Polyparabolic equations with two - dimensional time

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Abstract

The present article is devoted to an interesting and rapidly developing branch of partial differential equations: *partial differential equations with multidimensional time.* We introduce a new class of polyparabolic equations with two-dimensional time and study nonhomogeneous mixed problems for it. In particular, we obtain the uniqueness theorem for the problems satisfying the newly defined inductive polyparabolicity condition. Various equations of this type occur in the theory of probability, cosmology and physics.

Contents

Some Notation

Here is the list of the conventional notation to be used below:

$\mathbb{N}$ - the set of all positive integers.
$\mathbb{Z}$ - the set of all integers.
$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
$\mathbb{R}$ - the set of all real numbers.
$\mathbb{C}$ - the set of all complex numbers.
$\mathbb{R}^n$ - the standard real n-dimensional euclidean space.
$\mathbb{C}^m$ - the standard complex m-dimensional euclidean space.
$\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : x_n > \text{ (resp. } <) \ 0\}$; $\overline{\mathbb{R}^n_\pm}$ - the closures of the half-spaces $\mathbb{R}^n_\pm$ in $\mathbb{R}^n$; $(\overline{\mathbb{R}^n_+})^m = \overline{\mathbb{R}^n_+} \times \overline{\mathbb{R}^n_+} \times \cdots \times \overline{\mathbb{R}^n_+}$ $-$ m times.
$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ . \ . \ . \ , \frac{\partial}{\partial x_n}\right)$, where $(x_1, ..., x_n) \in \mathbb{R}^n$.
$D = i^{-1}\frac{\partial}{\partial x}$, $D_j = i^{-1}\frac{\partial}{\partial x_j}$, where $i = \sqrt{-1} \in \mathbb{C}$.
$\partial^l_{x_j} = \frac{\partial^l}{\partial x^l_j}$, $l \in \mathbb{Z}_+$.
$D^\alpha = D^{\alpha_1}_1 \ . \ . \ . \ D^{\alpha_n}_n$, where α is a multiindex, i.e. $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$.
$\xi^\alpha = \xi^{\alpha_1}_1 \ . \ . \ . \ \xi^{\alpha_n}_n$, where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$.
$x \cdot \xi = x_1\xi_1 + \cdot \cdot \cdot + x_n\xi_n$, where $x = (x_1, ..., x_n) = (x', x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, ..., \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$.
$|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, where α is a multiindex, $\alpha = (\alpha', \alpha_n)$.
$|||q|||^l = \sum_{i=1}^m |q_i|^l$, if $q = (q_1, ..., q_m) \in \mathbb{C}^m$; $M_q = \min_{i=1,...,m} |q_i|$.
$\prod_{j=1}^k E_j = E_1 \times E_2 \times \cdots \times E_k$, where E_j are Hilbert spaces.
$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $< x > = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$ for $x =$ $= (x_1, ..., x_n) \in \mathbb{R}^n$.
$\frac{l}{b} = \left\{\frac{l}{b_1}, ..., \frac{l}{b_m}\right\}$, where $l \in \mathbb{Z}_+$, $b \in \mathbb{N}^m$.
$\Delta = \Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdot \cdot \cdot + \frac{\partial^2}{\partial x_n^2}$ - the Laplacian in $\mathbb{R}^n$.
δ_{jk} - the Kroneker symbol ($\delta_{jk} = 0$ as $j \neq k$ and 1 as $j = k$).
$[s]$ - the integer part of s .
$\rhd$ means the end of the proof.
:= means "equals by definition".
$\subset\subset$ means the compact inclusion.
$\operatorname{supp} u$ means the support of the function u .
$\partial\Omega$ means the boundary of an open set $\Omega \subset \mathbb{R}^n$,
$\overline{\Omega}$ means its closure.
$C_0^\infty(\mathbb{R}^n)$ - the space of all infinitely smooth complex - valued functions with compact support.
$C^\infty(\overline{\Omega}, Y)$ - the space of smooth functions on Ω with the values in Y , which are continuous up to the boundary.
The direct and the inverse Fourier transformations F , F^{-1} and the

Fourier transformation in the variables $x' \in \mathbb{R}^{n-1}$ are defined by the formulae

$$\widehat{u}(\xi) \equiv F(u)(\xi) \equiv F_{x\to\xi}(u) = \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} u(x)\, dx \ ,$$

$$\widetilde{f}(x) \equiv F^{-1}(f)(x) \equiv F^{-1}_{\xi\to x}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} f(\xi)\, d\xi \ ,$$

$$\widehat{u}(\xi') \equiv F'(u)(\xi') \equiv F'{}_{x'\to\xi'}(u) = \int_{\mathbb{R}^{n-1}_{x'}} e^{-ix'\cdot\xi'} u(x')\, dx' \ .$$

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INTRODUCTION

0.1 History and previous results

The first multidimensional time problem to be considered was probably the Goursat problem for the Wave equation: $U_{t_1,t_2} = f(t_1,t_2)$, where $t_1, t_2 > 0$, $U\,|_{t_1=0} = \phi_1(t_2)$, $U\,|_{t_2=0} = \phi_2(t_1)$ and $\phi_1(0) = \phi_2(0)$.

The subsequent development of the Goursat problem theory was connected with the successive approximations method of Picard (see, for example, (Sobolev S.L. (1966))) being applied to the Goursat problem for Hyperbolic equation with bounded coefficients

$$U_{t_1,t_2} + a(t_1,t_2)U_{t_1} + b(t_1,t_2)U_{t_2} + c(t_1,t_2)U = f(t_1,t_2) , \tag{0.1.1}$$

$$U\Big|_{t_1=t_1^0} = \phi_1(t_2) , \quad U\Big|_{t_2=t_2^0} = \phi_2(t_1) ,$$

where $t_1^0 \le t_1 \le a$ and $t_2^0 \le t_2 \le b$.

The analytic Goursat problem for systems with two independent variables and the corresponding nonanalytic Goursat problem for systems of such kind with constant coefficients, but with boundary conditions of polynomial growth, were presented in the works (Mikhailov V.P. (1957a) , (1957b)). The next step in the study of multidimensional time problems was made by Friedman and Littman (see Friedman A. (1962), Friedman A., Littman W. (1963)).

In particular, Friedman A. and Littman W. (1963) have proved the existence and uniqueness of the solution to the following mixed problem with two-dimensional time :

$$U_{t_1 t_2} - LU = F(x,t) \tag{0.1.2}$$

$$U\,|_{t_1=0} = U\,|_{t_2=0} = U\,|_{x\in\partial D} = 0 ,$$

where L is a second order elliptic self-adjoint differential operator.

The further development of the theory was elaborated in the series of papers by Brish and Yurchuk (see Brish N.I., Yurchuk N.I. (1968), (1970), (1971), Yurchuk. N.I. (1968a), (1968b), (1969)). The Mixed and Goursat problems for hyperbolic equations of a special kind, and also Goursat problem for an abstract linear second order differential equation were studied in these works by the energy inequality method. We shall describe these results briefly, beginning from (Brish N.I., Yurchuk N.I. (1971)).

Let $J = (0,T_1) \times (0,T_2)$ be a bounded rectangle in the plane $\mathbb{R}^2$ with coordinates $t = (t_1,t_2)$ and let $\mathcal{E}$ be a Hilbert space. We consider the equation

$$\frac{\partial^2 U(t)}{\partial t_1 \partial t_2} + A_1(t)\frac{\partial U(t)}{\partial t_1} + A_2(t)\frac{\partial U(t)}{\partial t_2} + A_0(t)U(t) = f(t) , \tag{0.1.3}$$

where U and f are $\mathcal{E}$ - valued functions on J , A_i , $i = 0,1,2$ - are the given functions on J with values in the space of bounded linear operators in

$\mathcal{E}$. It was proved in (Brish N.I., Yurchuk N.I. (1971)) that equation (0.1.3) with initial conditions of the Goursat type (as in (0.1.1)) admits a unique solution in the corresponding functional spaces. A special case of equation (0.1.3) was introduced in the work (Brish N.I., Yurchuk N.I. (1968)), where

$$A_1(t) = a_1(x,t)\,, \quad A_2(t) = a_2(x,t)\,, \quad A_0(t) = a_0(x,t,D) \tag{0.1.4}$$

and a_1 , a_2 were operators of multiplication by functions and a_0 was a uniformly elliptic positive differential operator of order $2k$ with respect to $x \in \Omega \subset \mathbb{R}^n$. Different types of mixed boundary value problems for equation (0.1.3) under conditions (0.1.4) were studied in the works (Brish N.I., Yurchuk N.I. (1968), Yurchuk. N.I. (1968a), (1968b), (1969)). In these papers boundary conditions in x formed a system of boundary differential equations with operators $\{B_j\}_{j=1}^k$:

$$B_j U\big|_{\Gamma} = 0\,, \quad j = 1,..,m\,, \quad \Gamma = \partial\Omega \times J\ , \tag{0.1.5}$$

where

$$B_j = \sum_{|\alpha| \le k_j}^{r} b_{j,\alpha}(x) D^\alpha\,, \quad j = 1,..,m\ ,$$

are operators of orders $k_j \le 2k-1$ $(0 \le k_1 < ... < k_m \le 2k)$. It had been supposed that the system (0.1.5) was associated with operator A_0 through the well known Shapiro - Lopatinskij condition (see, for example, Egorov Yu.V., Shubin M.A. (1988)).

The existence and uniqueness theorems for the problems (0.1.3) - (0.1.5) with different initial conditions were proved in (Brish N.I., Yurchuk N.I. (1968), Yurchuk N.I. (1968b), (1969)) by the energy inequality method and in specially designed functional classes.

Finally, we refer to the increasing number of investigations containing different promising applications of equations with multidimensional time to mechanics, physics and cosmology (see Baez J.C., Segal I.E., Zhon Z.-F. (1990), Hillion P. (1990a), (1990b), Newton R.G. (1991), Rendall A.D. (1992), Uglum J. (1992), Polubinskij A.S. (1989), Fushchich V.I., Galitsyn A.S., Polubinskii A.S. (1990)). In particular, the last two works are devoted to the so called *biparabolic* equation

$$L_2 U = 0\ , \quad L_2 = L_1 \circ L_1\ , \quad L_1 = \frac{\partial}{\partial t} - k^2 \frac{\partial^2}{\partial x^2} \tag{0.1.6}$$

with interesting applications.

0.2 Description of the results

In this paper we consider equations of the kind

$$A\big(x,t,D_x,\frac{\partial}{\partial t}\big)\, u(x,t) \ =$$

$$= \frac{\partial^2 u}{\partial t_1 \partial t_2} - B(x,t,D_x)\frac{\partial u}{\partial t_1} - C(x,t,D_x)\frac{\partial u}{\partial t_2} + E(x,t,D_x)u \;=\; f(x,t) \quad (0.2.1)$$

in the domain $\Omega \times (0,T_1) \times (0,T_2)$, where $0 < T_j \leq \infty$, A, C and E are differential operators with coefficients being C^∞ - functions in $\overline{\Omega}$ $\times[0,T_1) \times [0,T_2)$ and of orders 2, 2 and 4 respectively. We attach to (0.2.1) Goursat initial conditions

$$u\big|_{t_1=0} = \phi_1(x,t_2) \ , \quad u\big|_{t_2=0} = \phi_2(x,t_1) \ , \qquad (0.2.2)$$

and boundary conditions on $\Gamma = \partial\Omega$

$$B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big)u(x,t)\big|_\Gamma \;=\; g_\mu(x',t) \ , \qquad (0.2.3)$$

where $x' \in \Gamma$, $\mu = 1,2$, and

$$B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big) \;=\; \sum_{|\alpha|+|\beta|\leq k_\mu} b_{\alpha\beta}(x,t)D_x^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta \ .$$

For simplicity we suppose $k_\mu = 0$ or 1. We can do it without loss of generality in the view of solvability conditions to be described now.

We assume that equation (0.2.1) is "parabolic" in some generalized sense and that together with boundary conditions (0.2.2) it satisfies to an appropriate analog of Shapiro - Lopatinskij condition. The problems (0.2.1) - (0.2.3) satisfying these conditions we call *polyparabolic*. Under these assumptions we obtain a general solvability theory for the problems (0.2.1) - (0.2.3) with homogeneous initial conditions $(\phi_1 = \phi_2 = 0)$.

In order to obtain the uniqueness in the case of nonhomogeneous initial conditions we introduce a new *Inductive Polyparabolicity Condition*. Roughly speaking, this condition means that the operators $\frac{\partial}{\partial t_2} - B$ and $\frac{\partial}{\partial t_1} - A$ with appropriate boundary conditions form parabolic boundary value problems .

The problem (0.2.1) - (0.2.3) largely differs from the case considered by Brish N.I., Yurchuk N.I. (1968). Here in (0.1.4) we have $A_1(t) = -B(x,t,D_x)$ and $A_2(t) = -C(x,t,D_x)$ which are no more functions, but differential operators .

Further, the method, based on the parametrix construction for the corresponding problem with parameters and on subsequent Laplace transformation, gives the opportunity to treat the class of so -called Ultraparabolic equations in the same way (see Brenner A. (to appear - 1)). These equations are of interest since the 30 - th, when they arose in the works of Kolmogorov A.N. (1933), (1934) connected with the theory of probability and Brownian motion. Also, Ultraparabolic equations appear in Cosmology, the theory of boundary layer, the theory of neutrons slowing - down etc. (see Chandrasekhar S. (1943), Genchev T.G. (1963), Vladimirov V.S., Drozhzhinov Yu.N. (1967), Drozhzhinov Yu.N. (1969), Marchuk G.I. (1958) and others).

We start the description of the results of the present work with operators with parameters which are of independent interest and have nice applications to the multiparameter spectral theory (see Brenner A. (to appear - 2)). We solve an elliptic boundary value problem with parameters corresponding to (0.2.1) - (0.2.3) in a bounded domain Ω :

$$Au \equiv A(x,D,q)\,u(x,q)$$

$$= \big(q_1^2q_2^2 - q_1^2B(x,D_x) - q_2^2C(x,D_x) + E(x,D_x)\big)\,u(x,q) = f(x,q)\ , \quad (0.2.4)$$

where $x \in \Omega \subset \mathbb{R}^n$, $q \in \mathbb{C}^2$, with boundary conditons

$$B_\nu u|_\Gamma \equiv B_\nu(x,D,q)u(x,q)\big|_{x=x'} = g_\nu(x',q)\ , \quad x' \in \Gamma\ ,\ \nu = 1,2\,, \quad (0.2.5)$$

on $\Gamma = \partial\Omega$, where

$$B_\nu(x,D,q) = \sum_{|\alpha|+|\beta|\le k_\nu} b_{\alpha\beta}(x)q^\beta D_x^\alpha\ ,$$

i.e. $B(x,D_x)$, $C(x,D_x)$, $E(x,D_x)$ and $B_\nu(x,D,q)$ are differential operators with coefficients being C^∞ - functions in $x \in \overline{G}$ and $B_\nu(x,D,q)$ are polynomials in q_i , $i = 1,2$, where

$$q_i \in Q_i = \{q_i \in \mathbb{C}^2\ :\ \arg q_i \in [\alpha_i,\beta_i]\}\ .$$

Thus $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees 4 and $k_\nu < 2$ $(\nu = 1,2)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to (ξ,q) being less then 4 and k_ν respectively.

We suppose the following conditions of the unique solvability of the problem (0.2.4) - (0.2.5) to be executed.

Condition 0.2.1 $\forall\, x \in \overline{\Omega}$, $\forall\, q \in Q = Q_1 \times Q_2$ *and* $\forall\, \xi \in \mathbb{R}^n$

$$A_0(x,\xi,q) \neq 0 \quad for \quad |\xi| + M_q \neq 0\ , \qquad (0.2.6)$$

where $M_q = \min_{i=1,2}|q_i|$.

Condition 0.2.2 *Let us fix a boundary point* $x' \in \Gamma$ *and suppose that all operators in (0.2.4) - (0.2.5) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the* x_n-*axis is looking inside* Ω *together with the inner normal to* Γ *).*

We consider the problem on the half-axis $x_n > 0$

$$A_0\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n) = 0\,, \quad x_n > 0\ , \qquad (0.2.7)$$

$$B_{0,\nu}\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n)\big|_{x_n=0}= h_\nu\,,\quad \nu=1,2\,, \tag{0.2.8}$$

smoothly depending on parameters ξ' *and* q , *and suppose that* $\forall\, q\in Q$ *for* $|\xi'|+M_q\neq 0$ *this problem has a unique solution in the space* $\mathcal{M}$ *of all stable solutions to the equation (0.2.7) for all* $h_\nu\in\mathbb{C}$.

Let us note that for $q=0$ Condition 0.2.1 turns into the ordinary ellipticity condition and Condition 0.2.2 - into the classical Shapiro - Lopatinskij condition. Here are the main examples.

Example 0.2.3 *Let* $A(x,D)$ *be an elliptic operator* ($\mathrm{ord}A=4$) *with a positive real leading part. Then the operator*

$$A(x,D)+q_1^2q_2^2$$

with Dirichlet boundary conditions

$$B_\nu\ =\ \Big(-i\frac{\partial}{\partial\vec{n}}\Big)^{\nu-1}\ ,\quad \nu=1,2,$$

satisfies Conditions 0.2.1 and 0.2.2 in the cone $Q=Q_1\times Q_2$, *where* $Q_j=\big\{|\arg q_j|\ \le\ \frac{\pi-\varepsilon}{4}\big\}$, $\varepsilon\in(0,\pi)$, $j=1,2$.

Example 0.2.4 *If* A_1 *and* A_2 $(\mathrm{ord}A_j=2,\ j=1,2)$ *are the two such operators as in Example 0.2.3, then the operator*

$$\big(A_1(x,D)+q_1^2\big)\,\big(A_2(x,D)+q_2^2\big)$$

with Dirichlet boundary conditions

$$B_\nu\ =\ \Big(-i\frac{\partial}{\partial\vec{n}}\Big)^{\nu-1}\ ,\quad \nu=1,2,$$

also satisfies Conditions 0.2.1 and 0.2.2, but in the cone $Q=Q_1\times Q_2$, *where* $Q_j=\big\{|\arg q_j|\ \le\ \frac{\pi-\varepsilon}{2}\big\}$, $\varepsilon\in(0,\pi)$, $j=1,2$.

The main result of Chapter 1 is the unique solvability of the problem (0.2.4) - (0.2.5) in the scale of Sobolev spaces equipped with special norms depending on parameters q for sufficiently large $|q_1|+|q_2|$ (and $M_q\ge q^{(0)}$ for some $q^{(0)}>0$) under Conditions 0.2.1 and 0.2.2 (Proposition 1.3.6 and Theorem 1.4.2). The proof is obtained via energy inequalities which follow in a standard manner from the parametrix construction. The difference with the one - parameter case is in the fact, that though $|q_1|+|q_2|$ is sufficiently large, some of q_j may be rather small (but not zero!).

Chapter 2 of the paper is devoted to the solution of the problem (0.2.1) - (0.2.3) in the multidimensional time case.

