A LOW CONCENTRATION ASYMPTOTIC EXPANSION FOR THE DENSITY OF STATES OF A RANDOM SCHRÖDINGER OPERATOR WITH POISSON DISORDER

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Abstract. In this paper, we study the density of states of a random Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$ where V_ω is a Poisson potential (i.e a Poisson random field) of concentration μ. We show that $N_\mu(d\lambda)$, the density of states of H_ω, admits an asymptotic expansion in μ when $\mu \to 0$. Then, we use this expansion to deduce the behaviour of the integrated density of states of H_ω in the energy interval $(-\infty, 0)$ when μ goes to 0.

Résumé. Dans ce travail, nous étudions la densité d'états d'un opérateur de Schrödinger aléatoire de la forme $H_\omega = -\Delta + V_\omega$ où V_ω est un potentiel de Poisson (i.e. un champs aléatoire de Poisson) de concentration μ. Nous démontrons l'existence d'un développement asymptotique en μ pour $N_\mu(d\lambda)$, la densité d'état de H_ω, ceci quand $\mu \to 0$. Puis nous utilisons ce développement pour estimer la taille de la densité d'états intégrée de H_ω dans l'intervalle d'énergie $(-\infty, 0)$ quand μ tend vers 0.

0) Introduction.

In the present paper, we study the density of states of a random Poisson Schrödinger operator defined on $L^2(\mathbb{R}^d)$. We prove that this density of state admits an asymptotic expansion in powers of μ when $\mu \to 0$, μ being the concentration parameter of the Poisson potential. We then use this expansion to study the behaviour of the integrated density of states of the random Poisson Schrödinger operator outside of the spectrum of $-\Delta$.

More precisely, let V_ω be a random potential obtained by putting impurities in $\mathbb{R}^d$ according to a Poisson process of concentration μ. This means that μ is the expectation of the number of impurities found in set of Lebesgue measure 1.

Consider the random Poisson Schrödinger operator $H_\omega = -\Delta + V_\omega$. One can define a density of states for H_ω (see [Pa-Fi]); we will denote it by $N_\mu(d\lambda)$. The support of N_μ is the almost sure spectrum of H_ω which does not depend on μ as is seen form the results of [Ki-Ma 2]; but, in the gaps of the spectrum of $-\Delta$, the density of states tends to 0 (in a weak sense) as μ tends to 0.

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We prove that $N_\mu(d\lambda)$ admits an asymptotic expansion (in the distributional sense) in powers of μ when μ tends to 0, i.e, there exists a sequence of distributions $(n_k)_{k\geq 0}$ such that

$$N_\mu(d\lambda) \underset{\substack{\mu\to 0\\ \mu>0}}{\sim} \sum_{k\geq 0} \mu^k n_k, \tag{0.1}$$

the precise meaning of this asymptotic expansion being given in Theorem 1.1.

We compute the distributions $(n_k)_{k\geq 0}$. n_0 is the density of states of the free Laplace operator. For $k\geq 1$, let Λ be a set of k sites (i.e k distinct points in $\mathbb{R}^d$), and H_Λ be the free Laplace operator perturbed by putting exactly one impurity at each site of Λ. This is a relatively compact perturbation of $H_0=-\Delta$; moreover one can define $\zeta(\lambda;\Lambda)$, the spectral shift function for the pair (H_Λ, H_0) (cf [Ya]). Then, for n_k for $k\geq 1$, we get the following formulae

$$n_1(\lambda) = -\zeta'(\lambda;\{0\}), \tag{0.2 a}$$

and, for $k\geq 2$,

$$n_k(\lambda) = \frac{(-1)^{k-1}}{k!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left(\sum_{A\subset\{0,x_1,\dots,x_{k-1}\}} (-1)^{\#A}\zeta'(\lambda;A)\right) dx_1\dots dx_{k-1}. \tag{0.2 b}$$

Here ζ' is the distributional derivative of ζ.

One can notice the analogy with the expansion found in [Kl] for the density of states of a random Bernoulli Schrödinger operator. Here we just have a continuous version of it.

In the case when the almost sure spectra of H_ω is the whole real axis, we use the asymptotic expansion (0.1) and (0.2) to show that one can divide the negative real axis in union of intervals $(I_k)_{k\geq 1}$ such that, loosely speaking, in each I_k, the integrated density is of size μ^k when $\mu\to 0$.

Formally the proof of the asymptotic expansion (0.1) is quite easy. In [Wa], a Taylor expansion in μ is proved at the point $\mu=0$ for the Poisson process of concentration μ. The coefficients of this expansion are combinations of Dirac masses. By [Pa-Fi], we know that the density of states may be expressed as the average of the spectral measure with respect to the Poisson process. So to get the expansion, we just have to integrate the spectral measure against each of the measure valued coefficients obtained in the Taylor expansion of the Poisson process. Unfortunately, this cannot be made rigorous in such a direct way. Instead, we choose to approximate the Poisson potential with Bernoulli potential, use the asymptotic expansion proved in [Kl], and show that all the terms are well enough behaved in the limit when the sequence of Bernoulli processes tends to the Poisson process.

One of the main new technical difficulties comes from the fact that, if the impurities take negative values, then with probability 1, the realizations of the Poisson potential are unbounded below. So the proof may be simplified greatly, if one assumes that the impurities are non negative.

The methods used here may be applied to obtain information on the density of states of other models of random Poisson Schrödinger operator; for example, in dimension 2, one may replace the free Laplace operator by a Laplace operator with a constant magnetic field.

I) The main results.

a) The random Poisson Schrödinger operator.

Let $m(\omega, dx)$ be a Poisson random measure of concentration μ on $\mathbb{R}^d$. By this we mean a function $m: \Omega \times \mathcal{B}(\mathbb{R}^d) \mapsto \mathbb{R}$ (here $\mathcal{B}(\mathbb{R}^d)$ denotes the set of bounded Borel subsets of $\mathbb{R}^d$, and $(\Omega, \mathbb{P}_\omega, \mathcal{T})$ is some probability space) such that:

a) for any $B \in \mathcal{B}(\mathbb{R}^d)$, $\omega \in \Omega \mapsto m(\omega, B)$ is an integer valued random variable with the following distribution

$$\mathbb{P}_\omega(\{\omega;\ m(\omega, B) = n\}) = e^{-\mu|B|}\frac{(\mu \mid B \mid)^n}{n!} \text{ for } n \in \mathbb{N},$$

b) for $B_1, \dots, B_n$ pairwise disjoint Borel sets in $\mathbb{R}^d$, the random variables $m(\omega, B_1), \dots, m(\omega, B_n)$ are independent.

As $\mathbb{E}_\omega\{m(\omega, B)\} = \mu \mid B \mid$, μ is called the concentration ($\mathbb{E}_\omega$ denotes the average with respect to $\mathbb{P}_\omega$).

Let $V \not\equiv 0$ be a continuous function satisfying

(H.1) for some $\eta > 0$, and some $C > 0$, $|V(x)| \leq Ce^{-\eta|x|}$ for $x \in \mathbb{R}^d$.

For $\omega \in \Omega$, define the Poisson random field $V_\omega(x) = \int_{\mathbb{R}^d} V(x-y)m(\omega, dy)$ and the Poisson random Schrödinger operator

$$H_\omega = -\Delta + V_\omega.$$

By our assumptions on V, H_ω is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ with probability 1 (cf [Pa-Fi] or [Ki-Ma 1] Proposition 2). Moreover H_ω is ergodic (cf [Pa-Fi]). Hence the spectrum of H_ω is almost surely indepedent of ω. We denote it by Σ. Using the criterion of [Ki-Ma 2], it is not difficult to see that:

a) if $\{x \in \mathbb{R}^d;\ V(x) < 0\} = \emptyset$ then $\Sigma = [0, +\infty)$,

b) if $\{x \in \mathbb{R}^d;\ V(x) < 0\} \neq \emptyset$ then $\Sigma = \mathbb{R}$.

b) The density of states.

For H_ω, one can define the density of states $N_\mu(d\lambda)$ (cf [Pa-Fi]) in the following way: let Λ_l be a cube in $\mathbb{R}^d$ centered at 0 and of sidelentgh l. Define $(H_\omega)_l^D = H_{\omega|\Lambda_l}$ i.e. H_ω restricted to Λ_l with Dirichlet boundary conditions. Define, for $\lambda \in \mathbb{R}$,

$$\mathcal{N}_l^\omega(\lambda) = \frac{1}{\mathrm{Vol}(\Lambda_l)}\#\{\lambda_n;\ \lambda_n \text{ is an eigenvalue of } (H_\omega)_l^D \text{ and } \lambda_n \leq \lambda\}$$

and $N_l^\omega(d\lambda) = \partial_\lambda \mathcal{N}_l^\omega$, the corresponding discrete measure on $\mathbb{R}$. Then, e.g. by Theorem 5.20 of [Pa-Fi], there exists a non-random measure $N_\mu(d\lambda)$ such that, with probability 1,

$$\lim_{l \to +\infty} N_l^\omega(d\lambda) = N_\mu(d\lambda),$$

and

$$N_\mu(d\lambda) = \mathbb{E}_\omega\{\mathrm{Tr}(\chi_0 E(\omega; d\lambda)\chi_0)\}$$

where $E(\omega; d\lambda)$ is the spectral resolution of H_ω, χ_0 is the characteristic function of the cube centered in 0 with sidelength 1, and $\mathrm{Tr}(Op)$ is the trace of Op.

Before we state our main result we need one more definition. Let A be a finite subset of $\mathbb{R}^d$. Define $H(A) = -\Delta + \sum_{y \in A} V_y$ where, for $y \in A$, $V_y(x) = V(x-y)$. $H(A)$ is relatively compact perturbation of $-\Delta$

Moreover, for $q > d/2$ and $z \notin \sigma(H(A))$, the operator $(z - H(A))^{-q} - (z + \Delta)^{-q}$ is trace class. Hence, we define $\zeta(\lambda; A)$ to be the *spectral shift function* for the pair of operators $H(A)$ and $-\Delta$; $\zeta(\lambda; A)$ is the distribution in $\mathcal{S}'(\mathbb{R})$ defined by, for $\phi \in \mathcal{S}(\mathbb{R})$,

$$Tr(\phi(H(A)) - \phi(-\Delta)) = \int_0^{+\infty} \zeta(\lambda; A)\phi'(\lambda)d\lambda$$

(see [Ya] chapter 8 section 9).

Our main result is

Theorem 1.1. *The density of states $N_\mu(d\lambda)$ admits an asymptotic expansion in $\mathcal{S}'(\mathbb{R})$ when μ tends to 0. More precisely there exists a sequence of distributions $(n_k)_{k\geq 0}$ such that*
(a) for any $k \geq 0$, $n_k \in \mathcal{S}'(\mathbb{R})$,
(b) for any $\mu_0 > 0$ and any $N > 0$, there exists $|\cdot\|$, a semi-norm in $\mathcal{S}(\mathbb{R})$ such that, for any $\varphi \in \mathcal{S}(\mathbb{R})$, one has

$$|\langle N_\mu(d\lambda), \varphi\rangle - \sum_{k=0}^{N} \mu^k \langle n_k, \varphi\rangle| \leq \mu^{N+1}|\varphi\| \quad \textit{for} \quad \mu \in [0, \mu_0].$$

Moreover the distributions $(n_k)_{k\in\mathbb{N}}$ are given by the following formulae:
(c) if $k = 0$: n_0 is the density of states for the unperturbed Laplace operator $-\Delta$,
(d) if $k = 1$: $n_1(\lambda) = -\zeta'(\lambda, 0)$, the derivative of the spectral shift function for the pair of operators $(-\Delta + V, -\Delta)$,
(e) if $k \geq 2$: n_k is given by the following convergent (in $\mathcal{S}'(\mathbb{R})$) integral:

$$n_k(\lambda) = \frac{(-1)^{k-1}}{k!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\sum_{A \subset \{0, x_1, \dots, x_{k-1}\}} (-1)^{\# A} \zeta'(\lambda; A) \right) dx_1 \dots dx_{k-1},$$

where $\zeta'(\lambda; A)$ is the derivative of $\zeta(\lambda; A)$ with respect to λ.

