Incompleteness of Sparse Coherent States

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1 Introduction

The Schrödinger reprentation of the Heisenberg group is the unitary representation defined by

$$\rho(p,q,t)f(x) = e^{\pi\imath pq}e^{2\pi\imath t}e^{2\pi\imath qx}f(x+p)$$

where $p, q, x \in \mathbf{R}^n$, $t \in \mathbf{R}$ and $f \in L^2(\mathbf{R}^n)$. The group law of the Heisenberg group is reflected in the formula

$$\rho(p,q,t)\rho(p',q',t') = \rho(p+p', q+q', t+t'+\frac{1}{2}(pq'-p'q)).$$

The unitary operator $\rho(p,q)f := \rho(p,q,0)f$ may be thought of as a phase-space translate of the function f. This paper is concerned with the completeness properties of the set

$$\mathcal{S}_{f,\Lambda} = \{\rho(p,q)f : (p,q) \in \Lambda\} \tag{1}$$

where $f \in L^2(\mathbf{R}^n)$ and Λ is a discrete set in $\mathbf{R}^{2n}$. It is a well known that the completeness of such sets depends on the density of the underlying set of phase space translates. In particular, if the density is sufficiently small, the set $\mathcal{S}_{f,\Lambda}$ should be incomplete. Examination of known special cases leads to the conjecture, made precise in the next paragraph, that the threshold density should be one.

Let Λ be a discrete subset of $\mathbf{R}^N$. Choose E, a measurable set of measure one. Denote by $\nu^+(r, E)$ the maximum of the number of points of Λ occurring in any translate of rE. The quantity $\nu^-(r, E)$ is the minimum of the number of points of Λ occurring in any translate of rE. The uniform upper and lower density of the set Λ are defined by

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\nu^+(r, \Lambda)}{r^N} \tag{2}$$

and

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{\nu^-(r, \Lambda)}{r^N} \tag{3}$$

respectively. If the two quantities above are the same, the set Λ is said to have *uniform density* given by

$$D(\Lambda) = D^+(\Lambda) = D^-(\Lambda).$$

It is a theorem of H.J. Landau [5] that these uniform densities are independent of the particular choice of set E with measure one.

Conjecture 1 *Let $\Lambda \subset \mathbf{R}^{2n}$ be a discrete set in phase space and $f \in L^2(\mathbf{R}^n)$ If $D^+(\Lambda) < 1$ then the set $\mathcal{S}_{f,\Lambda}$ is incomplete in $L^2(\mathbf{R}^n)$.*

In the case where $f(x)$ is a gaussian, the validity of this conjecture follows from the work of K. Seip and K. Seip and Wallstén; see [8] and [9]. The main thrust of the conjecture is that the imcompleteness phenomena is claimed for an arbitrary function $f \in L^2$. When Λ has the additional structure of a lattice in $\mathbf{R}^n$ this result follows from work of M. Rieffel [7]. His approach relies on the computation of the coupling constant of the von Neumann algebra generated by $\rho(p, q)$ where $(p, q) \in \Lambda$. Special cases of this result have been established without recourse to the machinery of operator algebras. One such approach, using the Zak transform, has been given by I. Daubechies [2] under the additional assumption that the density of the lattice is rational. In a recent preprint, A.J.E.M. Janssen [4] has shown that $S_{f,\Lambda}$ can never be a frame (see section 3 for the definition) provided Λ is a lattice with density less than one. In this paper, we shall give a direct proof of the conjecture when Λ is a lattice — our techniques do not use sophisticated concepts from operator algebras but rather an elementary analysis of the 'phase space content' of the functions in $\mathcal{S}_{f,\Lambda}$. It should be noted that the arguments presented also yield a necessary upper bound on the uniform lower density of a discrete set of coherent states that is also an Riesz basis for L^2. In addition, certain variants, where Λ is allowed to be more irregular, are also discussed. Necessary conditions on the density of Λ in order for $S_{f,\Lambda}$ to be a frame, have been given by Landau [6] provided both f and its Fourier transform have faster than linear decay. This type of result is shown to hold without the decay condition in theorem 3. It is hoped that the work presented here will provide the basis for proving the above conjecture.

2 A Comparison Theorem

The statement of the main result in this section requires the following definition.

