Preprint UGVA-DPT 1994/04-847

New aspects of field theory*

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Abstract. After a brief reminder of the Cartan 1-form and its application to the mechanics of point particles we review the application of Cartan 4-forms to field theory.

1 Introduction

The beginning of this century saw two revolutions in physics – relativity and quantum theory. In spite of this fact, which should have changed our fundamental conception of nature, consciously or unconsciously most physicists still hold to nineteenth Century ideas on the nature of the physical world. In this philosophy the world is nothing more than a collection of point particles moving in the void and only interacting by contact. Relativity is seen as just modifying the Newtonian laws of motion and quantum theory is regarded

* Talk presented at the Conference on Frontiers in Theoretical Physics at Erdine.

as forbidding precise measurement and so introducing probabilities and statistics. A clear image of physical reality disappears in a cloud of many hypothetical and probabilistic worlds, all in fact built in the same spirit as the schemes proposed by Leibnitz or Descartes.

Such a view, where one attempts to graft quantum theory and relativity onto a preconceived mechanistic worldview, must be forsaken once and for all – we must take quantum theory and relativity seriously (1). The empty space is not the void, it possesses a reality in of itself. Matter is not a system of point particles but is a collection of entities acting on space in a more or less local way (2). Such entities are created or annihilated individually but as a totality. One can describe entities first of all by integers such as leptonic, hadronic or charge quantum numbers and also by their non-local actions on the physical space. These actions are described by fields.

The new generation of experiments involving single neutrons, atoms, electrons or photons demonstrate the reality of non-local phenomena. Such phenomena are completely outside the current philosophy inherited from the nineteenth Century. They require completely new concepts since the notions of position and velocity of small particles of some substance are now definitively inadequate. Calling forth statistical effects or measurement imprecision will not save the situation.

It was the great merit of D. Aerts (3) to introduce the concepts of property and state in physics in a new way well adapted to this new aspect of reality. Given a system, a part of reality, we associate to each possible experimental project a particular property of the system. By experimental project we mean an experiment that can be performed on the system and where one has precisely defined in advance what is to be considered as a positive result. The corresponding property is said to be actual if, were one to realise the experiment, the result would be certainly positive. As we can see, an actual property is nothing more than what Einstein called an element of reality of the system.

We can give a very simple example of an experimental project which, while outside the usual physics, is very precise and adequate for our purposes. The regulations of golf state that the size of a golf ball should not be less that 1.680 inches in diameter. They go on to state that "This requirement will be satisfied if, under its own weight, a ball falls through a 1.680 inches diameter ring gauge in fewer than 25 out of 100 randomly selected positions, the test being carried out at a temperature of $23 \pm 1\ ^0$C." This is a practical example where the positive result is not just one event but a collection of events with a statistical criterion, illustrating very well the generality of the concept of experimental project.

The experimental project just described is non-destructive, one has the same ball after performing the experiment. The opposite case is illustrated by the following experimental project. Consider a chalk. The chalk is said to be breakable if I can snap it into two pieces. However the property of being breakable is a property of the chalk before we decide to perform the experiment. If we verify the breakability of a given chalk then after the experiment we only have pieces of it left.

As a third example let us consider a field in the vacuum. To say that there is an electric field in the vacuum means the following. If a charged particle were introduced at rest into the vacuum then it would be accelerated according to the direction and the intensity of the field. The element of reality, here the electric field, is a property of the vacuum in the absence of the charged particle and has nothing to do with the new complicated situation which results from the introduction of the charged particle.

2 The electromagnetic field

The electromagnetic field can be simply described in four dimensions in terms of differential forms (4). The quadripotential $A_\mu(x) = \big(-V(x), \vec{A}(x)\big)$, with $A_0(x) = -V(x)$ the potential and $\vec{A}(x)$ the potential vector, defines a 1-form

$$A = A_\mu(x) dx^\mu = -V(x)dt + A_i(x)dx^i.$$

Note that since $V(x)$ is like an energy and $\vec{A}(x)$ like a momentum, the negative sign in $A_0(x)$ comes from the thermodynamic convention that the energy increases when work is provided to the system.

The exterior derivative of A gives a 2-form

$$
\begin{aligned}
B = dA &= \partial_\mu A_\nu(x) dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) dx^\mu \wedge dx^\nu .
\end{aligned}
$$

Writing

$$
B = E_i(x) dx^i \wedge dt + B^i(x) dx^j \wedge dx^k + B^j(x) dx^k \wedge dx^i + B^k(x) dx^i \wedge dx^j
$$

we recognise the electric field $E_i = -\partial_i V(x) - \partial_t A_i(x)$ and the magnetic flux $B^k(x) = \partial_i A_j(x) - \partial_j A_i(x)$. From $dB = ddA = 0$ we then derive the first two Maxwell equations

$$
\partial_\rho B_{\mu\nu}(x) + \partial_\mu B_{\nu\rho}(x) + \partial_\nu B_{\rho\mu}(x) = 0,
$$

or equivalently

$$
\begin{aligned}
\partial_t \vec{B}(x) + \operatorname{curl} \vec{E}(x) &= 0 \\
\operatorname{div} \vec{B}(x) &= 0.
\end{aligned}
$$

