MAXIMUM ENTROPY ALGORITHM FOR SPECTRAL ESTIMATION PROBLEM WITH A PRIORI INFORMATION

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The power density estimation problem with a priori information about spectrum is considered. The problem is reduced to power density estimation problem with spectral gaps. The solution of the problem is studied in the context of the trigonometric moment problem with gaps. We establish the necessary and sufficient conditions for the existence of solution and describe all solutions in indeterminate case. A new definition of the entropy functional is given. The effective algorithm to construct maximum entropy solution is proposed. A number of explicit illustrations are presented.

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I. INTRODUCTION

The moment problem [1] is relevant to a variety of classical situations of the signal processing. There are several standard methods of dealing with this classical problem. A relative newcomer among these involves the introduction of the concept of entropy into the function space, selecting as preferred solution among those functions, which otherwise meet constrains imposed by the data which yields the maximum entropy (ME) value [2,3].

Following the original contributions, there has been a long debate concerning the conceptual foundations of the ME for a series of the mathematical physics problems [4] and for signal processing problems [5]. The debate is currently more meaningful than ever in view of the augmenting list of successful practical applications [6,7], which have become possible because of the steadily increasing computing power available today. While our aim is not to engage in further conceptual ramifications of the rationale maximum entropy, we shall attempt to sharpen its mathematical foundations to extend its applicability to various concrete problems encountered in signal processing.

The concept of entropy has been known for a long time in information theory [8,9], but its introduction into signal processing was due to Burg [7]. In 1967 he gave the new conception of spectral estimation and proposed the method by which one directly estimates the relative parameters γ_i , $i = 0,\ 1, \ldots,\ N$ from measurement data c_n , $n = 0,\ 1, \ldots,\ N$, and gets the power spectral density estimation (PSDE) by means of the equation

$$P(\omega) = \frac{c_0}{|\sum_{i=0}^{N} \gamma_i e^{-j\omega i}|^2}. \tag{1}$$

The theoretical advantage of the ME spectral analysis and the high-resolution property of Burg's algorithm attracted great attention. Many efforts have been made to explain the properties of ME estimation [5] and also to employ it in geophysics, speech and image reconstruction, radar, sonar and other areas [7,10]. Furthermore, various modifications of Burg's algorithm and some better spectral estimation methods have continuously been proposed [11,12].

In this paper we consider the PSDE problem in the most general statement and propose the novel method of its solution based on moment method technique. The statement of problem assumes using additional a priori information about spectrum. For example, the separate regions of spectrum or the original spectrum support may be known. In view of such information the higher-resolution method of spectral estimation is proposed.

We establish necessary and sufficient conditions of the solvability of the problem and for existence of the ME solution, and give the constructive way to compute this ME solution. To achieve the aim we describe all solutions of the trigonometric moment problem with gaps corresponding to the considered PSDE problem and introduce the new definition of the entropy functional. Next, we show that this definition coincides with the generally used one in classical cases. The new definition of entropy functional allows us to find the construction of the ME solution for nontrivial spectrum supports in terms of polynomials orthogonal with respect to given moment sequence. To compute the orthogonal polynomials corresponding to the known data (i.e., moments) we suggest using the modified Chebyshev algorithm [13] or Levinson algorithm [14]. A number of explicit illustrations are presented.

II. POWER DENSITY SPECTRAL ESTIMATION PROBLEM WITH A PRIORI INFORMATION

Consider the problem of estimating a power spectral density function given only that it is positive on the spectral support, zero outside and compatible with the known values of autocorrelation sequences. In addition a priori information about spectrum is assumed to be given, i.e., we assume the spectrum is unknown in full interval of frequencies, but one or several regions of spectrum are known, and consider the PSDE problem with such a priori information. The aim of spectral estimation in this case is to find the function which agrees with the set of the known values of autocorrelation function c_k, $k = 1,\ 2, \dots,\ N$ and given a priori information. To be rigorous we assume the original spectrum $S(\theta)$ is known on intervals $[r_1, t_1]$, $[r_2, t_2]$, $\dots$, $[r_m, t_m]$, where $-\pi \le r_1 < t_1 < r_2 < \dots < t_m \le \pi$, i.e., for all $\theta \in \cup_{i=1}^{m}[r_i, t_i] = I_m$. In view of the given a priori information, the original PSDE problem reduces to PSDE problem with a priori information about spectrum, that is to determine the nonnegative function $P(\theta)$ such that