Definition 1 *Suppose the set $\mathcal{S}_{f,\Lambda}$ is complete for some $f \in L^2(\mathbf{R}^n)$ and Λ a discrete subset of $\mathbf{R}^{2n}$. $\mathcal{S}_{f,\Lambda}$ has the* homogeneous approximation property *if given $g \in L^2(\mathbf{R}^n)$ and $\epsilon > 0$, there is a $R > 0$ such that for any $(p_0, q_0) \in \mathbf{R}^{2n}$, $\rho(p_0, q_0)g$ can be approximated to within ϵ in the L^2-norm by a vector in*

$$\{\rho(p,q)f : (p,q) \in \Lambda \cap B_R(p_0, q_0)\}.$$

Theorem 1 *Let Γ and Λ be discrete sets. Assume that there are two functions $\phi, f \in L^2(\mathbf{R}^n)$ such that $\mathcal{S}_{\phi,\Gamma}$ forms a Riesz bases for $L^2(\mathbf{R}^n)$ and that $\mathcal{S}_{f,\Lambda}$ is complete and has the homogeneous approximation property.*

1. *If Γ and Λ have uniform upper densities given by $D^+(\Gamma)$ and $D^+(\Lambda)$ then*
$$D^+(\Gamma) \leq D^+(\Lambda).$$
2. *If Γ and Λ have uniform lower densities given by $D^-(\Gamma)$ and $D^-(\Lambda)$ then*
$$D^-(\Gamma) \leq D^-(\Lambda).$$

PROOF: Let $(a,b) \in \mathbf{R}^{2n}$. Define V_r to be the span of

$$\{\rho(r_\alpha, s_\alpha)\phi : (r_\alpha, s_\alpha) \in B_r(a,b) \cap \Gamma\}$$

and let W_r be the span of

$$\{\rho(p_i, q_i)f : (p_i, q_i) \in B_r(a,b) \cap \Lambda\}.$$

A consequence of the preceding discussion is that any $\rho(p,q)\phi$ with $(p,q) \in B_r(a,b)$ can be approximated to within ϵ by a function in $W_{r+R}(a,b)$. Consider the self-adjoint operator from $T : V_r \to V_r$ defined as a composition of projections:

$$T = P_{V_r} \circ P_{W_{r+R}}.$$

Since $\mathcal{S}_{\phi,\Gamma}$ is an Riesz basis, there are functions $\psi_\alpha \in L^2(\mathbf{R}^n)$ such that

$$\langle \rho(p_\alpha, q_\alpha)\phi, \psi_\beta \rangle = \delta_{\alpha,\beta}.$$

The above arguments imply that,

$$\mathrm{Tr}(T) = \sum_{(r_\alpha,s_\alpha)\in B_r(a,b)\cap\Gamma} \langle T(\rho(r_\alpha, s_\alpha)\phi), \psi_\alpha \rangle \geq (1-2\epsilon)\ \mathrm{card}\left(B_r(0) \cap \Gamma\right).$$

Let

$$d = d((a,b);r) = \mathrm{card}\left(B_r(a,b) \cap \Gamma\right)$$

and denote the eigenvalues of T by $\lambda_1, \cdots, \lambda_d$. Since T is a composition of projections, $\lambda_i \in [0,1]$. It follows that the rank of T dominates the trace

$$\lambda_1 + \cdots + \lambda_d$$

The observation that the rank of T is no greater than the dimension of W_{r+R} produces the following estimate

$$d(1-2\epsilon) \leq \mathrm{Tr}(T) \leq \dim\ W_{r+R}(a,b).$$

The obvious upper bound for the dimension of $W_{r+R}(a,b)$ then produces the estimate

$$(1-2\epsilon)\mathrm{card}\left(B_r(a,b) \cap \Gamma\right) \leq \mathrm{card}\left(B_{r+R}(a,b) \cap \Lambda\right).$$

Clearly, this is equivalent to

$$(1-2\epsilon)\frac{\text{card}\,(B_r(a,b)\cap\Gamma)}{\text{vol}\,(B_r(a,b))} \leq \frac{\text{card}\,(B_{r+R}(a,b)\cap\Lambda)}{\text{vol}\,(B_{r+R}(a,b))}\frac{(r+R)^{2n}}{r^{2n}}. \tag{4}$$

Since the last term tends to one as $r \to \infty$, the two conclusions in the theorem are now straight-forward. Consider item 1. Fix $\delta > 0$, and note that for all $r > r_0$

$$\frac{\text{card}\,(B_{r+R}(a,b)\cap\Lambda)}{\text{vol}\,(B_{r+R}(a,b))} \leq (1+\delta)D^+(\Lambda)$$