In Maxwell's theory we also have another differential form, a 2-form of the second kind (an odd form), which changes sign with the sign of the Jacobian of the transformation:

$$
H = \tfrac{1}{4} H^{\mu\nu}(x) \varepsilon_{\mu\nu\rho\lambda} dx^\rho \wedge dx^\lambda ,
$$

where $H^{ij}(x) = -H^{ji}(x) = H_k(x)$ is the magnetic field and $H^{0i}(x) = -H^{i0}(x) = D^i(x)$ is the electric displacement. Here $\varepsilon_{\mu\nu\rho\lambda}$ is the completely antisymmetric tensor such that $\varepsilon_{0123} = 1$ in any coordinate system. Upon changing coordinates it is fourfold covariant and transforms with an extra factor, the inverse of the Jacobian of the transformation. The second pair of Maxwell equations can then be written

$$
dH = J,
$$

where $J = \frac{1}{6} J^\mu(x) \varepsilon_{\mu\nu\rho\lambda} dx^\nu \wedge dx^\rho \wedge dx^\lambda$ and $J^\mu(x) = \big(\rho(x), \vec{J}(x)\big)$ is the quadricurrent (or more simply the current). Here $\rho(x)$ is the charge density and $\vec{J}(x)$ the usual current density. The forms H and J must both be of the second kind in such a way that the total charge $\int \rho(x)\, \mathrm{d}v$ does not depend on some orientation which is not even defined, but is positive or negative depending on the nature of the charge.

The second pair of Maxwell equations can also be written

$$\partial_\nu H^{\mu\nu}(x) = J^\mu(x),$$

or equivalently

$$\begin{aligned} \operatorname{div} \vec{D}(x) &= \rho(x) \\ \operatorname{curl} \vec{H}(x) - \partial_t \vec{D}(x) &= \vec{J}(x). \end{aligned}$$

¿From $dJ = ddH = 0$ we derive the conservation of the charged current:

$$\partial_\mu J^\mu(x) = 0,$$

which is equivalent to

$$\partial_t \rho(x) + \operatorname{div} \vec{J}(x) = 0.$$

Finally, in the linear Maxwell theory the field $H^{\rho\lambda}(x)$ creates, via the vacuum, the field $B_{\mu\nu}(x)$ according to the phenomenological relations

$$B_{\mu\nu}(x) = \tfrac{1}{2} \mu_{\mu\nu\rho\lambda}(x) H^{\rho\lambda}(x)$$

or more explicitly

$$\begin{aligned} \vec{B}(x) &= \mu_0(x) \vec{H}(x), \\ \vec{E}(x) &= \varepsilon_0^{-1}(x) \vec{D}(x). \end{aligned}$$

This means that $\mu_{\mu\nu\rho\lambda}(x)$ is in fact a 6×6 diagonal matrix in the two pairs of indices $\mu\nu$ and $\rho\lambda$. It can be shown (see appendix I) that the phenomenological relations can also be written in an essentially unique way as

$$\begin{aligned} B_{\mu\nu}(x) &= \tfrac{1}{2}\left(\hat{g}_{\mu\rho}(x)\hat{g}_{\nu\lambda}(x) - \hat{g}_{\mu\lambda}(x)\hat{g}_{\nu\rho}(x)\right) H^{\rho\lambda}(x) \\ &= \hat{g}_{\mu\rho}(x)\hat{g}_{\nu\lambda}(x) H^{\rho\lambda}(x). \end{aligned}$$

It is important to remark that while any diffeomorphism of ^{4} leaves the Maxwell equations in the same form, only Lorentz conformal diffeomorphisms also conserve the phenomenological relations.

3 Mechanics in the Cartan formalism

The state space for a point particle in classical mechanics is given by $\Sigma = (\vec{p}, \vec{q}, t)$. A motion of the system is given by functions $t \mapsto \vec{p}(t)$ and $t \mapsto \vec{q}(t)$ satisfying the canonical equations of motion

$$\begin{aligned} \dot{p}_i(t) &= -\frac{\partial H}{\partial q^i}(\vec{p}(t), \vec{q}(t), t), \\ \dot{q}^i(t) &= +\frac{\partial H}{\partial p_i}(\vec{p}(t), \vec{q}(t), t), \end{aligned}$$

where H is the Hamiltonian. In this section we will show that the canonical equations can be written in a compact way in terms of differential forms.