Remark. Here $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing smooth functions in $\mathbb{R}$, and $\mathcal{S}'(\mathbb{R})$, the Schwartz space of temperate distributions.

Notice that, as $\mathrm{supp} N_\mu \subset \Sigma$, for any $k \geq 0$, one has $\mathrm{supp} n_k \subset \Sigma$. In the proof of Theorem 1.1, we also obtain an upper bound on the order of n_k which is $\mathrm{ord}(n_k) \leq (k+1)(d+1) + 2$. The same way one can give an estimate upon the order of the semi-norm used to control the remainder in the asymptotic expansion; if the expansion is made up to order N, then the remainder is at most of order $(N+1)d + N + 5$ (as an element in $\mathcal{S}'(\mathbb{R})$).

c) The behaviour of the integrated density of states for negative energies.

This question is interesting only if $\Sigma = \mathbb{R}$ i.e if $\{x \in \mathbb{R}^d;\ V(x) < 0\} \neq \emptyset$, for otherwise the support of N_μ is $[0, +\infty)$.

For $x_1, \dots, x_k$ be k points in $\mathbb{R}^d$, we define $\sigma(x_1, \dots, x_k)$ to be the spectrum of $H(x_1, \dots, x_k)$, $E(d\lambda; x_1, \dots, x_k)$ to be its spectral resolution. For B, a Borel subset of $\mathbb{R}$, let $E(B; x_1, \dots, x_k)$ be the spectral projection associated to the set of energies B and the operator $H(\{x_1, \dots, x_k\})$.

$H(\{x_1, \dots, x_k\})$ is jointly continuous in $(x_1, \dots, x_k)$ (in norm resolvent sense). Let B be an interval. So by [Re-Si 1], if the endpoints of B are not in the point spectrum of $H(x_1^0, \dots, x_k^0)$, $E(B; x_1, \dots, x_k)$ is continuous near $(x_1^0, \dots, x_k^0)$.

If B is some Borel subset of $(-\infty,0)$ such that $\overline{B}\subset(-\infty,0)$, then $E(B;x_1,\dots,x_k)$ is of finite rank (as $H(\{x_1,\dots,x_k\})$ is a relatively compact perturbation of $-\Delta$). The condition $\overline{B}\subset(-\infty,0)$ may be removed if the dimension $d\geq 3$ using the Lieb-Cwikel-Rosenblum bound (cf [Re-Si 2]).

Moreover, if B is open, then $(x_1,\dots,x_k)\mapsto \mathrm{Tr}(E(B;x_1,\dots,x_k)) = \mathrm{rank}((E(B;x_1,\dots,x_k))$ is lower semi-continuous, hence measurable. So, for any B interval in $(-\infty,0)$, $(x_1,\dots,x_k)\mapsto \mathrm{Tr}(E(B;x_1,\dots,x_k))$ is measurable.

For $k\geq 1$, B open such that $\overline{B}\subset(-\infty,0)$, we define

$$\mathfrak{n}_k(B)=\lim_{R\to+\infty}\left(\frac{1}{\mathrm{Vol}(\{\mid x\mid\leq R\})}\int_{|x_1|\leq R}\cdots\int_{|x_k|\leq R}\mathrm{Tr}(E(B;x_1,\dots,x_k))dx_1\dots dx_k\right).$$

Using the translation invariance of $-\Delta$, one has

$$\mathfrak{n}_1(B)=\sum_{\lambda\in B\cap\sigma(-\Delta+V)}\text{multiplicity of }\lambda=\mathrm{Tr}(E(B;0),$$

and

$$\mathfrak{n}_k(B)=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\mathrm{Tr}(E(B;0,x_1,\dots,x_{k-1}))dx_1\dots dx_{k-1}\text{ if }k\geq 2.$$

For $k\geq 2$, $\mathfrak{n}_k(B)$ may be infinite.

For $k\geq 1$, let $\mathfrak{B}_{k+1}$ be the closure of the union of the spectra of $H(\{x_1,\dots,x_k\})$ when $x_1,\dots,x_k$ run all over all possible points in $\mathbb{R}^d$ i.e, $\mathfrak{B}_0=\sigma(-\Delta)$, and for $k\geq 1$,

$$\mathfrak{B}_k=\overline{\bigcup_{x_1\in\mathbb{R}^d}\cdots\bigcup_{x_k\in\mathbb{R}^d}\sigma(\{x_1,\dots,x_k\})}.$$

Using the exponential decrease of eigenfunction in the negative spectrum of $H(x_1,\dots,x_k)$, it is not difficult to see that, for $j\leq k$, $\mathfrak{B}_j\subset\mathfrak{B}_k$. When $\{x\in\mathbb{R}^d;\ V(x)<0\}\neq\emptyset$, one easily proves that $\cup_{k\in\mathbb{N}}\mathfrak{B}_k=\mathbb{R}$. Moreover, if one supposes that, for almost every $x\in\mathbb{R}^d$, $V(x)\leq 0$, then one easily proves that, for $j\neq k$, $\mathfrak{B}_j\neq\mathfrak{B}_k$.

The behaviour of the integrated density of states for negative energies is given by the

Theorem 1.2. *Let I be an open interval such $\overline{I}\subset(-\infty,0)$. Then, for $k\geq 1$, if $I\cap\mathfrak{B}_k\neq\emptyset$ and for $1\leq j\leq k-1$, $\overline{I}\cap\mathfrak{B}_j=\emptyset$ then, $0<\mathfrak{n}_k(I)\leq\mathfrak{n}_k(\overline{I})<+\infty$ and*

$$\limsup_{\mu\to 0}\left[\mu^{-k}N_\mu(I)\right]=\limsup_{\mu\to 0}\left[\mu^{-k}\int_I N_\mu(d\lambda)\right]\leq\frac{\mathfrak{n}_k(\overline{I})}{k!},$$

and

$$\liminf_{\mu\to 0}\left[\mu^{-k}N_\mu(I)\right]=\liminf_{\mu\to 0}\left[\mu^{-k}\int_I N_\mu(d\lambda)\right]\geq\frac{\mathfrak{n}_k(I)}{k!}.$$

Remark. If $\mathfrak{n}_k(I)=\mathfrak{n}_k(\overline{I})$ then we get that $N_\mu(I)=\dfrac{\mathfrak{n}_k(I)}{k!}\mu^k(1+o(1))$ when $\mu\to 0$. The condition $\mathfrak{n}_k(I)=\mathfrak{n}_k(\overline{I})$ is a condition on the behaviour of the negative eigenvalues of $H(x_1,\dots,x_k)$. For this condition to be satisfied, one just needs that, for any such eigenvalue $\lambda(x_1,\dots,x_k)$, the level set $\lambda^{-1}(\lambda_0)$ is of measure 0 if λ_0 is one of the ends of the interval I.

Proof of Theorem 1.2. Let us first show some regularity result for $\mathfrak{n}_k(I)$ for I chosen as in Theorem 1.2. Assume $I = (a, b)$ and denote $I_{-\epsilon} = [a+\epsilon, b-\epsilon]$ and $I_{+\epsilon} = [a-\epsilon, b+\epsilon]$. Then $\lim_{\epsilon\to 0^+} \mathfrak{n}_k(I_{+\epsilon}) = \mathfrak{n}_k(\overline{I})$ and $\lim_{\epsilon\to 0^+} \mathfrak{n}_k(I_{-\epsilon}) = \mathfrak{n}_k(I)$.

Indeed, as for $1 \leq j \leq k-1$, $\overline{I} \cap \mathfrak{B}_j = \emptyset$ and the $\mathfrak{B}_j$ are closed, for ϵ small enough, the functions $(x_1, \ldots, x_k) \mapsto \mathrm{Tr}(E(I; x_1, \ldots, x_k))$, $(x_1, \ldots, x_k) \mapsto \mathrm{Tr}(E(\overline{I}; x_1, \ldots, x_k))$, $(x_1, \ldots, x_k) \mapsto \mathrm{Tr}(E(I_{+\epsilon}; x_1, \ldots, x_k))$ and $(x_1, \ldots, x_k) \mapsto \mathrm{Tr}(E(I_{-\epsilon}; x_1, \ldots, x_k))$ vanish outside a compact set. Moreover the regularity of the spectral measure tells us that, for any $(x_1, \ldots, x_k)$,

$$\mathrm{Tr}(E([a-\epsilon, a); x_1, \ldots, x_k)) \to 0 \text{ and } \mathrm{Tr}(E((a, a+\epsilon]; x_1, \ldots, x_k)) \to 0 \text{ when } \epsilon \to 0.$$

Hence the claim is a direct consequence of Lebesgue's Dominated Convergence Theorem.

Let χ_I be the characteristic function of I, and $\psi \in \mathcal{C}_o^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $[-1, 1]$, $\psi \equiv 0$ outside of $[-2, 2]$ and $\int_{\mathbb{R}} \psi = 1$. Pick $\delta > 0$. For $\epsilon > 0$, define $\chi_{-\epsilon}$ to be the characteristic function of $I_{-\epsilon}$ and $\chi_{+\epsilon}$ to be the characteristic function of $I_{+\epsilon}$. Now define $\psi_\mu(x) = \frac{1}{\mu^\delta}\psi(\frac{1}{\mu^\delta}x)$, $\chi_\mu = \chi_I * \psi_\mu$, $\chi'_\mu = \chi_{-2\mu^\delta} * \psi_\mu$ and $\chi''_\mu = \chi_{+2\mu^\delta} * \psi_\mu$. Then one has, $\chi'_\mu \leq \chi_I \leq \chi''_\mu$ hence, as N_μ is a positive measure,

$$\langle N_\mu, \chi'_\mu \rangle \leq \langle N_\mu, \chi_I \rangle = N_\mu(I) \leq \langle N_\mu, \chi''_\mu \rangle.$$

We then estimate $\langle N_\mu, \chi'_\mu \rangle$ and $\langle N_\mu, \chi''_\mu \rangle$ using the asymptotic expansion of Theorem 1.1. For μ small enough, for $1 \leq j \leq k-1$, $I_{+3\mu^\delta} \cap \mathfrak{B}_j = \emptyset$, so, for $1 \leq j \leq k-1$,

$$\langle n_j, \chi'_\mu \rangle = \langle n_j, \chi''_\mu \rangle = 0.$$

Moreover, one has, as for μ small enough,

$$\langle n_k, \chi'_\mu \rangle = \frac{1}{k!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} -\langle \zeta'(\lambda; \{0, x_1, \ldots, x_{k-1}\}), \chi'_\mu \rangle dx_1 \ldots dx_{k-1} \geq \mathfrak{n}_k(I_{-\mu^\delta}),$$

and

$$\langle n_k, \chi''_\mu \rangle = \frac{1}{k!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} -\langle \zeta'(\lambda; \{0, x_1, \ldots, x_{k-1}\}), \chi''_\mu \rangle dx_1 \ldots dx_{k-1} \leq \mathfrak{n}_k(I_{+3\mu^\delta})$$

as, for $\lambda < 0$ (i.e $\lambda \notin \sigma(-\Delta)$), $\zeta'(\lambda; \{0, x_1, \ldots, x_{k-1}\}) = -\mathrm{Tr}(E(d\lambda; x_1, \ldots, x_{k-1}))$

One easily checks that, for any $n \geq 0$, there exists $C_n > 0$ such that

$$\sup_{x\in\mathbb{R}} |\partial^n \chi''_\mu(x)| + \sup_{x\in\mathbb{R}} |\partial^n \chi''_\mu(x)| \leq C_n \mu^{\delta\cdot n}.$$

As we know an upper bound on the order needed to estimate the rest in the asymptotic expansion of Theorem 1.1 (see the remark following that theorem), we get, choosing δ enough small,

$$\left|\langle N_\mu, \chi'_\mu \rangle - \langle n_k, \chi'_\mu \rangle\right| + \left|\langle N_\mu, \chi''_\mu \rangle - \langle n_k, \chi''_\mu \rangle\right| \to 0 \text{ when } \mu \to 0.$$

This completes the proof of Theorem 1.2. $\square$

II) The proof of Theorem 1.1.

