THE SYSTEM OF TWO SPINNING DISKS IN THE TORUS.

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November 4, 1993

Abstract.

We study the system of two spinning disks in the torus. We show how to reduce this nonhamiltonian system to a 3 dimensional measure preserving reversible map. We establish that, in contrast to the case of elastic collisions, this system may have periodic orbits with all Floquet exponents on the unit circle.

We would like to thank Eugene Gutkin and Dave Levermore for helpful and enlightening discussions. The suggestions of the referee are also gratefully acknowledged.

§0.Introduction.

The Boltzman-Sinai system of hard balls, or two dimensional disks, moving in a torus and colliding elastically is nonuniformly hyperbolic ([S], also [S-Ch] and [W1]). In particular, the measure of the set of quasiperiodic motions is zero.

The elastic collision in such a system ignores the rotational degree of freedom of the balls. Physically it describes the case of massive particles with finite size but all the mass concentrated at the center. Broomhead and Gutkin in a recent paper [B-G] proposed to study the dynamics of spinning disks. They introduced the collision interaction which takes into account the rotation of the disks. It is based on the assumptions of no-slip friction and energy preservation.

In the present paper we consider the system of two spinning disks in the two dimensional torus. In the case of elastic collisions Sinai [S], using the natural first integrals, reduced this Hamiltonian system to the billiard system in the torus with a circular obstacle (see also [S-W]). The system of spinning disks is not Hamiltonian but it can also be reduced to the system of a point particle moving in the torus with a circular obstacle. The point particle has additionally an intrinsic (angular) velocity which affects the linear velocity after a collision with the obstacle. The system has a natural invariant measure and is reversible (in the sense that forward and backward dynamics are conjugate by an involution).

We establish that in our system Floquet exponents for some periodic orbits lie on the unit circle, if the disks and their moments of inertia are sufficiently large. Thus in the reduced system we have linearly stable orbits. As in the case of Hamiltonian systems the linear stability does not prevent the actual orbits starting in the neighborhood of our periodic solution from wandering away.

In Hamiltonian systems the information about the behavior of orbits in the vicinity of an elliptic periodic orbit is furnished by the KAM theory. Sevryuk [Sev] developed systematically the KAM theory for reversible systems. His results (Theorem 2.9) could be applied to our case , but the calculation of higher order terms and the checking of the nondegeneracy conditions is cumbersome, and we did not do it. Taking for granted these conditions, we would arrive at the conclusion that the union of quasiperiodic orbits in the vicinity of our periodic orbit has positive Lebesgue measure and thus our system is not ergodic.

The plan of the paper is as follows. In Section 1 we define the no-slip collision of spinning disks. In Section 2 we reduce the system of two spinning disks in the torus to a four dimensional flow. We also describe the simple periodic orbits in the system. In Section 3 we calculate when the Floquet exponents for these periodic orbits lie on the unit circle. We end with remarks about the impossibility of full hyperbolicity in the system of two spinning disks.

§1. Collisions of spinning disks.

Let us consider the system of two nonoverlapping disks in the plane $\mathbb{R}^2$ or the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $q_i \in \mathbb{R}^2$ be the positions of the centers of the disks and $v_i \in \mathbb{R}^2$ their linear velocities, $i = 1, 2$. By $\omega_i,\ i = 1, 2$, we denote the angular velocities of the disks with respect to the centers. We assume that the two particles are identical, in particular they have the same radius r, the same mass m, and their moments of inertia I are the same.

We require that the center of mass of the disk coincides with its center, but we do not restrict the distribution of mass in the disk, so that the moment of inertia I may assume any value between 0 (the mass concentrated in the center) and mr^2 (the mass concentrated at the perimeter). The uniform distribution of mass gives $I = \frac{1}{4}mr^2$.

Between collisions the linear velocities of the centers of the disks and their angular velocities stay constant. The total energy of the system is equal to

$$E = \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}I(\omega_1^2 + \omega_2^2).$$

At the collision of the disks, i.e., when

$$q_2 - q_1 = 2re \quad \text{where } e = \frac{q_2 - q_1}{\|q_2 - q_1\|},$$

(or simply $\|q_2 - q_1\| = 2r$), the velocities of the disks $(v_1, v_2, \omega_1, \omega_2)$ experience instantenous change – the collision interaction. Let $w \in \mathbb{R}^6$ denote the joint velocity vector of the two disks, $w = (v_1, v_2, \omega_1, \omega_2)$. By $w^- = (v_1^-, v_2^-, \omega_1^-, \omega_2^-)$ we denote the velocities immediately before the collision and by $w^+ = (v_1^+, v_2^+, \omega_1^+, \omega_2^+)$ the velocities immediately after the collision.