We start with the description of functional spaces. By definition the space $\mathbf{P}^{\alpha}\big(e^{-\gamma \cdot t}\big)$ (where $\alpha, \gamma, t \in (\overline{\mathbb{R}_+})^2$) consists of all functions $u(t) = u(t_1, t_2)$ supported in the positive quadrant $(\bar{\mathbb{R}}_+)^2$ and such that $e^{-\gamma \cdot t} u(t) \in H^{\vec{\alpha}}(\mathbb{R}^2)$, where $H^{\vec{\alpha}}(\mathbb{R}^2)$ is the anisotropic Sobolev space (see, for example, Slobodetskij L.N. (1958), Kudryavtsev L.D., Nikolskij S.M. (1988)).

The space of functions $\mathbf{E}_{\alpha}(\gamma)$ is defined as the set of functions $U(p) = U(p_1, p_2)$, where $p = \sigma + i\gamma = (\sigma_1 + i\gamma_1, \sigma_2 + i\gamma_2)$, which are holomorphic in the direct product of the right half-planes in $\Pi_{\gamma} = \{p \in \mathbb{C}^2 : \operatorname{Re} p_j > \gamma_j \,, \ j = 1, 2\}$. Obviously, we have to prove that Laplace transformation defines an isomorphism between the spaces $\mathbf{P}^{\alpha}\big(e^{-\gamma \cdot t}\big)$ and $\mathbf{E}_{\alpha}(\gamma)$. The key to the desired isomorphism is lying in the slight modification we give to Cauchy integral formula in several complex variables (see in this connection Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)).

After these preliminaries we turn to the problem (0.2.1) - (0.2.3) in a "polycylinder" domain $\Omega_I = \Omega \times (0, T_1) \times (0, T_2)$, where $0 < T_j \leq \infty$.

Suppose for the time being, that all $\phi_1 = \phi_2 \equiv 0$ and $T_1 = T_2 = +\infty$. Then Laplace transformation in the variables T_j transfers the problem (0.2.1) - (0.2.3) to the following "stationary" problem

$$A(x, D_x, p)\, U(x, p) \;=\; F(x, p) \;, \quad x \in \Omega \;, \tag{0.2.9}$$

$$B_{\mu}(x, D_x, p) U(x, p)\big|_{x = x'} \;=\; G_{\mu}(x', p) \;, \quad x' \in \Gamma \;, \tag{0.2.10}$$

$\mu = 1, 2$. Putting $p_j = q_j^2$, $j = 1, 2$, we see, that in the same time as p_j varies in the right half-plane $\operatorname{Re} p_j \geq 0$, q_j varies in its turn in the angle

$$Q_j \;=\; \big\{ q_j \in \mathbb{C} : \, |\arg q_j| \;\leq\; \frac{\pi}{4} \big\} \,. \tag{0.2.11}$$

The problem (0.2.1) - (0.2.3) is called *polyparabolic* if the problem (0.2.9) - (0.2.10) with p_j substituted by q_j^2 for $j = 1, 2$, satisfies in the cone $Q = Q_1 \times Q_2$ the Conditions 0.2.1 - 0.2.2.

In order to illustrate this definition we consider the inverse Laplace images of Examples 0.2.3, 0.2.4.

Example 0.2.5 *Let* $A_1(x, D)$ *and* $A_2(x, D)$ $(\operatorname{ord} A_j = 2 \,, \ j = 1, 2)$ *be two elliptic operators with positive real leading parts. Consider an operator*

$$\Big(\frac{\partial}{\partial t_1} + A_1(x, D) \Big) \Big(\frac{\partial}{\partial t_1} + A_2(x, D) \Big)$$

with Dirichlet boundary conditions

$$B_{\nu} \;=\; \Big(-i \frac{\partial}{\partial \vec{n}} \Big)^{\nu - 1} \;, \quad \nu = 1, 2,$$

and zero initial conditions (0.2.2).

This problem is polyparabolic in the cone $Q' = Q'_1 \times Q'_2$, $Q'_j = \{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi-\varepsilon}{2}\}$, *including the cone* $Q = Q_1 \times Q_2$, $Q_j = \{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi}{4}\}$, *for sufficiently small* ε .

Example 0.2.6 *Let* $A(x, D)$ *be an elliptic operator* $(\mathrm{ord} A = 4)$ *with a positive real leading part. Consider the operator (cf. Friedman A., Littman W. (1963))*

$$\frac{\partial^2}{\partial t_1 \partial t_2} + A(x, D)$$

with Dirichlet boundary conditions

$$B_\nu = \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} , \quad \nu = 1, 2,$$

and zero initial conditions.

This problem is not polyparabolic *in the above defined sense. Indeed, after substituting of* p_j *by* q_j^2 $(j = 1, 2)$ *in the corresponding system (0.2.9) - (0.2.10) we obtain that, if* p_j *varies in the right half-plane* $\mathrm{Re}\, p_j \ge 0$, q_j *varies in its turn in the angle* $Q_j = \{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi}{4}\}$. *And thus, the ellipticity condition (Condition 0.2.1) is violated for our problem is elliptic only in the cone* $Q' = Q'_1 \times Q'_2$, *where* $Q'_j = \{|\arg q_j| \le \frac{\pi-\varepsilon}{4}\}$.

In order to get the uniqueness and existence of the solution to the system (0.2.1) - (0.2.3) we have to improve results of Chapter 1 for the problem (0.2.4) - (0.2.5). We obtain (see Theorem 2.2.4) that there exists a vector γ with $\gamma_j > 0$ such that for $\mathrm{Re}\, p_j \ge \gamma_j$ the problem (0.2.9) - (0.2.10) possesses a unique solution $U(x, \cdot) \in \mathbf{E}_\alpha(\gamma)$ (suitably depending on x). Using Laplace isomorphism, we obtain (see Theorem 2.2.6) the unique solvability of the initial polyparabolic problem (0.2.1) - (0.2.3) (with $\phi_1 = \phi_2 \equiv 0$) in the scale of spaces $\mathbf{P}^\alpha(e^{-\gamma \cdot t})$ (suitably depending on x).

The rest part of Chapter 2 is devoted to the existence and uniqueness theorems for the problem (0.2.1) - (0.2.3) in the following cases:

A) nonhomogeneous initial conditions (Theorems 2.2.9 and 2.2.11);

B) finite ($T_1, T_2 < \infty$) polycylinder (Theorem 2.3.3);

C) the coefficients of operators A and B_μ are dependent of t (Theorem 2.4.1).

The main difficulty in A) -C) is lying in the uniqueness theorem for the case of nonhomogeneous initial conditions.

It happenes that usual Compatibility Condition 2.2.8 is sufficient only for the existence theorem. The uniqueness is guaranteed by the new *Inductive Polyparabolicity Condition* (see Condition 2.2.10) which is, roughly speaking, the condition of the unique solvability of certain additional boundary value problems determininig the functions $w_j = \frac{\partial u}{\partial t_j}\Big|_{t_j=0}$.

We note that though *Inductive Polyparabolicity Condition* is sufficient, it is not necessary, as it may be seen from the examples below.

Example 0.2.7 *Suppose operator* $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ *be as in (0.2.1), but with coefficients being independent of* t *:*

$$A\big(x, D_x, \frac{\partial}{\partial t}\big) \;=\; \frac{\partial^2}{\partial t_1 \partial t_2} - B(x, D_x)\frac{\partial}{\partial t_1} - C(x, D_x)\frac{\partial}{\partial t_2} + E(x, D_x) \;.$$

Also, we suppose that together with Dirichlet boundary conditions it forms a polyparabolic problem

$$\begin{aligned} A\big(x, D_x, \frac{\partial}{\partial t}\big)u \;&=\; f \;, \\ u\big|_{\Gamma} \;=\; h_0(t_1, t_2, x')\;, \quad \frac{\partial u}{\partial x_n}\Big|_{\Gamma} \;&=\; h_1(t_1, t_2, x')\;, \\ u\big|_{t_1=0} \;=\; \phi_1(t_2, x)\;, \quad u\big|_{t_2=0} \;&=\; \phi_2(t_1, x)\;. \end{aligned} \tag{0.2.12}$$

Then the respective systems for functions $w_j = \frac{\partial u}{\partial t_j}\big|_{t_j=0}$ *, are*

$$\begin{aligned} \Big(\frac{\partial}{\partial t_2} - B\Big)w_1 \;&=\; C\frac{\partial \phi_1}{\partial t_2} - E\phi_1 + f_1 \;:=\; \tilde{f}_1 \;, \\ w_1\big|_{\Gamma} \;=\; h_{0,t_1}(0, t_2, x')\;, \qquad w_1\big|_{t_2=0} \;&=\; \phi_{2,t_1}(0, x) \end{aligned} \tag{0.2.13}$$

and

$$\begin{aligned} \Big(\frac{\partial}{\partial t_1} - C\Big)w_2 \;&=\; B\frac{\partial \phi_2}{\partial t_1} - E\phi_2 + f_2 \;:=\; \tilde{f}_2 \;, \\ w_2\big|_{\Gamma} \;=\; h_{0,t_2}(t_1, 0, x')\;, \qquad w_2\big|_{t_1=0} \;&=\; \phi_{1,t_2}(0, x) \end{aligned} \tag{0.2.14}$$

for w_2 *, where* $f_1 = f(0, t_2, x)$ *,* $f_2 = f(t_1, 0, x)$ *and* z_{t_1} *means* $\frac{\partial z}{\partial t_1}$ *.*

If B *and* C *are elliptic operators then these systems are parabolic, and hence, it means that the initial problem (0.2.12) is inductively polyparabolic and possesses the unique solution (see Theorem 2.2.11).*

Though Example 0.2.6 gives us a nonpolyparabolic problem in the above sense, it shows, that Inductive Parabolicity Condition is sufficient, but not necessary. Indeed, in this case systems (0.2.13), (0.2.14) turn into Cauchy problems for ordinary differential equations with constant coefficients.

Example 0.2.8 *Consider the equation (cf. (0.1.2)):*

$$\frac{\partial^2 u}{\partial t_1 \partial t_2} \;+\; a(x, D)u \;=\; f \;,$$

where $a(x, D)$ *is an elliptic operator of order 4 with boundary and initial conditions*

$$\begin{aligned} u\big|_{\Gamma} \;=\; h_0(t_1, t_2, x')\;, \quad \frac{\partial u}{\partial x_n}\Big|_{\Gamma} \;&=\; h_1(t_1, t_2, x')\;, \\ u\big|_{t_1=0} \;=\; \phi_1(t_2, x)\;, \quad u\big|_{t_2=0} \;&=\; \phi_2(t_1, x)\;. \end{aligned}$$

The respective systems (0.2.13), (0.2.14) are

$$\frac{\partial w_1}{\partial t_2} \;=\; f_1 - a(x,D)\phi_1 \ , \qquad w_1\big|_{t_2=0} \;=\; \phi_{2,t_1}(0,x) \ ,$$

and

$$\frac{\partial w_2}{\partial t_1} \;=\; f_2 - a(x,D)\phi_2 \ , \qquad w_2\big|_{t_1=0} \;=\; \phi_{1,t_2}(0,x) \ .$$

Thus, we obtain Cauchy problems for simplest ordinary differential equations with respect to w_1 *and* w_2 *, i.e.* w_1 *and* w_2 *can be easily estimated through the functions* f *,* h_0 *,* h_1 *,* ϕ_1 *and* ϕ_2 *, possibly with incresed* γ *in appropriate* $\mathbf{P}^\alpha\left(e^{-\gamma\cdot t}\right)$.

In general, we observe that in the cases, where (0.2.13) and (0.2.14) are Cauchy problems for ordinary differential equations with constant coefficients, we obtain the desired uniqueness in our functional classes automatically. It is not true in the variable coefficients case, for example, if we obtain for $w_{1,2}$ first order linear equations of the form $\frac{\partial w}{\partial t_2} + a(x,t)w \;=\; f(x,t)$ (this is exactly the case of Brish - Yurchuk equation (0.1.3), (0.1.4)): we have to demand the function $a(x,t_1,t_2)$ to be bounded from below in t_2 uniformly in (x,t_1) , otherwise the solution does not fall into $\mathbf{P}^\alpha\left(e^{-\gamma\cdot t}\right)$ even with increased γ .

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Chapter 1

Differential operators with parameters

1.1 Main functional spaces and interpolation inequalities

1.1.1

We denote by $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ (see, for example, Egorov Yu.V., Shubin M.A. (1988)) the completion of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_s \;=\; \|u\|_{s,\mathbb{R}^n} \;=\; \left(\int_{\mathbb{R}^n_\xi} \big(1+|\xi|^{2s}\big)|Fu|^2\,d\xi\right)^{1/2}, \tag{1.1.1}$$

and also denote by $H^l(\mathbb{R}^n_+)$, $l \in \mathbb{Z}$ − the completion of the set of restrictions to $\mathbb{R}^n_+$ of the functions from $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{l,\mathbb{R}^n_+} \;=\; \left(\sum_{|\alpha|\le l}\int_{\mathbb{R}^n_+} |D^\alpha u|^2\,dx\right)^{1/2}, \tag{1.1.2}$$

or according to the equivalent norm

$$\left(\sum_{j+k\le l}\int_{\mathbb{R}^{n-1}_{\xi'}} |\xi'|^{2j}\int_0^\infty |D_n^k F' u|^2\,dx_n d\xi'\right)^{1/2}. \tag{1.1.3}$$

Note that, if $u \in H^l(\mathbb{R}^n)$ then $u\,|_{\mathbb{R}^n_+} \in H^l(\mathbb{R}^n_+)$ and

$$\|u\|_{l,\mathbb{R}^n_+} \;\le\; \|u\|_{l,\mathbb{R}^n}\,. \tag{1.1.4}$$

Inversely, there exists an extension operator (see Egorov Yu.V., Shubin M.A. (1988)) from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, such that

$$\|Lu(x)\|_{l,\mathbb{R}^n} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.5}$$

Here and further we denote by C an arbitrary positive constant, which is independent of $u(x)$ and the parameters $q\in\mathbb{C}^m$ (if there are any) and may differ from inequality to inequality.

Also, we shall use the trace inequality: let $u \in H^l(\mathbb{R}^n_+)$, $l > 1/2$, then there exists $u(x',0)\in H^{l-1/2}(\mathbb{R}^{n-1}_{x'})$ such that

$$\|u(x',0)\|^{'}_{l-1/2} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.6}$$

where we denote by $\|\cdot\|^{'}_s$ the norm in $H^s(\mathbb{R}^{n-1}_{x'})$.

1.1.2

The down-stated interpolation lemmae generalize the corresponding results of (Agranovich M.S., Dynin A.S. (1962), Agranovich M.S., Vishik M.I. (1964)) to the multiparameter case.

Lemma 1.1.1 *Let* $q\in\mathbb{C}^m$ *, then for* $0<k<l$ *the following interpolation inequalities are valid:*

$$\sum_{i=1}^m |q_i|^{l-k}\|u\|_k \;\le\; C_{kl}\left[\|u\|_l + \sum_{i=1}^m |q_i|^l\|u\|_0\right] , \tag{1.1.7}$$

$$\sum_{i=1}^m |q_i|^{2(l-k)}\|u\|_k^2 \;\le\; C'_{kl}\left[\|u\|_l^2 + \sum_{i=1}^m |q_i|^{2l}\|u\|_0^2\right] , \tag{1.1.8}$$

where the norms are taken over $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$ *, and the constants* C_{kl} *and* C'_{kl} *are independent of* $u(x)$ *and* q *.*

Lemma 1.1.2 *Let* $l=1/2$ *and* $u\in H^1(\mathbb{R}^n_+)$ *, then*

$$\sum_{i=1}^m |q_i|^{1/2}\,\|u(x',0)\|'_0 \;\le\; C\Big[\|u\|_1 + \sum_{i=1}^m |q_i|\,\|u\|_0\Big] \tag{1.1.9}$$

and

$$\sum_{i=1}^m |q_i|\,\|u(x',0)\|_0^{'\,2} \;\le\; C'\Big[\|u\|_1^2 + \sum_{i=1}^m |q_i|^2\|u\|_0^2\Big] , \tag{1.1.10}$$

where the norms are taken over $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$ *, and the constants* C *and* C' *are independent of* $u(x)$ *and* q *.*

Also we use the parameter-dependent norms

$$|||u|||_l^2 \ := \ \|u\|_l^2 + \sum_{i=1}^{m} |q_i|^{2l} \, \|u\|_0^2 \ = \ \|u\|_l^2 + |||q|||^{2l} \|u\|_0^2 \ , \qquad (1.1.11)$$

where the norms in the right - hand side are taken taken either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$. As follows from (1.1.8), the norms $\|\cdot\|_l$ and $|||\cdot|||_l$ are equivalent for the fixed $q \in \mathbb{C}^m$ and

$$|||u|||_l^2 \ \leq \ \sum_{k=0}^{l} \sum_{i=1}^{m} |q_i|^{2(l-k)} \, \|u\|_k^2 \ \leq \ C|||u|||_l^2 \ . \qquad (1.1.12)$$

Now we can rewrite (1.1.4) - (1.1.6) in the new terms

$$|||u(x)|||_{l,\mathbb{R}^n_+} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n} \ , \qquad (1.1.13)$$

$$|||Lu(x)|||_{l,\mathbb{R}^n} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n_+} \ , \qquad (1.1.14)$$

$$|||u(x',0)|||'_{l-1/2} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n_+} \ , \qquad (1.1.15)$$

where $l \in \mathbb{N}$.

Remark 1.1.3 *The inequalities of this paragraph remain true if instead of* $\mathbb{R}^n_+$ *we consider a bounded domain* $G \subset \mathbb{R}^n$ *with sufficiently smooth boundary* Γ , *which is an* $(n-1)-$*dimensional surface admitting in every point local straightenings under local coordinate transformations.*

In this case the Sobolev norms are defined in a standard way as follows: for $l \in \mathbb{N}$ the space $H^l(G)$ is defined as a completion of the space $C^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{l,G} \ = \ \left(\sum_{|\alpha| \leq l} \int_G |D^\alpha u|^2 \, dx \right)^{1/2} \ , \qquad (1.1.16)$$

or, equivalently,

$$|||u|||_{l,G}^2 \ = \ \|u\|_{l,G}^2 + |||q|||^{2l} \, \|u\|_{0,G}^2 \ , \qquad (1.1.17)$$

or with respect the equivalent norm which is defined by a partition of unity as follows: suppose $\{U_\alpha\}_{\alpha=1}^N$ is an open covering of G and $\sum_{\alpha=1}^N \varphi_\alpha(x) \equiv 1$ is the corresponding smooth partition of unity, then the norm (1.1.16) is equivalent to the norm $\left(\sum_{\alpha=1}^N \|\varphi_\alpha u\|_l^2 \right)^{1/2}$, where the norms are taken in local coordinates either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$, with the additional assumption in the last case that, if $\operatorname{supp} \varphi_\alpha \cap \Gamma \neq \emptyset$, then $\operatorname{supp} \varphi_\alpha \subset U_j$ for some U_j from the covering of the boundary Γ .

