords: *KAM theorem, dynamical systems, renormalization group, quantum field theory, multiscale decomposition, form factors*

1. Introduction

The KAM theorem proves in a indirect way that the formal perturbation series for the invariant tori of quasi integrable systems are convergent if some conditions are fulfilled by the hamiltonian, [11], [1], [12]. Of course one should be able to prove the convergence studing directly the perturbative series and this has been performed in recent times by Eliasson, [3]. However the work by Eliasson has not enjoyed a wide circulation, maybe because of the excessive generality with which the considered problem is attacked, (see also the regarding comments in [5]). Then in [4] the convergence of the perturbative expansion of the tori is proven by using the Eliasson's ideas in a special model, the *Thirring model*, [13], *with the further simplification that the interaction does not depend on the actions*: in this way the discussion becomes much less involved, and so is suitable for a more easy understanding. The proof is implemented with methods usual in the renormalization group approach to quantum field theory, like the multiscale decomposition of the propagator and the tree expansion. Some minor technicalities of the proof are resolved in [9], where the strong diophantine hypothesis used in [4] is completely relaxed, (see [4], [9], for a detailed review). In this paper the proof is extended to cover the case in which the interaction potential of the Thirring model does not depend only on the angle variables, as in [4], but also on the action variables. As a function of the angles, the potential is supposed to be still an even trigonometric polynomial of degree $N < \infty$, while, as a function of the actions, we assume that it is analytical, since this does yields no further technical work with respect to the case of a polynomial-like interaction. The idea is always to confine ourselves to a not too tangled case, in such a way to emphasize the interesting features of the method, and to point out the analogies with quantum field theory. We shall define a perturbative expansion for the KAM tori in terms of Feynman's graphs. However the "dimensional bounds" are not sufficient to prove the series convergence: then we perform a resummation so obtaining a multiple series, in terms of the coupling constant and of suitable functions called "form factors". If such functions are uniformly bounded, the series is convergent; endly the form

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factors boundedness is proven by using some cancellation mechanisms between diagrams operating to all perturbative orders. The analogy with quantum field theory is striking; for other results in this direction, see also [7], [8]. With respect to [4] and [9], the use of a smooth decomposition of the propagator and the introduction of a "renormalized expansion" via a localization operation makes simpler the proof.

The hamiltonian function of the Thirring model is

$$\frac{1}{2}J^{-1}\vec{A}\cdot\vec{A}\,+\,\varepsilon f(\vec{\alpha},\vec{A})\ , \tag{1.1}$$

where J is the non singular matrix of the inertia moments, $\vec{A}=(A_1,\ldots,A_l)\in\mathbf{R}^l$ are their angular momenta and $\vec{\alpha}=(\alpha_1,\ldots,\alpha_l)\in\mathbf{T}^l$ are the angles describing their positions. We shall consider a "rotation vector" $\vec{\omega}_0=(\omega_1,\ldots,\omega_l)\in\mathbf{R}^l$ verifying the *diophantine property* with diophantine constants $C_0,\tau>0$; this means that

$$C_0|\vec{\omega}_0\cdot\vec{\nu}|\geq|\vec{\nu}|^{-\tau}\ , \qquad\qquad \vec{0}\neq\vec{\nu}\in\mathbf{Z}^l\ , \tag{1.2}$$

and it is easy to see that the *diophantine vectors* have full measure in $\mathbf{R}^l$ if τ is fixed $\tau>l-1$.

We suppose f to be an even trigonometric polynomial of degree N in the angle variables and an analytic function in the angular momenta variables, *i.e.*

$$f(\vec{\alpha},\vec{A})=\sum_{|\vec{\nu}|\leq N}f_{\vec{\nu}}(\vec{A})\,\cos\vec{\nu}\cdot\vec{\alpha}\ ,\qquad f_{\vec{\nu}}(\vec{A})=f_{-\vec{\nu}}(\vec{A})\ , \tag{1.3}$$

with $f_{\vec{\nu}}(\vec{A})$ analytic in $\vec{A}$ in a domain $W(\vec{A}_0,\rho)=\{\vec{A}\in\mathbf{R}^l\ :\ |\vec{A}-\vec{A}_0|/|\vec{A}_0|\leq\rho\}$, for any $\vec{\nu}$, being $\vec{A}_0=J\vec{\omega}_0$. Finally, if J_j, $j=1,\ldots,l$ are the eigenvalues of the matrix J, we define

$$J_m=\min_{j=1,\ldots,l}J_j\ ,\qquad J_M=\max_{j=1,\ldots,l}J_j\ ,\qquad F=\max_{|\vec{\nu}|\leq N,\vec{A}\in W(\vec{A}_0,\rho)}f_{\vec{\nu}}(\vec{A})\ ,$$

The fundamental result of this work is the following one.

Theorem 1.1. *The hamiltonian model (1.1) admits an ε–analytic family of motions starting at $\vec{\alpha}=\vec{0}$ and having the form*

$$\vec{A}=\vec{A}_0+\vec{H}(\vec{A}_0,\vec{\omega}_0t;\varepsilon)+\vec{\mu}(\vec{A}_0,\varepsilon)\ ,\qquad\qquad \vec{\alpha}=\vec{\omega}_0t+\vec{h}(\vec{A}_0,\vec{\omega}_0t;\varepsilon)\ , \tag{1.4}$$

with $\vec{H}(\vec{A},\vec{\psi};\varepsilon)$, $\vec{h}(\vec{A},\vec{\psi};\varepsilon)$ analytic in $\vec{\psi}$ with $\mathrm{Re}\vec{\psi}\in\mathbf{T}^l$, and $|\mathrm{Im}\vec{\psi}_j|<\xi$, and in $\vec{A}\in W(\vec{A}_0,\rho)$, where $\vec{A}_0=J\vec{\omega}_0$, and with vanishing average in $\mathbf{T}^l$, and with $\vec{H}(\vec{A}_0,\vec{\psi};\varepsilon)$, $\vec{h}(\vec{A}_0,\vec{\psi};\varepsilon)$ and $\vec{\mu}(\vec{A}_0,\varepsilon)$ analytic for $|\varepsilon|<\varepsilon_0$ with a suitable ε_0 close to 0:

$$\varepsilon_0=E_0\,[J_MJ_m^{-2}C_0^2Fe^{\xi N}\rho^{-2}]^{-1}\ , \tag{1.5}$$

where E_0 is a dimensionless quantity depending only on N and l. This means that the set $(\vec{A},\vec{\alpha})$ described as $\vec{\psi}$ varies in $\mathbf{T}^l$ is, for ε small enough, an invariant torus for (1.1), which is run quasi periodically with angular velocity vector $\vec{\omega}_0$. It is a family of invariant tori coinciding, for $\varepsilon=0$, with the torus $\vec{A}=\vec{A}_0$, $\vec{\alpha}=\vec{\psi}\in\mathbf{T}^l$.

Remark 1. One recognizes a version of the KAM theorem. The proof that follows extends the one reported in [4] to the more general case in wich the interaction depends also on the angular momenta.

Remark 2. Note that, in distinction to [4], the result is *not uniform* in the twist rate T, defined as $T=J_M^{-1}$: this is known from the KAM theorem, and follows from the fact that the interaction depends also on the action variables. In other words the *twistless* property in [4] is a consequence of the special choice of the interaction potential, which is of the form (1.2), with $f_{\vec{\nu}}(\vec{A})$ replaced with $f_{\vec{\nu}}$ independent on $\vec{A}$.

Calling $\vec H^{(k)}$, $\vec h^{(k)}$, $\vec\mu^{(k)}$ the k-th order coefficients of the Taylor expansion of $\vec H$, $\vec h$, $\vec\mu$ in powers of ε, and writing the equation of motion as $\dot\alpha_j=(J^{-1}\vec A)_j+\varepsilon\partial_{A_j}f(\vec\alpha,\vec A)$, and $\dot A_j=-\varepsilon\partial_{\alpha_j}f(\vec\alpha,\vec A)$, $j=1,\ldots,l$, we get immediately recursion relations for $\vec H^{(k)},\vec h^{(k)}$; for $k=1$:

$$\begin{aligned}&\vec\omega_0\cdot\partial_{\vec\alpha}\,H_j^{(1)}=-\partial_{\alpha_j}f\ ,\\&\vec\omega_0\cdot\partial_{\vec\alpha}\,h_j^{(1)}=\Big(J^{-1}[\vec H^{(1)}+\vec\mu^{(1)}]\Big)_j+\partial_{A_j}f\ ,\end{aligned}\tag{1.6}$$

where $\vec\omega_0\cdot\partial_{\vec\alpha}=\sum_{i=1}^l\vec\omega_{0i}\partial_{\alpha_i}$, and, for $k>1$:

$$\begin{aligned}\vec\omega_0\cdot\partial_{\vec\alpha}\,H_j^{(k)}=&-\sum_{m>0}\ \sum_{\substack{p_1,\ldots,p_l,q_1,\ldots,q_l\\ \sum_{s=1}^l(p_s+q_s)=m}}\frac{1}{\prod_{s=1}^l p_s!\,q_s!}\ \cdot\\&\cdot\ \partial_{\alpha_j}\partial_{\alpha_1}^{p_1}\ldots\partial_{\alpha_l}^{p_l}\partial_{A_1}^{q_1}\ldots\partial_{A_l}^{q_l}f(\vec\omega_0t,\vec A_0)\ \cdot\\&\cdot\ \sum{}^*\prod_{s=1}^l\Big[\prod_{j=1}^{p_s}h_s^{(k_{sj})}\prod_{i=1}^{q_s}\Big(H_s^{(k'_{si})}+\mu_s^{(k'_{si})}\Big)\Big]\ ,\\\vec\omega_0\cdot\partial_{\vec\alpha}\,h_j^{(k)}=&\Big(J^{-1}[\vec H^{(k)}+\vec\mu^{(k)}]\Big)_j+\sum_{m>0}\ \sum_{\substack{p_1,\ldots,p_l,q_1,\ldots,q_l\\ \sum_{s=1}^l(p_s+q_s)=m}}\frac{1}{\prod_{s=1}^l p_s!\,q_s!}\ \cdot\\&\cdot\ \partial_{A_j}\partial_{\alpha_1}^{p_1}\ldots\partial_{\alpha_l}^{p_l}\partial_{A_1}^{q_1}\ldots\partial_{A_l}^{q_l}f(\vec\omega_0t,\vec A_0)\ \cdot\\&\cdot\ \sum{}^*\prod_{s=1}^l\Big[\prod_{j=1}^{p_s}h_s^{(k_{sj})}\prod_{i=1}^{q_s}\Big(H_s^{(k'_{si})}+\mu_s^{(k'_{si})}\Big)\Big]\ ,\end{aligned}\tag{1.7}$$

where the $\sum^*$ denotes summation over the integers $k_{sj}\ge1$, $k'_{si}\ge1$, with: $\sum_{s=1}^l(\sum_{j=1}^{p_s}k_{sj}+\sum_{i=1}^{q_s}k'_{si})=k-1$.