Let us define the 1-forms

$$\begin{aligned} \alpha_i &= dp_i + \frac{\partial H}{\partial q^i}(\vec{p}, \vec{q}, t)dt, \\ \beta^i &= dq^i - \frac{\partial H}{\partial p_i}(\vec{p}, \vec{q}, t)dt \end{aligned}$$

and choose $X = (\dot{p}_i(t), \dot{q}^i(t), 1)$ as tangent vector to the motion. Then since $dt(X) = 1 \neq 0$ the canonical equations are equivalent to

$$i_X\alpha_i \equiv \alpha_i(X) = dp_i(X) + \frac{\partial H}{\partial q^i}(\vec{p}(t), \vec{q}(t), t)dt(X)$$

$$
\begin{aligned}
&= \dot{p}_i(t) + \frac{\partial H}{\partial q^i}(\vec{p}(t), \vec{q}(t), t) = 0 \\
i_X\beta^i \equiv \beta^i(X) &= dq^i(X) - \frac{\partial H}{\partial p_i}(\vec{p}(t), \vec{q}(t), t)dt(X) \\
&= \dot{q}^i(t) - \frac{\partial H}{\partial p_i}(\vec{p}(t), \vec{q}(t), t) = 0.
\end{aligned}
$$

We can further simplify the expression for the canonical equations by defining the 2-form

$$
\Omega = \sum_i \alpha_i \wedge \beta^i.
$$

Then for X tangent to the motion we have that

$$
i_X\Omega(Y) = \Omega(X, Y) = \sum_i \big(\alpha_i(X)\beta^i(Y) - \alpha_i(Y)\beta^i(X)\big) = 0
$$

for all tangent vectors Y. In fact this condition is equivalent to the canonical equations. Let us write ∂_{p_1} for the tangent vector $(1, 0, 0, 0, 0, 0, 0)$, ∂_{q^1} for the tangent vector $(0, 0, 0, 1, 0, 0, 0)$ and so on. Then the condition $i_X\Omega(Y) = 0$ gives

$$
\begin{aligned}
i_X\Omega(\partial_{q^i}) &= \sum_j \big(\alpha_j(X)\beta^j(\partial_{q^i}) - \alpha_j(\partial_{q^i})\beta^j(X)\big) \\
&= \alpha_i(X) = 0, \\
i_X\Omega(\partial_{p_i}) &= \sum_j \big(\alpha_j(X)\beta^j(\partial_{p_i}) - \alpha_j(\partial_{p_i})\beta^j(X)\big) \\
&= -\beta^i(X) = 0.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\Omega &= \sum_i (dp_i + \frac{\partial H}{\partial q^i}dt) \wedge (dq^i - \frac{\partial H}{\partial p_i}dt) \\
&= \sum_i dp_i \wedge dq^i - \frac{\partial H}{\partial q^i}dq^i \wedge dt - \frac{\partial H}{\partial p_i}dp_i \wedge dt \\
&= \sum_i dp_i \wedge dq^i - dH \wedge dt \\
&= d\omega,
\end{aligned}
$$

where the 1-form ω, called the Cartan form, is given by

$$\omega = \sum_i p_i dq^i - H(\vec{p}, \vec{q}, t)dt.$$

To summarise, the canonical equations are equivalent to the requirement that we set the exterior derivative $d\omega$ of the Cartan form to zero on the tangent vector $X = (\dot{p}_i(t), \dot{q}^i(t), 1)$ to the motion.

This compact formalism has many advantages. For example, let us give a simple proof of Noether's theorem. First we need a preliminary definition. Let $\mu : \Sigma \to \Sigma$ be a diffeomorphism. Then we can define the pullback (reciprocal image) $\mu^*\alpha$ of a differential form α: for a function f we have that $\mu^* f = f \circ \mu$ and $\mu^*(df) = d(\mu^* f)$. If μ_λ is a one-parameter group of such diffeomorphisms then μ_λ will leave the motion invariant if, for example, $\mu_\lambda^*\omega = \omega$. It is easy to show that if Y is the generator of the one-parameter group then ω is invariant if and only if $\mathcal{L}_Y\omega = 0$ where $\mathcal{L}_Y = (i_Y d + d i_Y)$ is the Lie derivative. Let us consider a 1-parameter group of transformations generated by Y such that $\mathcal{L}_Y\omega = 0$. Then Noether's theorem states that $d i_Y\omega = 0$ on the motion. This is very easy to prove. We have that $d i_Y \omega = \mathcal{L}_Y\omega - i_Y d\omega$. However by hypothesis $\mathcal{L}_Y\omega = 0$ and on the motion $i_Y d\omega = 0$. Finally, as discussed in appendix II, the Cartan formalism provides a direct and rigorous proof of variational principles.

4 Field theory in the Cartan formalism

The same formalism can be adapted to the treatment of fields (5). In the case of mechanics we lifted one degree of freedom, the time, into the state space to provide the motion of the system. This was done with the use of a 1-form. In the case of field theory we must lift four degrees of freedom. We will therefore search a 4-form to provide the equations of motion.