Let $\{\varphi_\beta\}$, $(\beta = 1, ..., r)$ be all functions from $\{\varphi_\alpha\}$ such that $\operatorname{supp} \varphi_\beta \cap \Gamma \neq \emptyset$. For $l \in \mathbb{N}$ the Sobolev space $H^l(\Gamma)$ is defined as a completion of the space $C^\infty(\overline{\Gamma})$ with respect to the norm

$$\|u\|'_{l,G} \;=\; \left(\sum_{\beta=1}^{r} \|\varphi_\beta u\|'^2_{l,\mathbb{R}^{n-1}}\right)^{1/2} . \tag{1.1.18}$$

This space is independent of local coordinates and the partition of unity for sufficiently small coverings. Of course, we also need the equivalent norm

$$|||u|||'^2_{l,\Gamma} \;=\; \|u\|'^2_{l,\Gamma} \;+\; |||q|||^{2l}\, \|u\|'^2_{0,\Gamma} \;. \tag{1.1.19}$$

Further, we define the spaces $H^s(I)$, where $I \subset \mathbb{R}$ is an open interval in the real line. It can be the line $\mathbb{R}$ itself or the half-line $\mathbb{R}_+$. We introduce the space $H^s(I)$ as a completion of the space of restrictions to I of smooth functions on $\mathbb{R}$, relatively to the norm

$$\|u\|_s \;=\; \left(\sum_{k=0}^{[s]} \int_I \left|u^{(k)}(t)\right|^2 dt \;+\; \int_I \int_I \frac{\left|u^{[s]}(t) \;-\; u^{[s]}(\tau)\right|^2}{|t \;-\; \tau|^{2s-2[s]+1}}\, dt\, d\tau\right)^{1/2} , \tag{1.1.20}$$

The equivalent norm can be introduced in terms of the operator L (see (1.1.5) and (Slobodetskij L.N. (1958)) for details). Let $I = (a,b)$, $b < +\infty$ and $J = (a, +\infty)$, then

$$\langle\, u\, \rangle \;:=\; \inf \|v\|_{s,J} \;, \tag{1.1.21}$$

where $v = Lu$ and "inf" is taken over all $v \in H^s(J)$ such that $v = u$ on I almost everywhere.

At last, if I and J are the two such intervals, then the norm in the corresponding anisotropic Sobolev space (see Slobodetskij L.N. (1958)) $H^{l,k}(I \times J)$ is

$$\|u\|^2_{l,k,I\times J} \;=\; \int_I \left\|u(t_1,t_2)\right\|^2_{k,J} dt_1 \;+\; \int_J \left\|u(t_1,t_2)\right\|^2_{l,I} dt_2 \;, \tag{1.1.22}$$

and, in general, by induction, if $I = I_1 \times \cdots \times I_m$ (in particular, we shall consider $I = (\mathbb{R}_+)^m$), then the norm in the space $H^{\vec{\alpha}}(I)$ is defined by

$$\|u\|^2_{\vec{\alpha},I} \;=\; \sum_{i=1}^{m} \int_{I[i]} \left\|u(t)\right\|^2_{\alpha_i,I_i} dt[i] \;, \tag{1.1.23}$$

where $I[i] \;=\; I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_m$, $dt[i] \;=\; dt_1 ... dt_{i-1} dt_{i+1} ... dt_m$.

1.2 Some model examples of operators with constant coefficients

1.2.1 The case of $\mathbb{R}^n$.

Let us consider in $\mathbb{R}^n$ the equation

$$A(D,q)u(x,q) \;=\; f(x,q) \qquad x \in \mathbb{R}^n \;,\; q \in \mathbb{C}^m \;, \tag{1.2.1}$$

where

$$A(D,q) \;=\; q_1^2 q_2^2 - B(D)q_1^2 - C(D)q_2^2 + E(D)$$

and

$$B(D) = \sum_{|\alpha|=2} b_\alpha D_x^\alpha \;, \quad C(D) = \sum_{|\alpha|=2} c_\alpha D_x^\alpha \;, \quad E(D) = \sum_{|\alpha|=4} d_\alpha D_x^\alpha \;,$$

are homogeneous differential operators with constant coefficients.

We suppose that the symbol $A(\xi,q)$, which is obtained from $A(D,q)$ after substitution of D by ξ , is a homogeneous polynomial in (ξ,q) of degree s , and that each parameter q_i is varying in a closed angle

$$Q_i \;\subset\; \mathbb{C} \;:\; Q_i \;=\; \{q_i \in \mathbb{C} \;:\; \delta_i \le \arg q_i \le \gamma_i\} \tag{1.2.2}$$

with the vertex in the origin.

Proposition 1.2.1 *For* $l \ge 4$ *operator* $A(D,q) \,:\, H^l(\mathbb{R}^n) \to H^{l-4}(\mathbb{R}^n)$ *is bounded and*

$$|||f|||_{l-4} \;\le\; C|||u|||_l \;. \tag{1.2.3}$$

Proof. We easily obtain (1.2.3) applying (1.1.7) to the inequality

$$\left\|A(D,q)u\right\|_{l-4} \;\le\; C\sum_{k=0}^{4} |||q|||^{l-k}\|u\|_k \;.$$

▷

We now formulate the *ellipticity condition* for the operator $A(D,q)$.

Condition 1.2.2 $\forall\, q \in Q \;=\; Q_1 \times Q_2$ *and* $\forall\, \xi \in \mathbb{R}^n$

$$A(\xi,q) \;=\; q_1^2 q_2^2 - B(\xi)q_1^2 - C(\xi)q_2^2 + E(\xi) \ne 0 \quad \mathit{for} \quad |\xi| + M_q \ne 0 \;, \tag{1.2.4}$$

where $M_q \;=\; \min_{i=1,2} |q_i|$.

We note that, if $q \;=\; (q_1,q_2) \;=\; 0$, then (1.2.4) turns into the standard ellipticity condition for the operator $E(D)$.

Also, we denote by $Q_M \;=\; \{q \in Q \,:\, M_q > 0\}$.

Proposition 1.2.3 *Let* $l \geq 4$. *Then under Condition 1.2.2 ,* $\forall\, q \in Q_M$ *and* $f \in H^{l-4}(\mathbb{R}^n)$ *there exists the unique solution* $u = \mathcal{R}_0 f \in H^l(\mathbb{R}^n)$ *to the equation (1.2.1). Moreover for arbitrary* $q^{(0)} > 0$ *and arbitrary*

$$q \in Q_M^0 \;=\; \left\{ q \in Q_M : \; M_q \geq q^{(0)} \right\}$$

the following a priori estimate takes place

$$\||u\||_l \;=\; \||\mathcal{R}_0 f\||_l \;\leq\; C \||f\||_{l-s} \; . \tag{1.2.5}$$

Proof. Let $u \in H^l(\mathbb{R}^n)$, then $F\left[A(D,q)u\right] \;=\; A(\xi,q)Fu \;=\; Ff$ and it follows from (1.2.4) that $Fu \;=\; A^{-1}(\xi,q)Ff$. Then from the homogenity of A we obtain

$$\begin{aligned} \left(1 + |\xi|^{2l} + \||q\||^{2l}\right) |Fu|^2 \;&\leq\; C\, \frac{1 + |\xi|^{2l} + \||q\||^{2l}}{\left(|\xi| + \||q\||\right)^8}\, |Ff|^2 \\ &\leq\; C \left(1 + |\xi|^{2(l-4)} + \||q\||^{2(l-4)}\right) |Ff|^2 , \end{aligned} \tag{1.2.6}$$

where $q \in Q_M$, $M_q \geq q^{(0)}$ and the constants C are independent of u, ξ, q . Integrating over ξ we obtain (1.2.5) and the uniqueness follows. The existence is the consequence of the formula

$$u \;=\; \mathcal{R} f \;=\; F^{-1} A^{-1}(\xi,q) Ff \; , \tag{1.2.7}$$

which is valid $\forall\, q \in Q_M$, and, as follows from (1.2.6), it defines $\forall\, f \in H^{l-4}(\mathbb{R}^n)$ the solution $u \in H^l(\mathbb{R}^n)$ to the equation (1.2.1). $\triangleright$

Remark 1.2.4 *Let* $n > 1$ *and suppose (1.2.4) is satisfied, then* $A(\xi + \lambda\xi_0, q)$ *as a polynomial in* λ *has for* $q \in Q_M$, $|\xi| + M_q \neq 0$, ξ_0 *orthogonal to* ξ *and* $\xi_0 \neq 0$, *exactly* 2 *roots in the upper* $\lambda-$*half-plane and* 2 *roots in the lower half-plane, taking into account their multiplicities (for the proof see, for example, Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990)).*

Example 1.2.5 *The first main example of the above situation is the operator* $A(D) + q_1^2 q_2^2$, *where* $A(D)$ *is a homogeneous elliptic operator of order* 4 *with real coefficients and such that* $A(\xi) > 0$ *for* $\xi \neq 0$ *(for example,* $A = -\Delta$). *Then (1.2.4) is satisfied in the cone* $Q = Q_1 \times Q_2$, *where*

$$Q_i \;=\; \left\{ q_i \in \mathbb{C} : \; |\arg q_i| \leq \frac{\pi - \varepsilon}{4} \right\} ,$$

with arbitrary $\varepsilon > 0$.

Example 1.2.6 *The second example concerns the pair of operators* $A_1(D)$ *and* $A_2(D)$ *of the above type:*

$$\left(A_1(D) + q_1^2\right)\left(A_2(D) + q_2^2\right) ,$$

where $\operatorname{ord} A_1 \;=\; \operatorname{ord} A_2 \;=\; 2$. *Obviously, (1.2.4) is satisfied in the cone* $Q_1 \times Q_2$, *where*

$$Q_i \;=\; \left\{ q_i \in \mathbb{C} : \; |\arg q_i| \leq \frac{\pi - \varepsilon}{2} \right\} .$$

1.2.2 The case of $\mathbb{R}^n_+$.

Let us consider equation (1.2.1) in $\mathbb{R}^n_+$

$$A(D,q)u(x,q) \;=\; f(x,q) \;, \tag{1.2.8}$$

where $x = (x', x_n) \in \mathbb{R}^n_+$, $x_n \geq 0,\;\; q \in Q \subset \mathbb{C}^2$, with boundary conditions

$$B_\nu(D,q)u(x,q)\big|_{x_n=0} \;=\; g_\nu(x') \;, \quad \nu = 1,2. \tag{1.2.9}$$

As in (1.2.1) $A(D,q)$ and $B_\nu(D,q)$ are differential operators with constant coefficients such that after substitution of D by ξ we obtain polynomials $A(\xi,q)$ and $B_\nu(\xi,q)$ which are homogeneous in $(\xi_1,...,\xi_n,q_1,q_2) \in \mathbb{R}^n \times Q$ of orders 4 and $k_\nu < 2$ respectively. We denote by $\mathcal{A}$ a boundary value problem operator

$$\mathcal{A}u \;=\Big\{A(D,q)u,\; B_1(D,q)u\big|_{x_n=0}, ...,\; B_k(D,q)u\big|_{x_n=0}\Big\} \tag{1.2.10}$$

and by l_0 $-$ the $\max(4, k_\nu+1)$.

Proposition 1.2.7 *Let* $l \in \mathbb{Z}$, $l \geq l_0 = \max(4, k_\nu + 1)$. *Then the operator*

$$\mathcal{A}\,:\; H^l(\mathbb{R}^n_+) \;\longrightarrow\; H^{l-4}(\mathbb{R}^n_+) \,\times \prod_{\nu=1}^{2} H^{l-k_\nu-1/2}(\mathbb{R}^{n-1}) \tag{1.2.11}$$

is bounded and

$$|||f|||_{l-4} \;+\; \sum_{\nu=1}^{2} |||g_\nu|||'_{l-k_\nu-1/2} \;\leq\; C|||u|||_l \;. \tag{1.2.12}$$

Proof. For the operator A the proof is the same as in Proposition 1.2.1. Using (1.1.6) we obtain for B_ν that

$$|||B_\nu(D,q)u|||'_{l-k_\nu-1/2} \;\leq\; C|||B_\nu(D,q)u|||_{l-k_\nu} \;\leq\; C|||u|||_l \;,$$

where the last inequality is the analogue of (1.2.3) for B_ν , and the proof follows. $\triangleright$

We now formulate the analog of the Shapiro - Lopatinskij condition in our case, i.e. for the problem (1.2.8) - (1.2.9).

Suppose $f=0$ there. After the Fourier transformation F' in x' we obtain from (1.2.8) - (1.2.9) the boundary value problem for an ordinary differential equation on the half-axis:

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; 0\,, \quad x_n > 0 \;, \tag{1.2.13}$$

$$B_\nu\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n)\big|_{x_n=0} = h_\nu\,, \quad \nu = 1,2\,, \tag{1.2.14}$$

smoothly depending on parameters ξ' and q .

We denote by $\mathcal{M}$ the space of all stable solutions to the equation (1.2.13) for fixed ξ' and q . It follows from Remark 1.2.4 that $\dim \mathcal{M} = 2$ for $n > 1$. For $n = 1$ we additionally assume that the equation $A(\lambda, q) = 0$ has relatively to λ equally 2 solutions in the upper and the lower half-planes for $q \in Q_M$.

Condition 1.2.8 $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *the problem (1.2.13) - (1.2.14) has the unique solution in the space* $\mathcal{M}$ *for all* $h_\nu \in \mathbb{C}$.

Remark 1.2.9 *We do not lose the generality assuming* $\big(F'_{x'\to\xi'} f\big)(\xi', x_n) \equiv 0$ *in (1.2.13). Indeed, if for fixed* ξ' *it equals to* $g(x_n)$, $g(x_n) \in \mathcal{M}$, *we note that the equation*

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; g(x_n)$$

always has a particular solution $v \in \mathcal{M}$. *Subtracting it we return to (1.2.13) with* $g(x_n) \equiv 0$.

In the sequel we need the algebraic form of Condition 1.2.8 . Let $\omega_1(x_n, \xi', q)$, $\omega_2(x_n, \xi', q)$ be an arbitrary basis in $\mathcal{M}$. Then an arbitrary solution of (1.2.13) - (1.2.14) can be written in the form $v(x_n) \;=\; \alpha_1\omega_1 \,+\, \alpha_2\omega_2$. Substituting v into boundary conditions (1.2.14), we obtain the system of 2 linear algebraic equations with 2 unknowns α_1, α_2 . The condition of its unique solvability is

$$\det\Big\{ B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\, \omega_\nu(x_n, \xi', q)\Big|_{x_n=0}\Big\} \;\neq\; 0$$

for $|\xi'| + M_q \neq 0$, $q \in Q$ and $j, \nu = 1, 2$.

The crucial role in the following will be played by the canonical basis $\Omega_1(x_n, \xi', q)$, $\Omega_2(x_n, \xi', q)$ in $\mathcal{M}$ which is defined by the boundary conditions

$$B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\, \Omega_l(x_n, \xi', q)\Big|_{x_n=0} \;=\; \delta_{jl}\ .$$

Obviously, a solution to the problem (1.2.13) - (1.2.14) with arbitrary h_ν is expressed in terms of the canonical basis as follows: $v(x_n) \;=\; h_1\Omega_1 + h_k\Omega_k$. We also use the following integral representation for $\Omega_\nu(x_n, \xi', q)$ (see Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990) for the proof):

$$\Omega_\nu(x_n, \xi', q) \;=\; \int_{\gamma_+} e^{i\lambda x_n}\, A_+^{-1}(\xi', \lambda, q)\, N_\nu(\xi', \lambda, q)\, d\lambda\ , \tag{1.2.15}$$

for $|\xi'| + M_q \neq 0$, $q \in Q$ and $\nu = 1, 2$, where γ_+ is a smooth curve in the upper half-plane $\operatorname{Im}\lambda > 0$, depending on (ξ', q) and surrounding all λ -

zeroes of the polynomial

$$A_+(\xi', \lambda, q) \;=\; \prod_{j=1}^{2} \bigl(\lambda - \lambda_j^+(\xi', q)\bigr) \;, \tag{1.2.16}$$

where λ_j^+ - are all λ - zeroes of the polynomial $A(\xi', \lambda, q)$ with positive imaginary parts, $N_\nu(\xi', \lambda, q)$ are the C^∞-functions positively homogeneous of order $1 - k_\nu$ in all their variables and polynomial in λ .

Proposition 1.2.10 *Let $l \geq l_0$ and let Conditions 1.2.2 and 1.2.8 are fulfilled. Then, for arbitrary $q \in Q_M$, $f \in H^{l-4}(\mathbb{R}^n_+)$ and $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ there exists the unique solution to the problem (1.2.8) - (1.2.9).*

Moreover, for an arbitrary $q^{(0)} > 0$ and $\forall q \in Q^0_M$, we have the a priori estimate

$$|||u|||_l \;\leq\; C\bigl\{|||f|||_{l-4} \;+\; \sum_{\nu=1}^{2} |||g_\nu|||'_{l-k_\nu-1/2}\bigr\} \tag{1.2.17}$$

Proof. We follow here the scheme of the proof of Theorem 3.1 from (Agranovich M.S., Vishik M.I. (1964)).

1). First, we reduce the problem to the case $f = 0$. Let L be the extension operator (see (1.1.5)). Solving the equation $A(D, q)u \;=\; Lf$ in $\mathbb{R}^n$, we obtain in the view of (1.2.7) that

$$u_0 \;=\; F^{-1} A^{-1}(\xi, q) F L f \;, \tag{1.2.18}$$

where $q \in Q_M$, is its particular solution and $|||u_0|||_{l,\mathbb{R}^n_+} \;\leq\; C\,|||f|||_{l-4,\mathbb{R}^n_+}$ for $q \,:\, M_q \geq q^{(0)}$. We now suppose that $u = u_0 + u_1$, where $u_1 \in H^l(\mathbb{R}^n_+)$ is a solution to the problem

$$\begin{cases} A(D, q)u_1 \;=\; 0 & \text{in } \mathbb{R}^n_+ \;, \\ B_\nu(D, q)u_1\big|_{x_n=0} \;=\; g_\nu - g_{0\nu} & \text{on } \mathbb{R}^{n-1}_{x'} \;, \end{cases}$$

where $g_{0\nu} \;=\; B_\nu(D, q)u_0\big|_{x_n=0}$. It is sufficient to estimate u_1 as follows:

$$|||u_1|||_l \;\leq\; C \sum_{\nu=1}^{2} |||g_\nu - g_{0\nu}|||'_{l-k_\nu-1/2} \;. \tag{1.2.19}$$

¿From where the estimate (1.2.17) for u easily follows, and we are able to suppose $f = 0$.