Now choose (a,b) and $r > r_0$ so that

$$(1-\delta)D^+(\Gamma) \leq \frac{\text{card}\,(B_r(a,b)\cap\Gamma)}{\text{vol}\,(B_r(a,b))}.$$

By putting these estimates together with equation 4, we have

$$(1-\delta)(1-2\epsilon)D^+(\Gamma) \leq (1+\delta)D^+(\Lambda).$$

Since ϵ, δ were arbitrary positive numbers the first item in the theorem follows. The second item is established in an entirely similar manner.$\diamondsuit$

Theorem 2 *Suppose $\mathcal{S}_{f,\Lambda}$ is complete for some $f \in L^2(\mathbf{R}^n)$ and some lattice $\Lambda \in \mathbf{R}^{2n}$. Then $\mathcal{S}_{f,\Lambda}$ has the homogeneous approximation property.*

PROOF: Let $\Lambda = \{(p_i, q_i)\}$ and let D be a fundamental domain for the lattice Λ that has the origin in its boundary. Fix $\phi \in L^2$.

Fix $\epsilon > 0$ and note, by virtue of the completeness of $\mathcal{S}_{f,\Lambda}$, that for any $(p,q) \in \bar{D}$ there is a ball B centered at the origin (with radius perhaps depending on (p,q)) such that $\rho(p,q)\phi$ can be approximated to within $\epsilon/2$ in L^2-norm by a function in the span of

$$\{\rho(r,s)f : (r,s) \in B \cap \Lambda\}.$$

Denote the radius of this ball $R(p,q)$. The compactness of $\bar{D}$ and the L^2 continuity of the Schrödinger representation implies that there is a finite open cover of $\bar{D}$ by open sets $U_1, \cdots, U_k$ such that if $(a,b),(c,d) \in U_i$,

$$\|\rho(a,b)\phi - \rho(c,d)\phi\| < \epsilon/2.$$

Choose $(p_i, q_i) \in U_i$ and set $R = \max R(p_i, q_i)$. It is easy to check that *any* $\rho(p,q)\phi$ with $(p,q) \in \bar{D}$ can be approximated to within ϵ by a function in the span of

$$\{\rho(p,q)f : (p,q) \in B_R(0) \cap \Lambda\}.$$

Now any $(r,s) \in \mathbf{R}^{2n}$ can be expressed as

$$(r,s) = (p',q') + (r',s') \qquad (r',s') \in \Lambda, (p',q') \in \bar{D}.$$

We know $\rho(p',q')\phi$ can be approximated to within ϵ by a sum of the form

$$\sum_{n=1}^{N} c_n \rho(a_n, b_n) f$$

where $(a_n, b_n) \in B_R(0) \cap \Lambda$. Upon applying the unitary operator $\rho(r',s')$ and using the group law for the Heisenberg group, we get a sum of the form

$$\sum_{n=1}^{N} c'_n \rho(a_n + r', b_n + s') f$$

as an ϵ approximant to $\rho(r',s')\rho(p'q')\phi$. Now, in view of the group law for the Heisenberg group,

$$\rho(r',s')\rho(p'q')\phi = \exp(\pi\imath(r'q' - s'p'))\rho(r,s)\phi.$$

Therefore

$$\sum_{n=1}^{N} \exp(-\pi\imath(r'q' - s'p'))c'_n \rho(a_n + r', b_n + s') f$$

approximates $\rho(r,s)\phi$ to within ϵ in L^2-norm. Note that the points $(a_n + r', b_n + s')$ are in Λ and within distance R of (r,s) and the required property is established.$\diamondsuit$

Two corollaries follow immediately.

Corollary 1 *Let Λ be a lattice in $\mathbf{R}^{2n}$ with density smaller than one and $f \in L^2(\mathbf{R}^n)$. The set $\mathcal{S}_{f,\Lambda}$ is incomplete.*

PROOF: Take $\phi(x)$ to be the characteristic function of the unit cube $[0,1]^n$ and Γ to be the integer lattice. The set $\mathcal{S}_{\phi,\Gamma}$ is an orthonormal basis for L^2. If $\mathcal{S}_{f,\Lambda}$ is complete, theorems 2 and 1 yield the result that

$$1 \leq D^+(\Lambda).\diamondsuit$$

A proof, using von Neumann algebra theory, of the preceding result follows from the work of M. Rieffel [7]. A proof using the Zak transform has been given by I. Daubechies [2] in the case where the density of Λ is a rational number larger than one. A recent preprint of A.J.E.M. Janssen [4] gives a proof of the weaker statement that $\mathcal{S}_{f,\Lambda}$ can never be a frame under these conditions.