In fact from the equations of motion for the angular momenta, we obtain immediately the first recursive relation in (1.7). Then suppose that $\vec h^{(k)}(\vec\psi)$ and $\vec H^{(k)}(\vec\psi)$ are trigonometric polynomials of degree $\le kN$, respectively odd and even in t, for $1\le k<k_0$: we see immediately that the r.h.s. of the first equation in (1.7) is odd in t. Then the first equation in (1.7) can be solved for $k=k_0$. It yields an even function $\vec H^{(k_0)}(\vec\psi)+\vec\mu^{(k_0)}$ which is defined up to the constant $\vec\mu^{(k_0)}$, (which we call “counterterm”).[3] Such a constant, however, must be taken so that the equation for the angle variables, *i.e.* the second of (1.7), has zero average, in order to be soluble. Hence the equation for $\vec h^{(k)}$ can be solved and its solution is a trigonometric polynomial in $\vec\psi$, defined up to a constant: such a constant has to be chosen to be vanishing so that $\vec h^{(k_0)}$ is odd in t and the procedure can be iterated. Therefore the equations for $\vec\mu^{(k)}$ will have to be obtained recursively by imposing that, for all k's, the average over $\vec\psi$ of the r.h.s of the second equation in (1.7) is identically vanishing and requiring $\vec h^{(k)}_{\vec 0}\equiv\vec 0$, $\forall k$: then the trigonometric polynomial $\vec h^{(k)}(\vec\psi)$ will be completely determined (if possible at all) from the second of (1.7).

If, given a function $F(\vec\psi)$, we define by $F_{\vec\nu}$ its $\vec\nu$-th Fourier series component,

$$F_{\vec\nu}=\int_{\mathbf T^l}\frac{d\vec\psi}{(2\pi)^2}\,F(\vec\psi)\,e^{-i\vec\nu\cdot\vec\psi}\ ,\qquad\qquad F(\vec\psi)=\sum_{\vec\nu}F_{\vec\nu}\,e^{i\vec\nu\cdot\vec\psi}\ ,\tag{1.8}$$

one easily finds, for $k=1$, from (1.6)

$$\begin{aligned}h_{j\vec\nu}^{(1)}&=\big(-iJ^{-1}\vec\nu\big)_j\,[i\vec\omega_0\cdot\vec\nu]^{-2}\,f_{\vec\nu}(\vec A_0)+[i\vec\omega_0\cdot\vec\nu]^{-1}\,\partial_{A_j}f_{\vec\nu}(\vec A_0)\ ,\qquad\vec\nu\ne\vec 0\ ,\\H_{j\vec\nu}^{(1)}&=(-i\vec\nu_j)\,[i\vec\omega_0\cdot\vec\nu]^{-1}\,f_{\vec\nu}(\vec A_0)\ ,\qquad\vec\nu\ne\vec 0\ ,\\\mu_j^{(1)}&=-(J\partial_{\vec A})_jf_{\vec 0}(\vec A)\ .\end{aligned}\tag{1.9}$$

[3] Note that $\vec H^{(k)}$ has to have zero average over $\vec\psi$ by construction.

The (1.7) provides an algorithm to evaluate a formal power series solution to our problem. It has been remarked, [3], [4], [14], that (1.7) yields a *diagrammatic expansion* of $\vec{h}^{(k)}$, $\vec{H}^{(k)}$ and $\vec{\mu}^{(k)}$, (we simply "iterate" it until only $\vec{h}^{(1)}$ and $\vec{H}^{(1)}$, given by (1.9), and $\vec{\mu}^{(k')}$, $k'<k$, appear).

This paper is organized in the following way. In §2 the diagrammatic expansion for $\vec{h}$, $\vec{H}$ and $\vec{\mu}$ is defined by using the Feynman's graphs: like in field theory, the introduction of the Feynman's graphs is very useful in order to study a perturbative expansion. In §3 we introduce a class of *form factors* which are formal series in ε with the property that the expansion is convergent if such form factors are bounded. In §4 the boundedness of the form factors is verified by using some cancellation mechanisms in the perturbative series for them.

The introduction of the form factors could be avoided, but we prefer to proceed in this way in order to make more striking the connection with the renormalization methods in field theory in which the notion of form factors is widely used.

2. Diagrammatic expansion

It is convenient to use dimensionless quantities. Therefore we shall set

$$\vec{A}=(J_m/C_0)\,\vec{B}\;,\qquad \vec{\alpha}\equiv\vec{\beta}\;,$$

$$\mu_j^{(k)}=(J_m/C_0)\,\bar{\mu}_j^{(k)}\;,\qquad H_j^{(k)}=(J_m/C_0)\,\bar{H}_j^{(k)}\;,\qquad h_j^{(k)}\equiv\bar{h}_j^{(k)}\;,$$

$$f(\vec{\alpha},\vec{A})=F\,\varphi(\vec{\beta},\vec{B})\;,\qquad \vec{\omega}_0=C_0^{-1}\,\vec{\omega}\;,\qquad J/J_m=\eta_m\;,\qquad J/J_M=\eta_M\;,$$

where F, J_m and J_M are defined in §1, and C_0 is the diophantine constant introduced in (1.2). The dimensionless matrices η_m and η_M are so defined that $\|\eta_m^{-1}\|\le 1$ and $\|\eta_M\|\le 1$, where $\|\cdot\|$ denotes the usual matrix norm. In general the dimensionless quantities are all defined in terms of the relevant physical constants appearing in the theory, *i.e.* J_m, J_M, C_0 and F.

In terms of the above introduced dimensionless quantities and expressed in the Fourier space, the equations (1.6) and (1.7) and the right after discussion, give, for $\vec{\nu}\neq\vec{0}$,

$$\begin{aligned}
(i\vec{\omega}\cdot\vec{\nu})\,\bar{H}_{j\vec{\nu}}^{(k)} &= -\Big(\frac{F\,C_0^2}{J_m}\Big)\sum_{m>0}\ \sum_{\substack{p_1,\ldots,p_l,q_1,\ldots,q_l\\ \sum_{s=1}^{l}(p_s+q_s)=m}}\ \sum{}^{*}\ \frac{1}{\prod_{s=1}^{l}p_s!\,q_s!}\;\cdot\\
&\quad\cdot\ (i\vec{\nu}_0)_j\prod_{j=1}^{l}(i\vec{\nu}_0)_j^{p_j}\prod_{i=1}^{l}\partial_{B_i}^{q_i}\varphi_{\vec{\nu}_0}(\vec{B}_0)\;\cdot\\
&\quad\cdot\ \prod_{s=1}^{l}\Big[\prod_{j=1}^{p_s}\bar{h}_{s,\vec{\nu}_{sj}}^{(k_{sj})}\prod_{i=1}^{q_s}\Big(\bar{H}_{s,\vec{\nu}'_{si}}^{(k'_{si})}+\bar{\mu}_{s,\vec{\nu}'_{si}}^{(k'_{si})}\Big)\Big]\;,\\
(i\vec{\omega}\cdot\vec{\nu})^2\,\bar{h}_{j\vec{\nu}}^{(k)} &= (i\vec{\omega}\cdot\vec{\nu})\,(\eta_m^{-1}\bar{H}_{\vec{\nu}}^{(k)})_j\\
&\quad+(i\vec{\omega}\cdot\vec{\nu})\Big(\frac{F\,C_0^2}{J_m}\Big)\sum_{m>0}\ \sum_{\substack{p_1,\ldots,p_l,q_1,\ldots,q_l\\ \sum_{s=1}^{l}(p_s+q_s)=m}}\ \sum{}^{*}\ \frac{1}{\prod_{s=1}^{l}p_s!\,q_s!}\;\cdot\\
&\quad\cdot\ \partial_{B_j}\prod_{j=1}^{l}(i\vec{\nu}_0)_j^{p_j}\prod_{i=1}^{l}\partial_{B_i}^{q_i}\varphi_{\vec{\nu}_0}(\vec{B}_0)\;\cdot\\
&\quad\cdot\ \prod_{s=1}^{l}\Big[\prod_{j=1}^{p_s}\bar{h}_{s,\vec{\nu}_{sj}}^{(k_{sj})}\prod_{i=1}^{q_s}\Big(\bar{H}_{s,\vec{\nu}'_{si}}^{(k'_{si})}+\bar{\mu}_{s,\vec{\nu}'_{si}}^{(k'_{si})}\Big)\Big]\;,
\end{aligned}\tag{2.1}$$

and, for $\vec{\nu}=\vec{0}$,

$$
\begin{aligned}
\bar{\mu}_j^{(k)} = \Big(\frac{F\,J_M\,C_0^2}{J_m^2}\Big) \sum_{m>0} \sum_{\substack{p_1,\dots,p_l,q_1,\dots,q_l\\ \sum_{s=1}^{l}(p_s+q_s)=m}} & \sum\nolimits^{*} \frac{1}{\prod_{s=1}^{l} p_s!\, q_s!} \cdot \\
\cdot\ (\eta_M\partial_{\vec{B}})_j \prod_{j=1}^{l}(i\vec{\nu}_0)_j^{p_j} & \prod_{i=1}^{l} \partial_{B_i}^{q_i}\varphi_{\vec{\nu}_0}(\vec{B}_0)\,\cdot \\
\cdot \prod_{s=1}^{l}\Big[\prod_{j=1}^{p_s} \bar{h}_{s,\vec{\nu}_{sj}}^{(k_{sj})} & \prod_{i=1}^{q_s}\Big(\bar{H}_{s,\vec{\nu}'_{si}}^{(k'_{si})}+\bar{\mu}_{s,\vec{\nu}'_{si}}^{(k'_{si})}\Big)\Big]\,,
\end{aligned}
\tag{2.2}
$$

where the $\sum^*$ denotes summation over the integers $k_{sj}\ge 1$, $k'_{si}\ge 1$, with: $\sum_{s=1}^{l}(\sum_{j=1}^{p_s}k_{sj}+\sum_{i=1}^{q_s}k'_{si})=k-1$, and over the integers $\vec{\nu}_0$, $\vec{\nu}_{sj}$, $\vec{\nu}'_{si}$, $s=1,\dots,l$, $j=1,\dots,p_s$ and $i=1,\dots,q_s$, with: $\vec{\nu}_0+\sum_{s=1}^{l}(\sum_{j=1}^{p_s}\vec{\nu}_{sj}+\sum_{i=1}^{q_s}\vec{\nu}'_{si})=\vec{\nu}$.

For the time being we ignore the presence of the "constant part" of the angular momenta, *i.e.* $\vec{\mu}^{(k)}$, $k\ge 1$, *i.e.* we reason as if it was $\vec{\mu}^{(k)}=\vec{0}$, $\forall k\ge 1$: we shall see below how the discussion has to be modified when also such terms are taken into account.