$$P(\theta) \ge 0, \quad \text{for} \quad \theta \in [-\pi, \pi], \tag{2}$$

which is compatible with the known values c_k of the autocorrelation function:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta) e^{-ik\theta}\, d\theta = c_k, \quad k = 0,\ 1, \dots,\ N. \tag{3}$$

and with the given a priori information about spectrum

$$P(\theta) = S(\theta), \quad \text{for} \quad \theta \in I_m. \tag{4}$$

Let us note that the statement of problem (2)–(4) is the most general one of the PSDE problem. Really, this statement allows us to consider PSDE problem with a priori information about spectrum in its different modification as a uniform problem. Moreover, this statement contains both a traditional PSDE problem that corresponds to case $I_m = \emptyset$, and the PSDE problem with gaps, which is to determine the nonnegative function $P(\theta)$ with the gaps I_m, where I_m is the given system of intervals, i.e., the function $P(\theta)$, such that

$$P(\theta) \geq 0, \quad P(\theta) = 0 \quad \text{for} \quad \theta \in I_m,$$

and which is compatible with the known values c_k of the authocorrelation function:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) e^{-ik\theta} \, d\theta = c_k, \quad k = 0,\ 1, \ldots,\ N.$$

The solution of PSDE problem (2)–(4) means that we need spectrum only for frequencies outside I_m to be estimated.

One possible approach to solve the problem (2)–(4) is to refuse the given a priori information and estimate spectrum using one of known methods of spectral estimation, for example, maximum entropy method, maximum likelihood method or other. This approach possesses a fundamental disadvantage. First of all, the spectral estimation determined by means of a known method may not be in agreement with the given a priori information. Besides, when the spectral estimation with good resolution is necessary it seems unreasonable to refuse the additional information about spectrum, because the given a priori information can be used to improve the resolution.

An other possible approach is to take into account the given a priori information about spectrum and consider the PSDE problem with a priori information. In this case the problem arises of constructing a high-resolution method of spectral estimation which will use the given additional a priori information. Bellow we will consider this approach.

First of all let us show that PSDE problem with the given a priori information can be reduced to PSDE problem with gaps in spectrum.

Let $P(\theta)$ be any solution of the PSDE problem with a priori information (2)–(4). For convenience we define the function corresponding to a priori information

$$P_a(\theta) = \begin{cases} \text{S}(\theta) \\ \text{ if } \theta \in I_m \\ 0 \\ \text{otherwise} \end{cases} \tag{5}$$

and function with gaps I_m

$$P_g(\theta) = \begin{cases} 0 \\ \text{ if } \theta \in I_m \\ \text{P}(\theta) \\ \text{otherwise} \end{cases} \tag{6}$$

Then any solution $P(\theta)$ of the PSDE problem (2)–(4) can be represented by means of the sum of these functions, i.e., the representation

$$P(\theta) = P_a(\theta) + P_g(\theta)$$

holds for all $\theta \in [-\pi, \pi]$.

Therefore, for all $k = 0,\ 1,\ \ldots,\ N$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)e^{-ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(\theta)e^{-ik\theta}d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} P_g(\theta)e^{-ik\theta}d\theta \tag{7}$$

Due to relation (3) the integral on the left-side of the relation (7) is equal to c_k, $k = 0,\ 1,\ \ldots,\ N$. Next, the first integral on the right-side of the relation (7) can be calculated for all integer k because of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(\theta)e^{-ik\theta}d\theta = \frac{1}{2\pi}\int_{I_m} S(\theta)e^{-ik\theta}d\theta$$

and $S(\theta)$ is known on the set I_m.

