RANK ONE PERTURBATIONS AT INFINITE COUPLING

F. Gesztesy[1] and B. Simon[2]

Abstract. We discuss rank one perturbations $A_\alpha = A + \alpha(\varphi, \cdot)\varphi, \alpha \in \mathbb{R}, A \geq 0$ self-adjoint. Let $d\mu_\alpha(x)$ be the spectral measure defined by $(\varphi, (A_\alpha - z)^{-1}\varphi) = \int d\mu_\alpha(x)/(x-z)$. We prove there is a measure $d\rho_\infty$ which is the weak limit of $(1+\alpha^2)\, d\mu_\alpha(x)$ as $\alpha \to \infty$. If φ is cyclic for A, then A_∞, the strong resolvent limit of A_α, is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\rho_\infty)$. This generalizes results known for boundary condition dependence of Sturm-Liouville operators on half-lines to the abstract rank one case.

§1. Introduction

This paper is a contribution to the theory of rank one perturbations which in its natural format involves a self-adjoint operator, $A \geq 0$ in a complex separable Hilbert space $\mathcal{H}$, and a vector, $\varphi \in \mathcal{H}_{-1}(A)$, with $\mathcal{H}_s(A)$ the scale of spaces associated to A. Then $q_\varphi(\psi, \eta) = (\psi, \varphi)(\varphi, \eta)$ defines a quadratic form on $\mathcal{H}_{+1}(A)$ with q_φ a form-bounded perturbation of A with relative bound zero. Accordingly, $A_\alpha \equiv A + \alpha(\varphi, \cdot)\varphi, \alpha \in \mathbb{R}$ defines a self-adjoint operator with $\mathcal{H}_s(A_\alpha) = \mathcal{H}_s(A)$ for $|s| \leq 1$.

We will suppose that φ is cyclic for A in which case it is easy to see that φ is also cyclic for each A_α. If $d\mu_\alpha$ is the spectral measure for φ associated to A_α, then A_α is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_\alpha)$. Define

$$F_\alpha(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha(x)}{x - z}$$

where $\varphi \in \mathcal{H}_{-1}(A_\alpha)$ implies that

$$\int_{\mathbb{R}} \frac{d\mu_\alpha(x)}{|x| + 1} < \infty$$

[1]Department of Mathematics, University of Missouri, Columbia, MO 65211. E-mail: mathfg@mizzou1.missouri.edu

[2]Division of Physics, Mathematics and Astronomy, California Institute of Technology, 253-37, Pasadena, CA 91125. This material is based upon work supported by the National Science Foundation under Grant No. DMS-9101715. The Government has certain rights in this material.

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so that the integral defining F converges. One has the basic formula (with $F(z) \equiv F_{\alpha=0}(z)$)

$$F_\alpha(z) = \frac{F(z)}{1+\alpha F(z)}. \tag{1}$$

We are interested here in the case $\alpha = \infty$. By the monotone convergence theorem for forms ([3,6]), we have that $\operatorname*{s-lim}_{\alpha\to\infty}(A_\alpha - z)^{-1}$ exists (the existence also follows from the explicit formula for $(A_\alpha - z)^{-1}$, eq. (6) below) and can be described as follows. Let

$$\mathcal{H}_{+1}(A_\infty) = \{\psi \in \mathcal{H}_{+1} \mid (\varphi, \psi) = 0\}$$

and $\mathcal{H}(A_\infty) = \overline{\mathcal{H}_{+1}(A_\infty)}$. This is all of $\mathcal{H}$ if $\varphi \notin \mathcal{H}$ and a codimension one subspace if $\varphi \in \mathcal{H}$. Let A_∞ be the self-adjoint operator on $\mathcal{H}(A_\infty)$ defined by the closed quadratic form $\psi, \eta \mapsto (\psi, A\eta)$ on $\mathcal{H}_{+1}(A_\infty)$. If $\mathcal{H}(A_\infty) \neq \mathcal{H}$, extend $(A_\infty - z)^{-1}$ to all of $\mathcal{H}$ by setting it zero on $\mathcal{H}(A_\infty)^\perp$. Then $\operatorname{s-lim}(A_\alpha - z)^{-1} = (A_\infty - z)^{-1}$.