Corollary 2 *Let Γ be a discrete set in $\mathbf{R}^{2n}$ with $D^+(\Gamma) > 1$. Then there is no $\phi \in L^2(\mathbf{R}^n)$ for which $\mathcal{S}_{\phi,\Gamma}$ is an Riesz basis of $L^2(\mathbf{R}^n)$.*

Proof: Suppose a Γ and ϕ have the property that $\mathcal{S}_{\phi,\Gamma}$ is a Riesz basis. Then let f be the characteristic function of the unit cube and Λ the integer lattice. Then $\mathcal{S}_{f,\Lambda}$ is complete and has the homogeneous approximation property by theorem 2. Upon applying theorem 1,

$$D^+(\Gamma) \leq 1.\diamondsuit$$

3 Density of Frames

Recall that a sequence of vectors v_n in a Hilbert space H is a *frame* if there are constants $A, B > 0$ such that

$$A\|v\|^2 \leq \sum_n \langle v, v_n\rangle^2 \leq B\|v\|^2 \qquad \text{for all } v \in H.$$

This *frame inequality* insures that a frame is always a complete set. In this section, we will actually consider a somewhat weaker condition.

Definition 2 *A sequence v_n in a Hilbert space H is a* quasi-frame *with respect to a dense subspace D of H if*

$$A\|v\|^2 \leq \sum_n \langle v, v_n\rangle^2 < \infty$$

for every $v \in D$. We will call the constant A the quasi-frame constant.

In this section $H = L^2(\mathbf{R}^n)$ and D is the dense subspace consisting of finite linear combinations of functions of the form

$$\rho(p, q)\phi_0(x)$$

where $\phi_0(x) = 2^{n/4}\exp(-\pi x^2)$ and $(p,q) \in \mathbf{R}^{2n}$. Note that D is invariant under the action of the Heisenberg group. We will develop necessary conditions on a set Λ so that $S_{f,\Lambda}$ is a quasi-frame with respect to the subspace D. To this end, we need the following definition.

Definition 3 *A* uniformly *discrete set Λ of $\mathbf{R}^N$ is a discrete set for which there is a $\delta > 0$ satisfying*

$$|x_1 - x_2| > \delta \qquad \textit{for all distinct } x_1, x_2 \in \Lambda.$$

The constant δ is called the separation constant.

Let Λ be a uniformly discrete subset of $\mathbf{R}^{2n}$ with separation constant $\delta > 0$ and $f \in L^(\mathbf{R}^n)$. We first show that

$$\sum_{(p_0,q_0)\in\Lambda} |\langle \rho(p_0,q_0)f, g\rangle|^2 < \infty \tag{5}$$

for all $g \in D$. Let

$$g(x) = \sum_{i=1}^{N} c_i \rho(r_i, s_i)\phi_0.$$

For any function $h \in L^2$

$$|\langle h, g\rangle|^2 \leq \left(\sum_{i=1}^{N} |c_i|^2\right)\left(\sum_{i=1}^{N} |\langle h, \rho(r_i,s_i)\phi_0\rangle|^2\right). \tag{6}$$

Now fix $(p_0,q_0) \in \Lambda$ and set $h = \rho(p_0,q_0)f$.

We first claim that there is a constant $C > 0$, depending just on the separation constant, such that

$$|\langle \rho(p_0,q_0)f, \rho(r,s)\phi_0\rangle|^2 \leq C \int_{B_{\delta/2}(0,0)} |\langle \rho(p_0{+}p, q_0{+}q)f, \rho(s,t)\phi_0\rangle|^2\, dpdq. \tag{7}$$

To see this, note that the map

$$F : p + \imath q \mapsto \langle \rho(p_0 - s, q_0 - t)f, \rho(-p,-q)\phi_0\rangle \exp\left(\pi/2(p^2+q^2)\right)$$

is an analytic function in the variable $z = p + \imath q$ (see [1]). An application of a standard integral bound for analytic functions [3] gives:

$$|F(0)|^2 \leq C \int_{B_{\delta/2}(0,0)} |F(z)|^2.$$

The group law for the Heisenberg group can be used to compute that

$$|\langle \rho(p_0 - s, q_0 - t)f, \rho(-p, -q)\phi_0\rangle| = \langle \rho(p_0 + p, q_0 + q)f, \rho(s, t)\phi_0\rangle|.$$

The equation 7 follows.