Let c_k^a, $k = 0,\ 1,\ \ldots,\ N$ be the correlation coefficients that correspond to function $P_a(\theta)$, i.e., let

$$c_k^a = \int_{I_m} S(\theta)e^{-ik\theta}d\theta$$

Then from (7) we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_g(\theta)e^{-ik\theta}d\theta = c_k^g, \quad k=0,\ 1,\ \dots,\ N, \tag{8}$$

where

$$c_k^g = c_k - c_k^a$$

is known values.

Let us note that $P_g(\theta)$ is a nonnegative function with the gaps I_m. Hence due to relations (8) this function can be considered as a power spectral estimation corresponding to the correlation coefficients c_k^g, $k=0,\ 1,\ \dots,\ N$. Hence, the most general statement of the PSDE problem is reduced to PSDE problem with gaps.

So, to solve the PSDE problem with a priori information (2)–(4) it is sufficient to find the nonnegative function $P_g(\theta)$ with the gaps I_m, which satisfies the conditions (8), i.e., it is sufficient to solve the PSDE problem with gaps in spectrum.

In next sections we will consider the PSDE problem with some spectral gaps in context of trigonometric moment problem with gaps.

III. THE TRIGONOMETRIC MOMENT PROBLEM WITH GAPS

The trigonometric moment problem with gaps consists of a collection of complex numbers

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)e^{ik\theta}\,d\theta, \qquad k=0,\ 1,\dots,\ N, \tag{9}$$

where $P(\theta)$ is an unknown positive function of θ with the gaps I_m. Assuming that the moments c_k, $k=0,\ 1,\dots,\ N$ of $P(\theta)$ and the gaps I_m are exactly known, the problem is to determine the function $P(\theta)$. Not any set of numbers c_k, $k=0,\ 1,\dots,\ N$ can be the moments of nonnegative function with the gaps I_m, i.e., the trigonometric moment problem with the gaps I_m for arbitrary sequence of the moments c_k, $k=0,\ 1,\dots,\ N$ can be unsolvable. Below we will give the effective criterion of the solvability of the considered moment problem with the given gaps I_m.

Let $\{\mathcal{P}_C\}$ be a space of polynomials with complex coefficients on the unit circle $\mathbb{C} = \{z: |z| = 1\}$, i.e., a space of trigonometric polynomials, and let $<,>$ denote a scalar product on the space $\{\mathcal{P}_C\}$ with the property

$$< P, Q > = < P\overline{Q}, 1 > \qquad \text{for the polynomials} \quad P, Q \in \{\mathcal{P}_C\} \tag{10}$$

We shall assume that the Toeplitz determinants are nonzero:

$$\Delta_n = \det \begin{vmatrix} < 1, 1 > & < z, 1 > & \dots & < z^n, 1 > \\ \overline{< z, 1 >} & < 1, 1 > & \dots & < z^{n-1}, 1 > \\ \vdots & \vdots & \ddots & \vdots \\ \overline{< z^n, 1, >} & \overline{< z^{n-1}, 1 >} & \dots & < 1, 1 > \end{vmatrix} \neq 0, \qquad n = 0,\ 1, \dots.$$

It is well known (see for example [1]) that there exists a $<,>$-orthogonal system of polynomials $\{P_n(z)\}$ such that

1) $\deg P_n(z) = n$

2) the highest-degree coefficient is positive

3) $< P_n(z), P_m(z) > = 0$, if $n \neq m$.

We will consider the monic system of polynomials $P_k(z)$, which are characterized by the highest degree coefficient being equal to 1. The monic polynomials $\{P_n(z)\}$ satisfy the recurrence relation for the orthogonal sets of polynomials:

$$P_{n+1}(z) = zP_n(z) + b_{n+1}P_n^*(z), \tag{11}$$

where $R^*(z) \stackrel{\text{def}}{=} z^k\overline{R}(1/z)$ for any polynomial $R(z)$ of degree k; b_k are some constants for which formal analytic relations in determinant form can be obtained. Note that b_0 does not take part in the recursion (11), however, below it will be convenient to define

$$|b_0|^2 = 1 - < 1, 1 > .$$

¿From recurrence relations (11) the following three-term recurrence relation immediately follows:

$$b_{n+1}P_{n+2}(z) = (zb_{n+1} + b_{n+2})P_{n+1}(z) - b_{n+2}(1 - |b_{n+1}|^2)zP_n(z). \tag{12}$$