By (1), $d\mu_\alpha(x) \to 0$ weakly as $\alpha \to \infty$, so we do not have any obvious spectral measure of A_∞. Our main goal here is to prove that $(1+\alpha^2)\, d\mu_\alpha$ does have a weak limit as $\alpha \to \infty$ which is the spectral measure for a vector $\eta \in \mathcal{H}_{-2}(A_\infty)$. Explicitly, define

$$d\rho_\alpha(x) = (1+\alpha^2)\, d\mu_\alpha(x). \tag{2}$$

Then we will prove that

Theorem 1. *There exists a vector, $\eta \in \mathcal{H}_{-2}(A_\infty)$, cyclic for A_∞ so that if $d\rho_\infty(x)$ is the spectral measure for η with respect to A_∞, then*

$$\int_{\mathbb{R}} f(x)\, d\rho_\alpha(x) \to \int_{\mathbb{R}} f(x)\, d\rho_\infty(x) \tag{3}$$

for all continuous functions, f, of compact support.

Note that since $\eta \in \mathcal{H}_{-2}(A_\infty)$,

$$\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(|x|+1)^2} < \infty. \tag{4}$$

It may be that (4) fails if $(|x|+1)^{-2}$ is replaced by $(|x|+1)^{-1}$. We will see explicit examples in §5 where the integral diverges for $(|x|+1)^{-2+\epsilon}$. The proof will show that (3) holds if $f(x) = (|x|+1)^{-\alpha}$ with $\alpha > 2$. There will be examples when it fails if $\alpha = 2$.

Another major result we'll prove is that

$$d\rho_\infty(x) = \lim_{\epsilon\downarrow 0} \pi^{-1}\big[\operatorname{Im}((-F(x+i\epsilon))^{-1})\, dx\big].$$

The abstract theory appears in §2. We discuss boundary condition dependence of Schrö-dinger operators on the half-line in §3. In that case, $d\rho_\alpha$ is the Weyl spectral measure and $d\rho_\infty$ is the Dirichlet spectral measure. In §4, we consider the case when A is bounded. In §5, we discuss a further example.

§2. The Main Results

We begin by recalling some of the standard formulae for rank one perturbations [7]:

$$F_\alpha(z) = F(z)/[1+\alpha F(z)],$$

$$(A_\alpha - z)^{-1}\varphi = (1+\alpha F(z))^{-1}(A-z)^{-1}\varphi, \tag{5}$$

$$(A_\alpha - z)^{-1} = (A-z)^{-1} - \frac{\alpha}{1+\alpha F(z)}\,((A-\bar z)^{-1}\varphi,\cdot)(A-z)^{-1}\varphi, \tag{6}$$

$$\operatorname{Tr}[(A-z)^{-1} - (A_\alpha - z)^{-1}] = \int_{E_\alpha}^{\infty} (\lambda - z)^{-2}\xi_\alpha(\lambda)\,d\lambda, \quad E_\alpha = \min(0, \inf \operatorname{spec}(A_\alpha)),$$

where ξ_α is the Krein spectral shift [4] given by

$$\xi_\alpha(x) = \frac{1}{\pi}\operatorname{Arg}(1+\alpha F(x+i0)). \tag{7}$$

For $\alpha > 0$ we have $\operatorname{Arg}(\cdot) \in [0,\pi]$ and hence $0 \le \xi_\alpha \le 1$ in this case.

If $\|\varphi\| = \infty$, let $P = 0$, and if $\|\varphi\| < \infty$, let P be the projection onto $\{c\varphi \mid c \in \mathbb{C}\}$. Thus, $\mathcal{H}(A_\infty) = \operatorname{Ran}(1-P)$.