4.1 The scalar field

We start with the simple case of a free scalar field ϕ whose motion is given by the equation $(\; - m)\phi = 0$, where $\; = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu$ is the generalised Laplace-Beltrami operator acting on scalars. Here $g^{\mu\nu}$ is the inverse matrix of the symmetric metric tensor $g_{\mu\nu}$ and g the absolute value of the determinant of $g_{\mu\nu}$. If we introduce the auxiliary variables π^μ we can rewrite the equations of motion in the equivalent form

$$
\begin{aligned}
\partial_\mu\phi(x) - g^{-1/2} g_{\mu\nu}\pi^\nu(x) &= 0, \\
-\partial_\mu\pi^\mu(x) + m g^{1/2}\phi(x) &= 0.
\end{aligned}
$$

Note that as ϕ is a scalar π^μ is a contravariant density.

To find the corresponding Cartan form we must write these equations in terms of differential forms. For simplicity we introduce the notation

$$
\begin{aligned}
\eta &= \frac{1}{4!}\varepsilon_{\mu\nu\rho\lambda} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda, \\
\eta_\mu &= \frac{1}{3!}\varepsilon_{\mu\nu\rho\lambda} dx^\nu \wedge dx^\rho \wedge dx^\lambda
\end{aligned}
$$

and so on. With these definitions we can cast the equations of motion into the form

$$
\begin{aligned}
s^*(d\phi - g^{-1/2} g_{\mu\nu}\pi^\nu dx^\mu) &= 0, \\
s^*(-d\pi^\mu \wedge \eta_\mu + m g^{1/2}\phi\eta) &= 0.
\end{aligned}
$$

Here $s : {}^4 \to \Sigma$ is an immersion and s^* is the corresponding pullback. As s is an immersion we have that $s^*\eta \neq 0$ so that s provides a parameterisation of the motion.

As in the case of mechanics the equations of motion are then given by the requirement that $i_X\Omega = 0$, where here X is the geometrical object defined by the antisymmetric tensor product of four linearly independent vectors tangent to the motion and

$$
\Omega = (d\phi - g^{-1/2} g_{\mu\nu}\pi^\nu dx^\mu) \wedge (-d\pi^\mu \wedge \eta_\mu + m g^{1/2}\phi\eta)
$$

$$
\begin{aligned}
&= d\pi^\mu \wedge d\phi + (mg^{1/2}\phi d\phi - g^{-1/2}g_{\mu\nu}\pi^\nu d\pi^\mu) \wedge \eta \\
&= d\Big(\pi^\mu d\phi \wedge \eta_\mu + \tfrac{1}{2}(mg^{1/2}\phi^2 - g^{-1/2}g_{\mu\nu}\pi^\mu\pi^\nu)\eta\Big) \\
&\equiv d\omega.
\end{aligned}
$$

This condition is equivalent to the requirement that $s^*(i_Y\Omega) = 0$ for all tangent vectors Y since s is an immersion.

We generalise this Cartan 4-form to the complex scalar field by introducing new "conjugate" variables ϕ^* and $\pi^{*\mu}$. We define

$$
\omega = \pi^{*\mu} d\phi \wedge \eta_\mu + \pi^\mu d\phi^* \wedge \eta_\mu + (mg^{1/2}\phi^*\phi - g^{-1/2}g_{\mu\nu}\pi^{*\mu}\pi^\nu)\eta.
$$

We now discuss the nature of the 4-form ω. First, using the transformation properties of ϕ, π^*, η and η_μ it is easy to see that ω changes sign with the Jacobian of the transformation. It is thus a form of the second kind (an odd form). This is very important as the conserved currents obtained from Noether's theorem are then 3-forms of the second kind which can be integrated over a hypersurface. (As we will see later, the fact that ω must be of second kind can be exploited, for example, to find possible interaction terms between different fields). Secondly we have chosen an expression for ω which makes explicit the invariance of the Cartan 4-form under the changes of phase

$$
\phi \mapsto \mathrm{e}^{\mathrm{i}\lambda}\phi, \quad \phi^* \mapsto \mathrm{e}^{-\mathrm{i}\lambda}\phi^*, \quad \pi^\mu \mapsto \mathrm{e}^{\mathrm{i}\lambda}\pi^\mu \quad \text{and} \quad \pi^{*\mu} \mapsto \mathrm{e}^{-\mathrm{i}\lambda}\pi^{*\mu}.
$$

This is also very important as allows us to interpret such a field as a charged field with a conservation law via Noether's theorem. In fact, since the one parameter group of changes of phase is generated by the tangent vector

$$
Y = \mathrm{i}\phi\partial_\phi - \mathrm{i}\phi^*\partial_{\phi^*} + \mathrm{i}\pi^\mu\partial_{\pi^\mu} - \mathrm{i}\pi^{*\mu}\partial_{\pi^{*\mu}},
$$

the corresponding conserved current $J^\mu(x)\eta_\mu \equiv s^*(i_Y\omega)$ is given by

$$
\begin{aligned}
J^\mu(x)\eta_\mu &= \mathrm{i}s^*\big((\pi^{*\mu}\phi - \pi^\mu\phi^*)\eta_\mu\big) \\
&= \big(\pi^{*\mu}(x)\phi(x) - \pi^\mu(x)\phi^*(x)\big)\eta_\mu.
\end{aligned}
$$