The main idea of the proof of Theorem 1.1 is to approximate the Poisson random Schrödinger operator by suitably chosen Bernoulli random Schrödinger operators, to use the techniques and results of [Kl] to get an asymptotic expansion for these models and to show that these asymptotic expansions converge to some new asymptotic expansion when one lets the Bernoulli approximation tend to the Poisson model.

As a last remark, we point out that due to the many estimates needed and for the sake of simpleness, each estimate involving new constants, we chose to give the same name, C, to all these constants. So the value of C is changing from one equation to another.

a) The binomial approximation.

The main technical difficulty comes from the fact that the realizations of the Poisson potential are almost surely unbounded from below (at least if we assume $\{x \in \mathbb{R}^d;\ V(x) < 0\} \neq \emptyset$). To overcome this difficulty, we first approximate the Poisson potential by a potential that is semi-bounded below. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $\psi \geq 0$, $\psi \equiv 1$ in $[-1, 1]$ and $\mathrm{supp}\psi \subset [-2, 2]$. For $n \in \mathbb{N}^*$ and $\omega \in \Omega$, define

$$V_\omega^n(x) = V_\omega(x)\psi\left(\frac{1}{n}V_\omega(x)\right) \text{ and } H_\omega^n = -\Delta + V_\omega^n.$$

Then one has the

Proposition 2.1. *For any $\varphi \in \mathcal{C}_o^\infty(\mathbb{R})$,*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{Tr(\chi_0\varphi(H_\omega^n)\chi_0)\right\} = \mathbb{E}_\omega\left\{Tr(\chi_0\varphi(H_\omega)\chi_0)\right\}.$$

Now to compute an asymptotic expansion for $\mathbb{E}_\omega\left\{\mathrm{Tr}(\chi_0\varphi(H_\omega^n)\chi_0)\right\}$, we approximate V_ω^n by some truncated Bernoulli random potential constructed in the following way:

let $(t_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$ be a family of independent identically distributed Bernoulli random variables with common probability distribution $P(t_0 = 1) = p_m$ and $P(t_0 = 0) = 1 - p_m$ with $p_m = \mu m^{-d}$. Define

$$V_m(t;x) = V_m(x) = \sum_{\gamma\in\frac{1}{m}\mathbb{Z}^d} t_\gamma V(x-\gamma) \text{ and } V_m^n(t;x) = V_m^n(x) = V_m(x)\psi\left(\frac{1}{n}V_m(x)\right).$$

At last define,

$$H_m^n(t) = H_m^n = -\Delta + V_m^n.$$

Then one proves

Proposition 2.2. *For any $\varphi \in \mathcal{C}_o^\infty(\mathbb{R})$,*

$$\lim_{m\to+\infty} \mathbb{E}_t\left\{Tr(\chi_0\varphi(H_m^n)\chi_0)\right\} = \mathbb{E}_\omega\left\{Tr(\chi_0\varphi(H_\omega^n)\chi_0)\right\}.$$

Here $\mathbb{E}_t\{\cdot\}$ denotes the average with respect to the probability measure enduced by the random variables $(t_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$.

Then, to get Theorem 1.1, we compute an asymptotic expansion of $\mathbb{E}_t\left\{\mathrm{Tr}(\chi_0\varphi(H_m^n)\chi_0)\right\}$ for which we control the principal terms and the remainder when m and n go to ∞. This is done in the

Proposition 2.3. *Let $\mu_0 > 0$ and $N > 1$ be an integer. There exists $C_{\mu_0,N}$ and $|\cdot\|$ a semi-norm in $\mathcal{S}(\mathbb{R})$ of order at most $(N+1)d+N+5$ (depending only on μ_0 and N), such that, for any $n \geq 2C \cdot N$, for any $m \geq 1$ and for any $\varphi \in \mathcal{C}_o^\infty(\mathbb{R})$, one has*

$$\mathbb{E}_t\{\, Tr(\chi_0\varphi(H_m^n)\chi_0)\} = Tr((\chi_0\varphi(-\Delta)\chi_0) + \sum_{j=1}^{N}\frac{\mu^j}{j!}\langle n_j^m,\varphi\rangle + \mu^{N+1}R_{m,N}^n(\varphi),$$

where

$$\langle n_j^m,\varphi\rangle = m^{-jd}\sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\\ \#\Lambda=j}}\sum_{A\subset\Lambda}(-1)^{j-\#A}\, Tr(\chi_0\varphi(H_A)\chi_0)\ \text{ and } H_A = -\Delta + \sum_{\gamma\in A}V(x-\gamma),$$

and

$$|R_{m,k}^n(\varphi)| \leq C_{\mu_0,N}|\varphi\| \quad \text{for} \quad \mu\in[0,\mu_0].$$

b) The end of the proof of Theorem 1.1.

To get Theorem 1.1, we now just need to compute the limits of $\langle n_j^m,\varphi\rangle$ when $m\to+\infty$ for $\varphi\in\mathcal{C}_o^\infty(\mathbb{R})$. Therefore we prove

Proposition 2.4. *Let $(x_1,\dots,x_k)\in(\mathbb{R}^d)^k$. Define*

$$\begin{aligned}\langle Z(0,x_1,\dots,x_k),\varphi\rangle &= -\sum_{A\subset\{0,x_1,\dots,x_k\}}(-1)^{\#A}\langle\zeta'(A),\varphi\rangle\\ &= \sum_{A\subset\{0,x_1,\dots,x_k\}}(-1)^{\#A}\, Tr(\varphi(H_A)-\varphi(-\Delta)).\end{aligned}$$

Then $(x_1,\dots,x_k)\mapsto\langle Z(0,x_1,\dots,x_k),\varphi\rangle$ is continuous and is in $L^1((\mathbb{R}^d)^k)$. Moreover, for some $n>0$ and $C>0$, one has, for any $\varphi\in\mathcal{C}_o^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}|\langle Z(0,x_1,\dots,x_k),\varphi\rangle|dx_1\dots dx_k \leq C\sum_{j,l=0}^{n}\sup_{x\in\mathbb{R}}|(1+x^2)^{j/2}\partial_x^l\varphi(x)|.$$

Remark. The constant n in Proposition 2.4 may be bounded by $n\leq(k+1)(d+1)+2$.

Let $C(m)=\{\gamma=(\gamma_1,\dots,\gamma_d)\in\frac{1}{m}\mathbb{Z}^d;\ -1/2\leq\gamma_j<1/2 \text{ for } 1\leq j\leq d\}$, and χ_0^m be the characteristic function of a cube of center 0 and sidelength $1/m$. Then, using the translation invariance of $-\Delta$ and the commutation property of the trace, we compute

$$\begin{aligned}
\langle n_k^m, \varphi \rangle &= m^{-kd} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k}} \sum_{A \subset \Lambda} (-1)^{k-\#A} \mathrm{Tr}((\varphi(H_A) - \varphi(-\Delta))\chi_0) \\
&= m^{-kd} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k}} \sum_{\gamma \in C(m)} \sum_{A \subset \Lambda} (-1)^{k-\#A} \mathrm{Tr}((\varphi(H_A) - \varphi(-\Delta))\chi_0^m(\cdot - \gamma)) \\
&= m^{-kd} \sum_{\gamma \in C(m)} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{\beta \in \frac{1}{m}\mathbb{Z}^d} \sum_{A \subset \beta + \Lambda} (-1)^{k-\#A} \mathrm{Tr}((\varphi(H_A) - \varphi(-\Delta))\chi_0^m(\cdot - \gamma)) \\
&= m^{-kd} \sum_{\gamma \in C(m)} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{k-\#A} \sum_{\beta \in \frac{1}{m}\mathbb{Z}^d} \mathrm{Tr}((\varphi(H_A) - \varphi(-\Delta))\chi_0^m(\cdot - \gamma - \beta)) \\
&= m^{-(k-1)d} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \sum_{A \subset \Lambda} (-1)^{k-\#A} \mathrm{Tr}(\varphi(H_A) - \varphi(-\Delta))
\end{aligned}$$

as $\sum_{\beta \in \frac{1}{m}\mathbb{Z}^d} \chi_0^m(\cdot - \gamma - \beta) = 1$ for any $\gamma \in \frac{1}{m}\mathbb{Z}^d$ and $\#C(m) = m^d$. Hence

$$\langle n_k^m, \varphi \rangle = (-1)^k m^{-(k-1)d} \sum_{\substack{\Lambda \subset \frac{1}{m}\mathbb{Z}^d \\ \#\Lambda = k \text{ and } 0 \in \Lambda}} \langle Z(0, x_1, \dots, x_{k-1}), \varphi \rangle$$

so, by Proposition 2.4,

$$\lim_{m \to +\infty} \langle n_k^m, \varphi \rangle = (-1)^k \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \langle Z(0, x_1, \dots, x_{k-1}), \varphi \rangle dx_1 \dots dx_{k-1}.$$

This ends the proof of Theorem 1.1 by Propositions 2.1, 2.2 and 2.3.

III) The proofs of Proposition 2.1, 2.2, 2.3 and 2.4.

We will begin with the proof of Proposition 2.3.

Proof of Proposition 2.3. To find an expansion for $\mathbb{E}_t\{\mathrm{Tr}(\chi_0\varphi(H_m^n)\chi_0)\}$, we will proceed as in [Kl]. Let Λ_l be a cube in $\mathbb{R}^d$ centered at 0 and of sidelentgh l. For any realization of t, we define

$$H_{m,l}^n(t) = H + V_{m,l}^n \text{ where } V_{m,l}^n(t;x) = V_{m,l}(x) \cdot \psi\left(\frac{1}{n} V_{m,l}(x)\right) \text{ and } V_{m,l}(x) = \sum_{\gamma \in \Lambda_l} t_\gamma V_\gamma(x)$$

where ψ has been defined in section 2.