The following criterion of solvability of the moment problem (9) with the gaps I_m holds [15]:

Theorem 1 *The trigonometric moment problem (9) with the gaps* $I_m = \{z:\ z = e^{i\theta},\ \theta \in \cup_{i=1}^m [r_i, t_i],\ -\pi \le r_1 < t_1 < r_2 < \ldots < t_m \le \pi\}$ *is solvable if and only if the inequalities*

$$|a_k| < 1, \quad k = 0,\ 1, \ldots,\ N, \qquad |b_k^{(l)}| < 1, \quad k = 0,\ 1, \ldots,\ N-1, \quad l = 1, \ldots,\ m$$

hold, where $\{a_k\}_{k=0}^N$ *and* $\{b_k^{(l)}\}_{k=0}^{N-1}$ *are the coefficients of the recurrence relation of polynomials orthogonal with respect to the moment sequences* $\{c_k\}_{k=0}^N$ *and* $\{\mu_k^{(l)}\}_{k=0}^{N-1}$ $l = 1, \ldots,\ m$*, respectively,*

$$\mu_k^{(l)} = 2\cos\frac{t_l - r_l}{2} - e^{i\frac{r_l + t_l}{2}} c_{k+1} - e^{-i\frac{r_l+t_l}{2}} c_{k-1}, \qquad l = 1,\ 2, \ldots,\ m\ .$$

Let us now assume that the necessary conditions of solvability of the trigonometric moment problem with the gaps I_m are true. Then all functions $P(\theta)$, for which the relation (9) is fulfilled for all $k = 0,\ 1, \ldots, N$, can be described by means of the linear-fractional transformation.

Theorem 2 *Let* $P_N(z)$ *and* $Q_N(z)$ *be the monic polynomials of first and second type orthogonal with respect to given moment sequence* $\{c_k\}_{k=0}^N$*, and let* I_m *be the given gaps. Let also the moment problem (9) with the gaps* I_m *be indeterminate. Then there exist the linear-fractional functions*

$$\Phi_k(z, g) = \frac{x_k(z)g + y_k(z)}{u_k(z)g + v_k(z)}, \quad k = 0,\ 1, \ldots,\ m-1,$$

where $x_k(z)$*,* $y_k(z)$*,* $u_k(z)$ *and* $v_k(z)$ *are the polynomials, such that all solutions of the moment problem (9) with gaps can be described by means of the linear-fractional transformation*

$$P_\varepsilon(z) = \frac{c_0}{h_N} \cdot \Re e \frac{zQ_N(z)\varepsilon(z) + Q_N^*(z)}{zP_N(z)\varepsilon(z) - P_N^*(z)}, \quad z = e^{i\theta} \tag{13}$$

of the function $\varepsilon(z)$*, such that*

$$\Phi_k(z, \varepsilon(z)) \in B, \quad k = 0,\ 1, \ldots,\ m-1.$$

B *is a class of functions holomorphic in* $\mathbb{C}$ *and bounded there in modulus by 1;* $P_n^*(z) = z^n \overline{P}_n(1/z)$*,* $Q_n^*(z) = z^n \overline{Q}_n(1/z)$*,* $h_n = \frac{\Delta_n}{\Delta_{n-1}}$.

IV. MAXIMUM ENTROPY SOLUTION

One of the more interesting problem in the trigonometric moment problem with gaps is to find the extreme (i.e., ME) solution. Obviously, the consideration of the extreme problem is possible only for indeterminate moment problem.

Let $\Sigma_{I_m,P(\theta)}$ be the set of solutions of the indeterminate moment problem (9) with the gaps I_m. In correspondence with Theorem 2 the set $\Sigma_{I_m,P(\theta)}$ is described by means of a set B_{I_m} of the holomorphic function $\varepsilon(z)$.