If $g_{\mu\nu}(x)$ is constant then the Cartan 4-form ω is also invariant under translations, transformations generated by the four vectors ∂_{x^μ}. The corresponding four conserved currents $\tau_\mu{}^\nu(x)\eta_\nu \equiv s^*(i_{\partial_{x^\mu}}\omega)$ are given by

$$
\begin{aligned}
\tau_\mu{}^\nu(x)\eta_\nu = s^*(&-\pi^{*\nu} d\phi \wedge \eta_{\mu\nu} - \pi^\nu d\phi^* \wedge \eta_{\mu\nu} - (g^{-1/2} g_{\nu\rho}\pi^{*\nu}\pi^\rho - m g^{1/2}\phi^*\phi)\eta_\mu) \\
&= -\pi^{*\nu}(x)\partial_\mu\phi(x)\eta_\nu + \pi^{*\rho}(x)\partial_\rho\phi(x)\eta_\mu - \pi^\nu(x)\partial_\mu\phi^*(x)\eta_\nu + \pi^\rho(x)\partial_\rho\phi^*(x)\eta_\mu \\
&\quad - \big(g^{-1/2}(x) g_{\nu\rho}(x)\pi^{*\nu}(x)\pi^\rho(x) - m g^{1/2}(x)\phi^*(x)\phi(x)\big)\eta_\mu).
\end{aligned}
$$

This is the energy-momentum tensor for the scalar field. In particular $\tau_0{}^0(x)$ is the energy density, $\tau_i{}^0(x)$ the momentum density, $\tau_0{}^i(x)$ the energy flux and $\tau_i{}^j(x)$ the momentum flux.

4.2 The fermion field

Let us now consider the free fermion field. This is like the complex scalar field, but with the variables being a four component spinor ψ and its adjoint $\psi^\dagger$ By analogy the Cartan 4-form is given by

$$\omega = -\mathrm{i}\psi^\dagger\alpha^\mu d\psi \wedge \eta_\mu + \psi^\dagger u\psi\omega,$$

where the constant four-by-four matrices α^μ and u depend on the dynamical symmetry group that we consider. Since

$$\Omega = d\omega = -\mathrm{i}d\psi^\dagger \wedge \alpha^\mu d\psi \wedge \eta_\mu + d(\psi^\dagger u\psi) \wedge \eta$$

the equations of motion are then the following:

$$
\begin{aligned}
s^*(i_{\partial_{\psi^\dagger}} d\omega) &= s^*(-\mathrm{i}\alpha^\mu d\psi \wedge \eta_\mu + u\psi\eta) = 0, \\
s^*(i_{\partial_\psi} d\omega) &= s^*(\mathrm{i}d\psi^\dagger \wedge \alpha^\mu \eta_\mu + \psi^\dagger u\eta) = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
&-\mathrm{i}\alpha^\mu\partial_\mu\psi(x) + u\psi(x) = 0, \\
&\mathrm{i}\partial_\mu\psi^\dagger(x)\alpha^\mu + \psi^\dagger(x)u = 0.
\end{aligned}
$$

As for the complex scalar field the Cartan 4-form ω is invariant under the one-parameter change of phase $\psi \mapsto \mathrm{e}^{\mathrm{i}^{-1}\lambda}\psi$ and $\psi^\dagger \mapsto \mathrm{e}^{-\mathrm{i}^{-1}\lambda}\psi^\dagger$. Such an action is generated by the tangent vector $Y = \mathrm{i}^{-1}\psi\partial_\psi - \mathrm{i}^{-1}\psi^\dagger\partial_{\psi^\dagger}$, the corresponding conserved current being given by

$$\begin{aligned} J^\mu(x)\eta_\mu &\equiv s^*(i_Y\omega) \\ &= s^*(\psi^\dagger\alpha^\mu\psi\eta_\mu) \\ &= \psi^\dagger(x)\alpha^\mu\psi(x)\eta_\mu. \end{aligned}$$

Since α^μ and u do not depend on x the Cartan 4-form is also invariant under the translations generated by the four vectors ∂_{x^μ}. The four corresponding conserved quantities are given by

$$\begin{aligned} \tau_\mu{}^\nu(x)\eta_\nu &\equiv s^*(i_{\partial_{x^\mu}}\omega) \\ &= s^*(\mathrm{i}\psi^\dagger\alpha^\nu d\psi \wedge \eta_{\nu\mu} + \psi^\dagger u\psi\eta_\mu) \\ &= \mathrm{i}\psi^\dagger(x)\alpha^\nu\partial_\mu\psi(x)\eta_\nu - \mathrm{i}\psi^\dagger(x)\alpha^\rho\partial_\rho\psi(x)\eta_\mu + \psi^\dagger(x)u\psi(x)\eta_\mu \\ &= \big(\mathrm{i}\psi^\dagger(x)\alpha^\nu\partial_\mu\psi(x) + \delta_\mu{}^\nu\big(-\mathrm{i}\psi^\dagger(x)\alpha^\rho\partial_\rho\psi(x) + \psi^\dagger(x)u\psi(x)\big)\big)\,\eta_\nu. \end{aligned}$$