It is not difficult to see (cf [Kl]) that $\chi_0\varphi(H_{m,l}^n)\chi_0 \to \chi_0\varphi(H_m^n)\chi_0$ (in trace class norm) as $l \to +\infty$. Hence by Lebesgue's Dominated Convergence Theorem,

$$\mathbb{E}_t\{\mathrm{Tr}(\chi_0\varphi(H_m^n)\chi_0)\} = \lim_{l \to +\infty} \mathbb{E}_t\{\mathrm{Tr}(\chi_0\varphi(H_{m,l}^n)\chi_0)\}.$$

As $H^n_{m,l}$ only depends on a finite number of the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$, it is a polynomial in p_m. Hence, we may write its Taylor expansion. Using the computations done in [Kl], we get

$$\mathbb{E}_t\left\{\mathrm{Tr}(\chi_0\varphi(H^n_{m,l})\chi_0)\right\} = \sum_{j=0}^{k}\frac{f_l^{(j)}(\varphi,0)}{j!}(p_m)^j + \frac{(p_m)^{k+1}}{(k+1)!}\int_0^1 f_l^{(k+1)}(\varphi,p_m v)(1-v)^k dv \tag{3.1}$$

with

$$f_l^{(j)}(\varphi,0) = \sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\cap\Lambda_l\\ \#\Lambda=j}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\mathrm{Tr}(\chi_0\varphi(H^n_A)\chi_0)\} \tag{3.2}$$

where

$$H^n_A = H + V^n_A \text{ and } V^n_A(t;x) = V_A(x)\cdot\psi\left(\frac{1}{n}V_A(x)\right) \text{ and } V_A(x) = \sum_{\gamma\in A} t_\gamma V_\gamma(x),$$

and,

$$f_l^{(k+1)}(\varphi,p) = \sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\cap\Lambda_l\\ \#\Lambda=k+1}}\int_{[0,1]^\Lambda}\mathbb{E}_u\left\{\mathrm{Tr}\left(\chi_0\partial_\Lambda\left[\varphi(H^n_{m,\Lambda}(t,u))\right]\chi_0\right)\right\}dt_\Lambda, \tag{3.3}$$

where:
(a) $H^n_{m,\Lambda}(t,u) = H + V^n_{m,\Lambda}(t,u)$
(b) $V^n_{m,\Lambda}(t,u) = V_{m,\Lambda}(t,u)\psi(\frac{1}{n}V_{m,\Lambda}(t,u))$ and $V_{m,\Lambda}(t,u) = \sum_{\gamma\in\Lambda}t_\gamma V_\gamma + \sum_{\gamma\in(\Lambda_l\setminus\Lambda)}u_\gamma V_\gamma$,
(c) $(t_\gamma)_{\gamma\in\Lambda}$ are variables taking value in $[0,1]$, $\partial_\Lambda = \bigotimes_{\gamma\in\Lambda}\partial_{t_\gamma}$ et $dt_\Lambda = \bigotimes_{\gamma\in\Lambda}dt_\gamma$,
(d) $(u_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$ are i.i.d random variables with common probability distribution $p\delta_1 + (1-p)\delta_0$,
(e) $\mathbb{E}_u$ is the expectation taken with respect to these random variables.

If we choose $n > 2C\cdot(k+1)$ (C given in (H.1)) then, for $0\leq j\leq k$ and $\#A = j$, one has $\sup_{x\in\mathbb{R}}\mid V_A(x)\mid\leq n$, hence, $V^n_A = \sum_{\gamma\in A}t_\gamma V_\gamma(x)$. So, by [Kl], we know that

$$\lim_{l\to+\infty} f_l^{(j)}(\varphi,0) = \sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\\ \#\Lambda=j}}\sum_{A\subset\Lambda}(-1)^{\#(\Lambda\setminus A)}\mathrm{Tr}(\chi_0\varphi(H_A)\chi_0)\}.$$

The right hand side of this equation is a nice distribution in $\mathcal{S}'(\mathbb{R})$ as can be seen from [Kl]. It is of order at most $(j+1)(d+1)+2$.

To end the proof of Proposition 2.3, we then just are left with estimating the rest of expansion (3.1). Therefore, we recall formula (2.2) of [Kl] (see also [He-Sj]), that is

$$\varphi(H^n_{m,\Lambda}(t,u)) = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)(z-H^n_{m,\Lambda}(t,u))^{-1}d\overline{z}\wedge dz, \tag{3.4}$$

where $\tilde{\varphi}:\ \mathbb{C}\to\mathbb{C}$ is an almost analytic extension of φ. Derivating (3.4) with respect to t, we get

$$\partial_\Lambda\left[\varphi(H^n_{m,\Lambda}(t,u))\right] = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\partial_\Lambda\left[(z-H^n_{m,\Lambda}(t,u))^{-1}\right]d\overline{z}\wedge dz. \tag{3.5}$$

The interversion of teh derivation and the integration is easily justified using Lebesgue's Dominated Convergence Theorem, the fact that $\|\partial_\Lambda(z-H^n_{m,\Lambda}(t,u))^{-1}\|\leq C|\mathrm{Im} z|^{-k-1}$ and the vanishing of $\dfrac{\partial\tilde\varphi}{\partial\bar z}$ to any order on $\mathbb{R}$.

To estimate $f_l^{(k+1)}(\varphi,p_mv)$, we just need to estimate, for $z\in\mathbb{C}\setminus\mathbb{R}$ and $p=p_mv$,

$$f_l^{(k+1)}(z,p)=\sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\cap\Lambda_l\\ \#\Lambda=k+1}}\int_{[0,1]^\Lambda}\mathbb{E}_p\left\{\mathrm{Tr}\left(\chi_0\partial_\Lambda\left[(z-H^n_{m,\Lambda}(t,u))^{-1}\right]\chi_0\right)\right\}dt_\Lambda,$$

One computes

$$\partial_\Lambda\left[(z-H^n_{m,\Lambda}(t,u))^{-1}\right]=\sum_{\sigma\in\mathfrak{S}(\Lambda)}(z-H^n_{m,\Lambda}(t,u))^{-1}\prod_{\gamma\in\Lambda}\left(\partial_{t_{\sigma(\gamma)}}\left(V^n_{m,\Lambda}(t,u)\right)(z-H^n_{m,\Lambda}(t,u))^{-1}\right).$$

where $\mathfrak{S}(\Lambda)$ is the group of permutations of Λ.
By Lemma 4.1, $(z-H^n_{m,\Lambda}(t,u))^{-1}$ is in $\mathcal{T}_q$, the q-th Schatten class for $q>d/2$, hence as $k+1>d/2$, $\partial_\Lambda\left[(z-H^n_{m,\Lambda}(t,u))^{-1}\right]$ is trace class; so by (3.3), for $v\in[0,1]$,

$$|f_l^{(k+1)}(z,p_mv)|\leq\mathbb{E}_{p_m\cdot v}\left\{\Sigma^n_{m,l}\right\}, \tag{3.6}$$

where

$$\Sigma^n_{m,l}=\sum_{\substack{\Lambda\subset\frac{1}{m}\mathbb{Z}^d\\ \Lambda\subset\Lambda_l\\ \#\Lambda=k+1}}\sum_{\sigma\in\mathfrak{S}(\Lambda)}\sup_{t\in[0,1]^\Lambda}\left\|\chi_0(z-H^n_{m,\Lambda}(t,u))^{-1}\prod_{i=1}^{k+1}\left(\partial_{t_{\gamma_i}}V^n_{m,\Lambda}(t,u)(z-H^n_{m,\Lambda}(t,u))^{-1}\right)\chi_0\right\|_{\mathcal{T}_1}.$$

Now we want to estimate the right hand side of (3.6) uniformly in n and l that is, we want to take into account the fact that V_ω is almost surely unbounded. The crucial point therefore is to notice that nevertheless, loosely speaking, the probability that V_ω get very negative is small. We can estimate this probability; this is done in Lemma 4.2. Hence, as in (3.6) we compute an average, either V_ω is very negative which happens only with very small probability, or we have good estimates (given by Lemma 4.1) on $\Sigma^n_{m,l}$ uniformly in n and l. This will permit us to get the needed estimates for (3.6).

Let $(\gamma_i)_{1\leq i\leq k+1}$ are the points of Λ. We compute

$$\partial_{t_{\gamma_i}}\left(V^n_{m,\Lambda}(t,u)\right)=V_{\gamma_i}\cdot\left(\psi\left(\frac{1}{n}V_{m,\Lambda}(t,u)\right)+\frac{1}{n}V_{m,\Lambda}(t,u)\cdot\psi\left(\frac{1}{n}V_{m,\Lambda}(t,u)\right)\right).$$

Notice that, for some $C>0$, for any Λ, m and n,

$$\sup_{x\in\mathbb{R}^d}\left|\psi\left(\frac{1}{n}V_{m,\Lambda}(t,u)(x)\right)+\frac{1}{n}V_{m,\Lambda}(t,u)(x)\cdot\psi'\left(\frac{1}{n}V_{m,\Lambda}(t,u)(x)\right)\right|\leq C.$$

By Lemma 4.1, for $\nu' \in (0,1)$, if $|V^n_{m,\Lambda}(t,u)(x)| \leq |x|^\nu + N$ for all $x \in \mathbb{R}^d$, one has, for some $C > 0$ and some $\epsilon > 0$,

$$\|\chi_0(z - H^n_{m,\Lambda}(t,u))^{-1} \prod_{i=1}^{k+1} \left(\partial_{t_{\gamma_i}} \left(V^n_{m,\Lambda}(t,u)\right)(z - H^n_{m,\Lambda}(t,u))^{-1}\right) \chi_0\|_{\mathcal{T}_1}$$

$$\leq C \sum_{\beta_1 \in \mathbb{Z}^d} \cdots \sum_{\beta_{k+1} \in \mathbb{Z}^d} \|\chi_0(z - H^n_{m,\Lambda}(t,u))^{-1} \chi_{\beta_1}\|_{\mathcal{T}_q} \cdot$$

$$\cdot \left(\prod_{i=1}^{k} \|V_{\gamma_i}\chi_{\beta_i}\| \cdot \|\chi_{\beta_i}(z - H^n_{m,\Lambda}(t,u))^{-1}\chi_{\beta_{i+1}}\|_{\mathcal{T}_q} \right) \cdot$$

$$\cdot \|V_{\gamma_{k+1}}\chi_{\beta_{k+1}}\| \cdot \|\chi_{\beta_{k+1}}(z - H^n_{m,\Lambda}(t,u))^{-1}\chi_0\|_{\mathcal{T}_q}$$

$$\leq \frac{C}{\eta(z,N)^{k+2}} \sum_{\beta_1 \in \mathbb{Z}^d} \cdots \sum_{\beta_{k+1} \in \mathbb{Z}^d} \left(\prod_{i=1}^{k+1} e^{-\eta|\beta_i - \gamma_i|} \right) \cdot \left(\prod_{i=1}^{k+1} (1 + |\beta_i|)^\nu \right) \cdot$$

$$\cdot\, e^{-\epsilon \cdot \eta(z,N)\big(|1-\delta_\nu(\beta_1)|+\sum_{i=1}^{k} !rt\delta_\nu(\beta_i)-\delta_\nu(where} \delta_\nu(\cdot) = (1+(\cdot)^2)^{(1-\nu)/2}.$$

So, as $| \delta_n u(x) - \delta_n u(y) | \leq 2 \mid x - y \mid$,

1

$$\|\chi_0(z - H^n_{m,\Lambda}(t,u))^{-1} \prod_{i=1}^{k+1} \left(\partial_{t_{\gamma_i}} V^n_{m,\Lambda}(t,u)(z - H^n_{m,\Lambda}(t,u))^{-1}\right) \chi_0\|_{\mathcal{T}_1}$$

$$\leq \frac{C}{\eta(z,N)^{k+2}} \left(\prod_{i=1}^{k+1} e^{-\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))\delta_\nu(\gamma_i)} \right) \cdot$$

$$\cdot \left(\sum_{\beta_1 \in \mathbb{Z}^d} \cdots \sum_{\beta_{k+1} \in \mathbb{Z}^d} \left(\prod_{i=1}^{k+1} e^{-\frac{\eta}{4}|\beta_i - \gamma_i|}(1 + |\beta_i|)^\nu \right) \right)$$

$$\leq \frac{C}{\eta(z,N)^{k+2}} \left(\prod_{i=1}^{k+1} e^{-\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))\delta_\nu(\gamma_i)} (1 + |\gamma_i|)^\nu \right).$$