Let us consider the Burg's functional [16,17]

$$h[P,\zeta] = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\frac{(1-|\zeta|^2)}{|e^{i\theta}-\zeta|^2}\ln P(\theta)d\theta, \qquad (|\zeta|<1). \tag{14}$$

It is obvious that this functional is undetermined for all solutions of the moment problem (9) with the gaps $I_m \neq \{\emptyset\}$, i.e.,

$$h[P,\zeta] = +\infty \qquad P \in \Sigma_{I_m,P(\theta)},$$

and so the extreme problem corresponding to Burg's functional to find ME solution of the moment problem with gaps can not be considered.

But, due to Theorem 2 any solution $P(\theta)$ of the moment problem with gaps I_m can be expressed by means of linear-fractional transformation (13) with the holomorphic function $\varepsilon(z)$ corresponding to this solution. Using properties of orthogonal polynomials [1] it is easy to show that

$$P_\varepsilon(z) = \frac{1-|\varepsilon(z)|^2}{|zP_(z)\varepsilon(z) - P_n^*(z)|^2}, \qquad z = e^{i\theta}.$$

Then the Burg's functional $h[P,\zeta]$ can be written in the form

$$h[P,\zeta] = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\frac{(1-|\zeta|^2)}{|e^{i\theta}-\zeta|^2}\ln(1-|\varepsilon(e^{i\theta})|^2)d\theta + \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{(1-|\zeta|^2)}{|e^{i\theta}-\zeta|^2}\ln|zP_(z)\varepsilon(z) - P_n^*(z)|d\theta$$

Let us now introduce into consideration the functional

$$\hat{h}[\varepsilon,\zeta] = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\frac{(1-|\zeta|^2)}{|e^{i\theta}-\zeta|^2}\ln(1-|\varepsilon(e^{i\theta})|^2)d\theta, \qquad (|\zeta|<1), \tag{15}$$

on the set B_{I_m}. Obviously, this functional is defined for all function $\varepsilon(z) \in B_{I_m}$.

The following proposition holds:

Theorem 3 *Let (9) be indeterminate moment problem with the gaps* $I_m = \{z : \ z = e^{i\theta},\ \theta \in \cup_{i=1}^{m}[r_i, t_i],\ -\pi \le r_1 < t_1 < r_2 < \ldots < t_m \le \pi\}$*;* $\Sigma_{I_m,P(\theta)}$ *be the set of solutions of the moment problem (9);* B_{I_m} *be a set of the holomorphic functions corresponding to the set* $\Sigma_{I_m,P(\theta)}$ *due to relation (13) and let*

$$\varepsilon_0(z) = \arg \inf_{\varepsilon(z) \in B_{I_m}} |\varepsilon(z)| \ . \tag{16}$$

Then the inequality

$$\widehat{h}[\varepsilon, \zeta] \le \widehat{h}[\varepsilon_0, \zeta] \ . \tag{17}$$

holds for all functions $\varepsilon(z) \in B_{I_m}$.

Let us note, that for the moment problem without gaps (i.e., when $I_m = \{\emptyset\}$) we have

$$\varepsilon_0(z) \equiv 0$$

and, hence,

$$P_{\varepsilon_0}(z) = -\frac{c_0}{h_N} \Re e \frac{Q_N^*(z)}{P_N^*(z)} = \frac{1}{|P_N(z)|^2},$$

i.e., we get the well-known ME solution of the classical trigonometric moment problem without gaps. Moreover, in this case the following proposition holds:

Theorem 4 *Let* $I_m = \{\emptyset\}$*. Then for all* $\varepsilon(z) \in B$ *the relation*

$$h[P_\varepsilon, \zeta] = \widehat{h}[\varepsilon, \zeta] + C$$

is true, where the function $P_\varepsilon(z)$ *is determined by (13) and* C *is a constant.*

Thus, in spite of the fact that Burg's functional is undetermined for the solutions of the trigonometric moment problem with gaps the functional (15) is determined and in case of classical moment problem (i.e., $I_m \equiv \{\emptyset\}$) these functions have the same extreme solution taking into account of course the linear-fractional transformation (13).