It remains to give specific forms for the matrices α^μ and u. This choice is motivated by the required dynamical covariance. We obtain the usual Dirac equation with its dynamical Lorentz covariance by choosing

$$\alpha^0 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \qquad \alpha^i = c\begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}, \qquad u = mc^2\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

On the other hand, for the Schrödinger case with its dynamical Galilei covariance (6) we must consider two sets of matrices to be able to interpret the non-relativistic limit of the Dirac equation. We therefore choose

$$\alpha_I^0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \alpha_I^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}, \qquad u_I = 2m\begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix},$$

leading to the usual Schrödinger equation, or

$$\alpha^0_{II} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \qquad \alpha^i_{II} = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}, \qquad u_{II} = 2m \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

which leads to the "negative energy" equivalent of the usual Schrödinger equation.

4.3 The Maxwell field

The Maxwell theory of the electromagnetic field can also be derived from a Cartan 4-form. As discussed above, the electromagnetic field is described by the even 1-form $A = A_\mu(x)dx^\mu$ and the odd 2-form $H = \frac{1}{4}H^{\mu\nu}(x)\varepsilon_{\mu\nu\rho\lambda}dx^\rho \wedge dx^\lambda$. The four A_μ and the six $H^{\mu\nu} = -H^{\nu\mu}$ with the four x^μ are then the field variables. The derived even 2-form B is related to H via the phenomenological relations $B_{\mu\nu} = \frac{\partial}{\partial H^{\mu\nu}}L$, where L is the function $L(x, H) = \frac{1}{8}\mu_{\mu\nu\rho\lambda}(x)H^{\mu\nu}H^{\rho\lambda}$. Note that we could take a more general function L in order to formulate, for example, non-linear electrodynamics. Let $J = J^\mu(x)\eta_\mu$ be a given conserved current. Then Maxwell's equations can be derived from the Cartan 4-form

$$\omega = dA \wedge H - L\eta - eA \wedge J$$

since

$$d\omega = dA \wedge dH - B \wedge dH - edA \wedge J = (dA - B) \wedge (dH - eJ),$$

which leads directly to the Maxwell equations following the same reasoning as for particle mechanics.

If the function L does not depend explicitly on x and if $J^\mu(x) = 0$ (the free field) then the Cartan 4-form is invariant under translations generated by the four vectors ∂_{x^μ}. As for the scalar field the four corresponding conserved currents yield the energy momentum tensor. We have

$$\begin{aligned} \tau_\mu{}^\nu(x)\eta_\nu &\equiv s^*(i_{\partial_{x^\mu}}\omega) \\ &= B_{\rho\mu}(x)H^{\nu\rho}(x)\eta_\nu + \tfrac{1}{4}B_{\rho\lambda}(x)H^{\rho\lambda}(x)\eta_\mu. \end{aligned}$$

In particular, it is easy to show that the free-field energy density is given by

$$\tau_0{}^0(x) = \tfrac{1}{2}(E_i(x)D^i(x) + B(x)^i H_i(x))$$

with corresponding flux (the Poynting vector) $\tau_0{}^i(x) = \varepsilon^{ijk}E_j(x)H_k(x) = (\vec{E}(x) \times \vec{H}(x))^i$. We stress that one must not confuse the Poynting vector $\tau_0{}^i$ with the momentum density

$$\tau_i{}^0(x) = \varepsilon_{ijk}B^j(x)D^k(x) = (\vec{B}(x) \times \vec{D}(x))_i$$

with corresponding flux $\tau_i{}^j(x)$.

4.4 Electrodynamics

The current $J^\mu(x) = \psi^\dagger(x)\alpha^\mu\psi(x)$ is conserved. Hence we take $eA \wedge J = eA_\mu\psi^\dagger\alpha^\mu\psi\eta$ as the interaction giving the coupled Cartan 4-form

$$\omega = dA \wedge H - L\eta - eA \wedge J - \mathrm{i}\psi^\dagger\alpha^\mu d\psi \wedge \eta_\mu + \psi^\dagger u\psi\eta.$$

This leads not only to the Maxwell equations but also to the equations of motion of the coupled fermion fields

$$s^*(i_{\partial_{\psi^\dagger}} d\omega) = \alpha^\mu(-\mathrm{i}\partial_\mu - eA_\mu(x))\psi(x) + u\psi(x) = 0,$$
$$s^*(i_{\partial_\psi} d\omega) = (\mathrm{i}\partial_\mu - eA_\mu(x))\psi^\dagger(x)\alpha^\mu + \psi^\dagger(x)u = 0.$$

The invariance of the Cartan form under the one-parameter group of changes of phase leads to the conservation of the current J on the motion. Further, if L does not depend explicitly on x the Cartan form ω will also be invariant under translations. The corresponding energy momentum tensor is the sum of three terms. The first two are the energy-momentum tensors for the free fermion and free Maxwell fields respectively, whereas the third

$$s^*(i_{\partial_{x^\mu}} eA \wedge J) = e\psi^\dagger(x)\alpha^\nu\psi(x)A_\nu(x)\eta_\mu$$

arises from the interaction. We stress that it is only the total energy momentum tensor that is conserved.