2). Applying the Fourier transform F' with respect to x' in (1.2.8), (1.2.9), we arrive to the problem on the half-axis:

$$A(\xi', D_n, q)\, F'u \;=\; 0 \;, \tag{1.2.20}$$

$$B_\nu(\xi', D_n, q)\, F'u\big|_{x_n=0} \;=\; F'g_\nu \;, \tag{1.2.21}$$

where $\nu = 1, 2$. Since $u \in H^l(\mathbb{R}^n_+)$, then $F'u$ is a stable solution of (1.2.20) for almost all (ξ', q) . So we can write the decomposition of $F'u$ in the canonical basis Ω_ν (see (1.2.15)):

$$F'u \;=\; \sum_{\nu=1}^{2} F'g_\nu \times \Omega_\nu \;, \quad \text{or} \quad u \;=\; F'^{-1} \sum_{\nu=1}^{2} F'g_\nu \times \Omega_\nu \;. \tag{1.2.22}$$

In order to obtain the estimate of the H^l-norm of u , we have to estimate $D^j_{x_n}\Omega_\nu$. Differentiating (1.2.15) inside the integral, we get

$$|D^j_{x_n}\Omega_\nu|^2 \;\le\; \int_{\gamma_+} \big|\lambda^j e^{i\lambda x_n}\, A_+^{-1}(\xi', \lambda, q)\, N_\nu(\xi', \lambda, q)\big|\, |d\lambda|$$

$$\le\; C\left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu}\, \exp\Big\{-C_1 x_n \left(|\xi'|^2 + |||q|||^2\right)^{1/2}\Big\}$$

with C and C_1 independent of ξ', q, x_n and $M_q \ge q^{(0)}$ (we need it to estimate A_+^{-1}). Integrating over x_n we obtain

$$\int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \;\le C\left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu-1/2}$$

and thus

$$\sum_{j+j_1 \le l} |\xi'|^{2j_1} \int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \;+ |||q|||^{2l} \int_0^\infty |\Omega_\nu|^2\, dx_n$$

$$\le\; C\left(|\xi'|^2 + |||q|||^2\right)^{l-k_\nu-1/2} \;\le\; C\left(1 + |\xi'|^{2l-2k_\nu-1} \;+\; |||q|||^{2l-2k_\nu-1}\right) .$$

Multiplying the left- and the right-hand sides of the last inequality by $|F'g_\nu|^2$ and integrating over ξ' we obtain the desired estimate

$$\big|\big|F'^{-1}\{F'g_\nu \cdot \Omega_\nu\}\big|\big|^2_{l,\mathbb{R}^n_+} \;\le\; C|||g_\nu|||'^2_{l-k_\nu-1/2} \;. \tag{1.2.23}$$

The existence follows from the formulae (1.2.18) and (1.2.22) applied to u_0 and u_1 We put

$$u \;=\; u_0 \;+\; u_1 \;=\; F^{-1}A^{-1}(\xi, q)FLf \;+$$

$$+\; \sum_{\nu=1}^{2} F'^{-1}\Big\{F'\big[g_\nu - \big(B_\nu(D, q)F^{-1}A^{-1}(\xi, q)FLf\big)\big|_{x_n=0}\big] \cdot \Omega_\nu\Big\}$$

$$=\; \mathcal{R}_0 f \;+\; \sum_{\nu=1}^{2} \mathcal{R}_\nu\big[g_\nu - \big(B_\nu(D, q)\mathcal{R}_0 f\big)\big|_{x_n=0}\big] \;:=\; \mathcal{R}(f, g_1, g_2) \;, \tag{1.2.24}$$

where

$$\mathcal{R}_0 f \;:=\; F^{-1}A^{-1}(\xi, q)FLf \;\; \text{and} \;\; \mathcal{R}_\nu g_\nu \;:=\; F'^{-1}\{F'g_\nu \cdot \Omega_\nu\} \;. \tag{1.2.25}$$

For $f \in H^{l-4}(\mathbb{R}^n_+)$, $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ and $q \in Q_M$ these formulae define the solution $u \in H^l(\mathbb{R}^n_+)$ to the problem (1.2.8) - (1.2.9). ▷

1.3 Elliptic boundary value problems with parameters in a bounded domain

Let G be a bounded domain in $\mathbb{R}^n_x$ with boundary $\Gamma \in C^\infty$, which is an $(n-1)$ - dimensional surface admitting a local straightening with the help of a smooth nondegenerate coordinate transformations.

Let us consider in G the equation

$$Au \equiv A(x,D,q)\,u(x,q)$$

$$= \left(q_1^2q_2^2 - q_1^2B(x,D_x) - q_2^2C(x,D_x) + E(x,D_x)\right)u(x,q) = f(x,q) \qquad (1.3.1)$$

with boundary conditons on Γ :

$$B_\nu u|_\Gamma \equiv B_\nu(x,D,q)u(x,q)\big|_{x=x'} = g_\nu(x',q) \,, \qquad (1.3.2)$$

where $x' \in \Gamma$ and $\nu = 1,2$. Here $A(x,D,q)$ and $B_\nu(x,D,q)$ are differential operators with coefficients, which are C^∞- functions in $x \in \overline{G}$ and polynomials in q_i for $i=1,2$, where

$$q_i \in Q_i = \{q_i : \arg q_i \in [\alpha_i, \beta_i]\} \,.$$

Hence, $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees $s=4$ and $k_\nu < 2$ $(\nu = 1,2)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to the variables (ξ,q) being less than 4 and k_ν respectively.

The operators $A_0(x,D,q)$ and $B_{0,\nu}(x,D,q)$ are called the *leading terms* of the corresponding operators . By analogy with the previous section, we introduce the operator of the boundary value problem

$$\mathcal{A}u = \left\{Au,\ B_1u|_\Gamma,\ B_2u|_\Gamma\right\} : H^l(G) \longrightarrow H^{l-4}(G) \times \prod_{\nu=1}^{2} H^{l-k_\nu-1/2}(\Gamma)$$

and the number $l_0 = \max(4, k_\nu + 1)$.

Proposition 1.3.1 *Let $l \in \mathbb{Z}$, $l \geq l_0$. Then the operator $\mathcal{A}$ is bounded and*

$$|||f|||_{l-4} + \sum_{\nu=1}^{2} |||g_\nu|||'_{l-k_\nu-1/2} \leq C|||u|||_l \,, \qquad (1.3.3)$$

where the norms are taken over G and Γ (see (1.1.17) and (1.1.19)).

Proof does not differ from the proof of Proposition 1.2.7. $\triangleright$

Let us now formulate conditions of the unique solvability of the problem (1.3.1) - (1.3.2) which are analogous to Conditions 1.2.2 and 1.2.8 from the previous section.

Condition 1.3.2 $\forall\, x \in \overline{G}$, $\forall\, q \in Q \;=\; Q_1 \times Q_2$ *and* $\forall\, \xi \in \mathbb{R}^n$

$$A_0(x,\xi,q) \;\neq 0 \quad for \quad |\xi| + M_q \neq 0 \;, \tag{1.3.4}$$

where $M_q \;=\; \min_{i=1,2} |q_i|$.

In the case $n = 1$ we suppose in addition that the λ-roots of the polynomial $A_0(x, \lambda, q)$ are equally distributed between the upper and the lower half-planes.

Condition 1.3.3 *Let us fix a boundary point* $x' \in \Gamma$ *and suppose that the operators in (1.3.1) - (1.3.2) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the* x_n*-axis looks in one side with the inner normal to* Γ *).*

Consider the following problem on the half-axis $x_n > 0$:

$$A_0\big(0, \xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; 0\,, \quad x_n > 0 \;, \tag{1.3.5}$$

$$B_{0,\nu}\big(0, \xi', -i\frac{d}{dx_n}, q\big)\, v(x_n)\big|_{x_n=0} = \; h_\nu \;, \quad \nu = 1, 2 \;, \tag{1.3.6}$$

smoothly depending on parameters ξ' *and* q .

We suppose that $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *this problem has the unique solution in the space* $\mathcal{M}$ *of all stable solutions of the equation (1.3.5) for all* $h_\nu \in \mathbb{C}$.

Note that for $q = 0$ Condition 1.3.2 turns into the ordinary ellipticity condition as well as Condition 1.3.3 turns into the classical Shapiro - Lopatinskij condition.

Example 1.3.4 *Let* $A(x, D)$ *be an elliptic operator* $(\mathrm{ord}A = 4)$ *with a positive real leading part. Then the operator* $A(x, D) + q_1^2 q_2^2$ *with Dirichlet boundary conditions* $B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}$, $\nu = 1, 2,$ *satisfies Conditions 1.3.2 and 1.3.3 in the cone* $Q \;=\; Q_1 \times Q_2$, *where* $Q_j \;=\; \left\{|\arg q_j| \;\leq\; \frac{\pi-\varepsilon}{4}\right\}$, $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$.

Example 1.3.5 *If* A_1 *and* A_2 $(\mathrm{ord}A_j \;=\; 2\,,\; j \;=\; 1, 2)$ *are the two such operators as in Example 1.3.4, then the operator* $\left(A_1(x, D){+}q_1^2\right)\left(A_2(x, D){+}q_2^2\right)$ *with Dirichlet boundary conditions* $B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}$, $\nu = 1, 2,$ *also satisfies Conditions 1.3.2 and 1.3.3 in the cone* $Q \;=\; Q_1 \times Q_2$, *where* $Q_j \;=\; \left\{|\arg q_j| \;\leq\; \frac{\pi-\varepsilon}{2}\right\}$, $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$.

Proposition 1.3.6 *Let* $l \in \mathbb{Z}$, $l \geq l_0$, $q^{(0)} > 0$ *and suppose Conditions 1.3.2 and 1.3.3 being fulfilled. Then there exists a constant* $q^{(1)} > 0$ *such that for* *all* $q \in Q_M^0 = \{q \in Q_M : M_q \geq q^{(0)}\}$ *with the property* $|q_1| + |q_2| \geq q^{(1)}$, *the a priori estimate*

$$|||u|||_l \leq C\{|||f|||_{l-4} + \sum_{\nu=1}^{2} |||g_\nu|||'_{l-k_\nu-1/2}\} \qquad (1.3.7)$$

is valid for the solutions of (1.3.1) - (1.3.2), where the norms are taken over G *and* Γ *(see (1.1.17) and (1.1.19)).*

Proof. We begin with the local situation and then glue the required estimate from local ones using a partition of unity.

A). Suppose $x_0 \in U \subset \operatorname{int} G$, $u \in H^l(G)$ and $\operatorname{supp} u \subset\subset U$, where U is an open set. Then (1.3.7) is equivalent to the inequality $|||u|||_l \leq C|||A(x)u|||_{l-4}$, where for brevity we write $A(x) = A(x, D, q)$, and analogously $A(x_0) = A(x_0, D, q)$.

It follows from Proposition 1.2.3 that $|||u|||_l \leq C_1|||A_0(x_0)u|||_{l-4}$, where the norms are taken over G . And thus $|||u|||_l \leq C_1|||A(x)u|||_{l-4} + \mathcal{R}$, where

$$\mathcal{R} = C_1\big|\big|\big|\big(A(x) - A_0(x_0)\big)u\big|\big|\big|_{l-4}$$

$$\leq C_1\big|\big|\big|\big(A(x) - A_0(x)\big)u\big|\big|\big|_{l-4} + C_1\big|\big|\big|\big(A_0(x) - A_0(x_0)\big)u\big|\big|\big|_{l-4} .$$

We note that in $A(x) - A_0(x)$ the leading terms of the form $a(x)q^\alpha D^\beta$ $(|\alpha| + |\beta| = 4)$ are absent. Hence, it follows from (1.1.12) that

$$\big|\big|\big|\big(A(x) - A_0(x)\big)u\big|\big|\big|_{l-4} \leq C_2|||u|||_{l-1} .$$

Conversely, $A_0(x) - A_0(x_0)$ consists only of leading terms, but for sufficiently small U the coefficients $a(x)$ may be made sufficiently small. Thus

$$|||a(x)q^\alpha D^\beta u|||_{l-4} \leq \varepsilon|||u|||_l + C_3|||u|||_{l-1} .$$

Using (1.1.12), we obtain

$$|||q||| \cdot |||u|||_{l-1} = \sum_{i=1}^{2} |q_i|\, |||u|||_{l-1} \leq C_4|||u|||_l , \qquad (1.3.8)$$

and, hence, $\mathcal{R} \leq C_1\big[\varepsilon + (C_2 + C_3)C_4(|q_1| + |q_2|)^{-1}\big]\, |||u|||_l$, where C_i are independent of u and q . Now, for sufficiently small ε and sufficiently large $|||q||| = |q_1| + |q_2|$, $q \in Q_M^0$, we get $\mathcal{R} \leq 1/2|||u|||_l$ and thus, $|||u|||_l \leq C|||A(x)u|||_{l-4}$ for $C = 2C_1$.

B). We now suppose that $x_0 \in \Gamma$ and choose $U \ni x_0$ such that $U \subset U_\alpha$, where U_α is one of the boundary covering coordinate neighbourhoods. For the functions $u \in H^l(G)$ with $\operatorname{supp} u \subset U \cap \overline{G}$ we obtain the desired estimate

(1.3.7) in the same way as in part A) of the proof applying the trace inequality (1.1.6) as follows:

$$\big\|\big|\big(B_\nu(x) - B_{\nu,0}(x_0)\big)u\big|\big\|'_{l-k_\nu-1/2} \;\le\; \big\|\big|\big(B_\nu(x) - B_{\nu,0}(x_0)\big)u\big|\big\|_{l-k_\nu} \;.$$

C). In order to end the proof we have to arrange a standard procedure with a partition of unity. Let $\{U_\alpha\}$ be a covering of $\bar{G}$ such that the results of items A) and B) are valid in each U_α , and let $\{U_\mu\}_{\mu=1}^{\mu_0}$ be its finite subcovering. Denote by C and $q^{(1)}$ the largest local ones. Now (1.3.7) is valid for all $q:\ \||q|\| \ge q^{(1)}$, $q \in Q_M^0$ for all $u \in H^l(G)$ such that $\exists \mu \in \{1, ..., \mu_0\}$: $\operatorname{supp} u \subset U_\mu$.

We now take the corresponding to $\{U_\mu\}$ partition of unity $\{\varphi_\mu\}_{\mu=1}^{\mu_0}$. As for the function $\varphi_\mu u$ (1.3.7) is already proved, we have

$$\||u|\|_l \;\le\; \sum_{\mu=1}^{\mu_0} \||\varphi_\mu u|\|_l$$

$$\le\; C \sum_{\mu=1}^{\mu_0} \Big\{ \||A(x)[\varphi_\mu u]|\|_{l-4} \;+\; \sum_{\nu=1}^{2} \||B_\nu(x)[\varphi_\mu u]|\|'_{l-k_\nu-1/2} \Big\} \;,$$

but $A(x)[\varphi_\mu u] \;=\; \varphi_\mu A(x)u \;+ \dots$, where in the terms denoted by "..." φ_μ is differentiated at least once and thus

$$\||u|\|_l \;\le\; C \sum_{\mu=1}^{\mu_0} \Big\{ \||\varphi_\mu A(x)u|\|_{l-4} \;+\; \sum_{\nu=1}^{2} \||\varphi_\mu B_\nu(x)u|\|'_{l-k_\nu-1/2} \Big\} \;+$$

$$+\; C_1 \||u|\|_{l-1} \;.$$

Once again we use (1.3.8) (together with the fact that the multiplication by $\varphi_\mu \in C^\infty$ is a bounded operator in H^s and $\|\varphi u\|_l^2 \;\le\; \|u\|_l^2 \;+\; C_\varphi \|u\|_{l-1}^2$) and obtain

$$\||u|\|_l \;\le\; C_3 \sum_{\mu=1}^{\mu_0} \Big\{ \||A(x)u|\|_{l-4} \;+\; \sum_{\nu=1}^{2} \||B_\nu(x)u|\|'_{l-k_\nu-1/2} \Big\} \;+$$

$$+\; \frac{C_2}{\||q|\|} \||u|\|_l \;.$$

Finally, we suppose $q^{(1)} \ge 2C_2$ and the proof follows. ▷

Naturally, the *uniqueness of the solution to the problem (1.3.1) - (1.3.2)* follows from (1.3.7) for sufficiently large $\||q|\|$, $q \in Q_M^0$.

We can show that Conditions 1.3.2 and 1.3.3 are also necessary for the estimate (1.3.7) to take place for large $\||q|\|$, $q \in Q_M^0$.

Proposition 1.3.7 *Let* $l \in \mathbb{Z}$, $l \geq l_0$, $q^{(0)} > 0$ *and suppose there exists a large constant* $q^{(1)} > 0$ *such that the estimate (1.3.7) is true for all solutions from* $H^l(G)$ *to the problem (1.3.1) - (1.3.2) when* $|||q||| \geq q^{(1)}$ *and* $q \in Q_M^0$.

Then Condition 1.3.2 is satisfied in every interior point $x_0 \in G$ *and Condition 1.3.3 is satisfied in every boundary point* $x_0 \in \Gamma$ *where Condition 1.3.2 is already satisfied (if* $n = 1$ *, we assume additionally the regular distribution of* $\lambda-$*roots of the equation* $A_0(x, \lambda, q) = 0$*).*

Proof represents the corresponding one - parameter procedure due to Agmon S., Nirenberg L. (196) and Agranovich M.S., Vishik M.I. (1964). $\triangleright$

1.4 The existence theorems

We start with the theorem for operators with coefficients, sufficiently close to the constant ones, and denote by $\mathcal{A}$ an operator in $\mathbb{R}^n_+$ of the form

$$\mathcal{A}u = \Big\{ A(x,D,q)u,\ B_1(x,D,q)u\big|_{x_n=0},\ B_2(x,D,q)u\big|_{x_n=0} \Big\}.$$

Theorem 1.4.1 *A). Suppose* $l \in \mathbb{Z}$, $l \geq 4$ *and let*

$$A(x,D,q) = q_1^2 q_2^2 - q_1^2 B(x,D_x) - q_2^2 C(x,D_x) + E(x,D_x) ,$$

where $B(x,D_x)$, $C(x,D_x)$ *and* $E(x,D_x)$ *are homogeneous differential operators in* $\mathbb{R}^n$ *with smooth coefficients* $b_\alpha(x)$, $c_\alpha(x)$ *and* $e_\alpha(x)$ *. Suppose that* $A(0,D,q)$ *satisfies Condition 1.3.2. There exists* $\varepsilon > 0$ *such that, if*

$$\big|b_\alpha(x) - b_\alpha(0)\big| , \quad \big|c_\alpha(x) - c_\alpha(0)\big| , \quad \big|e_\alpha(x) - e_\alpha(0)\big| < \varepsilon , \qquad (1.4.1)$$

for all $x \in \mathbb{R}^n$ *, then, for* $q \in Q_M^0$ *with sufficiently large* $|||q|||$ *, the right inverse* $\mathcal{R}$ *to the operator* $A(x,D,q)$ *is defined and*

$$|||\mathcal{R}f|||_l \leq C|||f|||_{l-4} . \qquad (1.4.2)$$

B). Suppose $l \in \mathbb{Z}$, $l \geq \max(4, k_j + 1)$ *and let the operator* $A(x,D,q)$ *be as in part A) of the theorem with coefficients in* $C^\infty(\mathbb{R}^n_+)$ *, and*

$$B_j(x,D,q) = \sum_{|\alpha|+|\beta|=k_j} b_{j,\alpha,\beta}(x) q^\beta D^\alpha ,$$

where $b_{j,\alpha,\beta} \in C^\infty(\mathbb{R}^n_+)$ *. Suppose that* $A(0,D,q)$ *satisfies Condition 1.3.2, and together with* $B_j(0,D,q)$ $(j = 1,...,k)$ $-$ *Condition 1.3.3. Hence, there exists* $\varepsilon > 0$ *such that, if (1.4.1) is true together with*

$$\big|b_{j,\alpha,\beta}(x) - b_{j,\alpha,\beta}(0)\big| < \varepsilon \quad (|\alpha| + |\beta| = k_j) , \quad \forall\, x \in \mathbb{R}^n , \qquad (1.4.3)$$

then, for $q \in Q^0_M$ *with sufficiently large* $|||q|||$, *the right inverse* $\mathcal{R}$ *to the operator* $\mathcal{A}$ *is defined and*

$$|||\mathcal{R}(f,g_1,g_2)|||_l \ \leq \ C\big\{|||f|||_{l-4} \ + \ \sum_{\nu=1}^{2} |||g_\nu|||'_{l-k_\nu-1/2}\big\} \ . \qquad \textit{(1.4.4)}$$

Proof. We show how item A) follows from Proposition 1.2.3, item B) follows from Proposition 1.2.7 in the same way.