V. NUMERICAL PROCEDURE

To use the ME algorithm described in the previous section the effective numerical procedure is necessary. The main problem is to construct the polynomials $P_N(z)$ and $Q_N(z)$ and the holomorphic function $\varepsilon_0(z)$. But as follows from Theorem 3 the function $\varepsilon_0(z)$ can be expressed through the polynomials $\{P_n(z)\}$ and $\{Q_n(z)\}$ and the gaps I_m. Thus, the main problem is to propose the effective technique to get the set of polynomials $\{P_n(z)\}$, orthogonal with respect to given moment sequence, and polynomials $\{Q_n(z)\}$. Below the numerical procedure to construct the set of monic polynomial $\{P_n(z)\}$ is given.

Let $\{P_k(z)\}$ be the set of monic polynomials orthogonal with respect to given moment sequence $\{c_k\}_{k=0}^N$ with the recurrence relations (12) and let $\{\pi_k(z)\}$ be another set of monic orthogonal polynomials with the following recurrence relations

$$\beta_{n+1}\pi_{n+2}(z) = (z\beta_{n+1} + \beta_{n+2})\pi_{n+1}(z) - \beta_{n+2}(1 - |\beta_{n+1}|^2)z\pi_n(z). \tag{18}$$

The coefficients $\{\beta_n\}$ are assumed to be known and the coefficients $\{b_n\}$ in recurrence relation (12) are assumed to be unknown.

Definition 1 *The matrix $T_{P,\pi}$ of modified moments, where*

$$||T_{P,\pi}||_{k,l} = \sigma_{k,l} \stackrel{\text{def}}{=} < P_l, \pi_k >$$

will be called a Chebyshev matrix.

Note that $\sigma_{k,l} = 0$ for all $k > l$, i.e. the Chebyshev matrix is triangular.

The entries of the Chebyshev matrix $T_{P,\pi}$ and the coefficients of the three-term recurrence formulas (12) and (18) are connected by relations that can be considered as a recursion process to determine entries of the Chebyshev matrix and unknown elements b_k. Writing it down in a recursive manner, we have:

the initial conditions: $\sigma_{-1,l} = 0, \quad \sigma_{0,l} =< 1, \pi_l >, \quad l = 0,\ 1, \ldots, N$, and $b_0 = 1$

and for all $k = 1,\ 2, \ldots,\ N$

$$b_k = \frac{1}{b_{k-1}}[-\frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}} + \frac{\overline{\beta}_k}{\overline{\beta}_{k-1}} + (1-|b_{k-1}|^2)\frac{\sigma_{k-2,k-1}}{\sigma_{k-1,k-1}} - (1-|b_{k-1}|^2)\frac{\overline{\beta}_k}{\overline{\beta}_{k-1}}(1-|\beta_{k-1}|^2)\frac{\sigma_{k-2,k-2}}{\sigma_{k-1,k-1}}]$$

$$(m)-T_C$$

$$\sigma_{k,l}=\begin{cases} 0, & l=0,\ 1,\ldots,\ k-1; \\ \frac{\overline{\beta}_l}{\overline{\beta}_{l-1}}\sigma_{k,l-1}+\frac{b_k}{b_{k-1}}\sigma_{k-1,l}-\frac{b_k}{b_{k-1}}\frac{\overline{\beta}_l}{\overline{\beta}_{l-1}}\sigma_{k,l-1}+\sigma_{k-1,l-1} & \\ -\frac{\overline{\beta}_l}{\overline{\beta}_{l-1}}(1-|\beta_{l-1}|^2)\sigma_{k-1,l-2}-\frac{b_k}{b_{k-1}}(1-|b_{k-1}|^2)\sigma_{k-2,l-1} & \\ +\frac{b_k}{b_{k-1}}(1-|b_{k-1}|^2)\frac{\overline{\beta}_l}{\overline{\beta}_{l-1}}(1-|\beta_{l-1}|^2)\sigma_{k-2,l-2}, & l=k,\ k+1,\ldots,\ N \end{cases}$$