The collision interaction of spinning disks was recently studied by Broomhead and Gutkin in [B-G]; also Kozlov and Treschev discuss in [K-T] the limit case of the collision with a wall. We do not give here the derivation of the collision rules from basic principles of mechanics. The reader is referred to [B-G] for such a derivation. Instead we list the many natural properties which this interaction possess and finally present the formula.

(1) **Linearity.**

There is a linear map Λ independent of the velocities such that

$$w^+ = \Lambda w^-.$$

(2) **Reversibility.**

If $w^+ = \Lambda w^-$ then $-w^- = \Lambda(-w^+)$. It is equivalent to

$$\Lambda^2 = Id = \text{identity matrix}.$$

(3) **Energy preservation.**

$$E(w^+) = E(w^-).$$

It is equivalent to the orthogonality of the matrix Λ with respect to the scalar product defined by the quadratic form E.

(4) **Preservation of the linear momentum $U = mv_1 + mv_2$.**

$$v_1^+ + v_2^+ = v_1^- + v_2^-.$$

(5) **Normal and tangential components interact separately.**

Let us consider the orthonormal frame (e, f) at the point of contact. We define as before

$$e = \frac{q_2 - q_1}{\|q_2 - q_1\|}$$

and f is orthogonal to e and such that (e, f) has positive (counterclockwise) orientation. We will call the e-components of linear velocities normal and the f-components of the velocities tangential. Denoting by $\langle \cdot, \cdot \rangle$ the standard scalar product in tangent planes of $\mathbb{T}^2$, we have that, if $\langle v_i^-, e \rangle = 0$, then $\langle v_i^+, e \rangle = 0, i = 1, 2$ and if $\langle v_i^-, f \rangle = 0$, then $\langle v_i^+, f \rangle = 0, i = 1, 2$.

(6) **Preservation of the angular momenta $A = \pm mr\langle v, f \rangle + I\omega$.**

The angular momenta of the disks, $A_1 = -mr\langle v_1, f \rangle + I\omega_1$ and $A_2 = mr\langle v_2, f \rangle + I\omega_2$, calculated with respect to the point of contact at the time of the collision are preserved. We have

$$-mr\langle v_1^+, f \rangle + I\omega_1^+ = -mr\langle v_1^-, f \rangle + I\omega_1^- \quad \text{and}$$
$$mr\langle v_2^+, f \rangle + I\omega_2^+ = mr\langle v_2^-, f \rangle + I\omega_2^-.$$

Let us stress that the angular momentum is only a "local" integral; it cannot be properly defined for the system of disks moving in a torus.

(7) **Friction at the point of contact**

If tangential components of the linear velocities of the points in contact at the collision coincide, then they are not changed (because there is no friction to change them). More precisely, if

$$\langle v_1^-, f\rangle + r\omega_1^- = \langle v_2^-, f\rangle - r\omega_2^-,$$

then

$$\langle v_i^+ - v_i^-, f\rangle = 0, \quad \text{and} \quad \omega_i^+ = \omega_i^-, i = 1, 2.$$

(8) **Exchange of velocities**

The disks exchange the linear velocities of their points in contact at the collision, i.e.,

$$v_1^+ + r\omega_1^+ f = v_2^- - r\omega_2^- f,$$

and

$$v_2^+ - r\omega_2^+ f = v_1^- + r\omega_1^- f.$$

Expanding the linear velocity vectors v_1 and v_2 in the orthonormal basis (e, f) let us consider the vector $(\langle v_1, f\rangle, \langle v_2, f\rangle, \omega_1, \omega_2) \in \mathbb{R}^4$. The operator Λ projected to this 4 dimensional subspace has the following form