Hence, if u is such that $| \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} u_\gamma V_\gamma(x)| \leq | x |^\nu + N - k - 1$ then $\sup_t |V^n_{m,\Lambda}(t,u)(x)| \leq |x|^\nu + N$;

in this case we get

$$\Sigma^n_{m,l} \leq \frac{C}{\eta(z,N)^{k+2}} \left(\sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} e^{-\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))\delta_\nu(\gamma)} (1 + |\gamma|)^\nu \right)^{k+1}$$

$$\leq \frac{C}{\eta(z,N)^{k+2}} \left(\sum_{\gamma \in \mathbb{Z}^d} \sum_{\beta \in C(m)} e^{-\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))\delta_\nu(\gamma+\beta)} (1 + |\gamma + \beta|)^\nu \right)^{k+1}$$

$$\leq \frac{C \cdot m^{(k+1)d}}{\eta(z,N)^{k+1}} \left(\sum_{\gamma_1 \in \mathbb{Z}^d} e^{-\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))\delta_\nu(\gamma_i)} (1 + |\gamma_i|)^\nu \right)^{k+1}$$

$$\leq \frac{C \cdot m^{(k+1)d}}{\eta(z,N)^{k+1}} \cdot \frac{1}{\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z,N))^{(k+1)\frac{d+\nu}{1-\nu}}}.$$

We define the events E_k and, for $N > k$, E_N by

$$E_k = \left\{ u;\ \forall x \in \mathbb{R}^d \ \left| \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} u_\gamma V_\gamma(x) \right| \leq \mid x \mid^\nu + 1 \right\}.$$

and

$$E_N = \left\{ u;\ \exists x_0 \in \mathbb{R}^d \ \mid x_0 \mid^\nu + N - k \leq \left| \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} u_\gamma V_\gamma(x_0) \right| \right.$$
$$\left. \text{and } \forall x \in \mathbb{R}^d \ \left| \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} u_\gamma V_\gamma(x) \right| \leq \mid x \mid^\nu + N + 1 - k \right\}.$$

Using the Borel-Cantelli lemma and the estimate (1) of Lemma 4.2, we know that, for $v \in [0,1]$

$$\mathbb{P}_{p_m \cdot v}\left(\left\{ u; \exists N > 0 \text{ such that } \forall x \in \mathbb{R}^d \ \left| \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} u_\gamma V_\gamma(x) \right| \leq \mid x \mid^\nu + N \right\} \right) = 1.$$

This means that $\cup_{N \geq k} E_N$ is of total measure. So, to estimate (3.6), we compute, using Lemma 4.2

$$\begin{aligned}
|f_l^{(k+1)}(z, p_m v)| &\leq \mathbb{E}_{p_m \cdot v}\left\{ \Sigma^n_{m,l} \right\} \\
&\leq \sup_{E_k} \Sigma^n_{m,l} + \sum_{N \geq k+1} \mathbb{P}_{p_m \cdot v}(E_N) \sup_{E_N} \Sigma^n_{m,l} \\
&\leq \frac{C \cdot m^{(k+1)d}}{\eta(z, k+1)^{k+2}} \cdot \frac{1}{\inf(\frac{\eta}{4}, \frac{\epsilon}{k+1} \cdot \eta(z, k+1))^{(k+1)\frac{d+\nu}{1-\nu}}} + \\
&\quad + F_\nu(v\mu) \sum_{N \geq 1} \frac{(v\mu)^{\mathfrak{e}(N/c)}}{(\mathfrak{e}(N/c))!} \cdot \frac{C \cdot m^{(k+1)d}}{\eta(z, k+N)^{k+2}} \cdot \frac{1}{\inf(\frac{\eta}{4}, \frac{\epsilon}{k} \cdot \eta(z, k+N))^{(k+1)\frac{d+\nu}{1-\nu}}}
\end{aligned}$$

where for $x \in \mathbb{R}$, $\mathfrak{e}(x) = \inf\{n \in \mathbb{N};\ x \leq n\}$.

If $\epsilon \leq \dfrac{(k+1)\eta}{4}$, then $\inf(\dfrac{\eta}{4}, \dfrac{\epsilon}{k+1} \cdot \eta(z, k+N)) \geq \dfrac{\mathrm{Im}(z)}{\mid z \mid + k + N}$; hence

$$\begin{aligned}
|f_l^{(k+1)}(z, p_m v)| &\leq C \cdot m^{(k+1)d} \left(\frac{\mid z \mid + 1}{\mid \mathrm{Im} z \mid} \right)^{k+2+(k+1)\frac{d+\nu}{1-\nu}} \cdot \\
&\quad \cdot \left(1 + F_\nu(v\mu) \sum_{N > 1} \frac{(v\mu)^{\mathfrak{e}(N/c)}}{(\mathfrak{e}(N/c))!} (N+k)^{k+2+(k+1)\frac{d+\nu}{1-\nu}} \right) \\
&\leq C \cdot m^{(k+1)d} \left(\frac{\mid z \mid + 1}{\mid \mathrm{Im} z \mid} \right)^{k+2+(k+1)\frac{d+\nu}{1-\nu}}.
\end{aligned}$$

Plugging this into (3.4) and using the properties of almost analytic extensions (see [Kl]), we get that, for some $C>0$ (depending on ν, k) and some $|\cdot\|$, a semi-norm of $\mathcal{S}'(\mathbb{R})$ of order at most $k+4+(k+1)\frac{d+\nu}{1-\nu}$, such that for $m\geq 1$, one has, for some $C_{k,\nu}>0$,

$$\sup_{l>0}\sup_{v\in[0,1]}\sup_{n>0}|f_l^{(k+1)}(\varphi,p_m v)|\leq C_{k,\nu}\cdot m^{(k+1)d}|\varphi\|.$$

This ends the proof of Proposition 2.3 if one chooses ν close to 0. $\square$

Proof of Proposition 2.1. To prove this, we will use Lebesgue's Dominated Convergence Theorem. Pick $\nu\in(0,1)$ and $q>d/2$. For $N>0$, let

$$E_N=\{\omega;\ \forall x\in\mathbb{R}^d\quad |\,x\,|^\nu+N\leq|V_\omega(x)|\leq|\,x\,|^\nu+N+1\}$$

and

$$E_0=\{\omega;\ \forall x\in\mathbb{R}^d\ |V_\omega(x)|\leq|\,x\,|^\nu\}.$$

Using the Borel-Cantelli lemma and the estimate (2) of Lemma 4.2, we know that,

$$\mathbb{P}_\omega\left(\{\omega;\exists N>0\text{ such that }\forall x\in\mathbb{R}^d\ |V_\omega(x)|\leq|\,x\,|^\nu+N\}\right)=1.$$

So, for almost every ω, there exists $N>0$ such that for all $n>0$ and $x\in\mathbb{R}^d$, $|V_\omega^n(x)|\leq|\,x\,|^\nu+N$. For $n>N$, $z\in\mathbb{C}\setminus\mathbb{R}$ and $(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d$,

$$\begin{aligned}\|\chi_\alpha(z-H_\omega^n)^{-1}\chi_\beta-\chi_\alpha(z-H_\omega)^{-1}\chi_\beta\|_{\mathcal{T}_q}&=\|\chi_\alpha(z-H_\omega^n)^{-1}V_\omega\left(\psi\left(\frac{1}{n}V_\omega\right)-1\right)(z-H_\omega)^{-1}\chi_\beta\|_{\mathcal{T}_q}\\&\leq Cn\|\chi_\alpha(z-H_\omega^n)^{-1}\chi_{\{|x|\geq(n-N)^{1/\nu}\}}(z-H_\omega)^{-1}\chi_\beta\|_{\mathcal{T}_q}\end{aligned}$$

using the support properties of ψ. So Lemma 4.1 tells us that, for some $n_0>0$ and $\epsilon_0>0$,

$$\begin{aligned}\text{(3.7)}\quad&\|\chi_\alpha(z-H_\omega^n)^{-1}\chi_\beta-\chi_\alpha(z-H_\omega)^{-1}\chi_\beta\|_{\mathcal{T}_q}\leq\\&\leq Cn\left(\frac{|\,z\,|+N}{|\,\mathrm{Im}z\,|}\right)^{n_0}e^{-\epsilon\frac{|\mathrm{Im}z|}{|z|+N}(|\delta_\nu(\alpha)-n^{(1-\nu)/\nu}|+|\delta_\nu(\beta)-n^{(1-\nu)/\nu}|)}.\end{aligned}$$

Then using formula (3.4), if $\tilde\varphi$ is an almost analytic extension for the function $(i-x)^q\varphi(x)$, we get

$$\begin{aligned}&\mathrm{Tr}(\chi_0\varphi(H_\omega^n)\chi_0)-\mathrm{Tr}(\chi_0\varphi(H_\omega)\chi_0)\\&\quad=\frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde\varphi}{\partial\bar z}(z)\mathrm{Tr}[\chi_0((z-H_\omega^n)^{-1}(i-H_\omega^n)^{-q}-(z-H_\omega)^{-1}(i-H_\omega)^{-q}\chi_0)]d\bar z\wedge dz.\end{aligned}$$

By (3.7), computing the same kind of sums as in the proof of Proposition 2.3, we estimate, for some $n_0>0$,

$$\begin{aligned}&\|\chi_0\left((z-H_\omega^n)^{-1}(i-H_\omega^n)^{-q}-(z-H_\omega)^{-1}(i-H_\omega)^{-q}\right)\chi_0\|_{\mathcal{T}_1}\leq\\&\leq C\left(\frac{|\,z\,|+N}{|\,\mathrm{Im}z\,|}\right)^{n_0}\left(n\cdot e^{-\epsilon\frac{|\mathrm{Im}z|}{|z|+N}(n^{(1-\nu)/\nu})}\right).\end{aligned}$$

Then, as $\dfrac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)$ is vanishing at any order when $\mathrm{Im}z$ vanishes, we get that, for almost every ω,

$$\mathrm{Tr}(\chi_0\varphi(H^n_\omega)\chi_0)\} - \mathrm{Tr}(\chi_0\varphi(H_\omega)\chi_0) \to 0 \text{ when } n \to +\infty.$$

Using Lemma 4.1 and formula (3.4) in the same way as above, one shows that there exists $C_\nu > 0$ and $n_\nu > 0$ (depending on ν only) such that, for V a potential satisfying $\forall x \in \mathbb{R}^d$, $|V_(x)| \leq| x |^\nu + N$ (for some $N > 0$), one has

$$|\mathrm{Tr}(\chi_0\varphi(-\Delta + V)\chi_0)| \leq C_\nu(N+1)^{n_\nu}.$$

So, for almost every ω and for any $n > 0$

$$|\mathrm{Tr}(\chi_0\varphi(H^n_\omega)\chi_0)\}| \leq \sum_{N\geq 0} \chi_{E_N}(\omega)C_\nu(N+1)^{n_\nu}.$$