A tree diagram ϑ will consist of a family of lines (*branches* or *lines*) arranged to connect a partially ordered set of points (*vertices* or *nodes*), with the higher vertices to the right. The branches are naturally ordered as well; all of them have two vertices at their extremes, (possibly one of them is a top vertex), except the lowest or *first branch* which has only one vertex, the *first vertex* v_0 of the tree. The other extreme r of the first branch will be called the *root* of the tree and will not be regarded as a vertex; we shall call the first branch also *root branch*. If v_1 and v_2 are two vertices of the tree we say that $v_1<v_2$ if v_2 follows v_1 in the order established by the tree, *i.e.* if one has to pass v_1 before reaching v_2, while climbing the tree. Since the tree is partially ordered not every pair of vertices will be related by the order relation: we say that two vertices are comparable if they are related by the order relation (which we are denoting $\le$). Given a vertex v we denote by $\pi(v)$ the vertex immediately preceding v, *i.e.* the vertex from which the branch leading to v comes out. Given a tree ϑ with first vertex v_0, each vertex $v>v_0$ can be considered the first vertex of the tree consisting of the vertices following v: such a tree will be called a subtree of ϑ.

To each branch λ_v and to each vertex v we attach a finite set of labels: ζ_{λ_v}, R_{λ_v}, j_{λ_v}, and, respectively, δ_v, m_v, k_v, $\vec{\nu}_v$, and a *vertex function* $\mathcal{E}_v$, which are defined as follows.
(1) The label ζ_{λ_v} can assume the symbolic values $\zeta_{\lambda_v}=h,H$;
(2) $R_{\lambda_v}=1$ if $\zeta_{\lambda_v}=H$ and $R_{\lambda_v}=2$ if $\zeta_{\lambda_v}=h$, and it is called the *degree* of the line λ_v;
(3) $j_{\lambda_v}=1,\dots,l$;
(4) $\delta_v=0,1$, if $\zeta_{\lambda_v}=h$, and it is identically 0 if $\zeta_{\lambda_v}=H$;
(5) m_v is the number of branches emerging from v, and can be written as

$$
m_v=\sum_{i=1}^{3}\sum_{j=1}^{l} q_{v,j}^{i}\,,
$$

if $q_{v,j}^i$, $i=1,2,3$, are the branches leading to vertices w with $\pi(w)=v$ and carrying a ζ_{λ_w} label equal, respectively, to h, H, μ;
(6) the *order label* k_v is given by the number of vertices of the subtree with first vertex v;
(7) the *mode label* $\vec{\nu}_v$ is such that $\vec{\nu}_v\in\mathbf{Z}^l$, $|\vec{\nu}_v|\le N$;
(8) the vertex function is defined as

$$
\begin{aligned}
\mathcal{E}_v = & \frac{F\,C_0^2}{J_m}\Big\{\Big[\Big((-i(\eta_m)^{-1}\vec{\nu}_v)_{j_{\lambda_v}}(1-\delta_v)+(i\vec{\omega}\cdot\vec{\nu}(v))\partial_{B_{j_{\lambda_v}}}\delta_v\Big)\delta_{\zeta_{\lambda_v},h} \\
& +(-i\vec{\nu}_v)_{j_{\lambda_v}}\delta_{\zeta_{\lambda_v},H}\Big]\cdot \\
& \cdot \prod_{\substack{w:\pi(w)=v\\ \zeta_{\lambda_w}=h}} (i\vec{\nu}_v)_{j_{\lambda_w}} \prod_{\substack{w:\pi(w)=v\\ \zeta_{\lambda_w}=H}} \partial_{B_{j_{\lambda_w}}}\Big\}\,\varphi_{\vec{\nu}_v}(\vec{B})\Big|_{\vec{B}=\vec{B}_0}\,,
\end{aligned}
\tag{2.3}
$$

where $\delta_{\zeta_{\lambda_v},x}$ denotes the Kronecker's delta (which is 1 only if $\zeta_{\lambda_v}=x$), and the last m_v factors are missing if v is a top vertex, (in this case $m_v=0$). Each of the m_v+1 factors appearing in (2.3) will be called a *vertex factor*.

The labels, "decorating" the tree, will be called also decorations (*i.e.* the labels attached to the tree and the decorations are synonimous below). Finally to the branch λ_v leading from the vertex $\pi(v)$ to the vertex v we associate a "propagator" $g(\vec\omega\cdot\vec\nu_{\lambda_v})$, which is given by:

$$g(\vec\omega\cdot\vec\nu_{\lambda_v})=\frac{1}{[i\vec\omega\cdot\vec\nu_{\lambda_v}]^{R_{\lambda_v}}}\;, \tag{2.4}$$

where $\vec\nu_{\lambda_v}=\sum_{w\ge v}\vec\nu_w$ is defined as the *momentum* entering v, and R_{λ_v} is called the *degree of the line*. A possible tree is drawn in Fig.2.1.

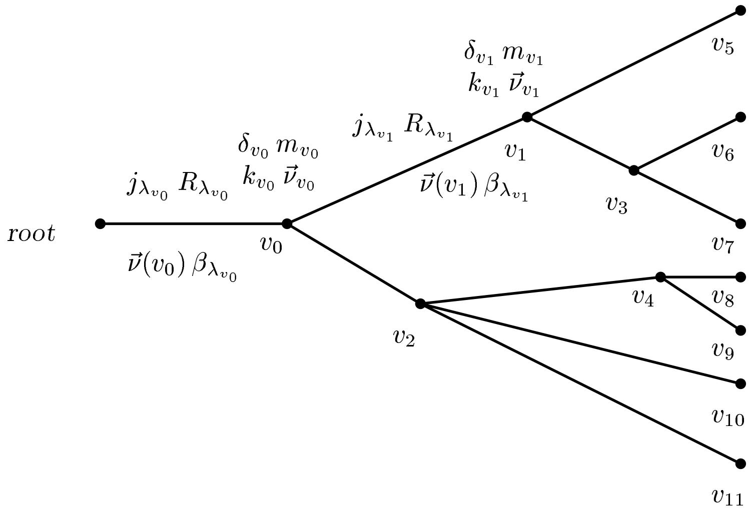

Fig.2.1: A tree ϑ with $m_{v_0}=2, m_{v_1}=2, m_{v_2}=3, m_{v_3}=2, m_{v_4}=2$ and $m=12$, $\prod_v m_v!=2^4\cdot 6$, and some decorations. The line numbers, distinguishing the lines, are not shown.

We imagine that all the tree lines have the same length (even though they are drawn with arbitrary length in Fig.2.1). A group acts on the set of trees, generated by the permutations of the subtrees having the same root. Two diagrams that can be superposed by the action of a transformation of the group will be regarded as identical, (the superpositon has to be such that all the decorations of the diagram match). To order k, not considering the decorations, the number of trees is bounded by 2^{2k}. We shall imagine that each branch carries also an arrow pointing to the root ("gravity" direction, opposite to the order).

Then, if $X^{(k)}_{j\vec\nu}(h)=\bar h^{(k)}_{j\vec\nu}$ and $X^{(k)}_{j\vec\nu}(H)=\bar H^{(k)}_{j\vec\nu}$, it follows that (2.1) can be rewritten as

$$X^{(k)}_{j\vec\nu}(\zeta)=\sum_{\vartheta}{}^{*}\prod_{v_0\le v\in\vartheta}\prod_{j=1}^{l}\frac{1}{q^1_{v,j}!q^2_{v,j}!}\,g(\vec\omega\cdot\vec\nu_{\lambda_v})\,\mathcal{E}_v\;, \tag{2.5}$$

where the sum runs over all the tree ϑ's of order k, with $\vec\nu_{\lambda_{v_0}}=\vec\nu$, $j_{\lambda_{v_0}}=j$ and $\zeta_{\lambda v_0}=\zeta$, and the $*$ recalls that the diagram ϑ can and will be supposed such that $\vec\nu_{\lambda_v}\ne\vec 0$ for all $v\in\vartheta$, (by the parity properties remarked in §1 and because the counterterms are assumed to be zero, so that $q^3_{v,j}\equiv 0$).

If also the contributions $\bar\mu^{(k)}_j$'s are considered, the above discussion has to be modified as follows. Some of the top vertices v are drawn as fat points, and represent the quantities $\bar\mu^{(k_v)}_{j_v}$: the corresponding ζ_{λ_v} label is set $\zeta_{\lambda_v}=\mu$, and the label k_v is such that $k_v<k_{\pi(v)}$; no other label but j_v is associated to such a vertex, and no propagator is associated to the branch leading to it.

The vertices which are not fat points will be called *free vertices*, while the fat points will be called *fruits*. Finally, given a fruit v, we define k_v the *fruit order*.

Then, for each $v > v_0$, which is not a top vertex, $\zeta_{\lambda_v} \neq \mu$, and (2.3) has to be replaced with

$$\begin{aligned}\mathcal{E}_v =& \frac{F\,C_0^2}{J_m}\Big\{\Big[\Big((-i(\eta_m)^{-1}\vec{\nu}_v)_{j_{\lambda_v}}(1-\delta_v) + (i\vec{\omega}\cdot\vec{\nu}_{\lambda_v})\partial_{B_{j_{\lambda_v}}}\,\delta_v\Big)\delta_{\zeta_{\lambda_v},h}\\ &+ (-i\vec{\nu}_v)_{j_{\lambda_v}}\delta_{\zeta_{\lambda_v},H}\Big]\cdot\\ &\cdot\prod_{\substack{w:\pi(w)=v\\ \zeta_{\lambda_w}=h}}(i\vec{\nu}_v)_{j_{\lambda_w}}\prod_{\substack{w:\pi(w)=v\\ \zeta_{\lambda_w}=H}}\partial_{B_{j_{\lambda_w}}}\prod_{\substack{w:\pi(w)=v\\ \zeta_{\lambda_w}=\mu}}\partial_{B_{j_{\lambda_w}}}\Big\}\,\varphi_{\vec{\nu}_v}(\vec{B})\Big|_{\vec{B}=\vec{B}_0}\,,\end{aligned} \tag{2.6}$$

while, if $v > v_0$ is a top vertex,

$$\begin{aligned}\mathcal{E}_v =& \frac{F\,C_0^2}{J_m}\Big\{\Big[\Big((-i(\eta_m)_j^{-1}\vec{\nu}_v)_{j_{\lambda_v}}(1-\delta_v) + (i\vec{\omega}\cdot\vec{\nu}(v))\partial_{B_{j_{\lambda_v}}}\,\delta_v\Big)\delta_{\zeta_{\lambda_v},h}\\ &+ (-i\vec{\nu}_v)_{j_{\lambda_v}}\delta_{\zeta_{\lambda_v},H} + \bar{\mu}^{(k_v)}_{j_{\lambda_v}}\delta_{\zeta_{\lambda_v},\mu}\Big]\Big\}\,\varphi_{\vec{\nu}_v}(\vec{B})\Big|_{\vec{B}=\vec{B}_0}\,,\end{aligned} \tag{2.7}$$

Then a formula analogous to (2.5) still holds, with the constraint that the label k is given by the number of the free vertices plus the sum of the orders k_v of all the leaves $v \in \vartheta$:

$$X^{(k)}_{j\vec{\nu}}(\zeta) = \sum_{\vartheta}{}^{*}\prod_{v_0\le v\in\vartheta}\prod_{i=1}^{3}\prod_{j=1}^{l}\frac{1}{q^i_{v,j}!}\,g(\vec{\omega}\cdot\vec{\nu}_{\lambda_v})\,\mathcal{E}_v\,, \tag{2.8}$$

where the sum runs over all the trees of order k, (with $\vec{\nu}_{\lambda_{v_0}} = \vec{0}$ and $j_{\lambda_{v_0}} = j$), having $k_0(\vartheta)$ free vertices and $\mathcal{N}_{\mathcal{F}}(\vartheta)$ fruits v_i of order k_i, $i = 1,\dots,\mathcal{N}_{\mathcal{F}}(\vartheta)$, with the constraint that their orders add to $k - k_0(\vartheta)$.