Let $\mathcal{R}_0$ be the inverse to $A(0,D,q)$ being built in Proposition 1.2.3 $\forall\, q \in Q^0_M$. As follows from (1.2.5)

$$|||\mathcal{R}_0 f|||_l \ \leq \ C|||f|||_{l-4} \ . \qquad (1.4.5)$$

Consider an operator

$$A(x,D,q)\mathcal{R}_0 \ = \ A(0,D,q)\mathcal{R}_0 \ + \ T \ = \ I \ + \ T \ ,$$

where $T \ = \ \big[A(x,D,q) \ - \ A(0,D,q)\big]\mathcal{R}_0$. By analogy with the proof of Proposition 1.3.6, part A), we see that

$$|||Tf|||_{l-4} \ \leq \ \eta(\varepsilon)|||\mathcal{R}_0 f|||_l \ + \ C|||\mathcal{R}_0 f|||_{l-1} \ ,$$

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, using the inequalities (1.3.8) and (1.4.5), we obtain

$$|||Tf|||_{l-4} \ \leq \ C_1\Big(\eta(\varepsilon) \ + \ \frac{C_3}{|||q|||}\Big)|||\ f|||_{l-4} \ .$$

Thus, for sufficiently large $|||q|||$ and sufficiently small ε , we obtain that

$$|||Tf|||_{l-4} \ \leq \ 1/2|||\ f|||_{l-4} \ ,$$

and thus, operator T has a small norm. It implies , that the inverse $(I+T)^{-1}$ to the operator $(I+T)$ exists and is bounded in the norm $|||\cdot|||_{l-4}$ and hence, in the norm $\|\cdot\|_{l-4}$. We now put $\mathcal{R} \ = \ \mathcal{R}_0(I+T)^{-1}$ and observe that, the inequality (1.4.2) is valid for $\mathcal{R}$ and $A(x,D,q)\mathcal{R}f \ = \ f\,;\quad f \in H^{l-4}(\mathbb{R}^n)$. $\rhd$

Now we are ready to formulate the main theorem about elliptic operators with parameters.

Theorem 1.4.2 *Let* $l \in \mathbb{Z}$, $l \geq l_0 = \max(4, k_j + 1)$ *and suppose the boundary value problem (1.3.1) - (1.3.2) satisfy Conditions 1.3.2 and 1.3.3.*

Then for all $q \in Q^0_M$ *with sufficiently large* $|||q|||$ *a bounded inverse*

$$\mathcal{A}^{-1} \ : \ H^{l-4}(G) \ \times \ \prod_{j=1}^{2} H^{l-k_j-1/2}(\Gamma) \ \longrightarrow \ H^l(G) \qquad \textit{(1.4.6)}$$

to the operator $\mathcal{A}$ *is defined.*

Proof. Using a sufficiently small partition of unity $\{\varphi_\nu\}_{\nu=1}^N$ such that $\varphi_\nu \in C_0^\infty(U_\nu)$ and $\sum_{\nu=1}^N \varphi_\nu(x) \equiv 1$ in a neighbourhood of G, we reduce the problem to the cases considered in Theorem 1.4.1.

Suppose that either $\operatorname{supp}\varphi_\nu \cap \Gamma = \emptyset$, or $\operatorname{supp}\varphi_\nu \subset U_j$, where $\bigcup_j U_j \supset \Gamma$. Moreover, for each U_ν we define an infinitely smooth function $\psi_\nu \in C_0^\infty(U_\nu)$ such that $\psi_\nu \equiv 1$ in a neighbourhood of $\operatorname{supp}\varphi_\nu(x)$, $D^\alpha\big[\varphi_\nu(x)\psi_\nu(x)\big] \equiv D^\alpha\varphi_\nu(x)$ and ψ_ν has the same support properties as $\varphi_\nu(x)$. Obviously,

$$\mathcal{A}u \;=\; \sum_{\nu=1}^{N} \varphi_\nu \mathcal{A}(\psi_\nu u) \;. \tag{1.4.7}$$

There are ingredients of two types in this sum: in those with $\operatorname{supp}\varphi_\nu, \psi_\nu \bigcap \Gamma \neq \emptyset$ we overcome to the local coordinates straightening the boundary in U_ν; if $\operatorname{supp}\varphi_\nu,\, \psi_\nu \subset \operatorname{int} G$, we associate $\mathcal{A}$ with A for here $B_j(\psi_\nu u)\big|_\Gamma = 0$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small the coefficients of the operator $\mathcal{A}$ on $\operatorname{supp}\psi_\nu$ are almost constant, and we construct an extension $\mathcal{A}_\nu$ of $\mathcal{A}$ from $\operatorname{supp}\psi_\nu$ to the whole $\mathbb{R}^n$ or $\mathbb{R}^n_+$ such that $\varphi_\nu\mathcal{A}(\psi_\nu u) = \varphi_\nu\mathcal{A}_\nu(\psi_\nu u)$, and its coefficients are almost constant too. We put

$$\mathcal{A}_\nu(x) \;=\; \alpha(x)\mathcal{A}(x) \;+\; [1-\alpha(x)]\mathcal{A}(x^{(\nu)}) \;,$$

where $\alpha(x) \in C_0^\infty$, $0 \le \alpha(x) \le 1$, $\alpha(x) \equiv 1$ on $\operatorname{supp}\psi_\nu$, $x^{(\nu)} \in \operatorname{supp}\psi_\nu$ and $x^{(\nu)} \in \Gamma$ if $\operatorname{supp}\psi_\nu \bigcap \Gamma \neq \emptyset$.

Operator $\mathcal{A}^0 u \;=\; ,\Big\{A_0 u,\ B_{1,0}u\big|_\Gamma,\ B_{2,0}u\big|_\Gamma\Big\}$ is called a leading part of operator $\mathcal{A}$. Here and further $\mathcal{A}^0_\nu, A_0, B_{j,0}$ are the leading parts of operators $\mathcal{A}_\nu, A, B_j$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small, then it follows from Theorem 1.4.1 that the right inverse $\mathcal{R}^0_\nu$ to $\mathcal{A}^0_\nu$ is defined for $q \in Q^0_M$ with sufficiently large $|||q|||$. For these q we define a bounded operator

$$\mathcal{R}(f,g) \;=\; \sum_{\mu=1}^{N} \psi_\mu \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g) \;:\; H^l(G,\Gamma) \longrightarrow H^l(G) \;, \tag{1.4.8}$$

where $g = (g_1, g_2)$ and $H^l(G,\Gamma) \;=\; H^{l-4}(G) \times \prod_{j=1}^2 H^{l-k_j-1/2}(\Gamma)$ is equipped with a norm

$$|||(f,g)|||_l \;=\; |||f|||_{l-4} \;+\; \sum_{j=1}^{2} |||g_j|||'_{l-k_j-1/2} \,. \tag{1.4.9}$$

As usual, we have to prove that $\mathcal{A}\mathcal{R} = I + T$, where $T = T(f,g)$ is an operator with the sufficiently small norm in $H^l(G,\Gamma)$. Thus we have

$$\mathcal{A}\mathcal{R}(f,g) \;=\; \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\nu \psi_\nu \psi_\mu \mathcal{R}_\mu(\varphi_\mu f, \varphi_\mu g) \;,$$

where we consider only the terms with $\psi_\nu \psi_\mu \neq 0$. We put

$$u_\mu \ = \ \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g)$$

and replace $\mathcal{A}_\nu$ by $\mathcal{A}_\mu$ in the last sum:

$$\mathcal{A}\mathcal{R}(f,g) \ = \ \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\mu \psi_\nu \psi_\mu u_\mu$$

$$= \ \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \ + \ T(f,g) \ ,$$

where

$$T(f,g) \ \ = \ \sum_\mu \sum_\nu \varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu \ +$$

$$+ \ \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu [\mathcal{A}_\mu \ - \ \mathcal{A}_{\mu,0}] u_\mu \ . \tag{1.4.10}$$

As $\mathcal{A}_{\mu,0}\mathcal{R}^0_\mu = I$, it follows that,

$$\sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \ = \ \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \varphi_\mu (f,g)$$

$$= \ \sum_\mu \sum_\nu \varphi_\nu \varphi_\mu (f,g) \ = \ (\sum_\nu \varphi_\nu)^2 (f,g) \ = \ (f,g) \ .$$

In order to estimate $T(f,g)$, we fix a pair of indices μ and ν and suppose $\mathcal{A}_\mu$ includes boundary operators (otherwize the work is easier).

For each addend in the first sum in (1.4.10) we have in terms of the norm (1.4.9)

$$\||\varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu|\|_l \ \leq \ \Big\{ \||[A\psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu A] u_\mu|\|_{l-4} \ +$$

$$+ \ \sum_{j=1}^{2} \||[B_j \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu B_j] u_\mu|\|_{l-k_j} \Big\} \ \leq \ C\||u_\mu|\|_{l-1} \ , \tag{1.4.11}$$

where as in the proof of Theorem 1.4.1 we use the fact that operators in square brackets do not contain the leading terms.

Now, it follows from (1.1.12) and (1.4.4) for $\mathcal{R}_\mu$ that

$$\||\varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu|\|_l \ \leq \ \frac{C}{\||q|\|} \||(\varphi_\mu f, \varphi_\mu g)|\|_l \ . \tag{1.4.12}$$

As $\|\varphi u\|^2_l \ \leq \ \|u\|^2_l \ + \ C_\varphi \|u\|^2_{l-1}$, we can get reed of φ_μ in the right - hand side of (1.4.12) and return to the initial coordinates .

The addends in the second sum in (1.4.10) we estimate in the same way, for the operator $\mathcal{A}_\mu - \mathcal{A}_\mu^0$ does not contain leading terms too. We arrive to inequality

$$|||T(f,g)\,|||_l \;\le\; \frac{C}{|||q|||}\,|||(f,g)|||_l\;.$$

Consequently, for $q \in Q_M^0$ with sufficiently large $|||q|||$ we obtain that $\|T\| \le 1/2$ and $\mathcal{AR} = I + T$ $\big(\,I$ is the identity map in $H^l(G,\Gamma)\,\big)$ possess the bounded inverse $(\mathcal{AR})^{-1}$. Finally, we define

$$\mathcal{A}^{-1} \;=\; \mathcal{R}(\mathcal{AR})^{-1}$$

being the bounded operator from $H^l(G,\Gamma)$ to $H^l(G)$ and the right inverse to $\mathcal{A}$. Thus, it follows, that $\mathcal{A}$ maps $H^l(G)$ onto $H^l(G,\Gamma)$, and is one-to-one for $q \in Q_M^0$ with sufficiently large $|||q|||$, and for these q , $\mathcal{A}^{-1}$ also is a left inverse to $\mathcal{A}$. $\triangleright$

We conclude this section with a few general remarks about possible generalizations and further results.

Remark 1.4.3 *A). If the problem (1.3.1) - (1.3.2) satisfies Conditions 1.3.2 and 1.3.2 on the rays* $\{\arg q_i = c_i\ \}$ *then, for all* $q \in Q_M^0$ *with sufficiently large* $|||q|||$ *, this is a Fredholm problem with zero index (this result follows directly from the work (Agranovich M.S., Dynin A.S. (1962))).*

B). It would be interesting to consider pseudodifferential boundary problems depending on a number of parameters. The one - parameter case was considered in the book (Grubb, G. (1986)).

C). We note that the parameter set $\{q \in \mathbb{C}^2\ :\ q \in Q_M^0,\ |||q||| \ge q^{(1)}\}$ *is a relatively complete Reinhart domain. It can be made logarithmically convex if* $q^{(1)}$ *is slightly increased (see Shabat B.V. (1985)).*

Chapter 2

Polyparabolic mixed problems

2.1 Main functional spaces

2.1.1

Let $\alpha = (\alpha_1, ..., \alpha_m)$, $\gamma = (\gamma_1, ..., \gamma_m)$ be multi-indices with nonnegative coordinates.

By definition the space $\mathbf{P}^\alpha\left(e^{-\gamma \cdot t}\right)$ consists of functions $u(t) = u(t_1, ..., t_m)$ supported in the positive "octant" $(\bar{\mathbb{R}}_+)^m$ and such that

$$e^{-\gamma \cdot t} u(t) \in H^{\vec{\alpha}}(\mathbb{R}^m) , \tag{2.1.1}$$

where $H^{\vec{\alpha}}(\mathbb{R}^m)$ denotes the anisotropic Sobolev space, defined as a completion of the space $C_0^\infty(\mathbb{R}^m)$ with respect to the norm

$$\|u\|_{\vec{\alpha}}^2 = \sum_{j=1}^{m} \int_{(\mathbb{R}_+)^{m-1}} \int_{\mathbb{R}_{\xi_j}} \left|(F_{t_j \to \xi_j} u)(t_1, ..., t_{j-1}, \xi_j, t_{j+1}, ..., t_m)\right|^2 \times$$

$$\times \left(1 + |\xi_j|^2\right)^{\alpha_j} d\xi_j dt[j], \tag{2.1.2}$$

where $dt[j] = dt_1 ... dt_{j-1} dt_{j+1} ... dt_m$. ¿From this definition and the embedding theorems (see, for example, Slobodeckij L.N. (1958), Kudryavtsev L.D., Nikolskij S.M. (1988)) follows that, if $u(t) \in \mathbf{P}^\alpha\left(e^{-\gamma \cdot t}\right)$ then the derivatives $\partial_{t_i}^{k_i} u(t)$ of orders $k_i \leq [\alpha_i] - 1$, for $\alpha_i \leq [\alpha_i] + 1/2$, and of orders $k_i \leq [\alpha_i]$, for $\alpha_i \geq [\alpha_i] + 1/2$, are continuous and thus, vanish when $t_i = 0$.

For the norm of a function $u(t)$ in the space $\mathbf{P}^\alpha\left(e^{-\gamma \cdot t}\right)$ we take the norm $\left\|e^{-\gamma \cdot t} u(t)\right\|_{\vec{\alpha}}$ of the function $e^{-\gamma \cdot t} u(t)$ in $H^{\vec{\alpha}}(\mathbb{R}^m)$.

We can give the "interior" definition of spaces $\mathbf{P}^\alpha\big(e^{-\gamma\cdot t}\big)$ for $\alpha_i \neq [\alpha_i]+1/2$, that is, the space $\mathbf{P}^\alpha\big(e^{-\gamma\cdot t}\big)$ consists of functions in positive "octant" $\hat{Q} = (\bar{\mathbb{R}}_+)^m$ such that

1) $e^{-\gamma\cdot t}u(t) \in H^{\vec{\alpha}}(\hat{Q})$

and

2) $e^{-\gamma\cdot t}u(t) \in H^{\vec{\alpha}}(\mathbb{R}^m)$ after zero extension into the domain, where at least one $t_i < 0$ (the continuation may be executed serially in $t_1, t_2, ..., t_m$ and the result is independent of the order of the operations).

In this case we take the value $\big\|e^{-\gamma\cdot t}u(t)\big\|_{\vec{\alpha},\hat{Q}}$ as the norm of the function $e^{-\gamma\cdot t}u(t)$ in the space $H^{\vec{\alpha}}(\hat{Q})$ (see (1.1.23)).

Condition 2) is equivalent to the following one:

2)′ for almost all $(t_1, ..., t_{i-1}, t_{i+1}, ..., t_m) = t[i]$ we have

$$\partial_{t_i}^{k_i} u(t)\big|_{t_i=0} = 0 \quad \text{for} \quad k_i \leq [\alpha_i]-1 \tag{2.1.3}$$

and

$$\int_0^1 t_i^{2(\alpha_i-[\alpha_i])}\big|\partial_{t_i}^{[\alpha_i]}u(t)\big|^2\,dt_i \;<\; +\infty \quad \text{for} \quad \alpha_i > [\alpha_i]\,. \tag{2.1.4}$$

The functional space $\mathbf{E}_\alpha(\gamma)$ is defined as the set of functions $U(p) = U(p_1, ..., p_m)$, where $p = \sigma + i\gamma$ and $p_j = \sigma_j + i\gamma_j$, for $j = 1, ..., m$, which are holomorphic in the direct product of right half-planes $\Pi_\gamma = \{p \in \mathbb{C}^m : \operatorname{Re} p_j > \gamma_j\,,\ j = 1, ..., m\}$ and have finite norms

$$\big\|U(p_1, ..., p_m)\big\|^2_{\mathbf{E}_\alpha(\gamma)} \;=\; \sum_{j=1}^{m} \sup_{\sigma\in\Pi_\gamma} \int_{\mathbb{R}^m} \big|U(\sigma+i\tau)\big|^2\,|\sigma_j + i\tau_j|^{2\alpha_j}\,d\tau\,. \tag{2.1.5}$$

Proposition 2.1.1 *Suppose* $p = (p_1, ..., p_m) \in \Pi_\gamma$ *with* $\gamma_j > 0$ *for* $j = 1, ..., m$. *Then the Laplace transformation*

$$Lu(t) \;:=\; \int_0^\infty \cdots \int_0^\infty e^{-p\cdot t}u(t)\,dt \;=\; U(p) \tag{2.1.6}$$

defines a one-to-one mutually continuous correspondence of spaces $\mathbf{P}^\alpha\big(e^{-\gamma\cdot t}\big)$ *and* $\mathbf{E}_\alpha(\gamma)$.

Proof. 1). We start with the case $\alpha = 0$.