In the case $\pi_k(z)=z^k$, i.e. $\beta_k=0$ for $k=0,\ 1,\ldots,$ the $(m)-T_C$-recurrence relations are transformed into the well-known Levinson algorithm [14]:

step 1: $P_0(z)=1, \qquad \sigma_{0,0}=<1,1>,$

step 2: for $k=1,\ 2,\ldots,\ N$

$$\begin{aligned} b_k &= -\frac{1}{\sigma_{k-1,k-1}}<zP_{k-1}(z),1>, \\ \sigma_{k,k} &= (1-|b_k|^2)\sigma_{k-1,k-1}, \\ P_k(z) &= zP_{k-1}(z)+b_kP^*_{k-1}(z). \end{aligned}$$

Let us note now that if the orthogonal polynomial $P_n(z)$ of the first type is known the orthogonal polynomial $Q_n(z)$ of the second type can be found by means of the relation

$$\begin{bmatrix} q_{n,0} \\ q_{n,1} \\ q_{n,2} \\ \vdots \\ q_{n,n} \end{bmatrix} = -\frac{1}{c_0}\begin{bmatrix} 0 & c_1 & c_2 & c_3 & \ldots & c_n \\ 0 & c_0 & 2c_1 & 2c_2 & \ldots & 2c_{n-1} \\ 0 & 0 & c_0 & 2c_1 & \ldots & 2c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & c_0 \end{bmatrix}\begin{bmatrix} p_{n,0} \\ p_{n,1} \\ p_{n,2} \\ \vdots \\ p_{n,n} \end{bmatrix}, \tag{19}$$

where $[q_{n,0},\ q_{n,1},\ldots,\ q_{n,n}]^T$ is the vector of coefficients of the polynomial $Q_n(z)=\sum\limits_{m=0}^{n}q_{n,m}z^m$, $[p_{n,0},\ p_{n,1},\ldots,\ p_{n,n}]^T$ is the vector of coefficients of the polynomial $P_n(z)=\sum\limits_{m=0}^{n}p_{n,m}z^m$ and $c_0,\ c_1,\ldots,\ c_N$ are the given moments.

VI. EXAMPLE

In this section we give an illustrative example and consider the PSDE problem with one gap $[r,t]$, i.e., further we will suppose $m=1$. In this case the above described numerical algorithm is transformed as follows:

Step 1 We find the system of monic polynomials $\{P_k(z)\}_{k=0}^N$ orthogonal with respect to given moment sequence by means of the $(m) - T_C$-recursion or Levinson algorithm.

Step 2 From relation (19) we find the polynomials $Q_{N-1}(z)$ and $Q_N(z)$.

Step 3 We find the functions

$$\mathcal{A}(z,\zeta) = Q_N(z)P_N(z,\zeta) - P_N(z)Q_N(z,\zeta),$$
$$\mathcal{B}(z,\zeta) = P_N^*(z)Q_N(z,\zeta) + Q_N^*(z)P_N(z,\zeta),$$

where

$$P_N(z,\zeta) = P_N(z)P_{N-1}(\zeta) - P_N(\zeta)P_{N-1}(z),$$
$$Q_N(z,\zeta) = Q_N(z)P_{N-1}(\zeta) - Q_{N-1}(z)P_N(\zeta)$$

for $\zeta = \zeta_r$ and $\zeta = \zeta_t$ ($\zeta_r = e^{-ir}$, $\zeta_t = e^{-it}$) and the functions

$$\alpha(z) = |z|^2[|z-\zeta_t|^2|\mathcal{A}(z,\zeta_r)|^2 - |z-\zeta_r|^2|\mathcal{A}(z,\zeta_t)|^2],$$
$$\beta(z) = \overline{z}[|z-\zeta_t|^2\overline{\mathcal{A}(z,\zeta_r)}\mathcal{B}(z,\zeta_r) - |z-\zeta_r|^2\overline{\mathcal{A}(z,\zeta_t)}\mathcal{B}(z,\zeta_t)],$$
$$\gamma(z) = |z-\zeta_t|^2|\mathcal{B}(z,\zeta_r)|^2 - |z-\zeta_r|^2|\mathcal{B}(z,\zeta_t)|^2.$$