If $\zeta_{\lambda_{v_0}} = \mu$, then (2.8) has to be replaced by the following equation:

$$\bar{\mu}^{(k)}_j = -\Big(\frac{J_M\,F\,C_0^2}{J_m^2}\Big)\sum{}^{*}(\eta_M\partial_{\vec{B}})_j\,\varphi_{\vec{\nu}}(\vec{B}_0)\prod_{v_0<v\in\vartheta}\prod_{i=1}^{3}\prod_{j=1}^{l}\frac{1}{q^i_{v,j}!}\,g(\vec{\omega}\cdot\vec{\nu}_{\lambda_v})\,\mathcal{E}_v\,, \tag{2.9}$$

Note that (2.8) and (2.9) can depend on $\bar{\mu}^{(k')}_{j'}$, only if $k' < k$, so that the coefficients $\bar{\mu}^{(k)}_j$ are recursively defined.

A tree decorated also with leaves is drawn in Fig.2.2. Note that no propagators are associated to the branches leading to the fruits.

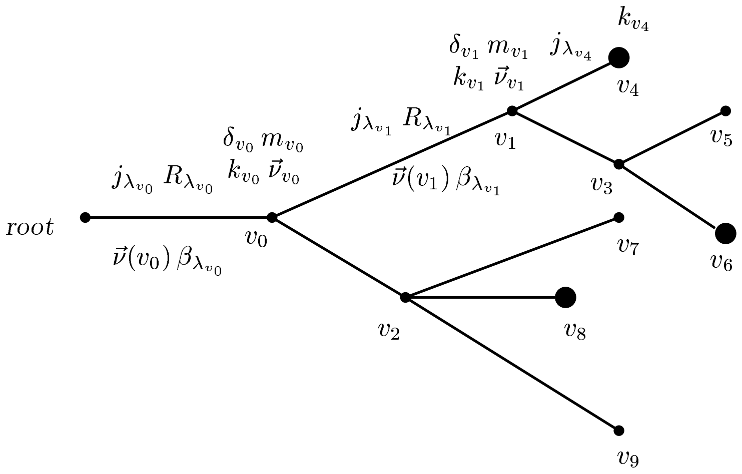

Fig.2.2: A tree ϑ with $\mathcal{N}_{\mathcal{F}}(\vartheta) = 3$ fruits, $m_{v_0} = 2, m_{v_1} = 2, m_{v_2} = 3, m_{v_3} = 2$ and some decorations; the branch label is defined to be $j_{\lambda_{v_0}} = j$. Each fat point represents a fruit.

¿From (2.8) and (2.9) we deduce that each tree can be considerd as representing a contribution to $X^{(k)}_{j\vec{\nu}}(\zeta)$, $\zeta=h,H$, or $\mu^{(k)}_j$: such a contribution will be called the *value of the tree*. Then, if v_0 is the first vertex of the tree, and $\zeta_{\lambda_{v_0}}=h$, the value of the label δ_{v_0} tells us which term we are considering among the two of the first equation in (2.1); the argument can be repeated for each following vertex. The interpretation of the other labels is obvious.

3. Dimensional bounds

The following Lemma shows that the estimates for the KAM tori can be reduced to the study of the contributions of the fruitless trees. The proof can be found in Appendix A1.

Lemma 3.1. *Suppose that we can prove that the contribution to $\bar\mu^{(k)}_j$ and the one to $X^{(k)}_{j\vec{\nu}}(\zeta)$, $\zeta=h,H$, arising from a single tree stripped of the fruits, (*i.e. *the contribution we obtain by deleting the fruits from the tree), is bounded by $D_1^{k_0}$ for some constant D_1, if k_0 is the number of the free vertices; then a bound B_0^k for the complete values $|\bar\mu^{(k)}_j|$ and $|X^{(k)}_{j\vec{\nu}}(\zeta)|$, $\zeta=h,H$, follows immediately, for a suitable constant $B_0=2^3\rho^{-1}B_1D_1$, being ρ defined after (1.3) and $B_1=2^5l(2N+1)^l$.*

Remark 1. The value of B_1 is found below, and is due to the trees counting, (see in particular the discussion after (3.5)): B_1 counts the possible decorated fruitless trees.

Remark 2. Note that, given a fruitless tree, contributing to $\bar\mu^{(k)}_j$, the bound we obtain for it contains an extra factor J_M/J_m with respect to the same bound we would obtain if it had contributed to $X^{(k)}_j(\zeta)$, $\zeta=h,H$, as a comparison between (2.2) and the first equation in (2.1) shows. Therefore we can confine ourselves to study the contributions to $X^{(k)}_{j\vec{\nu}}(\zeta)$, $\zeta=h,H$, arising only from fruitless trees, and, if a bound $\tilde B_0$ is found for them, then it will be $B_0=J_MJ_m^{-1}\tilde B_0$.

In order to bound the fruitless tree values, we introduce a multiscale decomposition of the propagator. Let $\chi(x)$ be a C^∞ not increasing function such that $\chi(x)=0$, if $|x|\ge 2$ and $\chi(x)=1$ if $|x|\le 1$, and let $\chi_n(x)=\chi(2^{-n}x)-\chi(2^{-(n-1)}x)$, $n\le 0$, and $\chi_1(x)=1-\chi(x)$: such functions realize a C^∞ partition of unity, for $x\in[0,\infty)$, in the following way. Let us write

$$1=\chi_1(x)+\sum_{n=-\infty}^{0}\chi_n(x)\equiv\sum_{n=-\infty}^{1}\chi_n(x)\ . \tag{3.1}$$

Then we can decompose the propagator in the following way:

$$g(\vec\omega\cdot\vec\nu_{\lambda_v})=\frac{1}{[i\vec\omega\cdot\vec\nu_{\lambda_v}]^{R_{\lambda_v}}}\equiv\sum_{n=-\infty}^{1}\frac{\chi_n(x)}{[i\vec\omega\cdot\vec\nu_{\lambda_v}]^{R_{\lambda_v}}}\equiv\sum_{n=-\infty}^{1}g^{(n)}(\vec\omega\cdot\vec\nu_{\lambda_v}) \tag{3.2}$$

where $g^{(n)}(\vec\omega\cdot\vec\nu_{\lambda_v})$ is the "propagator at scale n". If $n<0$, $g^{(n)}(\vec\omega\cdot\vec\nu_{\lambda_v})$ is a C^∞ compact support function different from 0 for $2^{n-1}<|\vec\omega\cdot\vec\nu_{\lambda_v}|\le 2^{n+1}$, while $g^{(1)}(\vec\omega\cdot\vec\nu_{\lambda_v})$ has support for $1<|\vec\omega\cdot\vec\nu_{\lambda_v}|$. In the domain where it is different from zero, the propagator verifies the bound

$$\Big|\frac{\partial^p}{\partial x^p}g^{(n)}(x)\Big|_{x=\vec\omega\cdot\vec\nu_{\lambda_v}}\le a_{R_{\lambda_v}}(p)2^{-n(R_{\lambda_v}+p)}\ ,\qquad p\in\mathbf{N}\ ,$$

where $a_{R_{\lambda_v}}(p)$ is a suitable constant, such that $a_{R_{\lambda_v}}(0)=2^{R_{\lambda_v}}$, which depends on the form of the function $\chi(x)$.[4]

Proceeding as in quantum field theory, see [5], given a tree ϑ we can attach a *scale label* n_{λ_v} to each branch λ_v in ϑ, which is equal to the scale of the propagator associated to the branch via (2.4) and (2.2).

[4] The consatnt $a_{R_{\lambda_v}}(p)$ has a bad dependence on p, but we shall see that in our bounds p does not increase ever beyond 2: see in particular (4.4) below and, especially, the right after remark.

Looking at such labels we identify the connected cluster T of vertices which are linked by a continuous path of branches with the same scale labels n_T or a higher one and which are maximal: we shall say that *the cluster* T *has scale* n_T. Therefore an inclusion relation is established between the clusters, in such a way that the innermost clusters are the clusters with the highest scale, and so on.

Each cluster can have an arbitrary number of branches entering it, (*incoming lines*), but only one line exiting, (*outgoing lines*); we use that the branches carry an arrow pointing to the root: this gives a meaning to the words "incoming" and "outgoing".

The multiscale decomposition (3.2) of the propagator allows us to rewrite (2.5) as

$$X_{j\vec{\nu}}^{(k)}(\zeta) = \sum_{\vartheta}^{*} \prod_{v_0 \le v \in \vartheta} \prod_{j=1}^{l} \prod_{i=1}^{3} \frac{1}{q_{v,j}^{i}!} g^{(n_{\lambda_v})}(\vec{\omega}\cdot\vec{\nu}_{\lambda_v})\, \mathcal{E}_v \;, \tag{3.3}$$

where the sum is over the labeled trees, and therefore, with respect to (2.8), we have the extra labels n_λ associated to the lines λ's: a (not optimal bound) of the number of terms appearing in the sum over the $\vec{\nu}_v$ and n_v labels is given by $[2(2N+1)^l]^k$, as $|\nu_i| \le N$, $\forall i = 1, \ldots, l$, and because of the support compact property of the propagators.

Definition 3.1 (Resonance). *Among the clusters we consider the ones with the property that there is only one incoming line, carrying the same momentum of the outgoing line, and we define them* resonances. *If* V *is one such cluster we denote by* λ_V *the incoming line and* $K(V)$ *the number of vertices contained in* V (resonance order). *The line scale* $n = n_{\lambda_V}$ *is lower than the lowest scale* $n' = n_V$ *of the lines inside* V*: we call* n_{λ_V} *the* resonance-scale, *and* λ_V *a* resonant line.

Remark. Note that a resonance V, as a cluster, has an its own scale n_V, which is higher than its resonance-scale n_{λ_V}, $n_V \ge n_{\lambda_V} + 1$.

Recall that R_λ is the *degree* of the line λ: it is the exponent of the propagator corresponding to the line, (see (2.2)). Given a resonance V, let us define the *resonance degree* $D_V = 1, 2$ as the degree $R_{\lambda_V} = 1, 2$ of the resonant line, *i.e.* $D_V = R_{\lambda_V}$.