$$\Lambda = \begin{pmatrix} \frac{1}{1+\xi^2} & \frac{\xi^2}{1+\xi^2} & \frac{-r\xi^2}{1+\xi^2} & \frac{-r\xi^2}{1+\xi^2} \\ \frac{\xi^2}{1+\xi^2} & \frac{1}{1+\xi^2} & \frac{r\xi^2}{1+\xi^2} & \frac{r\xi^2}{1+\xi^2} \\ \frac{-1}{r(1+\xi^2)} & \frac{1}{r(1+\xi^2)} & \frac{\xi^2}{1+\xi^2} & \frac{-1}{1+\xi^2} \\ \frac{-1}{r(1+\xi^2)} & \frac{1}{r(1+\xi^2)} & \frac{-1}{1+\xi^2} & \frac{\xi^2}{1+\xi^2} \end{pmatrix}$$

where $\xi = \sqrt{\frac{I}{mr^2}} \leq 1$. The description of Λ is completed by its action on the 2-dimensional subspace of vectors $(\langle v_1, e\rangle, \langle v_2, e\rangle)$

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Boltzman-Sinai system of hard disks is obtained in the limit as ξ goes to 0 as a factor system. It occurs when the mass is concentrated at the center. In this limit the tangential components of the linear velocities are not affected by the angular velocities. (It is interesting that in this limit the angular velocities are changed depending on the linear velocities, but they give no contribution to the energy.)

It is a consequence of the preservation of the linear and angular momenta in the collision that $\omega_1 - \omega_2$ is preserved, i.e.,

$$\omega_1^+ - \omega_2^+ = \omega_1^- - \omega_2^- .$$

The operator Λ has eigenvalues 1 with multiplicity 4 and -1 with multiplicity 2. The appearance of the double eigenvalue -1 is responsible for the **non**preservation of the symplectic structure.

§2. Reduction of the system.

In the case of the elastic collisions Sinai [S] (see also [S-W]) reduced the system of two disks to the billiard system in the torus with a circular obstacle. It turns out that a similar reduction is possible also in our case, notwithstanding the nonhamiltonian character of the system.

We introduce the relative position of the disks $q = q_2 - q_1$ which can be considered as a point in

$$\mathbb{T}_r^2 = \{q \in \mathbb{T}^2 \mid \|q + k\| \geq 2r \quad \text{for any integer vector} \quad k\}.$$

The relative velocity is $u = v_2 - v_1$. We have observed that $\Omega = \omega_1 - \omega_2$ is the first integral in our system. We introduce the following variable $z = r\xi(\omega_1 + \omega_2)$. We have

$$E = \frac{1}{4m}U^2 + \frac{I}{4}\Omega^2 + \frac{m}{4}\left(u^2 + z^2\right).$$

We can see that $u^2 + z^2$ has constant value. Between collisions both u and z are constant. It turns out that at the collision the relative velocities u and z change independently of the other variables. Hence we obtain the following factor system.

A point particle moves in $\mathbb{T}_r^2$ with a constant velocity u. It also carries an intrinsic scalar velocity z. When the point particle reaches the circular obstacle both of these velocities are instantaneously changed in the following way. Let e be the unit normal vector to the obstacle pointing outward, and let f be the unit vector orthogonal to e (i.e., tangent to the obstacle) and such that the frame (e, f) is positively oriented. Let further

$$u_n = \langle u, e \rangle,$$

$$u_t = \langle u, f \rangle$$

be the normal and tangential components of the velocity u respectively. At the collision the normal component is reversed ($u_n^+ = -u_n^-$) and the tangential component interacts with the intrinsic velocity z by the formula

$$\begin{pmatrix} u_t^+ \\ z^+ \end{pmatrix} = \begin{pmatrix} \frac{1-\xi^2}{1+\xi^2} & \frac{2\xi}{1+\xi^2} \\ \frac{2\xi}{1+\xi^2} & \frac{-1+\xi^2}{1+\xi^2} \end{pmatrix} \begin{pmatrix} u_t^- \\ z^- \end{pmatrix}.$$

Putting $\xi = \tan\alpha,\ 0 \le \alpha \le \frac{\pi}{2}$ we get

$$\begin{pmatrix} u_t^+ \\ z^+ \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} u_t^- \\ z^- \end{pmatrix}.$$

In this way we obtain the flow $\Phi^t : \mathbb{T}_r^2 \times \mathbb{S}^2 \to \mathbb{T}_r^2 \times \mathbb{S}^2, t \in \mathbb{R}$, where $\mathbb{S}^2 = \{ (u, z) \in \mathbb{R}^2 \times \mathbb{R} \mid u^2 + z^2 = 1 \}$ is the unit sphere of velocities. At the boundaries of $\mathbb{T}_r^2$ we identify the incoming velocities and the outgoing velocities according to the reflection law above.