5 Conclusion: a new aspect of the Dirac equation

If we write $\psi_I(x) = \begin{bmatrix} \phi_I(x) \\ \chi_I(x) \end{bmatrix}$ for the solutions of the first Schrödinger equation we have, supressing the dependence on x for simplicity,

$$\begin{aligned} \mathrm{i}\partial_0 \phi_I &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\chi_I - eA_0\phi_I \\ 0 &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\phi_I - 2m\chi_I. \end{aligned}$$

Substituting for χ_I we have

$$\mathrm{i}\partial_0 \phi_I = (\tfrac{1}{2m} g^{ij}(-\mathrm{i}\partial_i - eA_i)(-\mathrm{i}\partial_j - eA_j) + \tfrac{e}{2m}\sigma_i B^i - eA_0)\phi_I,$$

that is the Schrödinger-Pauli equation for a particle with magnetic moment $g = 2$.

On the other hand, writing the solutions of the second Schrödinger equation as $\psi_{II} = \begin{bmatrix} \phi_{II} \\ \chi_{II} \end{bmatrix}$ we have

$$\begin{aligned} 0 &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\chi_{II} + 2m\phi_{II} \\ \mathrm{i}\partial_0 \chi_{II} &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\phi_{II} - eA_0\chi_{II}. \end{aligned}$$

Substituting for ϕ_{II} gives

$$\mathrm{i}\partial_0 \chi_{II} = (-\tfrac{1}{2m} g^{ij}(-\mathrm{i}\partial_i - eA_i)(-\mathrm{i}\partial_j - eA_j) - \tfrac{e}{2m}\sigma_i B^i - eA_0)\chi_{II}.$$

This is the "negative-energy equivalent" of the usual Schrödinger-Pauli equation.

One can simply see how that non-relativistic limit arises in this formalism. If we write $\psi = \begin{bmatrix} \phi \\ c^{-1}\chi \end{bmatrix}$ and perform the transformation $\psi \mapsto \mathrm{e}^{-\mathrm{i}mc^2t/}\psi$ so that we label the energy in agreement with the Galilei convention, the Dirac equation takes the form

$$\begin{aligned} \mathrm{i}\partial_0 \phi &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\chi - eA_0\phi, \\ \mathrm{i}c^{-2}\partial_0 \chi &= \sigma^i(-\mathrm{i}\partial_i - eA_i)\phi + (-ec^{-2}A_0 - 2m)\chi. \end{aligned}$$

If the solution has positive energy the component χ is small with respect to ϕ and we can ignore the "mass" due to its "frequency and potential energies" with respect to $2m$. We then recover the first Schrödinger equation. Note that in the non-relativistic limit the small components do not approach zero, rather they approach a definite relationship with the large components: $\chi \rightarrow \frac{1}{2m}\sigma^i(-\mathrm{i}\partial_i - eA_i)\phi$ which is the same as in Schrödinger. If we have a solution of negative energy then the same argument allows us to recover the second Schrödinger equation.

In the first Schrödinger case the Hilbert space which describes the state of the particle must be constructed with the first two components of ψ since it is the integral of $J_I^0 = \phi_I^\dagger \phi_I$ over space which is conserved according to Noether's theorem. In this case the two other components only play a role in the dynamics. For the second Schrödinger equation with the opposite energy sign we must do the contrary. The Hilbert space must be built on the second two components and the first two only play a dynamical role.

The positive energy solutions for the Dirac case must be compared with solutions of the first Schrödinger equation and so the corresponding Hilbert space must be constructed with the first two components, the other two only playing a role in the dynamics. For negative energy solutions it is just the contrary. There is then a superselection variable between these two spaces since the primitive observables position, momentum and spin preserve this separation. In this way many paradoxes such as the Klein paradox and Zitterbewegung disappear. Further the spectroscopic convention of labelling eigenvectors by the quantum numbers of the first two components is then fully justified.