Given a tree ϑ, let us define N_n the number of lines with scale $n \le 0$, and N_n^j, $j = 1, 2$, the number of lines λ with scale $n \le 0$ and having $R_\lambda = j$, (*i.e.* with degree j). Then it is easy to check that the scaling properties of the propagators and the definitions (2.2) and (2.5) immediately imply that the contribution to $\vec{X}^{(k)}(\zeta)$ arising from a given tree ϑ can be bounded by:

$$\mathcal{C}^k \prod_{n \le 0} 2^{-(2nN_n^2 + nN_n^1)} \prod_{v} 2^{n_{\lambda_v}\delta_v} \;, \tag{3.4}$$

for a suitable constant $\mathcal{C}$; if we look at (2.6), we can write

$$\mathcal{C} = 2^2 J_m^{-1} C_0^2 N^2 F \rho^{-1} \;, \tag{3.5}$$

where C_0 is the diophantine constant introduced in (1.3), ρ is introduced after (1.3), N is a bound on the mode values $\vec{\nu}_v$ of the vertices $v \in \vartheta$, and the factor 2^2 is due to the definition of the compact support of the propagators. The last product in (3.4) arises because for each line λ_v of degree 2 such that $\delta_v = 1$ there is an extra factor $(i\vec{\omega}\cdot\vec{\nu}(v))$, (see (2.6)).

Therefore we only have to multiply (3.4) by the number of diagram shapes for ϑ having vertices with given bifurcation numbers m_v, $v \in \vartheta$, ($\le 2^{2k}$, see [10]), by the number of ways of attaching the labels, ($\le [2 \cdot 2 \cdot 2 \cdot l \cdot (2N+1)^l]^k$), so that the number of trees of order k can be bounded by B_1^k, if B_1 is defined as in Lemma 3.1.

The following bound holds for the number of lines with scale $n \le 0$:

$$N_n^1 + N_n^2 \equiv N_n \le \frac{2k}{E 2^{-n\tau^{-1}}} + \sum_{T, n_T = n} \sum_{j=1}^{2} m_T^j \;, \tag{3.6}$$

where m_T^j is the number of resonances V's inside the cluster T, having resonance-scale $n_{\lambda_V} = n_T$ and degree $D_V = j$, and E can be chosen $E = 2^{-3\tau^{-1}} N^{-1}$. This is a slightly modified version of the Brjiuno's lemma, [2]: a proof is in Appendix A2, and it is taken from [4], [9], with some minor changes.

Therefore if there was no resonance, *i.e.* if it was $m_T^j = 0$ for any T, then we would obtain a (not optimal) bound G_0^k for a suitable constant $G_0 > 0$; the labeled trees counting and (3.5) give

$$G_0 = b_1 2^7 l (2N+1)^l F \rho^{-1} J_m^{-1} C_0^2 N^2 \ , \tag{3.7}$$

where $b_1 = \exp[\sum_{n=1}^{\infty} 4(\ln 2) n E^{-1} 2^{-n\tau^{-1}}]$.

However *there are resonances*, and we have to deal with them.

Definition 3.2 (Resonance factor). *Let us consider a resonance V; let λ_V be the incoming line, as in Definition 3.1. We denote by w_0 the vertex which the outgoing line of V leads to, (recall that the ordering of the tree is opposite to the gravity direction), and by w_2 the vertex which the incoming line (resonant line) leads to: such lines are characterized by the labels, respectively, $\zeta_{\lambda_{w_0}}$ and $\zeta_{\lambda_{w_2}}$. Such a couple of values $(\zeta_{\lambda_{w_0}}, \zeta_{\lambda_{w_2}})$ can assume the values (H,H), (H,h), (h,H) and (h,h), and we can introduce a label s_V in order to distinguish the four above possible cases. Let us define the* resonance factor $\mathcal{V}_{s_V, j_{\lambda_{w_0}} j_{\lambda_{w_2}}}(\vec\omega \cdot \vec\nu_{\lambda_V}))$ *as the quantity*

$$\mathcal{V}_{s_V, j_{\lambda_{w_0}} j_{\lambda_{w_2}}}(\vec\omega \cdot \vec\nu(v_{\lambda_V})) = \mathcal{E}_{w_0} \prod_{w_0 < v \in V} g(\vec\omega \cdot \vec\nu_{\lambda_v})\, \mathcal{E}_v \ , \tag{3.8}$$

where the product is over all the $k(V) - 1$ vertices inside the resonance different from the vertex w_0 to which the outgoing line leads, and the value of the label $s_V = 1, \dots, 4$ is uniquely determined by the kind of resonance. The resonance order $k(V)$ is introduced in Definition 3.1.

Remark. Note that, given a resonance V, one has $\sum_{w \in V} \vec\nu_w = \vec 0$; moreover we can change the signs of the mode labels of all the vertices inside the resonance, *simultaneously*, without changing the values of all the other factors in (3.2) corresponding to vertices and lines outside V.

Definition 3.3 (Form factor). *Let us define the form factor $\sigma^n_{s,jj'}(\varepsilon; \vec\omega\cdot\vec\nu)$, $2^{n-1} < |\vec\omega\cdot\vec\nu| \le 2^{n+1}$, $j = 1, \dots, 4$, as*

$$\sigma^n_{s,jj'}(\varepsilon; \vec\omega\cdot\vec\nu) = \sum_{k=2}^{\infty} \varepsilon^k \sigma^{n(k)}_{s,jj'}(\varepsilon; \vec\omega\cdot\vec\nu) = \sum_{k=2}^{\infty} \varepsilon^k \sum_{\substack{V : \vec\nu_{\lambda_V} = \vec\nu \\ k(V) = k, n_{\lambda_V} = n}} \mathcal{V}_{s,jj'}(\vec\omega\cdot\vec\nu) g^{(n)}(\vec\omega\cdot\vec\nu) \ , \tag{3.9}$$

where j and j' are the j-labels, respectively, of the lines outgoing from and incoming into the resonance, and are supposed to be fixed, and $g^{(n)}(\vec\omega\cdot\vec\nu)$ is the propagator of the line entering *the resonance; note that $n_V \ge n+1$. Let us set*

$$\sigma \equiv \sigma(\varepsilon) = \max_{s=1,\dots,4} \max_{n \le 0} \left\{ \max_{\vec\nu} \|\sigma^n_s(\varepsilon; \vec\omega\cdot\vec\nu)\| \right\} \ , \tag{3.10}$$

where $\|\cdot\|$ denotes the matrix norm, and the maximum on $\vec\nu$ in (3.10) is over all the values of $\vec\nu$ such that $\vec\omega\cdot\vec\nu$ is on scale n.

If we take into account the resonances and bound term by term the sum (3.3), we do not find a convergent series, since some factorials appear. Then we look for a different arrangement of the series, performing some particular resummations in (3.3), aiming to single out the contributions which can give problems and so need a more careful analysis. This procedure is usual in field theory and it is called *renormalization*: we obtain a series over trees similar to those introduced before but in which *there are no resonances* and, on the contrary, the propagators are *dressed*. This means that to each line λ we associate a propagator of the form

$$\bar g^{(n_\lambda)}_{s,jj'}(\varepsilon; \vec\omega\cdot\vec\nu_\lambda) = g^{(n_\lambda)}(\vec\omega\cdot\vec\nu_\lambda)[(1 - \sigma^{n_\lambda}_s(\varepsilon; \vec\omega\cdot\vec\nu_\lambda))^{-1}]_{jj'} \ . \tag{3.11}$$

so that to such a line there correspond two labels $j_\lambda = j$ and $j'_\lambda = j'$, which have to be contracted with the labels of two vertex factors corresponding, respectively, to the vertex into which the line enters and to the vertex from which it exits. We set also $s_V = s_{\lambda_V}$.

We call the new trees *resummed trees*, and we denote by $\mathcal{T}_k$ the set of resummed trees of order k, (*i.e.* with k vertices), and with k dressed propagators. *Note that the resummed trees do not contain resonances.* The series for $X_{j\vec{\nu}}(\zeta)$ is given by

$$X_{j\vec{\nu}}(\zeta) = \sum_{k=1}^{\infty} \sum_{\bar\vartheta\in\mathcal{T}_k}{}^{*} \bar{X}^k_{j\vec{\nu}}(\bar\vartheta)\,\varepsilon^k \prod_{\lambda\in\bar\vartheta} [(1-\sigma_{s_\lambda}(\varepsilon;\vec{\omega}\cdot\vec{\nu}_\lambda))^{-1}]_{j_\lambda j'_\lambda} \;, \tag{3.12}$$

where the second sum runs over all the resummed trees $\bar\vartheta$ of order k, and with $\vec{\nu}_{\lambda_{v_0}} = \vec{\nu}$, $j_{\lambda_{v_0}} = j$ and $\zeta_{\lambda_{v_0}} = \zeta$, and $\bar{X}^k_{j\vec{\nu}}(\bar\vartheta)$ is given by

$$\bar{X}^k_{j\vec{\nu}}(\bar\vartheta) = \prod_{v_0\le v\in\bar\vartheta} g^{(n_{\lambda_v})}(\vec{\omega}\cdot\vec{\nu}_{\lambda_v})\,\mathcal{E}_v \;.$$

In the following we want to prove that the series (3.12) is dominated by a series of the form

$$\sum_{k=1}^{\infty} \sum_{\bar\vartheta\in\mathcal{T}_k}{}^{*} \bar\Xi^k \,[\varepsilon/(1-\sigma)]^k \;,$$

where σ is defined in (3.10), and $\bar\Xi^k$ satisfies the bound

$$\bar\Xi^k \le C^k \prod_{n\le 0} 2^{-(2nN^1_n + nN^1_n)} \prod_{v\in\bar\vartheta} 2^{n_{\lambda_v}\delta_v} \;, \tag{3.13}$$

where N^j_n is the number of (nonresonant) dressed propagators on scale n and degree j in $\bar\vartheta$, so that we obtain the series convergence for $|\varepsilon(1-\sigma)^{-1}| < (G_0)^{-1}$, being G_0 defined in (3.7), *i.e.* for ε such that $|\sigma(\varepsilon)| < 1 - G_0\varepsilon$. Obviously such a bound can be correct only if the formal series (3.9) can be proven to be convergent for $|\varepsilon| < \varepsilon_0$, for some $\varepsilon_0 > 0$: to such an aim the next section is devoted.

4. Boundedness of the form factors and convergence of the perturbative series

In this section we prove that $|\sigma^{n(k)}_{s,jj'}(\vec{\omega}\cdot\vec{\nu})| \le C^k$, for some $C > \varepsilon_1^{-1}$ and any value of $\vec{\omega}\cdot\vec{\nu}$. This will be done by writing the form factor as a sum over diagrams which can be thought as resonances with their incoming line, (see Definition 3.3), so that, in order to obtain the contribution arising from a single diagram, we have to compute the resonance factor times the propagator associated to its incoming line. The resonance factor is expressed in terms of the original trees, (*not of the resumed trees*): the corresponding resonance V can be interpretated as a tree having an endpoint w_2, (see Definition 3.2), from which a mode $\vec{\nu}_{w_2} + \vec{\nu}_{\lambda_V}$ is emitted, instead of a mode $\vec{\nu}_{w_2}$ simply.