But, by Lemma 4.2 (2), for $N > 0$,

$$\mathbb{E}_\omega(\chi_{E_N}(\omega)) \leq \mathbb{P}(\{\omega; \exists x \in \mathbb{R}^d \quad | x |^\nu + N \leq |V_\omega(x)|\} \leq \frac{\mu^{\mathfrak{e}(N/C)}}{\mathfrak{e}(N/C)!}G_\nu(\mu),$$

So the function $\omega \mapsto \sum_{N\geq 0} \chi_{E_N}(\omega)C_\nu(N+1)^{n_\nu}$ is $L^1(\Omega, \mathbb{P}_\omega)$. Having found an integrable majorant for $|\mathrm{Tr}(\chi_0\varphi(H^n_\omega)\chi_0)\}|$, we apply Lebesgue's Dominated Convergence Theorem to get Proposition 2.1. $\square$

Proof of Proposition 2.2. To prove this proposition we will use general facts about the convergence of probability measures. We introduce as shortly as possible the needed notions. More details may be found in [Ne] (essentially in chapter I) where our presentation comes from. Let $\mathcal{M}_+(\mathbb{R}^d)$ be the set of positive Radon measures on $\mathbb{R}^d$ (endowed with the topology of weak convergence). Then the Poisson process and the family of random variables $(t_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$ define two probability measures on $\mathcal{M}_+(\mathbb{R}^d)$. A precise definition of the relevant σ-algebra can be found in [Ne]. These probability measures may characterised by their Laplace transforms denoted by Ψ_ω for the Poisson process and Ψ_t for the Bernoulli process. The Laplace transforms are defined in the following way, for f a positive borelian function on $\mathbb{R}^d$,

$$\Psi_\omega(f) = \mathbb{E}_\omega\left(e^{-\int_{\mathbb{R}^d} f(x)dm(\omega,dx)}\right) \text{ and } \Psi_t(f) = \mathbb{E}_t\left(e^{-\sum_{\gamma\in\frac{1}{m}\mathbb{Z}^d} f(\gamma)}\right).$$

Ψ_ω may be computed in terms of the intensity μ (i.e the concentration) (see [Ne] p. 260). It is easy to see that, for f being the characteristic function of some cube in $\mathbb{R}^d$, one has $\Psi_t(f) \to \Psi_\omega(f)$ as $m \to +\infty$. This is just the usual theorem about the convergence in law of the sum of n independent identical Bernoulli trials (with a suitably chosen probability law) to a Poisson law.

Hence, approximating any positive borelian function by a sequence of positive linear combinations of characteristic functions of cubes, one gets that, for any f, positive borelian function on $\mathbb{R}^d$, one has $\Psi_t(f) \to \Psi_\omega(f)$ as $m \to +\infty$. By Proposition I.16 of [Ne], this implies that for any bounded weakly continuous function F on $\mathcal{M}_+(\mathbb{R}^d)$, one has $\mathbb{E}_t[F(dm(t,dx))] \to \mathbb{E}_\omega[F(dm(\omega,dx))]$ when $m \to +\infty$.

To prove Proposition 2.2, let us first assume that V is compactly supported. This assumption will be removed later on. Then, for any $\mathfrak{m}$, a positive Radon measure on $\mathbb{R}^d$, we define

$$V_\mathfrak{m}(x) = \int_{\mathbb{R}^d} V(x-y)d\mathfrak{m} \text{ and } V^n_\mathfrak{m}(x) = V_\mathfrak{m}(x)\psi\left(\frac{1}{n}V_\mathfrak{m}(x)\right) \text{ and } H_\mathfrak{m} = -\Delta + V^n_\mathfrak{m},$$

here ψ is the function defined in section 2.

If we show that, for $\varphi \in \mathcal{C}_o^\infty(\mathbb{R})$, $\mathfrak{m} \mapsto \mathrm{Tr}(\chi_0\varphi(H_\mathfrak{m})\chi_0)$ is weakly continuous and bounded on $\mathcal{M}_+(\mathbb{R}^d)$, Proposition I.16 of [Ne] will show that $\mathbb{E}_l\{\mathrm{Tr}(\chi_0\varphi(H_m^n)\chi_0)\} \to \mathbb{E}_\omega\{\mathrm{Tr}(\chi_0\varphi(H_\omega^n)\chi_0)\}$ when $m \to +\infty$.

As $\sup_\mathfrak{m}\sup_{x\in\mathbb{R}^d}|V_\mathfrak{m}^n(x)| \leq n$, for any $\mathfrak{m}$, the operator $H_\mathfrak{m}$ is lower semi-bounded by $-n$. Hence, we only need to consider test function φ supported in $[-n,+\infty)$. Then, for such a test function, we define $\Phi(x) = (n+1+x)^q\varphi(x)$; obviously, $\Phi \in \mathcal{C}_o^\infty(\mathbb{R})$. Using (3.4) for $\tilde{\varphi}$ is an analytic extension of Φ and the fact that $\varphi(H_\mathfrak{m}) = \Phi(H_\mathfrak{m})(n+1+H_\mathfrak{m})^{-q}$, we get

$$\mathrm{Tr}(\chi_o\varphi(H_\mathfrak{m})\chi_o) = \frac{i}{2\pi}\int_\mathbb{C}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\mathrm{Tr}(\chi_0(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0)d\overline{z}\wedge dz \tag{3.8}$$

as $\chi_0(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0$ is trace class (we take $q > d/2$). Then using the bound on $V_\mathfrak{m}^n$, we can upper-bound $\mathrm{Tr}(\chi_0(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0)$ uniformly in $\mathfrak{m}$ by some polynomial in $\dfrac{|\,z\,|+1}{|\,\mathrm{Im}z\,|}$ for any $z \in \mathbb{C}\setminus\mathbb{R}$ (using either Lemma 4.1 or the section 4 of [Kl]). This gives us a uniform bound on $\mathrm{Tr}(\chi_0\varphi(H_\mathfrak{m})\chi_0)$ for $\mathfrak{m} \in \mathcal{M}_+(\mathbb{R}^d)$ (using the infinite order of vanishing on the real line for almost analytic extensions).

We then need to show that $\mathfrak{m} \mapsto \mathrm{Tr}(\chi_0\varphi(H_\mathfrak{m})\chi_0)$ is weakly continuous in $\mathfrak{m}$. That is, if $(\mathfrak{m}_l)_{l\geq 0}$ is sequence of Radon measure converging weakly to $\mathfrak{m}$ (i.e, for any f continuous with compact support, $\int_{\mathbb{R}^d}f(x)d\mathfrak{m}_l \to \int_{\mathbb{R}^d}f(x)d\mathfrak{m}$ when $l \to +\infty$) then $\mathrm{Tr}(\chi_0\varphi(H_{\mathfrak{m}_l})\chi_0) \to \mathrm{Tr}(\chi_0\varphi(H_\mathfrak{m})\chi_0)$ when $l \to +\infty$.

Let $(\mathfrak{m}_l)_{l\geq 0}$ be a sequence of Radon measure converging weakly to $\mathfrak{m}$. Then, for any $x \in \mathbb{R}^d$, $V_{\mathfrak{m}_l}^n(x) \to V_\mathfrak{m}^n(x)$ when $l \to +\infty$ and $\sup_{l\geq 0}\sup_{x\in\mathbb{R}^d}|V_{\mathfrak{m}_l}^n(x)| \leq n$. Using (3.8) and the $\mathfrak{m}$-uniform estimates on the traces we consider , we see that by Lebesgue's Dominated Convergence Theorem, we only need to show that, for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\mathrm{Tr}(\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-q}\chi_0) \to \mathrm{Tr}(\chi_0(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0) \text{ when } l \to +\infty. \tag{3.9}$$

We evaluate

$$\begin{aligned}
&|\mathrm{Tr}(\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-q}\chi_0) - \mathrm{Tr}(\chi_0(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0)| \leq \\
&\quad\leq |\mathrm{Tr}(\chi_0(z-H_{\mathfrak{m}_l})^{-1}(V_{\mathfrak{m}_l}^n - V_\mathfrak{m}^n)(z-H_\mathfrak{m})^{-1}(n+1+H_\mathfrak{m})^{-q}\chi_0)| + \\
&\qquad + \sum_{p=1}^{q}|\mathrm{Tr}(\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-p}(V_{\mathfrak{m}_l}^n - V_\mathfrak{m}^n)(n+1+H_\mathfrak{m})^{-q+p-1}\chi_0)|.
\end{aligned} \tag{3.10}$$

Pick $q > d$. Pick $1 \leq p \leq q$. Let us prove that

$$|\mathrm{Tr}(\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-p}(V_{\mathfrak{m}_l}^n - V_\mathfrak{m}^n)(n+1+H_\mathfrak{m})^{-q+p-1}\chi_0)| \to 0 \text{ when } n \to +\infty. \tag{3.11}$$

The proof of this fact is the same for the first term of the right hand side of (3.10).

We write

$$\begin{aligned}
&\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-p}(V_{\mathfrak{m}_l}^n - V_\mathfrak{m}^n)(n+1+H_\mathfrak{m})^{-q+p-1}\chi_0 = \\
&\quad = \sum_{\alpha\in\mathbb{Z}^d}\chi_0(z-H_{\mathfrak{m}_l})^{-1}(n+1+H_{\mathfrak{m}_l})^{-p}\chi_\alpha \cdot \\
&\qquad\qquad \cdot \chi_\alpha(V_{\mathfrak{m}_l}^n - V_\mathfrak{m}^n)(n+1+H_\mathfrak{m})^{-1}\cdot\chi_\beta \\
&\qquad\qquad\qquad \cdot \chi_\beta(n+1+H_\mathfrak{m})^{-q+p}\chi_0.
\end{aligned}$$

Either $\chi_0(z - H_{\mathtt{m}_l})^{-1}(n + 1 + H_{\mathtt{m}_l})^{-p}\chi_\alpha$ or $\chi_\beta(n + 1 + H_{\mathtt{m}})^{-q+p}\chi_0$ are trace class. Assume $\chi_\beta(n+1+H_{\mathtt{m}})^{-q+p}\chi_0$ is trace class. Moreover, using (4.1) and classical estimates on their operator norm, we get, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\|\chi_\beta(n + 1 + H_{\mathtt{m}})^{-q+p}\chi_0\|_{\mathcal{T}_1} \leq e^{-\epsilon\delta_{1/2}(\beta)},$$

and

$$\sup_{l\geq 1} \|\chi_0(z - H_{\mathtt{m}_l})^{-1}(n + 1 + H_{\mathtt{m}_l})^{-p}\chi_\alpha\| \leq P(z)e^{-|\alpha|/P(z)}$$

where $\epsilon > 0$ and $P(z)$ is some polynomial in $\dfrac{|\ z\ | + 1}{|\ \mathrm{Im} z\ |}$.

Then if we show that $\chi_\alpha(V^n_{\mathtt{m}_l} - V^n_{\mathtt{m}})(n + 1 + H_{\mathtt{m}})^{-1}\chi_\beta \to 0$ as a bounded operator when $l \to +\infty$, as it is uniformly bounded, using Lebesgue's Dominated Convergence Theorem, we get (3.11). As χ_α is compactly supported, $\chi_\alpha(V^n_{\mathtt{m}_l} - V^n_{\mathtt{m}}) \to 0$ in any $L^p(\mathbb{R}^d)$ for $1 \leq p < +\infty$. Hence also as a bounded operator from $L^q(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ for any $q > 2$. But by [Si] section B.2, $(n+1+H_{\mathtt{m}})^{-1}\chi_\beta$ is bounded from $L^2(\mathbb{R}^d)$ to $L^{2+\delta}(\mathbb{R}^d)$ for some $\delta > 0$. This ends the proof of (3.9), hence the proof of Proposition 2.2 in the case when V is compactly supported.