Proposition 2. *There exists $\eta \in \mathcal{H}_{-2}(A_\infty)$ so that for all $z \in \mathbb{C}$:*

$$(A_\infty - z)^{-1}\eta = \lim_{\alpha\to\infty} \alpha(1-P)(A_\alpha - z)^{-1}\varphi. \tag{8}$$

If φ is cyclic for A, then η is cyclic for A_∞.

Proof. By (5), the limit on the right side of (8) exists, call it $\psi(z)$, and is given by

$$\psi(z) = F(z)^{-1}(1-P)(A-z)^{-1}\varphi. \tag{9}$$

We have that

$$(A_\alpha - z)^{-1}\varphi - (A_\alpha - w)^{-1}\varphi = (z-w)(A_\alpha - z)^{-1}(A_\alpha - w)^{-1}\varphi. \tag{10}$$

Multiply by α, take $\alpha \to \infty$, and note that if $\|\varphi\| < \infty$, then $P(A_\infty - w)^{-1}\varphi = 0$. We conclude that

$$\psi(z) - \psi(w) = (z-w)(A_\infty - z)^{-1}\psi(w)$$

or

$$\psi(z) = [1 + (z-w)(A_\infty - z)^{-1}]\psi(w). \tag{11}$$

Note that $\psi(z) \in \mathcal{H}(A_\infty)$ (because of the $1-P$) so we can define $\eta(z) \equiv (A_\infty - z)\psi(z)$ in $\mathcal{H}_{-2}(A_\infty)$. (11) precisely says that $\eta(z) = \eta(w)$, that is, it is independent of z; call it η. Cyclicity follows from (9) since if $\{(A-z)^{-1}\varphi\}$ is total in $\mathcal{H}$, then clearly $\{(1-P)(A-z)^{-1}\varphi\}$ is total in $(1-P)\mathcal{H} = \mathcal{H}(A_\infty)$. $\square$

Remark. In §4, we'll prove that when A is bounded, then $\eta = -(1-P)A\varphi$.

Theorem 3. *Let $d\rho_\infty$ be the spectral measure for η. Then*

$$\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2} = F(z)^{-2}\,\frac{dF}{dz} - \frac{1}{\|\varphi\|^2}. \tag{12}$$

Proof. For simplicity, suppose z is real and negative. By definition of η:

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2} &\equiv (\eta, (A_\infty - z)^{-2}\eta) \\
&= \bigl(\varphi, (A-z)^{-1}(1-P)(A-z)^{-1}\varphi\bigr)/F(z)^2 \\
&= \left[(\varphi, (A-z)^{-2}\varphi) - \frac{1}{\|\varphi\|^2}\,\langle\varphi, (A-z)^{-1}\varphi\rangle^2\right]/F(z)^2
\end{aligned}$$

since $P = \|\varphi\|^{-2}(\varphi,\cdot)\varphi$. But this is precisely the right side of (12). (12) for general z follows by analyticity. $\square$

Recall that $d\rho_\alpha$ is defined by (2). Then

Theorem 4. (i)

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^2} = \frac{1}{\|\varphi\|^2} + \int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2}.$$

(ii)

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^3} = \int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^3}.$$

(iii) *For any continuous f of compact support*

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} f(x)\,d\rho_\alpha(x) = \int_{\mathbb{R}} f(x)\,d\rho_\infty(x).$$

Proof. (ii) implies (iii) by a Stone-Weierstrass type argument. (i) implies (ii) by using the fact that both sides are analytic in z on $\mathbb{C}\backslash\mathbb{R}$ so their derivatives in z converge. To prove (i), use Theorem 3 and the calculation

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^2} &= (1+\alpha^2)(\varphi, (A_\alpha - z)^{-2}\varphi) \\
&= \frac{(1+\alpha^2)}{(1+\alpha F)^2}\,(\varphi, (A-z)^{-2}\varphi) \\
&= \frac{1+\alpha^2}{(1+\alpha F)^2}\,\frac{dF}{dz}.
\end{aligned} \tag{13}$$