The first step in order to prove the boundedness of the form factor is to note that $\sum_V \mathcal{V}^{n_{\lambda_V}}_{s_V,jj'}(\vec{\omega}\cdot\vec{\nu}_{\lambda_V}) = O((\vec{\omega}\cdot\vec{\nu}_{\lambda_V})^{R_{\lambda_V}})$, if R_{λ_V} is the degree of the propagator in (3.2). This is in fact the case, as the following lemma shows.

Lemma 4.1. *The form factor introduced in Definition 3.3 through (3.9) can be written in the following way:*

$$\sigma^{n(k)}_{s,jj'}(\vec{\omega}\cdot\vec{\nu}) = \sum_{\substack{V:\vec{\nu}_{\lambda_V}=\vec{\nu}\\ k(V)=k,\,n_{\lambda_V}=n}} \int_0^1 dt\; t^{R_{\lambda_V}-1} \left[\frac{\partial^{R_{\lambda_V}}}{\partial(\vec{\omega}\cdot\vec{\nu})^{R_{\lambda_V}}}\, \mathcal{V}^{n_{\lambda_V}}_{s,jj'}(t\vec{\omega}\cdot\vec{\nu}) \right] \chi_n(\vec{\omega}\cdot\vec{\nu}) \;. \tag{4.1}$$

This means that the first R_{λ_V} terms of the Taylor expansion of the resonance factor $\mathcal{V}^{n_{\lambda_V}}_{s,jj'}(\vec{\omega}\cdot\vec{\nu})$ in powers of $\vec{\omega}\cdot\vec{\nu}$ add to zero when summed to give the form factor.

We have taken into account explicitly the expression giving the propagator on scale n, $g^{(n)}(\vec{\omega}\cdot\vec{\nu}) = \chi_n(\vec{\omega}\cdot\vec{\nu})[i\vec{\omega}\cdot\vec{\nu}]^{-R_{\lambda_V}}$, (see (3.2)). The proof of Lemma 4.1 is given in Appendix A3.

In order to prove that $|\sigma^{n(k)}_{s,jj'}(\vec\omega\cdot\vec\nu)|\le C^k$, for some constant C, we modify the rules how to construct the trees by splitting each resonance factor $\mathcal V$ as $\mathcal V=\mathcal{LV}+(1-\mathcal L)\mathcal V$, where

$$
\begin{aligned}
\mathcal{LV}^n_{1,jj'}(\vec\omega\cdot\vec\nu) &= \mathcal V^n_{1,jj'}(0)\;,\\
\mathcal{LV}^n_{2,jj'}(\vec\omega\cdot\vec\nu) &= \mathcal V^n_{2,jj'}(0)+[\vec\omega\cdot\vec\nu]\,\dot{\mathcal V}^n_{2,jj'}(0)\;,\\
\mathcal{LV}^n_{3,jj'}(\vec\omega\cdot\vec\nu) &= \mathcal V^n_{3,jj'}(0)\;,\\
\mathcal{LV}^n_{4,jj'}(\vec\omega\cdot\vec\nu) &= \mathcal V^n_{4,jj'}(0)+[\vec\omega\cdot\vec\nu]\,\dot{\mathcal V}^n_{2,jj'}(0)\;,
\end{aligned}
\tag{4.2}
$$

where $\dot{\mathcal V}^n_{s,jj'}(0)$ denotes the first derivative of $\mathcal V^n_{s,jj'}$ with respect to $\vec\omega\cdot\vec\nu$, computed in $\vec\omega\cdot\vec\nu=0$. Note that the resonant factors depend on $\vec\omega\cdot\vec\nu$ only through the propagators, (see (3.2)). Then, for each line λ inside the resonance, the momentum flowing in it is given by $\vec\nu_\lambda\equiv\vec\nu^0_\lambda+\varepsilon_\lambda\vec\nu$, where $\vec\nu^0_\lambda$ is the sum of the mode labels corresponding to the vertices following λ but inside the resonance, and $\varepsilon_\lambda=0,1$, so that, even if we set $\vec\omega\cdot\vec\nu=0$, (*i.e.* $\vec\omega\cdot\vec\nu_\lambda=\vec\omega\cdot\vec\nu^0_\lambda$ for each λ inside the resonance), no too small divisor appears because of the presence of the compact support functions $\chi_{n'}(\vec\omega\cdot\vec\nu_\lambda)$, $n'>n$.

Given a tree, on any cluster the $\mathcal L$ or $1-\mathcal L\equiv\mathcal R$ operators apply; however for the cancellations seen in Lemma 4.1 the sum over the trees of order k containing one or more resonances on which the $\mathcal L$ operator applies is vanishing, so that we can rule out all such contributions and consider simply the trees with resonances on which the operator $\mathcal R$ applies.

It is convenient to write the effect of $\mathcal R$ on a resonance V as

$$
\begin{aligned}
\mathcal{RV}^n_{s,jj'}(\vec\omega\cdot\vec\nu) &= (\vec\omega\cdot\vec\nu)\int_0^1 dt\;\dot{\mathcal V}^n_{s,jj'}(t\vec\omega\cdot\vec\nu) &&\text{(first order zero : } s=1,3)\;,\\
\mathcal{RV}^n_{s,jj'}(\vec\omega\cdot\vec\nu) &= (\vec\omega\cdot\vec\nu)^2\int_0^1 dt\;t\;\ddot{\mathcal V}^n_{s,jj'}(t\vec\omega\cdot\vec\nu) &&\text{(second order zero : } s=2,4)\;,
\end{aligned}
\tag{4.3}
$$

where $\ddot{\mathcal V}^n_{s,jj'}$ denotes the second derivative with respect to the variable $\vec\omega\cdot\vec\nu$.

As there are resonances enclosed in other resonances the above formula can suggest that there are propagators derived up to $\approx k$ times, if k is the order of the graph. This would be of course a source of problems, as $a_{R_{\lambda_V}}(p)>p!$, where $a_{R_{\lambda_V}}(p)$ is defined after (4.2). However it is not so: in fact the propagators are derived at most two times. This can be seen as follows. Let n be the resonance-scale of the maximal resonance V, and let us define V_0 as the collection of lines and vertices in V not contained in any resonance internal to V. Then we can write $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_V})$, (we do not write explicitly the labels of the resonance factor), as

$$
\mathcal R\Big(\prod_{\lambda\in V_0} g^{(n_\lambda)}_\lambda \prod_{V'\subset V}[\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})]\prod_{v\in V_0}\mathcal E_v\Big)\;,
\tag{4.4}
$$

being the second product over the resonances $V'\subset V$ which are maximal; in (4.4), for any resonance $V'\subseteq V$, $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$ can be written either as in (4.3) or as a difference $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})=\mathcal V(\vec\omega\cdot\vec\nu_{\lambda_{V'}})-\mathcal{LV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$, in according to which expression turns out to be more convenient to deal with.

Then the first step is to write the action of $\mathcal R$ on the maximal cluster as in (4.3), leaving the other terms $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$ written as differences: so (4.4) can be written by the Leibniz's rule as a sum of terms, and the derivatives of $\mathcal R$ apply either on some propagator $g^{(n)}_\lambda$ or on some $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$. In the end there can be either no derivative, or one derivative, or two derivatives applied on each $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$. If only one derivative acts on $\mathcal{RV}(\vec\omega\cdot\vec\nu)$, $\vec\nu=\vec\nu_{\lambda_{V'}}$, and, e.g. , $s=2,4$, then we write, when such a term is not vanishing,

$$
\partial\mathcal{RV}(\vec\omega\cdot\vec\nu)=\partial\mathcal V(\vec\omega\cdot\vec\nu)-\dot{\mathcal V}(0)=(\vec\omega\cdot\vec\nu)\int_0^1 dt\,\ddot{\mathcal V}(t\vec\omega\cdot\vec\nu)\;,
$$

because in such a case the derivative with respect to $\vec\omega\cdot\vec\nu$ is equal to the dervative with respect to $\vec\omega\cdot\vec\nu_{\lambda_V}$, while if two derivatives act on $\mathcal{RV}(\vec\omega\cdot\vec\nu_{\lambda_{V'}})$, then we write

$$
\partial^2\mathcal{RV}(\vec\omega\cdot\vec\nu)=\ddot{\mathcal V}(\vec\omega\cdot\vec\nu)\;.
$$

The case $s = 1,3$ is easier, and can be discussed in the same way. Then no more than two derivatives can act on each resonance V' in any case, and the procedure can be iterated, since the resonances V' can be dealt with as the resonance V.

The effect of the $\mathcal{R}$ operator is to obtain a gain factor either $2^{n-n'}$ or $2^{n-n'}2^{n'-n''}$, where n' and n'' are the scales of two lines λ' and λ'' contained in some clusters T' and T'' inside V; the line λ'' can coincide with λ', or also be absent, if $s_V = 1,3$. So we can rewrite, *e.g.* , the first factor as $2^{n-n'} = 2^{n-n_1}\dots 2^{n_q-n'}$, where n_i is the scale of the cluster $T_i \supset T_{i+1}$, with $T_0 = V$ and $T_{q+1} = T'$. Analogous considerations hold for n'', so that we can conclude that: (1) no more than two derivatives can ever act on any propagators; (2) a gain $2^{D_{V'}(n_{\lambda_{V'}}-n_{V'})}$ is obtained for any resonance $V' \subseteq V$; (3) the total number of terms generated by the derivation operations is bounded by $k(V)^2$, if $k(V)$ is the order of the resonance V, (see Definition 3.1).

Therefore, for the value of the diagram formed by the resonance plus its incoming line, we find the bound

$$2^{-D_V n_V}\Big[\tilde{\mathcal{C}}^k \prod_v{}^{*} 2^{n_{\lambda_v}\delta_v} \prod_{n\ge n_V} 2^{-(2nN_n^2+nN_n^1)}\Big]\cdot \\ \cdot\Big[\prod_{n\ge n_V}\prod_{\substack{T\\ n_T=n}}\prod_{j=1}^{2}\prod_{i=1}^{m_T^j} 2^{D_{V_i}(n-n_{V_i})}\Big]\,, \tag{4.5}$$

where n_V is the scale of the resonance, D_V is the degree of the resonance, the product with $*$ is over all the lines not exiting from any resonance,[5] and the second square bracket is the part coming from the resummations, and follows from the above discussion about the gain factors. The constant $\tilde{\mathcal{C}}$ differs from $\mathcal{C}$ in (3.5) as it takes into account the bound on the derivatives of the propagators: we can set $\tilde{\mathcal{C}} = \mathcal{C}\,e^2\,a_2(2)$, as the sum over all the outer resonances V's of the factors $[2k(V)]^2$ can be bounded by e^{2k}, and $a_R(p) \le a_2(2)$, for any $R = 1,2$, and $0 \le p \le 2$.