I: The phenomenological relations

In this appendix we show that there is an essentially unique symmetric matrix $\hat{g}_{\mu\nu}$ such that $B_{\mu\nu} = \frac{1}{2}(\hat{g}_{\mu\rho}\hat{g}_{\nu\lambda} - \hat{g}_{\mu\lambda}\hat{g}_{\nu\rho})H^{\rho\lambda}$. The equations $B_{ij} = \frac{1}{2}(\hat{g}_{i\rho}\hat{g}_{j\lambda} - \hat{g}_{i\lambda}\hat{g}_{j\rho})H^{\rho\lambda} = \mu_0 H^{ij}$ and $B_{0i} = \frac{1}{2}(\hat{g}_{0\rho}\hat{g}_{i\lambda} - \hat{g}_{0\lambda}\hat{g}_{i\rho})H^{\rho\lambda} = \varepsilon_0^{-1}H^{i0}$ give first the following two sets of relations

(i) $\hat{g}_{ii}\hat{g}_{jj} - \hat{g}_{ij}^2 = \mu_0$ for $i \neq j$

(ii) $\hat{g}_{00}\hat{g}_{ii} - \hat{g}_{0i}^2 = -\varepsilon_0^{-1}$

and for the off-diagonal part we find

(iii) $\hat{g}_{0j}\hat{g}_{ik} - \hat{g}_{0k}\hat{g}_{ij} = 0$

(iv) $\hat{g}_{00}\hat{g}_{ij} - \hat{g}_{0i}\hat{g}_{0j} = 0$ for $i \neq j$.

¿From (iii) we have that $\hat{g}_{0j}^2\hat{g}_{ik}^2 = \hat{g}_{0k}^2\hat{g}_{ij}^2$. If we eliminate the off-diagonal $\hat{g}_{\mu\nu}$ using (i) and (ii) we find that

$$(\hat{g}_{00}\hat{g}_{ii} + \varepsilon_0^{-1})(\hat{g}_{ii}\hat{g}_{kk} - \mu_0) = (\hat{g}_{00}\hat{g}_{kk} + \varepsilon_0^{-1})(\hat{g}_{ii}\hat{g}_{jj} - \mu_0).$$

Expanding and regrouping leads to

$$(\hat{g}_{ii} - \hat{g}_{kk})(\varepsilon_0^{-1}\hat{g}_{jj} + \mu_0\hat{g}_{00}) = 0.$$

Hence either $\hat{g}_{ii} = \hat{g}_{kk}$ or $\hat{g}_{jj} = -\varepsilon_0\mu_0\hat{g}_{00}$. Thus at least three of the values $\alpha = \hat{g}_{11}$, $\beta = \hat{g}_{22}$, $\gamma = \hat{g}_{33}$ and $\delta = -\varepsilon_0\mu_0\hat{g}_{00}$ must be equal.

Suppose that $\alpha = \beta = \gamma$ (the other cases being treated by permutation). Then from (i) the three $\hat{g}_{ij}^2$ with $i \neq j$ must be equal and from (iii) so must the three $\hat{g}_{0i}^2$. Now, setting $i = j$ in (iii) we have

$$\hat{g}_{0i}^2\hat{g}_{ik}^2 = \hat{g}_{0k}^2\hat{g}_{ii}^2.$$

However from (i) we see that $g_{ik}^2 \neq g_{ii}^2$ since μ_0 does not vanish and so $\hat{g}_{0i} = 0$. Since ε_0^{-1} does not vanish either, from (ii) we find that $\hat{g}_{00} \neq 0$. Using (iv) we then have that $\hat{g}_{ij} = 0$

for $i \neq j$. Finally (i) and (ii) give $\hat{g}_{ii} = \pm\mu_0^{1/2}$ and $\hat{g}_{00} = \mp\varepsilon_0^{-1}\mu_0^{-1/2}$. It is trivial to verify that this does indeed give a solution. Hence there is a diagonal matrix $\hat{g}_{\mu\nu}$, unique up to a sign, such that $\mu_{\mu\nu\rho\lambda} = (\hat{g}_{\mu\rho}\hat{g}_{\nu\lambda} - \hat{g}_{\mu\lambda}\hat{g}_{\nu\rho})$.

II: Variational principles

It is well known that the canonical equations for particle mechanics can be derived from the variational principle: among all curves Γ between two points P_1 and P_2 of the state space the one which realises the motion (if such a path exists) renders extremal (stationary) the integral

$$s(\Gamma) = \int_\Gamma \omega = \int_\Gamma \left(p_i dq^i - H(p,q,t)dt\right).$$

The proof is very simple in the Cartan formalism. If Γ' is another curve between P_1 and P_2 obtained by moving each point of Γ along an infinitesimal tangent vector Y then the stationarity of $s(\Gamma)$ follows directly from

$$\int_{\Gamma'} \omega - \int_\Gamma \omega = \int_\Xi d\omega = 0$$

since for X tangent to the motion $d\omega(X,Y) = 0$ for all tangent vectors Y. Note that Ξ is the infinitesimal surface bounded by $\Gamma' - \Gamma$.

It is clear that such a variational theorem can also be formulated in field theory, where the motions are characterised by $s^*(i_Y d\omega) = 0$ for the Cartan 4-form ω. We do not utilise such principles since the Cartan condition leads directly to the equations of motion and the Noether theorems.

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