An application of equation 7 to each term in the sum

$$A = \sum_{(p,q)\in\Gamma} |\langle \rho(p,q)f, \rho(r_i, s_i)\phi_0\rangle|^2$$

produces the bound

$$A \leq C \int_{\mathbf{R}^{2n}} |\langle \rho(p,q)f, \rho(r_i, s_i)\phi_0\rangle|^2 \, dpdq$$

The integral on the right is finite by Moyal's formula. It is useful to actually glean a little more from this argument. Clearly,

$$\sum_{(p,q)\in\Gamma\setminus B_r} |\langle \rho(p,q)f, \rho(r_i, s_i)\phi_0\rangle|^2 \leq$$
$$C \int_{B^c_{r-\delta}} |\langle \rho(p,q)f, \rho(r_i, s_i)\phi_0\rangle|^2 \, dpdq = I(r) \tag{8}$$

with $I(r) \to 0$ as $r \to \infty$. This observation, together with equation 6 readily yields the following lemma.

Lemma 1 *Let g be any finite linear combination of phase space translates of the gaussian ϕ_0. Given $\epsilon, \delta > 0$ there is a conmpact set K such that*

$$\sum_{(p,q)\in\Gamma\setminus K} |\langle \rho(p,q)f, g\rangle|^2 \leq \epsilon$$

for any uniformly discrete set Γ with separation constant $\delta > 0$.

Our objective is the following theorem.

Theorem 3 *Let $\Lambda \in \mathbf{R}^{2n}$ be a uniformly discrete subset with $D^+(\Lambda) < \infty$. Suppose that for $f \in L^2(\mathbf{R}^n)$, $\mathcal{S}_{f,\Lambda}$ is a quasi-frame. Then $\mathcal{S}_{f,\Lambda}$ has the homogeneous approximation property.*

When taken together with theorem 1, this yields the following strengthening of a result of H. J. Landau [6].

Corollary 3 *Let $\Lambda \in \mathbf{R}^{2n}$ be a uniformly discrete set. Suppose that for $f \in L^2(\mathbf{R}^n)$, $\mathcal{S}_{f,\Lambda}$ is a quasi-frame. Then*

$$D^-(\Lambda) \geq 1.$$

We first develop some preliminary results necessary for the proof of theorem 3.

Definition 4 *The Frechet distance $[Q, R]$ between two closed sets Q, R of $\mathbf{R}^n$ is the smallest $t \geq 0$ such that*

$$Q \subset R_t \qquad \textit{and} \qquad R \subset Q_t$$

where Q_t is the set of all points within the distance t of Q. Q_n converges wealky to Q if

$$[Q_n \cap K, Q \cap K] \to 0$$

for every compact set K.

Lemma 2 *Let Λ be a uniformly discrete set and $f \in L^2(\mathbf{R}^n)$ such that $S_{f,\Lambda}$ is a frame. Let Λ' be a uniformly discrete set that is a weak limit of the sequence of translates of Λ. Then $S_{f,\Lambda'}$ is a frame with the same frame bounds as $S_{f,\Lambda}$.*

Lemma 3 *Let Λ_n be a sequence of translates of a uniformly discrete set, Λ with $D^+(\Lambda) < \infty$. Then there is a sub-sequence n_i and a uniformly discrete set Λ' such that $D^+(\Lambda') < \infty$ and Λ_{n_i} converges weakly to Λ'.*

We defer the proofs of these two lemmas till after the proof of the above theorem.