The flow preserves the measure μ equal to the product of the standard Lebesgue measures in $\mathbb{T}_r^2$ and $\mathbb{S}^2$. Moreover it is a reversible system in the sense that, if we consider the map $R : \mathbb{T}_r^2 \times \mathbb{S}^2 \to \mathbb{T}_r^2 \times \mathbb{S}^2$

$$R(x, u, z) = (x, -u, -z),$$

then R^2 is the identity map and $\Phi^t \circ R = R \circ \Phi^{-t}$ for all times t.

Figure 1. The periodic orbit.

This flow has many periodic orbits. Indeed, let us consider a segment in $\mathbb{T}_r^2$ with both endpoints on the boundary of the obstacle and perpendicular to the boundary, see Fig.1. This segment carries a periodic orbit of our flow in the sense that, if the velocity u is parallel to the segment and $z = 0$ then we have a periodic orbit. Also every nearby parallel segment carries a periodic orbit when we choose z so that the velocity u is reversed at the collisions.

§3. Linear stability of the periodic orbit.

Let us to introduce a section of the flow. As it is done in the case of the Sinai billiards, we consider the dynamics from collision to collision. Our reduced phase space is the solid torus $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^2_+$, where $\mathbb{S}^1$ is the boundary of the obstacle and $\mathbb{S}^2_+$ is the semisphere of velocities with the u-component pointing outwards. We obtain the map $\Psi : \mathcal{M} \to \mathcal{M}$ which is a local diffeomorphism with discontinuities produced by the orbits tangent to the obstacle. We introduce coordinates $(s,\, u_t,\, z)$ into $\mathcal{M}$, where s is the arc length along the boundary of the obstacle and u_t is the tangential component of the velocity u, so that $u_t^2 + z^2 \le 1$. The measure $ds \wedge du_t \wedge dz$ is invariant under the map Ψ. We introduce the involution $\Upsilon : \mathcal{M} \to \mathcal{M}$,

$\Upsilon(s,\, u_t,\, z) = (s,\, -\cos 2\alpha u_t - \sin 2\alpha z,\, -\sin 2\alpha u_t + \cos 2\alpha z)$. The map Ψ is reversible with respect to this involution, i.e., $\Upsilon \circ \Psi = \Psi^{-1} \circ \Upsilon$.

In the calculation of the derivative of the Poincaré map for our periodic orbit we will use the ideas developed for other "flows with collisions" [W2]. It is also advantageous to extend the dynamics to the large ambient space $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}$. Let

$$\widetilde{\Phi}^t : \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}$$

be the flow describing the free motion of a point particle in $\mathbb{T}^2$, i.e.,

$$\widetilde{\Phi}^t(q,\, u,\, z) = (q + tu,\, u,\, z),$$

where $q \in \mathbb{T}^2$, $u \in \mathbb{R}^2$, $z \in \mathbb{R}$.

Let further

$$\Gamma : \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{R}$$

be the following "collision map" $\Gamma(s^-,\, u^-,\, z^-) = (s^+,\, u^+,\, z^+)$, where

$$\begin{aligned} s^+ &= s^-, \\ u^+ &= -\langle u^-,\, e\rangle e + \left(\cos 2\alpha \langle u^-,\, f\rangle + \sin 2\alpha z^-\right) f, \\ z^+ &= \sin 2\alpha \langle u^-,\, f\rangle - \cos 2\alpha z^-, \end{aligned}$$

s is the arc length parameter and $(e,\, f) = (e(s),\, f(s))$ is the positively oriented orthonormal frame at the boundary $\mathbb{S}^1$ of the obstacle, e is the unit normal vector pointing outward and f is the unit tangent vector.

Let us denote by $\tau : \mathbb{S}^1 \times \mathbb{S}^2_+ \to \mathbb{R}$ the return time to the section $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^2_+$. Now the section map $\Psi : \mathcal{M} \to \mathcal{M}$ can be described as the following composition

$$\Psi = \Gamma \circ \widetilde{\Phi}^\tau_{|\mathcal{M}}.$$

This composition corresponds to the natural separation of the dynamics into the free motion followed by the collision with the obstacle. We will be able to find the derivative of Ψ, if we learn how to differentiate $\widetilde{\Phi}^\tau$ and Γ.