In Appendix A2, we show that, if $N_n^j(V)$ is the number of lines on scale n and of degree j contained inside a resonance V, then the following bound holds:

$$\sum_{j=1}^{2} jN_n^j(V) \le \frac{8k(V)}{E\,2^{-n\tau^{-1}}} + \sum_{\substack{T\subseteq V\\ n_T=n}}\Big[-2+\sum_{j=1}^{2} j\,m_T^j\Big]\,. \tag{4.6}$$

Substituting (4.6) into (4.5), we see that the jm_T^j is taken away by the first factor in $2^{D_{V_i}n}$ $2^{-D_{V_i}n_{V_i}}$, being $n = n_{\lambda_{V_i}}$, while the remaining $2^{-D_{V_i}n_{V_i}}$ are compensated by the -2 before the $\sum_j m_T^j$ in (4.6), taken from the factors with $T = V_i$, (note that there are always enough -2's); in particular we can get rid of the factor $2^{-D_V n_V}$ in (4.5). Hence (4.5) is bounded by

$$a_2(2)^k\mathcal{C}^k e^{2k}\prod_n 2^{-8nkE^{-1}2^{n/\tau}} \le D_1^k\,, \tag{4.7}$$

if $k = k(V)$, with a suitable constant D_1; the previous bounds give

$$D_1 = b_1^2\,e^2\,a_2(2)\,\mathcal{C} = 2^2\,a_2(2)\,e^2\,F\,J_m^{-1}\,C_0^2\,N^2\,\rho^{-1}\,\exp[\sum_{n=1}^{\infty} 8\,n(\ln 2)\,E^{-1}\,2^{-\tau^{-1}n}]\,.$$

As in the preceding section, the number of trees contributing to $\sigma_{s,jj'}^{n(k)}(\vec\omega\cdot\vec\nu)$ is bounded by B_1^k, so that, if we reacall the Remark 2 after Lemma 3.1, we can bound $|\sigma_{s,jj'}^{n(k)}(\vec\omega\cdot\vec\nu)|$ by B_0^k, for a suitable constant B_0 given by

$$B_0 \equiv 2^3\rho^{-1}J_M J_m^{-1}B_1D_1 = 2^{10}\,e^2\,a_2(2)\,b_1^2\,l\,(2N+1)^l\,F\,\rho^{-2}\,J_M\,J_m^{-2}\,C_0^2\,N^2\,. \tag{4.8}$$

[5] If a line λ_{w_0} comes out from a resonance, and $\delta_{w_0} = 1$, then the factor $(i\vec\omega\cdot\vec\nu(w_0))$ appearing in the first vertex factor corresponding to w_0, (see (3.6)), is used in order to implement the cancellation of the form factor, (as proof of Lemma 4.1 shows, see Appendix A3), and then the bound improvement (4.4); therefore we have no more the factor $2^{n_{\lambda_{w_0}}}$ in (3.4) corresponding to such a line.

Obviously an analogous bound holds also for the contributions to $\bar\mu_j^{(k)}$ and to $X_{j\vec\nu}^{(k)}(\zeta)$, $\zeta=h,H$, so that the proof of Theorem 1.1 is complete, and the value (1.5) for the KAM tori convergence radius is obtained. ■

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Appendix A1. Proof of Lemma 3.1

Let us consider first the contribution to $\bar\mu_j^{(k)}$. A value $\bar\mu_j^{(k)}$ can be represented as a sum of tree values, as shown in (2.9). Then each tree whose value contributes to $\bar\mu_{j_i}^{(k_i)}$ can be enclosed inside a bubble. Since $\bar\mu_j^{(k)}$ depends on other fruits, we can iterate the procedure until no fruit is left: at each step we obtain some new bubbles which are contained in a bubble drawn in the previous step.

We consider a single tree appearing in the multiple sums obtained through the above procedure: we note that the values arising from the different bubbles factorize, since the momentum flowing in the first branch of the maximal tree inside a bubble is identically vanishing, and therefore the momenta running in any tree branches do not depend on the bubbles encircling the following vertices. Then for each innermost bubble b_0, which does not contain any bubbles, a bound $D_1^{\tilde k_{b_0}}$, if $\tilde k_{b_0}$ is the number of (free) vertices of the tree inside the bubble, is obtained, by assumption. Hence we pass to the next to the innermost bubbles: for each such bubble b_1 a bound $D_1^{\tilde k_{b_1}}\ 2^{\tilde k_{b_1}}$ $\prod_{b_0\subset b_1}2\rho^{-1}$ is found, where $\tilde k_{b_1}$ now is the number of vertices contained in the considered bubble, (*i.e.* is the number of free vertices of the tree inside the considered bubble but outside the inner bubbles). In fact, for each vertex inside b_1 we can bound

$$\left\{\frac{\prod_{j=1}^l \partial_{B_j}^{q^2_{v,j}+q^3_{v,j}}\varphi(\vec B_0)}{q^2_{v,j}!q^3_{v,j}!}\right\}\le\left[\prod_{j=1}^l 2^{q^2_{v,j}+q^3_{v,j}}\rho^{-q^3_{v,j}}\right]\rho^{-q^2_{v,j}}\ ,$$

where $\rho^{-q^2_{v,j}}$ can be interpretated as a bound for $[q^2_{v,j}!]^{-1}\prod_{j=1}^l\partial_{B_j}^{q^2_{v,j}}\varphi(\vec B_0)$, because of the assumption of analiticity in the angular momenta variables of the interaction potential.

And so on, until in the end a bound $\prod_b[(2D_1)^{\tilde k_b}2\rho^{-1}]$ follows for the considered tree value, being the product over the bubbles b and $\tilde k_b$ being the number of vertices inside the bubble b but outside the inner bubbles. This means that the value of a single tree contributing to $\bar\mu_j^{(k)}$ is bounded by $[4\rho^{-1}D_1]^k$, as $\sum_b\tilde k_b\le k$.

Then we have only to perform the sum over the trees, but from the above discussion we conclude that such a sum is arranged in the following way: we sum over all the trees with k vertices, and over all the possible ways to put bubbles around the vertices of the tree ϑ. The first sum is bounded by B_1^k, (as it is proven in §3, see the discussion after (3.5)), while the second one is trivially bounded by 2^k, as there can be at worst one bubble per vertex. Then, if we take $B_0=2^3\rho^{-1}B_1D_1$, the stated result follows.

If the trees contribute to $X_{j\vec\nu}^{(k)}(\zeta)$, $\zeta=h,H$, the proof remains the same: simply we do not draw any bubble around the entire tree. ■

Appendix A2. Resonant Siegel-Brjiuno bound.

Let us call $N_n^*\equiv N_n^*(\vartheta)$ the number of non resonant lines of a tree ϑ carrying a scale label $\le n$, *i.e.* $N_n^*+\sum_T(m_T^1+m_T^2)=\sum_{n'\le n}N_n$. We shall prove first that $N_n^*\le 2k(E2^{-n/\tau})^{-1}-1$ if $N_n>0$.

If ϑ has the root line with scale $>n$ then calling $\vartheta_1,\vartheta_2,\ldots,\vartheta_m$ the subdiagrams of ϑ emerging from the first vertex of ϑ and with $k_j>E2^{-n/\tau}$ lines, it is $N_n^*(\vartheta)=N_n^*(\vartheta_1)+\ldots+N_n^*(\vartheta_m)$ and the statement is inductively implied from its validity for $k'<k$ provided it is true that $N_n^*(\vartheta)=0$ if $k<E2^{-n/\tau}$, which is is certainly the case if E is chosen as in (4.1).[6]

In the other case it is $N_n^*\le 1+\sum_{i=1}^m N_n^*(\vartheta_i)$, and if $m=0$ the statement is trivial, or if $m\ge 2$ the statement is again inductively implied by its validity for $k'<k$.

If $m=1$ we once more have a trivial case unless the order k_1 of ϑ_1 is $k_1>k-2^{-1}E\,2^{-n/\tau}$. Finally, and this is the real problem as the analysis of a few examples shows, we claim that in the latter case the root line of ϑ_1 is either a resonant line or it has scale $>n$.

Accepting the last statement we have: $N_n^*(\vartheta)=1+N_n^*(\vartheta_1)=1+N_n^*(\vartheta'_1)+\ldots+N_n^*(\vartheta'_{m'})$, with ϑ'_j being the m' subdiagrams emerging from the first node of ϑ'_1 with orders $k'_j>E\,2^{-n/\tau}$: this is so because the root line of ϑ_1 will not contribute its unit to $N_n^*(\vartheta_1)$. Going once more through the analysis the only non trivial case is if $m'=1$ and in that case $N_n^*(\vartheta'_1)=N_n^*(\vartheta''_1)+\ldots+N_n^*(\vartheta''_{m''})$, *etc.*, until we reach a trivial case or a diagram of order $\le k-2^{-1}E\,2^{-n/\tau}$.

It remains to check that if $k_1>k-2^{-1}E\,2^{-n/\tau}$ then the root line of ϑ_1 has scale $>n$, unless it is entering a resonance.

Suppose that the root line of ϑ_1 has scale $\le n$ and is not entering a resonance. Note that $|\vec\omega\cdot\vec\nu(v_0)|\le 2^{n+1}, |\vec\omega\cdot\vec\nu(v_1)|\le 2^{n+1}$, if v_0,v_1 are the first vertices of ϑ and ϑ_1 respectively. Hence $\delta\equiv|(\vec\omega\cdot(\vec\nu(v_0)-\vec\nu(v_1))|\le 2\,2^{n+1}$ and the diophantine assumption implies that $|\vec\nu(v_0)-\vec\nu(v_1)|>(2\,2^{n+1})^{-1/\tau}$, or $\vec\nu(v_0)=\vec\nu(v_1)$. The latter case being discarded as we are not considering the resonances, it follows that $k-k_1<2^{-1}E\,2^{-n/\tau}$ is inconsistent: it would in fact imply that $\vec\nu(v_0)-\vec\nu(v_1)$ is a sum of $k-k_1$ vertex modes and therefore $|\vec\nu(v_0)-\vec\nu(v_1)|<2^{-1}NE\,2^{-n/\tau}$ hence $\delta>2^3\,2^n$ which is contradictory with the above opposite inequality. ∎

Analogously, we can prove that, if $N_n>0$, then the number $p_n(\vartheta)$ of clusters of scale n verifies the bound $p_n(\vartheta)\le 2k\,(E2^{-n/\tau})^{-1}-1$. In fact this is true for $k\le E2^{n/\tau}$, (see footnote 6). Otherwise, if the first tree vertex v_0 is not in a cluster of scale n, it is $p_n(\vartheta)=p(\vartheta_1)+\ldots+p_n(\vartheta_m)$, with the above notation, and the statement follows by induction. If v_0 is in a cluster on scale n we call $\tilde\vartheta_1,\ldots,\tilde\vartheta_m$ the subdiagrams emerging from the cluster containing v_0 and with orders $k_j>E2^{-n/\tau}$, $j=1,\ldots,m$. It will be $p_n(\vartheta)=1+p(\tilde\vartheta_1)+\ldots+p_n(\tilde\vartheta_m)$. Again we can assume $m=1$, the other cases being trivial. But in such a case there will be only one branch entering the cluster T on scale n containing v_0 and it will have a momentum of scale $n'\le n-1$. Therefore the cluster T must contain at least $E2^{-n/\tau}$ vertices, (otherwise, if λ is a line on scale n contained in T, and $\vec\nu^0_\lambda$ is the sum of the mode labels corresponding to the vertices following v_0 but inside T, we would have $|\vec\omega\cdot\vec\nu_\lambda|\le 2^{n+1}$ and, simultaneously, $|\vec\omega\cdot\vec\nu_\lambda|\ge 2^{n+3}-2^{n-1}>2^{n+2}$, which would lead to a contradiction). This means that $k_1\le k-E2^{-n/\tau}$. ∎

Let us consider now a resonance V, and let us call $N_n(V)$ and $N_n^*(V)$ the number of lines on scale n in V, and, respectively, the number of non resonant lines inside V carrying a scale label $\le n$. Again a bound $N_n^*(V)\le 2k(V)(E2^{-n/\tau})^{-1}-1$ holds, if $k(V)$ is the order of the resonance V; analogously $p_n(V)\le 2k(V)(E2^{-n/\tau})^{-1}-1$, if $p_n(V)$ is the number of clusters on scale n contained in V. The proofs of such statements can be easily adapted from the previous ones, by noting that $N_n(V)\ne 0$ requires $k(V)>E2^{-n/\tau}$. We give them explicitly, only for completeness.