Let $u(t) \in \mathbf{P}^0\big(e^{-\gamma\cdot t}\big)$. Then

$$Lu(t) \;=\; U(p) \;=\; \int_0^\infty \cdots \int_0^\infty e^{-i\tau\cdot t}\big[e^{-\sigma\cdot t}u(t)\big]\,dt \tag{2.1.7}$$

$$= \; F_{t\to\tau}\big[e^{-\sigma\cdot t}u(t)\chi(t)\big]\,,$$

where $\chi(t) = \prod_{j=1}^m \chi(t_j)$ and $\chi(t_j)$ is Heaviside function (i.e. the characteristic function of the ray $0 \leq t_j < \infty$).

For all $\sigma \in \overline{\Pi}_\gamma$ we have $e^{-\sigma\cdot t}u(t) \in L_2(\hat{Q})$, consequently $U(p) \in L_2$ in τ and in the view of Parseval equality

$$\frac{1}{(2\pi)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|U(\sigma+i\tau)\right|^2 d\tau \;=\; \int_0^{\infty}\cdots\int_0^{\infty} e^{-2\sigma\cdot t}\left|u(t)\right|^2 dt \quad (2.1.8)$$

with $\sigma \in \overline{\Pi}_\gamma$, we obtain

$$\sup_{\sigma\in\Pi_\gamma}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|U(\sigma+i\tau)\right|^2 d\tau \;=\; \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|U(\gamma+i\tau)\right|^2 d\tau$$

$$= \; (2\pi)^m \left\|e^{-\gamma\cdot t}u(t)\right\|_0^2 . \quad (2.1.9)$$

Note that for $\sigma = \gamma$ the integral in (2.1.7) is defined as the L_1-limit of integrals over $\prod_{j=1}^m [0, N_j]$ as $N_j = N \to \infty$, the same conserns the case where some of $\sigma_j = \gamma_j$. If $\sigma_j > \gamma_j$ for all j then the integral in (2.1.7) is absolutely convergent and $U(p)$ is holomorphic in each p_j . Thus, if $\operatorname{Re} p = \sigma \in \operatorname{int}\Pi_\gamma$ (i.e. $\sigma_j > \gamma_j$ for all $j \in \{1, ..., m\}$) then $U(p)$ is holomorphic in each p_j and, in the view of Hartogs theorem, is indeed holomorphic in Π_γ .

Thus, we have proved that L maps continuously the space $\mathbf{P}^0\left(e^{-\gamma\cdot t}\right)$ into $\mathbf{E}_0(\gamma)$.

2). We show now that there exists L^{-1} , which is defined on $\mathbf{E}_0(\gamma)$ as a whole and maps continuously $\mathbf{E}_0(\gamma)$ into $\mathbf{P}^0\left(e^{-\gamma\cdot t}\right)$.

In order to prove it, we need a slight modification of Cauchy integral formula in several complex variables (see, for example, Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)).

Lemma 2.1.2 *Suppose* $\sigma_j > \sigma_j^0 > \gamma_j$, *for each* $j = 1, ..., m$ *and* $U(p) \in \mathbf{E}_0(\gamma)$, *then*

$$U(p) \;=\; \frac{1}{(2\pi i)^m}\int_{\sigma_1^0-i\infty}^{\sigma_1^0+i\infty}\cdots\int_{\sigma_m^0-i\infty}^{\sigma_m^0+i\infty}\frac{U(\zeta)}{(p_1-\zeta_1)\cdots(p_m-\zeta_m)}\,d\zeta_1\cdots d\zeta_m . \quad \textit{(2.1.10)}$$

Proof. We start with Cauchy integral formula in a polycircle domain (see Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)) $\Gamma^N = \Gamma_1^N \times \cdots \times \Gamma_m^N$, where Γ_j^N is a rectangle with the sides $\sigma_j = \sigma_j^0$, $\sigma_j = \sigma_j^1$, $\tau_j = N_j$, $\tau_j = -N_j$, $\sigma_j^1 > \sigma_j^0$. Integrating over Shilov boundary (i.e. the frame) of Γ^N , we obtain Cauchy formula

$$U(p) \;=\; \frac{1}{(2\pi i)^m} \int_{\Gamma^N} \frac{U(\zeta)}{(\zeta - p)}\, d\zeta \;, \tag{2.1.11}$$

where $d\zeta \;=\; d\zeta_1 \cdots d\zeta_m$, and $(\zeta - p) \;=\; \prod_{j=1}^{m} (\zeta_j - p_j)$. Integrating (2.1.11) over $N_j \in [N_j', N_j' + 1]$, for the fixed $j \in \{1, ..., m\}$, we obtain the integrands of the two kinds. In those with the integration over $[\sigma_j^0 \pm iN_j, \sigma_j^1 \pm iN_j]$ we obtain intagrals of the form

$$\mp \int_{N_j'}^{N_j'+1} dN_j \int_{\Gamma'} \frac{d\zeta'}{(\zeta' - p')} \int_{\sigma_j^0}^{\sigma_j^1} \frac{U(..., \sigma_j \pm iN_j, ...)\, d\sigma_j}{p_j - (\sigma_j \pm iN_j)} \;, \tag{2.1.12}$$

where the integral over ζ' denotes the rest integrations. Changing the order of integration in (2.1.12) and applying Schwartz inequality, we obtain in the view of (2.1.5) that this term vanishes as $N_j' \to \infty$. Consequently, after integration over all N_j , only the terms with the integrations over $[\sigma_j^0 + iN_j, \sigma_j^0 - iN_j]$ or over $[\sigma_j^1 - iN_j, \sigma_j^1 + iN_j]$ (before passing to the limit as $N_j' \to \infty$) are left in the right-hand side of (2.1.11). Finally, after passing to the limits as all $N_j' \to \infty$ we obtain

$$\frac{1}{(2\pi i)^m} \int_{\sigma_1^{0\vee 1} \mp i\infty}^{\sigma_1^{0\vee 1} \pm i\infty} \cdots \int_{\sigma_m^{0\vee 1} \mp i\infty}^{\sigma_m^{0\vee 1} \pm i\infty} \frac{U(\zeta)\, d\zeta}{(\zeta - p)} \;,$$

where for $\sigma_j = \sigma_j^0$ we integrate over $(\sigma_j^0 + i\infty, \sigma_j^0 - i\infty)$ and for $\sigma_j = \sigma_j^1$ respectively over $(\sigma_j^1 - i\infty, \sigma_j^1 + i\infty)$. If we have at least one integration over $(\sigma_j^1 - i\infty, \sigma_j^1 + i\infty)$, we again estimate the appropriate integral with the help of Schwartz inequality and, in the view of (2.1.5) we see that, this term vanishes as $\sigma_j^1 \to \infty$ in the same way as the integral

$$\int_{\sigma_j^1 - i\infty}^{\sigma_j^1 + i\infty} \frac{d\tau_j}{|\sigma_j^1 + i\tau_j - p_j|^2} \;.$$

Finally, the only term which is left in (2.1.11) after the procedures of passing to the limits as $\sigma_j^1 \to \infty$ for all j , is the one with only σ_j^0 left:

$$U(p) \;=\; \frac{1}{(2\pi i)^m} \int_{\sigma_1^0 + i\infty}^{\sigma_1^0 - i\infty} \cdots \int_{\sigma_m^0 + i\infty}^{\sigma_m^0 - i\infty} \frac{U(\zeta)\, d\zeta}{(\zeta - p)} \;.$$

Changing the order of the limits we obtain (2.1.10). ▷

Estimating (2.1.10) we easily obtain the following

Lemma 2.1.3 *Let* $U(p) \in \mathbf{E}_0(\gamma)$ *. Then*

$$U(p) \to 0 \quad \text{as} \quad |\tau| \to \infty$$

uniformly in σ *as* $\sigma_j \geq \sigma_j^1$ *for arbitrary* $\sigma_j^1 > \gamma_j$, $\forall j \in \{1, ..., m\}$.

The last lemma gives the opportunity to consider the operator M such that

$$MU(p) \;=\; \frac{1}{(2\pi i)^m} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \cdots \int_{\sigma_m - i\infty}^{\sigma_m + i\infty} U(p)\, e^{p \cdot t}\, dp \tag{2.1.13}$$

$$=\; \frac{1}{(2\pi i)^m} \operatorname*{l.i.m.}_{N \to \infty} \int_{\sigma_1 - iN}^{\sigma_1 + iN} \cdots \int_{\sigma_m - iN}^{\sigma_m + iN} U(p)\, e^{p \cdot t}\, dp \ ,$$

where $\sigma_j > \gamma_j$, for all $j \in \{1, ..., m\}$. Operator M is defined on the whole $\mathbf{E}_0(\gamma)$, is independent of the choise of $\sigma_j > \gamma_j$ and maps continuously from $\mathbf{E}_0(\gamma)$ into $\mathbf{P}^0\big(e^{-\gamma \cdot t}\big)$ in the view of Parseval inequality (2.1.8). Further, $M = L^{-1}$ on smooth functions and thus, by continuity, we obtain from (2.1.8) that L and M are mutually inverse and provide the isomorphism between $\mathbf{P}^0\big(e^{-\gamma \cdot t}\big)$ and $\mathbf{E}_0(\gamma)$.

3). Whence the case with $\alpha = 0$ is proved, the case $\alpha > 0$ follows in a standard way (see Agranovich M.S., Vishik M.I. (1964)). $\triangleright$

2.1.2

We are ready now to define main functional spaces in "Polycylinder" domains .

Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma \in C^\infty$, and let $I = \prod_{j=1}^m [T_j, T_j']$, where $-\infty \leq T_j < T_j' \leq +\infty$. Denote by Ω_I the "polycylinder" domain $G \times I$, and by Ω_I' its "side" surface $\Gamma \times I$.

We use the notation $H^{k,\vec{\alpha}}\big(\Omega_I\big)$ for a completion of the space $C_0^\infty\big(\Omega_I\big)$ with respect to the norm

$$\big\|u(x,t)\big\|_{k,\vec{\alpha}}^2 \;=\; \int_I \big\|u(x,t)\big\|_{k,G}^2\, dt \;+\; \int_G \big\|u(x,t)\big\|_{\vec{\alpha},I}^2\, dx \ , \tag{2.1.14}$$

where $dt = dt_1 ... dt_m$, and $\|\cdot\|_{\vec{\alpha},I}$ is the norm in $H^{\vec{\alpha}}(I)$ (see (1.1.23)).

Also, the norm in $H^{k,\vec{\alpha}}\big(\Omega_I'\big)$ is defined by

$$\big\|u(x,t)\big\|_{k,\vec{\alpha}}'^{\,2} \;=\; \int_I \big\|u(x,t)\big\|_{k,\Gamma}'^{\,2}\, dt \;+\; \sum_\beta \int_\Gamma \big\|\varphi_\beta(x') u(x',t)\big\|_{\vec{\alpha},I}^2\, dx' \ , \tag{2.1.15}$$

where $\{\varphi_\beta\}$ with the property $\operatorname{supp} \varphi_\beta \bigcap \Gamma \neq \emptyset$ are the functions from the partition of unity on G defining the norm $\|\cdot\|_{k,\Gamma}'$ (see (1.1.18)). Certainly, these spaces are the particular cases of anisotropic Sobolev spaces, being considered in (Slobodeckij L.N. (1958)). The down-stated Remark contains their properties in the special case of $\alpha_j = \frac{l}{2b_j}$.

Remark 2.1.4 *Hereafter we consider the spaces*

$$H^{l,\frac{l}{2\vec{b}}}(\Omega_I) \ := \ H^{l,\{\frac{l}{2b_1},\ldots,\frac{l}{2b_m}\}}(\Omega_I) \ \text{ and } \ H^{\lambda,\frac{\lambda}{2\vec{b}}}(\Omega_I') \ := \ H^{\lambda,\{\frac{\lambda}{2b_1},\ldots,\frac{\lambda}{2b_m}\}}(\Omega_I') \ , \tag{2.1.16}$$

($l+1, \lambda+1/2, b_j \in \mathbb{N}$), *with the following properties (see Slobodeckij L.N. (1958)):*

1). Any given function $u \in H^{l,\frac{l}{2\vec{b}}}(\Omega_I)$ *has generalized derivatives*

$$v(x,t) \ = \ D_x^\alpha \partial_t^\beta u(x,t) \ \in \ H^{r,\frac{r}{2\vec{b}}}(\Omega_I) \ , \tag{2.1.17}$$

where $r = l - |\alpha| - 2b \cdot \beta \geq 0$, *and*

$$\|v\|_{r,\frac{r}{2\vec{b}}} \ \leq \ C\,\|u\|_{l,\frac{l}{2\vec{b}}} \ . \tag{2.1.18}$$

2). For $r \geq 1$ *the function* $v(x,t)$ *has boundary values* $v(x',t)$ *belonging to* $H^{\lambda,\frac{\lambda}{2\vec{b}}}(\Omega_I')$ *with* $\lambda = r - 1/2$ *and*

$$\|v(x',t)\|'_{\lambda,\frac{\lambda}{2\vec{b}}} \ \leq \ C\,\|v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ . \tag{2.1.19}$$

3). For $r \geq b_j$ *there exists a trace* $v(x,t[j],t_j^0)$ *of the function* $v(x,t)$ *on the hyperplane* $\Omega_I \bigcap \{t_j = t_j^0\}$, *where* $t_j^0 \in [T_j, T_j']$. *In this case* $v(x,t[j],t_j^0)$ *belongs to* $H^{r-b_j,\frac{r-b_j}{2\vec{b}}[j]}(\Omega_I \bigcap \{t_j = t_j^0\})$ *and*

$$\|v(x,t[j],t_j^0)\|_{r-b_j,\frac{r-b_j}{2\vec{b}}[j]} \ \leq \ C\,\|v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ , \tag{2.1.20}$$

where we denote by $(t[j],t_j^0)$ *the vector* $(t_1,\ldots,t_{j-1},t_j^0,t_{j+1},\ldots,t_m)$ *and by* $\frac{l}{2\vec{b}}[j]$ *the vector* $\left(\frac{l}{2b_1},\ldots,\frac{l}{2b_{j-1}},\frac{l}{2b_{j+1}},\ldots,\frac{l}{2b_m}\right)$.

In particular, if $t_j = 0 \in [T_j, T_j']$, *the following initial conditions are defined for a function* $u(x,t) \in H^{l,\frac{l}{2\vec{b}}}(\Omega_I)$:

$$u\Big|_{t_j=0} \ = \ \phi_{j,0}(x,t[j]) \ , \quad \frac{\partial u}{\partial t_j}\Big|_{t_j=0} \ = \ \phi_{j,1}(x,t[j]) \ ,\ldots,$$

$$\frac{\partial^{\kappa_j-1}u}{\partial t_j^{\kappa_j-1}}\Big|_{t_j=0} \ = \ \phi_{j,\kappa_j}(x,t[j]) \ , \tag{2.1.21}$$

where

$$\kappa_j \ = \ \begin{cases} \left[\frac{l}{2b_j}\right] \ , & \text{if } \frac{l}{2b_j} - \left[\frac{l}{2b_j}\right] \ \leq \ 1/2 \ , \\ \left[\frac{l}{2b_j}\right] + 1 \ , & \text{if } \frac{l}{2b_j} - \left[\frac{l}{2b_j}\right] \ > \ 1/2 \ . \end{cases}$$

In this case

$$\phi_{j,\nu_j}(x,t[j]) \ \in \ H^{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j]}(\Omega_I \bigcap \{t_j = 0\}) \ ,$$

where $l_{\nu,j} = l - 2b_j\nu_j - b_j > 0$, $0 \le \nu_j \le \kappa_j - 1$ *and*

$$\|\phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]} \ \le \ C\,\|u\|_{l,\frac{l}{2\bar{b}}} \ . \tag{2.1.22}$$

4). Conversely, for a given set of functions

$$\phi_{j,\nu_j}(x,t[j]) \ \in \ H^{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]}\big(\Omega_I \bigcap \{t_j = 0\}\big) \ ,$$

where $0 \le \nu_j \le \kappa_j - 1$, *there exists* $u \in H^{l,\frac{l}{2\bar{b}}}(\Omega_I)$ *such that (2.1.21) are valid and*

$$\|u\|_{l,\frac{l}{2\bar{b}}} \ \le \ C\sum_{j=1}^{m}\sum_{\nu_j=0}^{\kappa_j-1} \|\phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]} \ . \tag{2.1.23}$$

5). Suppose $I = \Pi_{j=1}^m [a_j, b_j]$, *where* $b_j < +\infty$ *and* $J = \Pi_{j=1}^m [a_j, +\infty]$. *Then there exists an extension operator* L *(resp.* L' *) which is bounded from* $H^{l,\frac{l}{2\bar{b}}}(\Omega_I)$ *to* $H^{l,\frac{l}{2\bar{b}}}(\Omega_J)$ *(resp.* $H^{\lambda,\frac{\lambda}{2\bar{b}}}(\Omega'_I)$ *to* $H^{\lambda,\frac{\lambda}{2\bar{b}}}(\Omega'_J)$*) . We construct the operator* L *by induction over the number of times; for the one-dimensional time it is described in (Agranovich M.S., Vishik M.I. (1964)).*

In the same way as for spaces $\mathbf{P}^{\alpha}(e^{-\gamma\cdot t})$ and $\mathbf{E}_{\alpha}(\gamma)$ we introduce the spaces $\mathbf{P}^{l,\frac{l}{2\bar{b}}}(e^{-\gamma\cdot t})$ and $\mathbf{E}_{l,\frac{l}{2\bar{b}}}(\gamma)$ in "Polycylinder" domains (for simplicity we bound ourselves with the case $l \in \mathbb{N}$).

By definition, $u \in \mathbf{P}^{l,\frac{l}{2\bar{b}}}(e^{-\gamma\cdot t})$, if $u(x,t)$ is defined in the domain $\Omega = G \times \mathbb{R}^m$, $(I = \mathbb{R}^m)$, $u = 0$ if at least one of t_j is negative and $e^{-\gamma\cdot t}u \in H^{l,\frac{l}{2\bar{b}}}(\Omega_I)$. We take the value $\left\|e^{-\gamma\cdot t}u\right\|_{l,\frac{l}{2\bar{b}}}$ for the norm in $\mathbf{P}^{l,\frac{l}{2\bar{b}}}(e^{-\gamma\cdot t})$.

Equivalently, $u(x,t) \in \mathbf{P}^{l,\frac{l}{2\bar{b}}}(e^{-\gamma\cdot t})$ if, for almost all $x \in G$, the function $u(x,t) \in \mathbf{P}^{\frac{l}{2\bar{b}}}(e^{-\gamma\cdot t})$ relatively to t and, for almost all $t \in (\bar{\mathbb{R}}_+)^m$, it belongs to $H^l(G)$ relatively to x and, finally, the norm $\left\|e^{-\gamma\cdot t}u(x,t)\right\|_{l,\frac{l}{2\bar{b}}}$ is finite.

Suppose $\lambda + 1/2 \ge 1$ is an integer. We define the space ${\mathbf{P}'}^{\lambda,\frac{\lambda}{2\bar{b}}}(e^{-\gamma\cdot t})$ as a space of functions $u(x',t)$ defined in the "Polycylinder" $\Omega' = \Gamma \times \mathbb{R}^m$, equal to zero if at least one of t_j is negative and such that $e^{-\gamma\cdot t}u \in H^{\lambda,\frac{\lambda}{2\bar{b}}}(\Omega'_I)$.