(13) follows from (5). $\square$

Theorem 5.

$$d\rho_\infty(x) = \pi^{-1} \lim_{\epsilon\downarrow 0} \operatorname{Im}\left[-\frac{1}{F(x+i\epsilon)}\right] dx.$$

Proof. We start with (12) and integrate, noting that $F'/F^2 = \frac{d}{dz}(-1/F)$ to get

$$\int_{\mathbb{R}} d\rho_\infty(x)\left(\frac{1}{x-z} - \frac{1}{x+1}\right) = -\frac{1}{F(z)} + \frac{1}{F(-1)} - (z+1)\,\frac{1}{\|\varphi\|^2}.$$

The theorem then follows by the standard relations between a measure and the boundary values of its Borel transform. □

§3. Variation of Boundary Condition

As an example of the general theory, we consider the case of boundary conditions variation for Schrödinger operators on $L^2(0,\infty)$. The formulae that result are well-known (see, e.g., [1,2,5,8]). The point is that they fit into a more general framework. Let V be continuous and bounded below on $[0,\infty)$. Let H_θ be the operator on $L^2([0,\infty), dx)$ formally given by $-\frac{d^2}{dx^2} + V(x)$ with $u(0)\cos\theta + u'(\theta)\sin\theta = 0$ boundary conditions. One defines the Weyl m-function, $m_\theta(z)$, and Weyl spectral measure, $d\rho_\theta(x)$, so that for $\theta \neq 0$:

$$m_\theta(z) = \cot(\theta) + \int_{\mathbb{R}} \frac{d\rho_\theta(x)}{x-z} \tag{14}$$

and $d\rho_\theta \to d\rho_{\theta=0}$ as $\theta \downarrow 0$. Moreover,

$$m_{\theta=0}(z) = -1/m_{\theta=\pi/2}(z). \tag{15}$$

For $\theta \neq 0$, the Green's function, $G_\theta(0,0;z)$ is related to $m_\theta(z)$ by

$$G_\theta(0,0;z) = \sin^2(\theta)[-\cot\theta + m_\theta(z)]. \tag{16}$$

This fits into the general framework by taking $A = H_{\theta=\pi/2}$ and $\varphi = \delta_0$, the delta function at 0. Then for $\theta \neq 0$,

$$H_\theta = A - \cot(\theta)(\varphi,\cdot)\varphi$$

and $F_{-\cot(\theta)}(z) = G_\theta(0,0;z)$. By (14) and (16), $d\rho_\theta$ is just $(1+\alpha^2)\,d\mu_\alpha$ where $\alpha = -\cot\theta$ and $\lim\limits_{\theta\to 0} d\rho_\theta = d\rho_0$ is just what we found in the last section. (15) is just Theorem 5.

We want to identify the vector η. Let $\psi_+(x,z)$ be the solution of $(-\frac{d^2}{dx^2} + V(x) - z)\psi = 0$ which is L^2 at infinity normalized any way that is convenient. Then from the Wronskian formula for $G_{\theta=\pi/2}$, we get

$$((A-z)^{-1}\varphi)(x) = \frac{\psi_+(x,z)}{\psi_+'(0,z)}$$

and

$$F(z) = -\psi_+(0,z)/\psi'_+(0,z).$$

It follows that

$$F(z)^{-1}(A-z)^{-1}\varphi = \psi_+(x,z)/\psi_+(0,z)$$

which, by the Wronskian formula for $G_{\theta=0}$, is just

$$(A_\infty - z)^{-1}\delta'(x),$$

that is, η is δ' (note that $P=0$ in this case) and $d\rho_\infty$ is the spectral function for the vector δ'.

We note that it is well known that $\int\limits_0^\mu d\rho_\infty(x) \sim C\mu^{3/2}$ as $\mu \to \infty$ so that $\int\limits_0^\infty \frac{d\rho_\infty(\lambda)}{(1+|\lambda|)^k} < \infty$ if and only if $k > \frac{3}{2}$. In particular, $\eta \notin \mathcal{H}_{-1}(A_\infty)$.