Step 4 We calculate the function

$$\varepsilon_0(z) = \left(\frac{|\beta(z)|}{\alpha(z)} - R(z)\right)e^{i\frac{\Im m\beta(z)}{\Re e\beta(z)}},$$

where

$$O(z) = \frac{\beta(z)}{\alpha(z)}, \qquad R(z) = \frac{\sqrt{|\beta(z)|^2 - \alpha(z)\gamma(z)}}{\alpha(z)},$$

for all $z = e^{i\theta}$, $\theta \in [-\pi,\pi]$.

Step 5 We find the PSDE by

$$P_{\varepsilon_0}(z) = \frac{c_0}{h_N}\cdot \Re e\frac{zQ_N(z)\varepsilon_0(z) + Q_N^*(z)}{zP_N(z)\varepsilon_0(z) - P_N^*(z)}, \quad z = e^{i\theta}, \quad \theta \in [-\pi,\pi].$$

Let us note that the functions $P_N(z,\zeta)$, $Q_N(z,\zeta)$, $\mathcal{A}(z,\zeta)$, $\mathcal{B}(z,\zeta)$, $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ from Steps 1–3 are polynomials and, therefore, it is sufficient to perform all calculations only with the coefficients of these polynomials, but not with their values. The computation of values of the functions $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ for $z = e^{i\theta}$ is necessary only in Step 4 to calculate the function $\varepsilon_0(z)$ for $z = e^{i\theta}$ and then the PSDE $P_{\varepsilon_0}(z)$ corresponding to the function $\varepsilon_0(z)$.

Now we will consider the elementary numerical example for correlation sequence

$$c_k = \sum_{l=1}^{L} q_l \frac{\alpha_l}{4\alpha_l^2 - k^2} \frac{\sin \frac{\pi k}{2\alpha_l}}{\frac{\pi k}{2\alpha_l}} e^{-ika_l} + \epsilon\delta_k,$$

where q_l, a_l and α_l, $l = 1,\ 2,\ \ldots,\ L$ and L are given values, ϵ is a parameter of regularization. Let us note, that the original spectral power density has the gaps $[a_l + \frac{\pi}{2\alpha_l}, a_{l+1} - \frac{\pi}{2\alpha_l}]$, $l = 0, \ldots,\ L-1$ with the widths $\Delta\theta = a_{l+1} - a_l - \frac{\pi}{2}\left(\frac{1}{\alpha_l} + \frac{1}{\alpha_{l+1}}\right)$. The original power spectral density for $L = 2$, $a_1 = -\pi/7, a_2 = \pi/9, q_1 = 10, q_2 = 100, \alpha_1 = 16.1, \alpha_2 = 2.37$ is shown in Fig. 1

We assume that finite number of moments is known, for example $N = 21$. The PSDE obtained by means of proposed technique for the pointed out parameters and different values of the regularization parameter ϵ is presented in Fig. 2.We used the algorithm for $m = 1$ and $r = a_1 + \frac{\pi}{2\alpha_1} + 0.15$, $t = a_2 - \frac{\pi}{2\alpha_2} - 0.15$. As one can see from the figures the proposed algorithm helps to get the PSDE with good resolution and to keep gaps in the PSDE.

VII. CONCLUSION

We considered the spectral power density estimation problem with gaps. The maximum entropy (ME) approach developed in the paper to solve the problem is universal and can be applied to the solution of other problems. The specificity of PSDE problem prompts us to consider trigonometric moment problem and construct the ME algorithm to find the extreme solution of trigonometric moment problem with gaps. In complete analogy the classic moment problem can be considered and the ME algorithm to find the extreme solution of moment problem with gaps can be obtained.

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