We have

$$D\widetilde{\Phi}^t = (dq + tdu,\, du,\, dz)\,. \tag{1}$$

The derivative of $\widetilde{\Phi}^\tau$ is obtained by composing (1) with the projection on $\mathcal{M}$ in the direction of the velocity vector of the flow (equal to $(u,\, 0,\, 0)$). In the case when the orbit is orthogonal to the obstacle (u parallel to e) we get

$$D\widetilde{\Phi}^\tau_{|\mathcal{M}} = (dq + \tau\langle du,\, f\rangle f,\, du,\, dz)\,. \tag{2}$$

We proceed with the differentiation of Γ. Let us stress that Γ is linear in the **variable** frame. We have

$$\begin{aligned} de &= \kappa ds f, \\ df &= -\kappa ds e \end{aligned} \tag{3}$$

where $\kappa = (2r)^{-1}$ is the curvature of the obstacle $\mathbb{S}^1$.

The differentiation of Γ yields $ds^+ = ds^-$, and

$$\begin{aligned} du^+ =& -\langle du^-,\, e\rangle e - \langle u^-,\, de\rangle e - \langle u^-,\, e\rangle de + \\ & \left(\cos 2\alpha\langle du^-,\, f\rangle + \cos 2\alpha\langle u^-,\, df\rangle + \sin 2\alpha dz^-\right) f + \left(\cos 2\alpha\langle u^-,\, f\rangle + \sin 2\alpha z^-\right) df, \\ dz^+ =& \sin 2\alpha\langle du^-,\, f\rangle + \sin 2\alpha\langle u^-,\, df\rangle - \cos 2\alpha dz^-, \end{aligned}$$

We restrict our attention to orbits perpendicular to the obstacle, which allows us to use $\langle u^-,\, f\rangle = 0$ and $z^- = 0$. Taking this and (3) into account we get

(4)

$$\begin{aligned} du^+ &= -\langle du^-,\, e\rangle e + \left(-\langle u^-,\, e\rangle\kappa ds + \cos 2\alpha\langle du^-,\, f\rangle - \cos 2\alpha\langle u^-,\, e\rangle\kappa ds + \sin 2\alpha dz^-\right) f, \\ dz^+ &= \sin 2\alpha\langle du^-,\, f\rangle - \sin 2\alpha\langle u^-,\, e\rangle\kappa ds - \cos 2\alpha dz^-. \end{aligned}$$

Let us consider the periodic orbit carried by the segment perpendicular to the obstacle at its both endpoints, as described in the previous section (Fig.1). The

Poincaré section map for this periodic orbit is Ψ^2. On this orbit $\langle u, f\rangle = 0$, $z = 0$ and $\tau\langle u, e\rangle = l$, the length of the orbit. Using the formulas (2) and (4) we can obtain the derivative of Ψ on this orbit. In the coordinates $(ds,\, \langle du, e\rangle,\, \langle du, f\rangle,\, dz) \in \mathbb{R}^4$ we get the product of the following 4×4 matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ c(1+\cos 2\alpha) & 0 & \cos 2\alpha & \sin 2\alpha \\ c\sin 2\alpha & 0 & \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} -1 & 0 & -\tau & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $c = \kappa l \tau^{-1}$. Dropping the redundant component $\langle du,\, e\rangle$ we get the following product of 3×3 matrices

$$D\Psi = \begin{pmatrix} 1 & 0 & 0 \\ c(1+\cos 2\alpha) & \cos 2\alpha & \sin 2\alpha \\ c\sin 2\alpha & \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} -1 & -\tau & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

To obtain the eigenvalues of this matrix, let us observe that it is reversible (as expected in view of the reversibility of Ψ). The determinant of the matrix is -1. Hence one of the eigenvalues is -1 and the product of the other two λ_1, λ_2 is equal to 1. It follows that λ_1 and λ_2 are on the unit circle if and only if the trace of the matrix is between -3 and 1. Calculating the trace of the matrix we get

$$-2 \le -(1+\cos 2\alpha)\kappa l - 2\cos 2\alpha \le 2. \tag{6}$$

Since in our situation the curvature κ is positive, it follows that $D\Psi$ and $D\Psi^2$ have eigenvalues on the unit circle if and only if

$$\kappa l \le 2\xi^2 \tag{7}$$

With these formulas we can address the question of linear stability in the system of a spinning disk inside a planar domain. It was established in [B-G] and [K-T] that the collisions of a spinning disk with a wall are the same as in the reduced system above. Let us consider a periodic orbit carried by the segment perpendicular to the boundary of the domain at both endpoints.