Now let $l \in \mathbb{N}$ and $\gamma_j \ge 0$. Denote by $\mathbf{E}_{l,\frac{l}{2\bar{b}}}(\gamma)$ a set of functions $U(x,p)$ defined for almost all $x \in G$ and $p :\ \mathrm{Re}\,p_j \ \ge \ \gamma_j$, $j = 1,...,m$, and possessing the following properties:

1). For all $p = \sigma + i\tau$ with $\mathrm{Re}\,p_j \ > \ \gamma_j$, $j = 1,...,m$ (i.e. $p \in \Pi_\gamma$), and for almost all p on hyperplanes $\{\sigma_j = \gamma_j\}$, $j = 1,...,m$ (i.e. $p \in \partial\Pi_\gamma$), we have $U(\cdot,p) \in H^l(G)$.

2). For almost all $x \in G$ we have $U(x,\cdot) \in \mathbf{E}_{\frac{l}{2\bar{b}}}(\gamma)$, and its generalized derivatives $D_x^\alpha U$ belong to $\mathbf{E}_0(\gamma)$ for $|\alpha| \le l$ (in particular, it means that

$U(x,p)$ is holomorphic in domain $\Pi_\gamma = \{p \in \mathbb{C}^m : \operatorname{Re} p_j >, \gamma_j , \ j=1,...,m\}$ for these x and has limit boundary values almost everywhere on $\partial\Pi_\gamma$).

3). The norm in $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ is defined by the formula

$$\| |U(x,p)| \|_{l,\frac{l}{2b}} = \int_{\Pi_{j=1}^m \{\sigma_j=\gamma_j\}} \Big[\|U\|^2_{l,G} + \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} \|U\|^2_{0,G} \Big] \, d\tau \ , \qquad (2.1.24)$$

where $p_j = \sigma_j + i\tau_j$.

For $\lambda = [\lambda] + 1/2$ the space $\mathbf{E}'_{\lambda,\frac{\lambda}{2b}}(\gamma)$ is defined as the space of functions $U(x',p)$, $x' \in \Gamma$ with the properties analogous to 1) -3).

The following Proposition is the analog of Proposition 2.1.1.

Proposition 2.1.5 *Suppose $\gamma \in \mathbb{R}^m$ is a vector with positive coordinates. Then Laplace transformation L (see (2.1.6)) defines a one-to-one mutually continuous correspondence of the spaces $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ and $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ $(l \in \mathbb{N} \cup \{0\})$, and the spaces $\mathbf{P}'^{\lambda,\frac{\lambda}{2b}}(e^{-\gamma\cdot t})$ and $\mathbf{E}'_{\lambda,\frac{\lambda}{2b}}(\gamma)$ $(\lambda + 1/2 \in \mathbb{N})$.*

We conclude this Section with some remarks about the properties of the above - defined spaces which are used below.

Remark 2.1.6 *The spaces $\mathbf{P}^\alpha(e^{-\gamma\cdot t})$, $\mathbf{E}_\alpha(\gamma)$, $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$, $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ are complete.*

Remark 2.1.7 *Let $U \in \mathbf{E}_0(\gamma)$. Then*

$$U(\sigma + i\tau) \longrightarrow U(\sigma_0 + i\tau)$$

in $L_2(\mathbb{R}^m)$ as $\sigma \to \sigma_0$, where some of (or even all) the coordinates of σ_0 may be equal to γ_j .

Remark 2.1.8 *Now let $U \in \mathbf{E}_\alpha(\gamma)$, then*

$$\|U\|^2_{\mathbf{E}_\alpha(\gamma)} = \sum_{j=1}^m \int \big|U(\gamma + i\tau)\big|^2 \big|\gamma_j + i\tau_j\big|^{2\alpha_j} \, d\tau \ . \qquad (2.1.25)$$

In the following Remark we reformulate the properties (2.1.17) - (2.1.23) of spaces $H^{l,\frac{l}{2b}}(\Omega_I)$, $H^{\lambda,\frac{\lambda}{2b}}(\Omega'_I)$ (see Remark 2.1.4).

Remark 2.1.9 *Let $u \in \mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ and $|\alpha| + 2b \cdot \beta \le l$, then*

1). Any given function $u \in \mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ has generalized derivatives

$$v(x,t) = D_x^\alpha \partial_t^\beta u(x,t) \in \mathbf{P}^{r,\frac{r}{2b}}(e^{-\gamma\cdot t}) \ , \qquad (2.1.26)$$

where $r = l - |\alpha| - 2b \cdot \beta \geq 0$, *and*

$$\|e^{-\gamma\cdot t} v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ \leq\ C\,\|e^{-\gamma\cdot t} u(x,t)\|_{l,\frac{l}{2\vec{b}}} \ , \qquad (2.1.27)$$

and moreover $\forall\,\alpha:\ |\alpha| \leq l$, *we have* $D_x^\alpha u \in \mathbf{P}^0\big(e^{-\gamma\cdot t}\big)$ *for almost all* $x \in G$.

2). The same properties are valid for boundary classes $\mathbf{P}'^{\,\lambda,\frac{\lambda}{2\vec{b}}}\big(e^{-\gamma\cdot t}\big)$, *where* $\lambda \geq 1/2$ *and* $r = \lambda + 1/2$. *A function* $v(x,t) \in \mathbf{P}^{r,\frac{r}{2\vec{b}}}\big(e^{-\gamma\cdot t}\big)$ *has boundary values* $v(x',t)$ *belonging to* $\mathbf{P}'^{\,\lambda,\frac{\lambda}{2\vec{b}}}\big(e^{-\gamma\cdot t}\big)$ *and*

$$\|e^{-\gamma\cdot t} v(x',t)\|'_{\lambda,\frac{\lambda}{2\vec{b}}} \ \leq\ C\,\|e^{-\gamma\cdot t} v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ . \qquad (2.1.28)$$

We end this Section with some Remarks concerning the properties of spaces $\mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ and $\mathbf{E}'_{\lambda,\frac{\lambda}{2\vec{b}}}(\gamma)$.

Remark 2.1.10 *Suppose* $U = U(x,p) \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$, *then*

$$\big\||U(x,p)|\big\|_{l,\frac{l}{2\vec{b}}} \ =\ \int_{\Pi_{j=1}^m\{\sigma_j=\gamma_j\}} \Big[\|U\|^2_{l,G} \,+\, \sum_{i=1}^m |p_i|^{\frac{l}{b_i}}\,\|U\|^2_{0,G}\Big]\,d\tau$$

$$=\ \sup_{p^o\in\Pi_\gamma} \int_{\Pi_{j=1}^m\{\sigma_j=\sigma_j^o\}} \Big[\|U\|^2_{l,G} \,+\, \sum_{i=1}^m |p_i|^{\frac{l}{b_i}}\,\|U\|^2_{0,G}\Big]\,d\tau \ , \qquad (2.1.29)$$

where $p_j = \sigma_j + i\tau_j$, $p_j^o = \sigma_j^o + i\tau_j^o$. *Also, this statement is valid for the spaces with primes. The norms in (2.1.29) in this case have to be taken with primes too.*

Remark 2.1.11 *As follows from the proof of Proposition 2.1.5 and Remark 2.1.9 the Laplace image of the generalized derivative in* t_j *is the multiplication by* p_j *and the operations of taking generalized derivatives in* x *and traces on* Γ *commute with Laplace transform (2.1.6). We conclude that for* $U \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ *with* $|\alpha| + 2b\cdot\beta \leq l$ *and* $l \in \mathbb{Z}$,

$$V(x,p) \ =\ p^\beta D_x^\alpha U(x,p) \ \in\ \mathbf{E}_{r,\frac{r}{2\vec{b}}}(\gamma) \ ,$$

where $r = l - |\alpha| - 2b\cdot\beta \geq 0$, *and*

$$\|V(x,p)\|_{r,\frac{r}{2\vec{b}}} \ \leq\ C\,\|U(x,p)\|_{l,\frac{l}{2\vec{b}}} \ ,$$

moreover,

$$\|V(x',p)\|'_{\lambda,\frac{\lambda}{2\vec{b}}} \ \leq\ C\,\|V(x,p)\|_{r,\frac{r}{2\vec{b}}} \ ,$$

where $\lambda = r - 1/2$ *and* $r \geq 1$.

Remark 2.1.12 *Suppose* $U = U(x,p) \in \mathbf{E}_{l,\frac{l}{2\overline{b}}}(\gamma)$, *then*

A). For $\operatorname{Re} p_j > \gamma_j$ *and* $\forall\, j \in 1,...,m$

$$\left\| U(x,p+\Delta p) - U(x,p) \right\|_{l,G} \longrightarrow 0 \quad \text{as} \quad \Delta p \to 0 \ . \tag{2.1.30}$$

B). Suppose there exists at least one j_0 *such that in* $p = \sigma^o + i\tau$ *we have* $\sigma^o_{j_0} = \operatorname{Re} p^o_{j_0} = \gamma_{j_0}$, *then there exists a sequence* $\sigma^{(n)} \to \sigma^o$ *such that*

$$\left\| U(x,\sigma^{(n)} + i\tau) - U(x,\sigma^o + i\tau) \right\|_{l,G} \longrightarrow 0 \quad \text{as} \quad n \to \infty \ . \tag{2.1.31}$$

2.2 Polyparabolic mixed problems in the spaces $\mathbf{P}^{l,\frac{l}{2\overline{b}}}(e^{-\gamma\cdot t})$

We consider an equation

$$A\big(x, D_x, \frac{\partial}{\partial t}\big)\, u(x,t)$$

$$= \frac{\partial^2 u}{\partial t_1 \partial t_2} - B(x,D_x)\frac{\partial u}{\partial t_1} - C(x,D_x)\frac{\partial u}{\partial t_2} + E(x,D_x)u \ = \ f(x,t) \tag{2.2.1}$$

in a "polycylinder" domain $\Omega_I = G \times \prod_{j=1}^{m}(0,T_j)$, where $0 < T_j \le \infty$, with boundary conditions

$$B_\mu\big(x, D_x, \frac{\partial}{\partial t}\big) u(x,t)\big|_{x=x'} \ = \ g_\mu(x',t) \tag{2.2.2}$$

on its "side surface" $\Omega'_I = \Gamma \times \prod_{j=1}^{m}(0,T_j)$, where $(x',t) \in \Omega'_I$, $\mu = 1,2$, A and B_μ are differential operators with coefficients being C^∞ - functions in $x \in \overline{G}$, and $\operatorname{ord} E = 4$, $\operatorname{ord} B = \operatorname{ord} C = 2$, $\operatorname{ord} B_\mu = k_\mu < 2 \ (\mu = 1,2)$. As usual, the order of an operator is defined as the highest of the orders of its terms and the order of the term

$$D^\alpha_x \partial^\beta_t u \ = \ D^{\alpha_1}_{x_1} \cdots D^{\alpha_n}_{x_n} \, \partial^{\beta_1}_{t_1} \partial^{\beta_2}_{t_2} \, u$$

is computed in the view of the rule $|\alpha| + 2\cdot\beta = \alpha_1 + \cdots + \alpha_n \ + 2(b_1\beta_1 + b_2\beta_2)$.

Also, we add initial conditions to the equations (2.2.1) - (2.2.2)

$$u\big|_{t_1=0} = \phi_1(x,t_2) \qquad u\big|_{t_2=0} = \phi_2(x,t_1) \ . \tag{2.2.3}$$

Suppose for the time being that $\phi_j = 0$ and $T_j = +\infty$ for $j = 1,2$. Then Laplace transformation in T_j transfers the problem (2.2.1) - (2.2.3) into the "stationary" problem

$$A(x,D_x,p)\, U(x,p) \ = \ F(x,p) \ , \quad x \in G \ , \tag{2.2.4}$$

$$B_\mu(x,D_x,p) U(x,p)\big|_{x=x'} \ = \ G_\mu(x',p) \ , \quad x' \in \Gamma \ , \tag{2.2.5}$$

where $\mu = 1,2$. Putting $p_j = q_j^2$, $j = 1,2$, we see that while p_j varies in the right half-plane $\operatorname{Re} p_j \geq 0$, q_j varies in its turn in the angle

$$Q_j \; = \; \left\{ q_j \in \mathbb{C} \, : \, |\arg q_j| \, \leq \, \frac{\pi}{4} \right\} . \tag{2.2.6}$$

Definition 2.2.1 *The problem (2.2.1) - (2.2.3) is called "polyparabolic" if the problem (2.2.4) - (2.2.5) with p_j substituted by into q_j^2 for $j = 1,2$ satisfies in the cone $Q = Q_1 \times Q_2$ Conditions 1.3.2 - 1.3.3. In this case we call the problem (2.2.4) - (2.2.5) "semibounded" (following Agranovich M.S., Vishik M.I. (1964)).*

In order to illustrate this definition we continue our Examples 1.3.4, 1.3.5.

Example 2.2.2 *Let $A_1(x,D)$ and $A_2(x,D)$ $(\operatorname{ord} A_j = 2, \; j = 1,2)$ be two elliptic operators with positive real leading parts. Consider an operator $\left(\frac{\partial}{\partial t_1} + A_1(x,D)\right)\left(\frac{\partial}{\partial t_1} + A_2(x,D)\right)$ with Dirichlet boundary conditions $B_\nu = \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}$, $\nu = 1,2$, and zero initial conditions (2.2.3).*

This problem is polyparabolic as the corresponding problem (2.2.4)-(2.2.5) (with p_j replaced by q_j^2 for $j = 1,2$) satisfies Conditions 1.3.2 - 1.3.3 in the cone $Q' = Q_1' \times Q_2'$, where $Q_j' = \left\{ q_j \in \mathbb{C} : |\arg q_j| \leq \frac{\pi - \varepsilon}{2} \right\}$, which includes the cone $Q = Q_1 \times Q_2$, where $Q_j = \left\{ q_j \in \mathbb{C} : |\arg q_j| \leq \frac{\pi}{4} \right\}$, for sufficiently small ε .

Example 2.2.3 *Let $A(x,D)$ be an elliptic operator* $(\operatorname{ord} A = 4)$ *with a positive real leading part. Consider an operator $\frac{\partial^2}{\partial t_1 \partial t_2} + A(x,D)$ with Dirichlet boundary conditions $B_\nu = \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}$, $\nu = 1,2$, and zero initial conditions (2.2.3).*

This problem is not polyparabolic *in the sense of Definition 2.2.1. Indeed, after substituting of p_j by q_j^2 $(j = 1,2)$ in the corresponding system (2.2.4) - (2.2.5) we obtain that, if p_j varies in the right half-plane $\operatorname{Re} p_j \geq 0$, q_j varies in its turn in the angle $Q_j = \left\{ q_j \in \mathbb{C} : |\arg q_j| \leq \frac{\pi}{4} \right\}$. And thus the ellipticity condition (Condition 1.3.2) is violated because our problem is elliptic only in the cone $Q' = Q_1' \times Q_2'$, where $Q_j' = \left\{ |\arg q_j| \leq \frac{\pi - \varepsilon}{4} \right\}$.*

In this section we consider "polyparabolic" mixed problems with $T_j = +\infty$ for $j = 1,2$ and with zero initial conditions (2.2.3), i.e. in the spaces $\mathbf{P}^{l,\frac{l}{2}}\left(e^{-\gamma \cdot t}\right)$. We start with elliptic "semibounded" problems (2.2.4) - (2.2.5) in the spaces $\mathbf{E}_{l,\frac{l}{2}}(\gamma)$.

Theorem 2.2.4 *Let $l \in \mathbb{Z}$ and $l \geq l_0 = \max(4, k_\mu + 1)$. Then there exists a vector $\gamma : \gamma_j > 0$, such that the "semibounded" boundary value problem*

(2.2.4) - (2.2.5) possesses the unique solution $U(x,p) \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$ *for all right-hand sides* $F(x,p) \in \mathbf{E}_{r,\frac{r}{2}}(\gamma)$ *and* $G_\mu(x',p) \in \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2}}(\gamma)$ *, where* $r = l-4$ *and* $\lambda_\mu = l - k_\mu - 1/2$ *. The solution* $U(x,p)$ *satisfies the two-sided a priori estimate*

$$C_1 \|U\|_{l,\frac{l}{2}} \;\le\; \|F\|_{r,\frac{r}{2}} \;+\; \sum_{\mu=1}^{2} \|G_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;\le\; C_2 \|U\|_{l,\frac{l}{2}} \;, \tag{2.2.7}$$

where the constants C_1 *and* C_2 *are independent of functions.*

Remark 2.2.5 *Under a solution* $U(x,p) \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$ *we understand a function from this space satisfying the problem for all* $p \in \Pi_\gamma$ *and for almost all* $p \in \partial\Pi_\gamma$ *.*

Proof (of Theorem 2.2.4). A). We start with the proof of estimate (2.2.7). Let $U \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$, so for all $p \in \operatorname{int}\Pi_\gamma$ and for almost all $p \in \partial\Pi_\gamma$ we have $U(\cdot,p) \in H^l(G)$ and we are in position to apply Propositions 1.3.1 and 1.3.6. Thus, for an arbitrary γ from the set

$$\mathcal{N} \;=\; \left\{\gamma \in (\mathbb{R}_+)^2 \;:\; \gamma_j^{\frac{1}{2}} \;\ge\; q^{(0)}\,,\; \sum_{j=1}^{2} \gamma_j^{\frac{1}{2}} \;\ge\; q^{(1)}\right\}$$

(here $q^{(0)} > 0$ can be chosen infinitely small, though $q^{(1)} > 0$ must be sufficiently large, cf. Proposition 1.3.6), the following estimate is valid

$$C_1 \|U\|_l^2 \;+\; \||p|||^l \|U\|_0^2 \le \|F\|_r^2 \;+\; \||p|||^r \|F\|_0^2$$

$$+ \sum_{\mu=1}^{2} \left[\|G_\mu\|'^2_{\lambda_\mu} \;+\; \||p|||^{\lambda_\mu} \|G_\mu\|'^2_0\right] \;\le\; C_1 \|U\|_l^2 \;+\; \||p|||^l \|U\|_0^2 \,. \tag{2.2.8}$$

Further, according to Remark 2.1.11,

$$F(x,p) \;=\; A(x,D_x,p)\,U(x,p) \;\in\; \mathbf{E}_{r,\frac{r}{2}}(\gamma) \tag{2.2.9}$$

and

$$G_\mu(x',p) \;=\; B_\mu(x,D_x,p)U(x,p)\big|_{x=x'} \;\in\; \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2}}(\gamma) \,. \tag{2.2.10}$$

Finally, integrating (2.2.8) over τ in the plane $\{\sigma_1 = \gamma_1, \sigma_2 = \gamma_2\}$ we obtain the inequalities which are equivalent to (2.2.7) in the view of Remark 2.1.10. As usual, the uniqueness of the solution is the consequence of (2.2.7).