Proof of Theorem 3: Suppose the set $S_{f,\Lambda}$ *does not* satisfy the homogeneous approximation property. Fix $\epsilon > 0$. Then there is a $g \in L^2(\mathbf{R}^n)$ and sequences (p_n, q_n) and $r_n \to \infty$ such that the distance of $\rho(p_n, q_n)g$ to the span of

$$\{\rho(p, q)f : (p, q) \in \Lambda \cap B_{r_n}(p_n, q_n)\}$$

is at least ϵ. Set $\Lambda_n = \Lambda - (p_n, q_n)$. By applying the unitary tranformation $\rho(p_n, q_n)$ and using the group law for the Heisenberg group, we observe that g has distance at least ϵ to the span of

$$\{\rho(p, q)f : (p, q) \in \Lambda_n \cap B_{r_n}(0, 0)\}.$$

Lemma 3 assures us that, by passing to a subsequence, Λ_n converges weakly to a uniformly discrete set Λ'. Fix $r > 0$. Given $\kappa > 0$, there is an $N > 0$ such that every $(p, q) \in \Lambda' \cap B_r(0, 0)$ is within κ distance of a point from Λ_n for all $n \geq N$. Consequently, the unit vectors in the span of

$$\{\rho(p, q)f : (p, q) \in \Lambda' \cap B_r(0, 0)\}$$

can be made uniformly close to the span of

$$\{\rho(p, q)f : (p, q) \in \Lambda_n \cap B_{r_n}(0, 0)\}$$

provided n is sufficiently large. Hence the distance between g and the span of

$$\{\rho(p, q)f : (p, q) \in \Lambda' \cap B_r(0, 0)\}$$

cannot be less than ϵ. Since this statement is independent of r, g cannot be in the span of

$$S_{f,\Lambda'} = \{\rho(p, q)f : (p, q) \in \Lambda'\}.$$

But this last set is a quasiframe and, hence, complete. Contradiction.$\diamondsuit$

Proof of lemma 2: Let Λ_n be a sequence of translates of Λ that converges weakly to Λ'. It is clear that there is a $\delta > 0$ that serves as a separation constant for both the Λ_n and Λ'. Let $g(x)$ be a linear combination of a finite number of phase space translates of the gaussian

$$2^{n/4} \exp(-\pi x^2).$$

Fix $\epsilon > 0$. Let K be a compact subset of $\mathbf{R}^{2n}$ satisfying the conditions of lemma 1. For all n sufficiently large, we must have that

$$\left| \sum_{(p,q)\in\Lambda'\cap K} |\langle f, \rho(p, q)g\rangle|^2 - \sum_{(p,q)\in\Lambda_n\cap K} |\langle f, \rho(p, q)g\rangle|^2 \right| \leq \epsilon.$$

By putting this together with the estimate in lemma 1, we obtain the estimate

$$\left| \sum_{(p,q)\in\Lambda'} |\langle f, \rho(p, q)g\rangle|^2 - \sum_{(p,q)\in\Lambda_n} |\langle f, \rho(p, q)g\rangle|^2 \right| \leq 3\epsilon$$

for all n sufficiently large. The sets $S_{\Lambda_n,f}$ are all quasi-frames with the same quasi-frame bound. Hence, $S_{\Lambda',f}$ must also have the same quasi-frame bound.$\diamondsuit$

Proof of lemma 3: Decompose $\mathbf{R}^{2n}$ as the union of non-overlapping closed unit cubes:

$$\mathbf{R}^{2n} = \cup_k Q_k$$

By passing to a subsequence, if necessary, we may assume that the number of points of Λ_n in a particular cube Q_{k_0} eventually stabilizes. This can be done because of the uniformly discrete nature of the Λ_n. A diagonalization argument shows that — by passing to a subsequence, if necessary — for any Q_k there is an $N = N(k)$ such that the number of points in $\Lambda_n \cap Q_k$ stabilizes for all $n \geq N$. Each cube in the decomposition is compact. For each k and $n \geq N(k)$, list the points in $\Lambda_n \cap Q_k$ by $a_{1k}^{(n)}, \cdots, a_{l_k k}^{(n)}$. Note that by passing to a subsequence, we may assume that each $a_{ik}^{(n)}$ converges as $n \to \infty$. A diagonalization argument allows us to assume, by passing to a subsequence if necessary, that this is true for each k. Note by a_{ik} the limit of $a_{ik}^{(n)}$. Set Λ' to be the collection of these limit points. It is easy to verify that Λ' is a uniformly discrete set with finite upper uniform density.$\diamondsuit$

We conclude this paper with the following direct consequence to corollary 2 and corollary 3.

Corollary 4 *Let Λ be a uniformly discrete set and $f \in L^2(\mathbf{R}^{2n})$ such that $S_{f,\Lambda}$ is a Riesz basis. Then Λ must have a uniform density of one:*

$$D^+(\Lambda) = D^-(\Lambda) = 1.$$

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