To get the result when V is not compactly supported but satisfies only assumption (H.1), we approximate V by $V^r = V\phi_r$ where $\phi_r(x) = \phi(x^2/r)$ and $\phi \in \mathcal{C}^\infty_o(\mathbb{R})$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $|\ x\ | \leq 1$. Define

$$V^r_\omega(x) = \int_{\mathbb{R}^d} V^r(x - y)m(\omega, dy),\ \ V^{n,r}_\omega(x) = V^r_\omega(x)\psi\left(\frac{1}{n}V^r_\omega(x)\right) \text{ and } H^r_\omega = -\Delta + V^{n,r}_\omega,$$

and

$$V^r_m(x) = \sum_{\gamma\in\frac{1}{m}\mathbb{Z}^d} t_\gamma V^r(x - \gamma),\quad V^{n,r}_m(x) = V^r_m(x)\psi\left(\frac{1}{n}V^r_m(x)\right) \text{ and } H^r_m = -\Delta + V^{n,r}_m.$$

We then have to show that

$$\mathbb{E}_\omega\left\{\mathrm{Tr}(\chi_0\varphi(H^r_\omega)\chi_0)\right\} \to \mathbb{E}_\omega\left\{\mathrm{Tr}(\chi_0\varphi(H_\omega)\chi_0)\right\} \text{ when } r \to +\infty$$

and

$$\mathbb{E}_t\left\{\mathrm{Tr}(\chi_0\varphi(H^r_m)\chi_0)\right\} \to \mathbb{E}_t\left\{\mathrm{Tr}(\chi_0\varphi(H_m)\chi_0)\right\} \text{ uniformly in } m \text{ when } r \to +\infty.$$

Let us do it in the case of the Bernoulli random variables. The Poisson case is done in the same way. By (3.8), it enough to show that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\sup_{m\geq 1} |\mathbb{E}_t\left\{\mathrm{Tr}(\chi_0(z - H^r_m)^{-1}(n + 1 + H^r_m)^{-q}\chi_0)\right\} - \mathbb{E}_t\left\{\mathrm{Tr}(\chi_0(z - H_m)^{-1}(n + 1 + H_m)^{-q}\chi_0)\right\}| \to 0 \quad \text{when } r \to +\infty.$$

Using the decomposition (3.10) and the uniform (in m and in the realization of t) bounds known for the trace class norm and operator norm of $\chi_\beta(z - H^r_m)^{-p}\chi_0$ and $\chi_\beta(z - H_m)^{-p}\chi_0$ (these bounds are sub-exponentially decreasing in β), we just have to show that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\sup_{m\geq 1} |\mathbb{E}_t\left\{\|\chi_\alpha\left[(z - H^r_m)^{-1} - (z - H_m)^{-1}\right]\chi_\beta\|\right\}| \to 0 \text{ when } r \to +\infty$$

here $\|\cdot\|$ denotes the operator norm.
For $x \in \mathbb{R}^d$, by assumption (H.1) on V, one has

$$|V_m^r(x) - V_m(x)| \leq \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} t_\gamma |V(x-\gamma)| \cdot (1-\phi_r(x-\gamma)) \leq e^{-\eta r/2} \sum_{\gamma \in \frac{1}{m}\mathbb{Z}^d} t_\gamma e^{\eta|x-\gamma|/2}.$$

Then, mimicking the proof of Lemma 4.2, one proves that, for some $C>0$ and $\epsilon>0$ such that $C\epsilon e^{r\eta/2} > 2$

$$\mathbb{P}(\{t; \exists x \in \mathbb{R}^d \text{ such that } |V_m^r(x) - V_m(x)| \geq \epsilon |x|\}) \leq \frac{\mu^{\mathfrak{e}(C\epsilon e^{r\eta/2})}}{\mathfrak{e}(C\epsilon e^{r\eta/2})!} F(\mu) \tag{3.12}$$

where F is increasing and positive.
Define the event $\mathcal{E}_r = \{t; \forall \mid x \mid \leq e^{\eta r/8}, |V_m^r(x) - V_m(x)| \leq e^{-\eta r/8}\}$. Choosing in (3.12), $\epsilon = e^{-\eta r/4}$, we get, for r large enough and some $C>0$,

$$\mathbb{P}(\mathcal{E}_r) \geq 1 - Ce^{-\eta r/4}.$$

Then, for some $C>0$,

$$\begin{aligned}
|\mathbb{E}_t \{\| \left[\chi_\alpha (z - H_m^r)^{-1} - (z-H_m)^{-1}\right] \chi_\beta \|\}| &= |\mathbb{E}_t \{\|\chi_\alpha (z-H_m^r)^{-1}(V_m^{n,r} - V_m^n)(z-H_m)^{-1}\chi_\beta)\|\}| \\
&\leq C\mathbb{P}({}^c\mathcal{E}_r) + \left| \int_{\mathcal{E}_r} \|\chi_\alpha (z-H_m^r)^{-1}(V_m^{n,r} - V_m^n)(z-H_m)^{-1}\chi_\beta)\| d\mathbb{P}_t \right| \\
&\leq Ce^{-\eta r/8} + \left| \int_{\mathcal{E}_r} \|\chi_\alpha (z-H_m^r)^{-1}\chi_{|x|>e^{\eta r/8}}(V_m^{n,r} - V_m^n)(z-H_m)^{-1}\chi_\beta)\| d\mathbb{P}_t \right| + \\
&\qquad + \left| \int_{\mathcal{E}_r} \|\chi_\alpha (z-H_m^r)^{-1}\chi_{|x|\leq e^{\eta r/8}}(V_m^{n,r} - V_m^n)(z-H_m)^{-1}\chi_\beta)\| d\mathbb{P}_t \right|,
\end{aligned}$$

where χ_E is the characteristic function of $E \subset \mathbb{R}$.
We know that, for some $C>0$ and $\epsilon>0$, $\|\chi_\alpha(z-H_m^r)^{-1}\chi_{|x|>e^{\eta r/8}}\| \leq Ce^{-\epsilon||\alpha|-e^{\eta r/8}|}$ and $\|\chi_{|x|>e^{\eta r/8}}(z-H_m)^{-1}\chi_\beta)\| \leq Ce^{-\epsilon||\beta|-e^{\eta r/8}|}$. Moreover, for some $C>0$, $\mid V_m^{n,r}(x) - V_m^n(x) \mid \leq C \mid V_m^r(x) - V_m(x) \mid$. Hence, for some $C>0$ and $\epsilon>0$, we get

$$\begin{aligned}
&\sup_{m\geq 1} \left(|\mathbb{E}_t \{\| \left[\chi_\alpha (z-H_m^r)^{-1} - (z-H_m)^{-1}\right] \chi_\beta \|\}| \right) \leq \\
&\qquad \leq C \left(e^{-\eta r/8} + n \cdot e^{-\epsilon(||\alpha|-e^{\eta r/16}|+||\beta|-e^{\eta r/16}|)} + e^{-\eta r/4} \right).
\end{aligned}$$

This ends the proof of Proposition 2.2. $\square$

Proof of Proposition 2.4. As, for $\#A = k$, H_A is semi-bounded below by $-k\cdot C$ (C given in (H.1)), we rewrite (3.8) in the following form, for $n \geq k\cdot C$,

$$\varphi(H_A) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)(z-H_A)^{-1}(n+1+H_A)^{-q} d\overline{z}\wedge dz.$$

where $\tilde{\varphi}$ is an almost analytic extension of $\Phi(x) = (n+1+x)^q\varphi(x)$.

So, we get, for $q > d/2$, (see [Kl])

(3.13)

$$\langle Z(0,x_1,\dots,x_k),\varphi\rangle =$$

$$= \frac{i}{2\pi}\sum_{A\subset\{0,x_1,\dots,x_k\}}(-1)^{\#A}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\mathrm{Tr}\left((z-H_A)^{-1}(n+1+H_A)^{-q}-\right.$$
$$\left.-(z+\Delta)^{-1}(n+1-\Delta)^{-q}\right)d\overline{z}\wedge dz$$
$$= \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\mathrm{Tr}\left(\sum_{A\subset\{0,x_1,\dots,x_k\}}(z-H_A)^{-1}(-1)^{\#A}(n+1+H_A)^{-q}\right)d\overline{z}\wedge dz$$
$$= \frac{i}{2\pi}\int_{[0,1]^k}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\mathrm{Tr}\left(\partial_A\left((z-H_t)^{-1}(n+1+H_t)^{-q}\right)\right)d\overline{z}\wedge dzdt$$

where $H_t = -\Delta + t_0V + \sum_{l=1}^k t_lV_l$ où $(t_l)_{0\leq l\leq k}\in[0,1]^{k+1}$ et $\partial_A = \bigotimes_{0\leq l\leq k}\partial_{t_l}$.

Then as V is continuous, using the estimates of Lemma 4.1, for $z\in\mathbb{C}\setminus\mathbb{R}$, $(x_1,\dots x_k)\mapsto(z-H_t)^{-1}(n+1+H_t)^{-q}$ is continuous in $\mathcal{T}_1$ uniformly in t. Hence, $(x_1,\dots x_k)\mapsto\langle Z(0,x_1,\dots,x_k),\varphi\rangle$ is continuous using (3.13) and the well controlled $\mathcal{T}_1$-norm estimates for $(z-H_t)^{-1}(n+1+H_t)^{-q}$.

Now estimating $\partial_A\left((z-H_t)^{-1}(n+1+H_t)^{-q}\right)$ as in [Kl], we get, for some $\epsilon>0$, $C>0$, $N>0$ and $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\|\partial_A\left((z-H_t)^{-1}(n+1+H_t)^{-q}\right)\|\leq C\left(\frac{|\,z\,|+1}{|\,\mathrm{Im}z\,|}\right)^N e^{-\epsilon\frac{|\mathrm{Im}z|}{|z|+1}\sum_{l=1}^k|x_l|}.$$

Hence using the properties of almost analytic extensions, we get that $\langle Z(0,x_1,\dots,x_k),\varphi\rangle$ is in $L^1((\mathbb{R}^d)^k)$. This ends the proof of Proposition 2.4. □

IV) Some technical lemmas.

a) Resolvent estimates.

Here we estimate the norm of the truncated resolvent in appropriate Schatten classes for some operators with unbounded potentials.

Lemma 4.1. *Let $\nu\in(0,1)$ and $q>d/2$. Then, there exists $C_{\nu,q}>0$ and $\epsilon_{\nu,q}>0$ such that, for any V, a function that satisfies for some $N>0$,*

$$|\,V(x)\,|\leq|x|^\nu+N \text{ for all } x\in\mathbb{R}^d$$

and, for any $(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d$ and $z\in\mathbb{C}\setminus\mathbb{R}$, one has

$$\|\chi_\alpha(z-(-\Delta+V))^{-1}\chi_\beta\|_{\mathcal{T}_q}\leq C_{\nu,q}\frac{(1+|\beta|)^\nu}{\eta(z,N)}e^{-\epsilon_{\nu,q}\cdot\eta(z,N)|\delta_\nu(\alpha)-\delta_\nu(\beta)|}$$

$$\text{where } \delta_\nu(x)=(1+x^2)^{(1-\nu)/2} \text{ and } \eta(z,N)=\frac{dist(z,\sigma(-\Delta+V))}{|z|+N+1}$$

here $\|\cdot\|_{\mathcal{T}_q}$ denotes the norm in the q-th Schatten class and $dist(z,z')$ denotes the distance in $\mathbb{C}$.