Given a subdiagram T, let us denote by $k(T)$ the number of vertices contained in T, and by $N_n^*(T)$ the number of non resonant lines inside T with a scale label $\le n$. We prove by induction that $N_n^*(T)\le 2k(T)(E2^{-n/\tau})^{-1}-1$, each time only one line on scale $n'<n$ enters the subdiagram T and it is $n_\lambda>n'$ for every line λ inside T, (note that the resonance V satisfies such a requirement, but it is not necessary that T is a cluster to make the statement to hold).

Let us consider a subdiagram T, verifying the above described properties, *e.g.* a resonance V on scale n_V, with $n'\le n_V-1$. By assumption there is only one line entering the subdiagram T, and essentially by definition there is only one line exiting; by analogy to Definition 3.2, let us call w_0 and w_2 the two vertices which the two lines, respectively, lead to. Let us call T_1, …, T_m the subdiagrams emerging from the vertex w_0 and with $k(T_j)>E2^{-n/\tau}$: by construction, one of such diagram, say the first one, contains the vertex $\pi(w_2)$, while the other ones are trees.

---

[6] Note that if $k\le E\,2^{-n/\tau}$ it is, for all momenta $\vec\nu$ of the lines, $|\vec\nu|\le NE\,2^{-n/\tau}$, i.e. $|\vec\omega\cdot\vec\nu|\ge(NE\,2^{-n/\tau})^{-\tau}=2^3\,2^n$ so that there are *no* clusters T with $n_T=n$ and $N_n=0$. The choice $E=N^{-1}2^{-3/\tau}$ is convenient in order to simplify the analysis: but this, as well as the whole lemma, remains true if 3 is replaced by any number larger than 1.

Let us consider the first branch of the subdiagram T_1. If the considered branch scale label is $> n$, then the assertion follows by induction, by using also the previous results on $N_n^*(\vartheta)$ for trees, and the fact that $k(T) > E2^{-n/\tau}$ if $N_n(T) \neq 0$.

Otherwise, it is $N_n^*(T) = 1 + \sum_{i=1}^m N_n^*(T_i)$; the case $m \geq 2$ can be again inductively studied and the statement easily follows.

If $m = 1$, then $N_n^*(T) = 1 + N_n^*(T_1)$, and, if $v_1 \in T_1$ is the vertex to which the outgoing line of T_1 leads to, we consider the m' subdiagrams emerging from v_1: again one of them, say the first one, contains the vertex $\pi(w_2)$ of V, while the other ones are trees. If $m' \geq 1$, again an inductive proof can be performed. If $m' = 1$, we have once more a trivial case unless it is $k(T_1) > k(T) - 2^{-1}E2^{-n/\tau}$. But in this case we can reason as along the proof of the bound on $N_n^*(\vartheta)$, and check that there are only two possibilities: either the line leading to v_1 is a resonant line on scale n, or it has scale $> n$. The proof of such a statement can be carried out exactly in the same way as to $N_n(\vartheta)$, so that we do not repeat it here. This means that we can write: $N_n^*(T) = 1 + N_n^*(T_1) = 1 + N_n^*(T_1') + \ldots + N_n^*(T_{m''}')$, and the above analysis can be iterated as many times as it is needed to reach either a trivial case or a subdiagram $\tilde{T}$ such that $k(\tilde{T}) \leq k(T) - 2^{-1}E2^{-n/\tau}$.

This proves the statement about the number of lines on scale n contained in a resonance, (note the bound we have obtained can be replaced by zero if the scale is $n < n_V$, because it is not possible to have a line with such a scale label inside a resonance on scale n_v, by construction); a similar, far easier, induction can be used to prove the statement concerning the number of clusters on scale n contained inside the resonance. ∎

Thus (3.6) and (4.6) are proven, noting that $\sum_{T, n_T = n} 1 = p_n(\vartheta)$, and $\sum_{T \subseteq V, n_T = n} 1 = p_n(V)$.

Appendix A3: Proof of Lemma 4.1

Given a resonance V, consider a line λ_w in V, (*i.e.* a line leading to a vertex $w > w_0$, $w \in V$), and let us study its dependence on the mode labels. We see from (3.3) that, if $R_{\lambda_w} = 2$ we can associate to such a line a *line factor* which is given by the product of a factor linear in the mode labels arising from the vertex w, times a factor $(i\vec{\nu}_{\pi(w)})_{j_{\lambda_w}}$ arising from the vertex $\pi(w)$, times a propagator $g^{(n_{\lambda_w})}(\vec{\omega} \cdot \vec{\nu}(w))$; if $R_{\lambda_w} = 1$ we associate to it a *line factor* which is given by the product of a factor linear in the mode labels arising from the vertex w, times a factor independent on the the mode labels arising from the vertex $\pi(w)$, times a propagator $g^{(n_{\lambda_w})}(\vec{\omega} \cdot \vec{\nu}(w))$. Then, for each line inside V, the line factor is a homogeneous function of even degree in the mode labels. To the first vertex w_0 we associate a factor $(-i\vec{\nu}_{w_0})_{j_{\lambda_{w_0}}}(1 - \delta_{w_0}) + (i\vec{\omega} \cdot \vec{\nu}(w_0))\partial_{B_{j_{\lambda_{w_0}}}} \delta_{w_0}$. Since the function $\varphi_{\vec{\nu}}(\vec{B}_0)$ is supposed to be even in $\vec{\nu}$, no other factor has to be considered in order to obtain the behaviour of the resonance, when $\vec{\omega} \cdot \vec{\nu}_{\lambda_V} = 0$ and the signs of the mode labels of all the vertices in V are simultaneously changed. When such an operation is performed we see that, if $\zeta_{\lambda_{w_0}} = H$, (recall that $\delta_{w_0} \equiv 0$ if $\zeta_{\lambda_{w_0}} = H$), or $\zeta_{\lambda_{w_0}} = h$, with $\delta_{w_0} = 0$, the overall sign of the resonance factor changes, while, if $\zeta_{\lambda_{w_0}} = h$, with $\delta_{w_0} = 1$, the overall sign of the resonance factor does not change.

Now, let us consider separately the possible kinds of resonance, see (3.8) above.

(1) If $\zeta_{\lambda w_0} = \zeta_{\lambda_{w_2}} = H$, then we deduce from the above discussion that the sign of $\mathcal{V}_1^{n_{\lambda_V}}(\vec{\omega} \cdot \vec{\nu}_{\lambda_V})$ changes when all the signs of the mode labels of the vertices in V are changed; then, fixed a set of compatible values of $\vec{\nu}_w$, $w \in V$, if we sum together the two contributions $\{\vec{\nu}_w\}_{w \in V}$ and $\{-\vec{\nu}_w\}_{w \in V}$, we obtain zero.

(2) If $\zeta_{\lambda_{w_0}} = H$ and $\zeta_{\lambda_{w_2}} = h$, we consider all the trees we obtain by detaching from the resonance the subtree with first vertex w_2, then reattaching it to all the remaining vertices $w \in V$, and we add also the contributions obtained by the previous ones by an overall sign reversal of the mode labels $\vec{\nu}_w$: if $\vec{\omega} \cdot \vec{\nu}_{\lambda_V} = 0$, no propagator changes, and the only effect of our operation is that one of the vertex factors changes by taking successively the values $(\vec{\nu}_w)_{j_{\lambda_{w_2}}}$, $w \in V$. Then we build in this way a quantity proportional to $\sum_{w \in V} (\vec{\nu}_w)_{j_{\lambda_{w_2}}} = [\vec{\nu}(w_2) - \vec{\nu}(w_0)]_{j_{\lambda_{w_2}}} \equiv 0$. If we sum also on a overall change of sign of the $\vec{\nu}_w$'s, and we take into account the parity in the mode labels of the resonance factor, we obtain a second order zero.

(3) If $\zeta_{\lambda_{W_0}} = h$ and $\zeta_{\lambda_{W_2}} = H$ we note that, when $\delta_{w_0} = 0$, then the difference from the contribution with $\zeta_{\lambda_{w_0}} = H$ reduces to the label $R_{\lambda_{w_0}}$ which now is 2 instead of 1: then the results of the the

first item still hold. If $\delta_{w_0}=1$, then $\mathcal{V}_3^{n_{\lambda_V}}(\vec\omega\cdot\vec\nu_{\lambda_V})$ vanishes to first order, as it contains a factor $i\vec\omega\cdot\vec\nu_{\lambda_V}$, see (3.4).
(4) If $\zeta_{\lambda_{W_0}}=\zeta_{\lambda_{W_2}}=h$ we note that, when $\delta_{w_0}=0$, then the difference from the contribution with $\zeta_{\lambda_{w_0}}=H$ reduces to the label $R_{\lambda_{w_0}}$ which now is 2 instead of 1: then the results of the the second item still hold. If $\delta_{w_0}=1$, then $\mathcal{V}_2^{n_{\lambda_V}}(0)\equiv 0$, as it contains a factor $\vec\omega\cdot\vec\nu_{\lambda_V}$, (as to the first order in the previous item for $\delta_{w_0}=1$), and the first derivative with respect to $\vec\omega\cdot\vec\nu_{\lambda_V}$ in $\vec\omega\cdot\vec\nu_{\lambda_V}=0$ is still vanishing for parity reasons analogous to those of the case $\mathcal{V}_2^{n_{\lambda_V}}(\vec\omega\cdot\vec\nu_{\lambda_V})$ in fact the only difference is that a factor $(-i\vec\nu_{w_0})_{j_{\lambda_{w_0}}}$ is missing, (replaced by a derivative $\partial_{B_{j_{\lambda_{w_0}}}}$), and a factor $(i\vec\nu_{\pi(w_2)})_{j_{\lambda_{w_2}}}$ replaces $\partial_{B_{j_{\lambda_w}}}$.

Then from the above discussion and from the definition of factor form given in Definition 3.3, (see in particular (3.9)), the results stated in Lemma 4.1 are proven. ■

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