If the curvature κ of the boundary at these endpoints is the same, then the necessary and sufficient condition for the Floquet exponents to be on the unit circle is given by the condition (6). If the boundary is locally concave at the collision points (nonnegative κ) then this condition is equivalent to (7).

In the case of convex boundary (negative κ) we get from (6) the condition

$$-\kappa l \leq 2, \tag{8}$$

which coincides with the condition of linear stability of an ordinary billiard trajectory (it has to be the case since now the parameter ξ is absent).

Let us note that when $\kappa = 0$ we cover the case of the system, considered by Broomhead and Gutkin [B-G], of the disk between two parallel lines. They established the (nonlinear) stability of the periodic orbit perpendicular to the lines. Our conditions (7) and (8) show how much we can curve the boundaries without making the orbit linearly unstable.

For the system of a spinning disk inside a planar circular domain, we obtain that the periodic orbits passing through the center of the disk have all Floquet exponents equal to one ($\kappa l = -2$) but the orbit is unstable. Indeed, the system is completely integrable and we can describe the orbits in the following way. Because of the symmetry of the domain, for a segment of the orbit between collisions, the tangential component of the velocity is the same at the departure from the boundary and at the next arrival. The normal component is constant in absolute value. Hence, although after the reflection the tangent component gets changed, it will assume the original value after the next reflection. Consequently every second segment is tangent to the same concentric circle. We obtain a "double caustic" and all motions are quasiperiodic, Fig.2.

Figure 2. The double caustic.

Let us finally consider the case of the periodic orbit perpendicular to the boundary with different curvatures κ_1, κ_2 at the two endpoints. To get a hold of the eigenvalues of $D\Psi^2$ we use again reversibility of the product of the two matrices (5) with different values of c. Since the determinant of this product is equal to 1 we can conclude that one eigenvalue is equal to 1 and the other two eigenvalues are on the unit circle if and only if the trace of the matrix is between -1 and 3. After a lengthy but straightforward calculation we obtain that this condition is equivalent to

$$0 \leq (\kappa_1 l + 1 - \xi^2)(\kappa_2 l + 1 - \xi^2) \leq (1 + \xi^2)^2. \tag{9}$$

When $\xi = 0$ the condition (9) coincides with the condition of linear stability of the ordinary billiard orbit (cf. [W3], Proposition 3).

§4. Concluding remarks.

By the condition (7) we have linear stability of the periodic orbit when ξ is sufficiently close to 1 (away from the elastic collisions) and the length of the orbit is sufficiently short (the obstacle, or the disks are large). The orbit seems to be degenerate (it has a one parameter family of periodic orbits in its vicinity), but it is only the minimal degeneration caused by the reversibility. One would like to apply the KAM theory to the neighborhood of the periodic orbit. The KAM theory for reversible maps was developed systematically by Sevryuk [Sev]. We found the task of calculating the higher order terms in the expansion of our map prohibitively cumbersome and so we were unable to establish for what values of the parameters the nondegeneracy conditions of Theorem 2.9 of [Sev] hold. If and when this is the case, then most of the nearby orbits are quasi periodic and will stay close forever. In particular we would establish that the system is not ergodic.

Our dynamical system $\Psi : \mathcal{M} \to \mathcal{M}$ cannot have all Lyapunov exponents different from zero almost everywhere and be ergodic. Indeed the number of positive (or negative) exponents is an invariant function. If the system is ergodic the number of positive and the number of negative exponents must be constant almost everywhere. These numbers are also equal because of reversibility. Hence if the system is ergodic, there may be at most one positive and one negative Lyapunov exponent and at least one equal to zero.

Because of this observation we do not expect to find systems of this type with hyperbolicity in all of the phase space.

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