Proof. By [Si] section B.9, we know that $\chi_\alpha\cdot(z-(-\Delta+V))^{-1}\cdot\chi_\beta\in\mathcal{T}_q$. To prove the estimate on its norm we will use the idea used in [Si]. For $\epsilon>0$, one has,

$$e^{-\epsilon\delta_\nu}\Delta e^{\epsilon\delta_\nu}=\Delta-2\epsilon\nabla\delta_\nu\cdot\nabla+(\epsilon\nabla\delta_\nu)^2-\epsilon\Delta\delta_\nu,$$

hence, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$e^{\epsilon\delta_\nu}(z-(-\Delta+V))^{-1}e^{-\epsilon\delta_\nu} = \left(z+\Delta-V-2\epsilon\nabla\delta_\nu\cdot\nabla+(\epsilon\nabla\delta_\nu)^2-\epsilon\Delta\delta_\nu\right)^{-1}$$
$$= \left(1-(z-(-\Delta+V))^{-1}\left(2\epsilon\nabla\delta_\nu\cdot\nabla-(\epsilon\nabla\delta_\nu)^2+\epsilon\Delta\delta_\nu\right)\right)^{-1}(z-(-\Delta+V))^{-1}.$$

Let us estimate

$$\|(z-(-\Delta+V))^{-1}\left(2\epsilon\nabla\delta_\nu\cdot\nabla-(\epsilon\nabla\delta_\nu)^2+\epsilon\Delta\delta_\nu\right)\| \leq \epsilon^2\|(z-(-\Delta+V))^{-1}(\nabla\delta_\nu)^2\|+$$
$$+\epsilon\|(z-(-\Delta+V))^{-1}\Delta\delta_\nu\|+\epsilon\|(\Delta-1)^{-1}\nabla\delta_\nu\cdot\nabla\|+$$
$$+\epsilon\|(z-(-\Delta+V))^{-1}(z+1-V)(\Delta-1)^{-1}\nabla\delta_\nu\cdot\nabla\|.$$

As $-\Delta+V$ is self-adjoint and $\|V\nabla\delta_\nu\| \leq N$, using [Si] Theorem B.6.5, we get, for some $C>0$,

$$\|(z-(-\Delta+V))^{-1}\left(2\epsilon\nabla\delta_\nu\cdot\nabla-(\epsilon\nabla\delta_\nu)^2+\epsilon\Delta\delta_\nu\right)\| \leq C\epsilon\left(1+\frac{1}{\eta(z,N)}\right).$$

So if we choose ϵ of the form $\pm\epsilon_\nu\eta(z,N)$ with $\epsilon_\nu C<1/8$, we get

$$\|e^{\epsilon\delta_\nu}\chi_\alpha(z-(-\Delta+V))^{-1}\chi_\beta e^{-\epsilon\delta_\nu}\|_{\mathcal{T}_q} \leq 4\|(z-(-\Delta+V))^{-1}\chi_\beta\|_{\mathcal{T}_q}. \tag{4.1}$$

Using the growth assumption on V and standard estimates for the free Laplace operator, we get

$$\|(z-(-\Delta+V))^{-1}\chi_\beta\|_{\mathcal{T}_q} \leq \|(z-(-\Delta+V))^{-1}(z-V+1)(\Delta-1)^{-1}\chi_\beta+(\Delta-1)^{-1}\chi_\beta\|_{\mathcal{T}_q}$$
$$\leq \sum_{\alpha\in\mathbb{Z}^d}\|(z-(-\Delta+V))^{-1}(z-V+1)\chi_\alpha\|\cdot\|\chi_\alpha(\Delta-1)^{-1}\chi_\beta\|_{\mathcal{T}_q}+$$
$$+\|(\Delta-1)^{-1}\chi_\beta\|_{\mathcal{T}_q}$$
$$\leq C\frac{(1+|\beta|)^\nu}{\eta(z,N)},$$

then using (4.1) for ϵ of the form $\pm\epsilon_\nu\eta(z,N)$, we get the announced result. □

a) A probabilistic estimate.

Let V_ω be the Poisson potential defined in section 1, and $\mathcal{P}_\omega$ be the associated probability measure. Let $V_m(t;x) = \sum_{\gamma\in\frac{1}{m}\mathbb{Z}^d} t_\gamma V_\gamma$ where $(t_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$ be a family of independent identically distributed Bernoulli random variables with common probability distribution $P(t_0=1)=p_m$ and $P(t_0=0)=1-p_m$ with $p_m=\mu m^{-d}$. Let $\mathbb{P}_m$ denote the probability measure induced on the space of potentials by these random variables. For $x>0$, we define $\mathfrak{e}(x)=\inf\{n\in\mathbb{N};\ n\geq x\}$. Then one has

Lemme 4.2. *For $\nu\in(0,1)$ and $N\in\mathbb{N}^*$, for some $C>0$, one has*

$$\sup_{m\geq 1}\left(\mathbb{P}_m\left(\{t;\ \exists x\in\mathbb{R}^d \text{ such that } |V_m(t;x)|\geq|x|^\nu+N\}\right)\right) \leq \frac{\mu^{\mathfrak{e}(N/C)}}{\mathfrak{e}(N/C)!}F_\nu(\mu), \tag{1}$$

and

(2) $$\mathbb{P}_\omega\left(\{\omega;\ \exists x\in\mathbb{R}^d \text{ such that } |V_\omega(x)|\geq |x|^\nu+N\}\right)\leq \frac{\mu^{\mathfrak{e}(N/C)}}{\mathfrak{e}(N/C)!}G_\nu(\mu),$$

where F_ν and G_ν are positive increasing functions of μ that do not depend on N.

Proof. Let us first prove (1). By assumption (H.1), one has, for some $C>0$ and $x\in\mathbb{R}^d$,

$$|V_m(t;x)|\leq C\sum_{\beta\in\mathbb{Z}^d}e^{-\eta|x-\beta|}T_\beta \text{ where } T_\beta=\sum_{\gamma\in C(m)}t_{\beta+\gamma} \text{ for } \beta\in\mathbb{Z}^d.$$

Hence,

$$\begin{aligned}\mathbb{P}_m&\left(\{t;\exists x\in\mathbb{R}^d \text{ such that } |V_m(t;x)|\geq|x|^\nu+N\}\right)\\ &\leq \mathbb{P}_m\left(\{t;\exists x\in\mathbb{R}^d\ \exists\beta\in\mathbb{Z}^d \text{ such that } T_\beta\geq\frac{1}{C}(|x|^\nu+N)e^{\eta|x-\beta|}\}\right)\\ &\leq \mathbb{P}_m\left(\{t;\exists\beta\in\mathbb{Z}^d \text{ such that } T_\beta\geq\frac{1}{C}(|\beta|^\nu+N)\}\right).\end{aligned}$$

By the definition of $C(m)$ and the assumptions on the $(t_\gamma)_{\gamma\in\frac{1}{m}\mathbb{Z}^d}$, $(T_\beta)_{\beta\in\mathbb{Z}^d}$ is a family of i.i.d random variables that satisfies the following estimates, for $M>0$, for any $\beta\in\mathbb{Z}^d$, for any $m\geq1$,

$$\mathbb{P}_m(\{t;T_\beta\geq M\})=\sum_{n=\mathfrak{e}(M)}^{m^d}\binom{m^d}{n}(\mu\cdot m^{-d})^n(1-\mu\cdot m^{-d})^{m^d-n}\leq\frac{\mu^{\mathfrak{e}(M)}}{(\mathfrak{e}(M))!}e^\mu.$$

So, for any $m\geq1$,

$$\begin{aligned}\mathbb{P}_m\left(\{t;\exists x\in\mathbb{R}^d \text{ such that } |V_m(t;x)|\geq|x|^\nu+N\}\right)&\leq\sum_{\beta\in\mathbb{Z}^d}\mathbb{P}_m\left(\{t;\ T_\beta\geq\frac{1}{C}(|\beta|^\nu+N)\}\right)\\ &\leq e^\mu\sum_{\beta\in\mathbb{Z}^d}\frac{\mu^{\mathfrak{e}((|\beta|^\nu+N)/C)}}{(\mathfrak{e}((|\beta|^\nu+N)/C))!}\\ &\leq\frac{\mu^{\mathfrak{e}(N/C)}}{(\mathfrak{e}(N/C))!}F_\nu(\mu).\end{aligned}$$

This ends the proof of (1).

To get (2), using (H.1), we know that, for some $C>0$,

$$\{\omega;\ \exists x\in\mathbb{R}^d \text{ such that } |V_\omega(x)|\geq|x|^\nu+N\}\subset\bigcup_{\gamma\in\mathbb{Z}^d}\{\omega;\ |V_\omega(\gamma)|\geq\frac{1}{C}(|\gamma|^\nu+N)\}.$$

Moreover, if $C_\beta(1)$ denotes the cube centered in β of sidelength 1, then

$$|V_\omega(\gamma)|\leq C\sum_{\beta\in\mathbb{Z}^d}e^{-\eta|\beta-\gamma|}m(\omega;C_\beta(1))=C\sum_{\beta\in\mathbb{Z}^d}e^{-\eta|\beta|}m(\omega;C_{\beta+\gamma}(1)).$$

Hence, for some $C > 0$, one has

$$\{\omega;\ |\ V_\omega(\gamma)| \geq \frac{1}{C}(|\gamma|^\nu + N)\} \subset \bigcup_{\beta\in\mathbb{Z}^d} \{\omega;\ m(\omega; C_{\beta+\gamma}(1)) \geq \frac{1}{C}(|\gamma|^\nu + N)e^{\eta|\beta|}\}.$$

So, using the definition of the Poisson process, we get

$$\begin{aligned}\mathbb{P}_\omega\left(\{\omega;\ |V_\omega(\gamma)| \geq \frac{1}{C}(|\gamma|^\nu + N)\}\right) &\leq \sum_{\beta\in\mathbb{Z}^d} \mathbb{P}_\omega\left(\{\omega;\ m(\omega; C_{\beta+\gamma}(1)) \geq \frac{1}{C}(|\gamma|^\nu + N)e^{\eta|\beta|}\}\right)\\
&\leq \sum_{\beta\in\mathbb{Z}^d} \frac{\mu^{\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu+N)e^{\eta|\beta|})}}{(\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu + N)e^{\eta|\beta|}))!}\\
&\leq \sum_{\beta\in\mathbb{Z}^d} \frac{1}{(\mathfrak{e}(e^{\eta|\beta|}) - 1)!}\left(\frac{\mu^{\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu+N))}}{\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu + N))!)}\right)^{e^{\eta|\beta|}}\\
&\leq \frac{\mu^{\mathfrak{e}((|\gamma|^\nu+N)/C)}}{\mathfrak{e}((|\gamma|^\nu + N)/C)!} G(\mu),\end{aligned}$$

so

$$\begin{aligned}\mathbb{P}_m\left(\{t; \exists x \in \mathbb{R}^d \text{ such that } |V_\omega(x)| \geq |x|^\nu + N\}\right) &\leq \sum_{\gamma\in\mathbb{Z}^d} \frac{\mu^{\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu+N))}}{\mathfrak{e}(\frac{1}{C}(|\gamma|^\nu + N))!} G(\mu)\\
&\leq \frac{\mu^{\mathfrak{e}(N/C)}}{\mathfrak{e}(N/C)!} G_\nu(\mu).\end{aligned}$$

This ends the proof of Lemma 4.2. $\square$

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