§4. Bounded Operators

One gets insight into the general theory by considering the case where A is bounded. Since $\|\varphi\|$ is then finite, we'll suppose $\|\varphi\| = 1$. We'll also get a better understanding of the $\frac{1}{\|\varphi\|^2}$ term in Theorem 4(i). We first note:

Theorem 6. *If A is bounded and $\|\varphi\| = 1$, then $\eta = -(1-P)A\varphi$.*

Proof. If A is bounded, then A_∞ is just $(1-P)A(1-P)$. Thus

$$\begin{aligned}\eta &= F(z)^{-1}(1-P)(A-z)(1-P)(A-z)^{-1}\varphi \\ &= F(z)^{-1}(1-P)(A-z)(A-z)^{-1}\varphi - F(z)^{-1}[(1-P)(A-z)\varphi]F(z).\end{aligned}$$

The first term is zero since $(1-P)\varphi = 0$. The second is $-(1-P)A\varphi$ since $(1-P)z\varphi = 0$. $\square$

Since $\|\varphi\| = 1$, $(\varphi, \cdot)\varphi$ is just a projection, P. Instead of $A + \alpha P$, look at

$$P + \alpha^{-1}A = B_\alpha.$$

P has an isolated simple eigenvalue at 1 with eigenvector φ. Thus by regular perturbation theory [3], B_α has the eigenvalue at $1 + (\varphi, A\varphi)\alpha^{-1} + O(\alpha^{-2})$ with eigenvector

$$\psi_\alpha = \varphi + \alpha^{-1}(1-P)A\varphi + O(\alpha^{-2}) = \varphi + \alpha^{-1}\eta + O(\alpha^{-2}).$$

The first order term is standard perturbation theory where the reduced resolvent $(H_0 - E)^{-1}(1-P)$ is just $-(1-P)$ since H_0 is P is 0 on $\mathrm{Ran}(1-P)$.

Thus, with respect to $A + \alpha P = \alpha B_\alpha$, the measure $(1+\alpha^2)\,d\mu_\alpha$ has a pole of weight $(1+\alpha^2)$ at $E_\alpha = \alpha + (\varphi, A\varphi) + O(\alpha^{-1})$ plus the spectral measure of η for the operator A_∞ plus an error of order α^{-1}. If $\nu > 2$, the pole at E_α makes no asymptotic contribution to $\int\limits_{\mathbb{R}} \frac{d\rho_\alpha(x)}{|x-z|^\nu}$ as $\alpha \to \infty$ but for $\nu = 2$, it makes a contribution of $(1+\alpha^2)/E_\alpha^2 \to 1 = 1/\|\varphi\|^2$.

§5. A Further Example

Let $0 < \gamma < 1$. Let $d\mu_0(x) = \pi^{-1}|x|^{-\gamma}\sin(\pi\gamma)\,dx$ on $[0,\infty)$. Let A be multiplication by x on $L^2([0,\infty), d\mu_0(x))$ and $\varphi \equiv 1$. Then $\int_0^\infty \frac{d\mu_0(x)}{|x|+1} < \infty$ so $\varphi \in \mathcal{H}_{-1}(A_\infty)$.

$$F(z) = \int_0^\infty \frac{d\mu_0(x)}{x-z} = (-z)^{-\gamma}$$

(the easiest way to see this is to compute the imaginary part of $(-z)^{-\gamma}$ for $z = x + i\epsilon$ with $\epsilon \to 0$). Then, by Theorem 5,

$$d\rho_\infty(x) = \pi^{-1}|x|^{\gamma}\sin(\pi\gamma)\,dx.$$

It follows that $\int_0^\infty d\rho_\infty(x)/(|x|+1)^k < \infty$ only if $k > 1+\gamma$. Thus, we cannot conclude in general that $\int_0^\infty d\rho_\infty(x)/(|x|+1)^k < \infty$ for any $k < 2$.

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