B). We start the proof of the existence for smooth right-hand sides F and G_μ being described in Remark 2.1.6 D).

We fix $l \ge \max\{4, k_j+1\}$ and put $l_1 = l + [n/2] + 1$. For $\gamma \in \mathcal{N}$, we obtain in the view of Theorem 1.4.2 that, for all $p \in \overline{\Pi}_\gamma$, the problem (2.2.4)

- (2.2.5) possesses an unique solution $U \in H^l(G)$ for all right-hand sides $F \in H^{l-4}(G)$, $G_\mu \in H^{l-k_\mu-1/2}(\Gamma)$, and an unique solution $U \in H^{l_1}(G)$ for all $F \in H^{l_1-4}(G)$, $G_\mu \in H^{l_1-k_\mu-1/2}(\Gamma)$, with the estimate (2.2.8) being valid for l and l_1 .

We now prove that, if $U(\cdot,p) \in H^{l_1}(G)$ is the solution to (2.2.4) - (2.2.5) with the right-hand sides as in Remark 2.1.6 D), than it also belongs to $\mathbf{E}_{l,\frac{l}{2}}(\gamma)$.

It is sufficient to prove, that $U(x,\cdot) \in \mathbf{E}_{\frac{l}{2}}(\gamma)$ and $D_x^\alpha U(x,\cdot) \in \mathbf{E}_{\vec{0}}(\gamma)$ as $|\alpha| \le l$. First, we prove that the norm (2.1.24) is finite and then, that U and $D_x^\alpha U$ are holomorphic in Π_γ .

Evidently, for all $p \in \overline{\Pi}_\gamma$, it follows from $U \in H^{l_1}(G)$ that $U \in H^l(G)$ and hence $\|U\|_{l,\frac{l}{2}} \le \|U\|_{l_1,\frac{l_1}{2}} \le +\infty$, where the last inequality is obtained by integration of (2.2.8) (with l substituted by l_1) over the plane $\{\sigma_1 = \gamma_1, \sigma_2 = \gamma_2\}$.

Now it follows from Sobolev embedding theorem (see, for example, Egorov Yu.V., Shubin M.A. (1988)), that for $l_1 > l + n/2$ the functions U and $D_x^\alpha U$ ($|\alpha| \le l$) are continuous and $|U| \le C\|U\|_\nu$, $|D_x^\alpha U| \le C\|U\|_{l_1}$, where $\nu = l_1 - l$. Thus, for $p \in \overline{\Pi}_\gamma$ and $x \in G$ we obtain using (1.1.12) that

$$\int |||p|||^{\frac{l}{b}} |U|^2 \, d\tau \;\le\; C \int |||p|||^{\frac{l}{b}} \|U\|_k^2 \, d\tau$$

$$\le\; C \int \left[\|U\|_{l_1}^2 \;+\; |||p|||^{\frac{l_1}{b}} \|U\|_0^2 \right] d\tau \; .$$

Now, from (2.2.8) being integrated over the plane $\{\operatorname{Re} p_1 = \sigma_1, \operatorname{Re} p_2 = \sigma_2\}$ (and with $l = l_1$), we obtain the desired inequality:

$$\int \left|D_x^\alpha U\right|^2 d\tau \;\le\; C_1 \int \|U\|_{l_1}^2 \, d\tau \;\le\; C_2 \left[|||F|||_{r_1,\frac{r_1}{2}}^2 \;+\; \sum_{\mu=1}^{2} |||G_\mu|||'^{\,2}_{\lambda_{1,\mu},\frac{\lambda_{1,\mu}}{2}} \right] ,$$

where the last inequality holds in the view of Remark 2.1.8. Here $r_1 = l_1 - 4$ and $\lambda_{1,\mu} = l_1 - k_\mu - 1/2$. The derivatives $D_x^\alpha U$ are estimated in the same way.

C). In order to prove that U and $D_x^\alpha U$ ($|\alpha| \le l$) are holomorphic in Π_γ , it is enough to state their holomorphicity in each p_j in the view of Hartogs theorem (see Shabat B.V. (1985)). Differentiating (2.2.9) and (2.2.10) with respect to p_j , we obtain the appropriate problem for the unknown function $U_j' = \partial U / \partial p_j$:

$$A(x, D_x, p)\, U_j'(x,p) \;=\; F_j'(x,p) \;-\; A_j'(x, D_x, p)\, U(x,p) \qquad (2.2.11)$$

and

$$B_\mu(x, D_x, p) U_j'(x,p)\big|_\Gamma \;=\; G_\mu(x',p) \;-\; B'_{\mu_j}(x, D_x, p) U(x,p)\big|_\Gamma \;, \qquad (2.2.12)$$

where $\mu = 1, 2$, and by X'_j we denote the derivative $\partial X / \partial p_j$, and U is the above found solution to the problem (2.2.4) - (2.2.5).

As $U \in H^{l_1}(G)$, it follows from Proposition 1.3.1 that

$$A'_j U \in H^{r_1}(G) \quad B'_{\mu_j} U\big|_\Gamma \in H^{\lambda_{1,\mu}}(\Gamma) \, ,$$

and thus, the problem (2.2.11) - (2.2.12) has a unique solution $U'_j(x,p) \in H^{l_1}(G)$ for all $p \in \overline{\Pi}_\gamma$. We have to prove that the functions U'_j and $D^\alpha_x U'_j$ are really the derivatives of U and $D^\alpha_x U$ for all $p \in \Pi_\gamma$. It can be checked by definition and, hence, the functions $D^\alpha_x U$ are holomorphic for $p \in \Pi_\gamma$.

D). We now overcome to arbitrary right - hand sides F and G_μ from $\mathbf{E}_{r,\frac{r}{2}}(\gamma)$ and $\mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2}}(\gamma)$ respectively. In the view of Remark 2.1.6 D) there exist sequences of smooth functions F_n and G_{μ_n} converging to F and G_μ in respective spaces.

Let $U_n(x,p) \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$ be a solution to the problem (2.2.4) - (2.2.5) with right - hand sides F_n and G_{μ_n} . It follows from (2.2.7) that U_n form a fundamental sequence in $\mathbf{E}_{l,\frac{l}{2}}(\gamma)$. As this space is complete we have $U_n \to U \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$.

On the other hand, for all $p \in \overline{\Pi}_\gamma$ we can find a solution $U_0(x,p)$ with right - hand sides F and G_μ and such that $U_0(\cdot,p) \in H^l(G)$. Then the function $U_n - U_0$ is the solution to (2.2.4) - (2.2.5) with the right - hand sides $F_n - F$ and $G_{\mu_n} - G_\mu$. Applying the estimate (2.2.8) integrated over τ , we obtain

$$\int_{\sigma_1=\sigma_1^o} \int_{\sigma_2=\sigma_2^o} \left[\|U_n - U_0\|_l^2 \, + \, \||p|\|^l \|U_n - U_0\|_0^2 \right] d\tau \;\longrightarrow\; 0$$

as $n \to \infty$, where $\sigma^o = \operatorname{Re} p^o$ and $p^o \in \overline{\Pi}_\gamma$. Thus

$$\int_{\sigma_1=\sigma_1^o} \int_{\sigma_2=\sigma_2^o} \|U_n - U_0\|_0^2 \, d\tau \;\longrightarrow\; 0$$

as $n \to \infty$, and by definition

$$\int_{\sigma_1=\sigma_1^o} \int_{\sigma_2=\sigma_2^o} \|U_n - U\|_0^2 \, d\tau \;\longrightarrow\; 0$$

as $n \to \infty$, for this integral is not larger than the norm $\|U_n - U\|^2_{l,\frac{l}{2}}$ vanishing itself as $n \to \infty$.

Thus, for almost all $p \in \overline{\Pi}_\gamma$, we have $U(x,p) = U_0(x,p)$ and the above - consructed function U is the solution to the problem (2.2.4) - (2.2.5) for almost all $p \in \overline{\Pi}_\gamma$.

Finally, Remark 2.1.12 gives us an opportunity to state that $U(x,p)$ is the solution to the problem (2.2.4) - (2.2.5) for all $p \in \Pi_\gamma$ and for almost all p

from its boundary $\partial\Pi_\gamma$. $\triangleright$

We now overcome to the problem of solvability of our initial polyparabolic problem (2.2.1) - (2.2.2) in the spaces $\mathbf{P}^{l,\frac{l}{2}}\left(e^{-\gamma\cdot t}\right)$, i.e. with zero initial conditions (2.2.3).

Theorem 2.2.6 *Let* $l \in \mathbb{Z}$, *where* $l \geq l_0 = \max(4, k_\mu + 1)$. *Then there exists a vector* $\gamma : \gamma_j > 0$ *such that the "polyparabolic" problem (2.2.1) - (2.2.2) with zero initial conditions (2.2.3) has the unique solution* $u \in \mathbf{P}^{l,\frac{l}{2}}\left(e^{-\gamma\cdot t}\right)$ *for all right-hand sides*

$$f \in \mathbf{P}^{r,\frac{r}{2}}\left(e^{-\gamma\cdot t}\right) \ , \ \textit{and} \ \ g_\mu \in \mathbf{P}'^{\lambda_\mu,\frac{\lambda_\mu}{2}}\left(e^{-\gamma\cdot t}\right) \ ,$$

where $r = l - 4$, $\lambda_\mu = l - k_\mu - 1/2$. *The solution* $u(x,t)$ *satisfies the two-sided a priori estimate*

$$C_1 \left\|e^{-\gamma\cdot t}u\right\|_{l,\frac{l}{2}} \ \leq \ \left\|e^{-\gamma\cdot t}f\right\|_{r,\frac{r}{2}} \ + \ \sum_{\mu=1}^{2} \left\|e^{-\gamma\cdot t}g_\mu\right\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \ \leq \ C_2 \left\|e^{-\gamma\cdot t}u\right\|_{l,\frac{l}{2}} \ , \tag{2.2.13}$$

where the norms are taken over $G \times (\mathbb{R}_+)^2$ *and* $\Gamma \times (\mathbb{R}_+)^2$ *and the constants are independent of functions.*

Proof. Suppose $u(x,t) \in \mathbf{P}^{l,\frac{l}{2}}\left(e^{-\gamma\cdot t}\right)$ is a solution of (2.2.1) - (2.2.2). Then it follows from Remark 2.1.9 that in (2.2.1) - (2.2.2) $f \in \mathbf{P}^{r,\frac{r}{2}}\left(e^{-\gamma\cdot t}\right)$ and $g_\mu \in \mathbf{P}'^{\lambda_\mu,\frac{\lambda_\mu}{2}}\left(e^{-\gamma\cdot t}\right)$, where $r = l-4$ and $\lambda_\mu = l - k_\mu - 1/2$. Applying Laplace transform (2.1.6) in t to (2.2.1) - (2.2.2) we obtain in the force of Proposition 2.1.5 and Remark 2.1.11 the boundary value problem (2.2.4) - (2.2.5) with

$$U = Lu \in \mathbf{E}_{l,\frac{l}{2}}(\gamma) \ , \quad F = Lf \in \mathbf{E}_{r,\frac{r}{2}}(\gamma) \ , \quad G_\mu = Lg_\mu \in \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2}}(\gamma) \ . \tag{2.2.14}$$

The polyparabolicity of (2.2.1) - (2.2.2) means the "semiboundedness" of (2.2.4) - (2.2.5). Thus from Theorem 2.2.4 follows the existence of $\gamma \in (\mathbb{R}_+)^2$ such that under condition (2.2.14) the problem (2.2.4) - (2.2.5) possesses the unique solution $U \in \mathbf{E}_{l,\frac{l}{2}}(\gamma)$ satisfying the estimate (2.2.7). Applying to it the inverse Laplace transform and using once again Proposition 2.1.5 we overcome to the estimate (2.2.13) and the uniqueness of the solution of (2.2.1) - (2.2.2) follows.

The existence is the straightforeward consequence of the existence of the solution to the problem (2.2.4) - (2.2.5) and $u = L^{-1}U$ in the view of Proposition 2.1.5. $\triangleright$

We now overcome to the case of *nonhomogeneous initial conditions* (2.2.3). We denote by Ω the "two-sided polycylinder" $G \times \mathbb{R}^2$, by Ω_+ the "one-sided polycylinder" $G \times (\mathbb{R}_+)^2$ and by Ω' and Ω'_+ their "side-surfaces". We look for solutions $u(x,t)$ to the problem (2.2.1) - (2.2.3) such that $e^{-\gamma\cdot t}u(x,t) \in H^{l,\frac{l}{2}}(\Omega_+)$. With this end of view we suppose that

$$e^{-\gamma\cdot t}f(x,t) \in H^{\lambda,\frac{\lambda}{2}}(\Omega_+) , \quad \lambda = l-4 , \tag{2.2.15}$$

$$e^{-\gamma\cdot t}g_\mu(x',t) \in H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega'_+) , \quad \lambda_\mu = l-k_\mu-1/2 , \tag{2.2.16}$$

$$e^{-\gamma_2 t_2}\phi_1(x,t_2) , \quad e^{-\gamma_1 t_1}\phi_2(x,t_1) \in H^{l-1,\frac{l-1}{2}[j]}(G\times\mathbb{R}_+) . \tag{2.2.17}$$

Clearly that on the planes $t_j=0$ the right-hand sides of (2.2.1) - (2.2.3) have to satisfy the compatibility conditions. The simplest compatibility conditions are the ones with "zero":

Condition 2.2.7 *(Condition of Compatibility with zero).*

Suppose $\phi_1 = \phi_2 \equiv 0$. We demand that after the zero extension of the functions $f(x,t)$ and $g_\mu(x',t)$ into the domain where at least one of t_j is negative (i.e. $\mathbb{R}^2\setminus(\mathbb{R}_+)^2$), the following inclusions are satisfied:

$$e^{-\gamma\cdot t}f(x,t) \in H^{\lambda,\frac{\lambda}{2}}(\Omega) , \quad e^{-\gamma\cdot t}g_\mu(x',t) \in H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega') . \tag{2.2.18}$$

The above - formulated Condition means that the zero extension of the functions f and g_μ does not disturb their smoothness. For nonhomogeneous initial conditions we pose the following *General Compatibility Condition:*

Condition 2.2.8 *There exists a function $u_0(x,t)$ such that the following relations take place:*

$$e^{-\gamma\cdot t}u_0(x,t) \in H^{l,\frac{l}{2}}(\Omega_+) , \tag{2.2.19}$$

$$u_0\big|_{t_1=0} = \phi_1(x,t_2) \quad \text{and} \quad u_0\big|_{t_2=0} = \phi_2(x,t_1) . \tag{2.2.20}$$

where ϕ_j are the functions from (2.2.3) and satisfy (2.2.17). Moreover, if we put

$$f_0(x,t) = Au_0 \text{ in } \Omega_+ \text{ and } g_{\mu_0}(x',t) = B_\mu u_0\big|_{\Omega'_+}$$

for $\mu=1,2$, then after zero extension into $\mathbb{R}^2\setminus(\mathbb{R}_+)^2$, the functions $f-f_0$ and $g_\mu - g_{\mu_0}$ being the new right-hand sides of (2.2.1) - (2.2.3) are compatible with zero, i.e.

$$e^{-\gamma\cdot t}(f-f_0) \in H^{\lambda,\frac{\lambda}{2}}(\Omega) \tag{2.2.21}$$

and

$$e^{-\gamma\cdot t}(g_\mu - g_{\mu_0}) \in H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega') . \tag{2.2.22}$$

Theorem 2.2.9 *Let (2.2.1) - (2.2.3) be a polyparabolic problem,* $l \in \mathbb{Z}$ *such that* $l \geq l_0 = \max(4, k_\mu + 1)$ *and* $l/2 \in \mathbb{Z}$ *for* $j = 1, 2$ *. Then there exists a vector* γ *with* $\gamma_j > 0$ *such that for arbitrary right-hand sides* f, g_μ, ϕ_{j,ν_j} *satisfying the relations (2.2.15) - (2.2.17) and General Compatibility Condition, there exists such a solution* $u(x,t)$ *to the problem (2.2.1) - (2.2.3) that*

$$e^{-\gamma \cdot t} u(x,t) \in H^{l, \frac{l}{2}}(\Omega_+) .$$

Proof follows trivially from Theorem 2.2.6 and Condition 2.2.8. Indeed, suppose $v(x,t) = u(x,t) - u_0(x,t)$, then for the function v we have to deal with the problem

$$Av = f - f_0 \quad \text{in } \Omega_+ , \tag{2.2.23}$$

$$B_\mu v = g_\mu - g_{\mu 0} \quad \text{on } \Omega'_+ , \tag{2.2.24}$$

$$v\big|_{t_1=0} = v\big|_{t_2=0} = 0 . \tag{2.2.25}$$

As $f - f_0 \in \mathbf{P}^{l, \frac{l}{2}}(e^{-\gamma \cdot t})$ and $g_\mu - g_{\mu_0} \in \mathbf{P}'^{\lambda_\mu, \frac{\lambda_\mu}{2}}(e^{-\gamma \cdot t})$, then in the view of Theorem 2.2.6 there exists the unique function $v \in \mathbf{P}^{l, \frac{l}{2}}(e^{-\gamma \cdot t})$ satisfying (2.2.23) - (2.2.25). Finally, $u = v + u_0$ is the solution to our initial problem (2.2.1) - (2.2.3). ▷

In order to formulate the uniqueness theorem for the problem (2.2.1) - (2.2.3), we have to introduce the new additional condition.

In this way we deduce from (2.2.1) - (2.2.3) the boundary value problems for the operators $A_1 = \frac{\partial}{\partial t_2} - B(x, D_x)$ and $A_2 = \frac{\partial}{\partial t_1} - C(x, D_x)$:

$$A_j\big(x, D_x, \frac{\partial}{\partial t}[j]\big)\, w(x, t[j]) = \tilde{f}(x, t[j]) , \tag{2.2.26}$$

$$B_{\mu,j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big) w(x, t[j])\big|_{x=x'} = \tilde{g}_{\mu,j}(x', t[j]) , \tag{2.2.27}$$

$$w\Big|_{t_i=0} = \tilde{\phi}_i(x) \text{ ,for } \quad i \neq j , \tag{2.2.28}$$

where $i, j, \mu = 1, 2$, and we use the notation $w = w_j = \frac{\partial u}{\partial t_j}\big|_{t_j=0}$. Also, we denote by $\tilde{f}$ the sum of $f(x,0)$ and the replaced terms of $A\big(x, D_x, \frac{\partial}{\partial t}\big)u$, which do not contain the t_j-derivative $\frac{\partial u}{\partial t_j}$ with the subsequent restriction of these terms to the plane $t_j = 0$ and the substitution of the appropriate derivatives of u by $(x, t[j])-$ derivatives of ϕ_j in the way that

$$e^{-\gamma[j] \cdot t[j]} \tilde{f} \in H^{\lambda - 1, \frac{\lambda - 1}{2}[j]}\big(\Omega \cap \{t_j = 0\}\big) .$$

Firther, we denote by $\tilde{g}_{\mu,j}$ a linear combination of $(x, t[j])-$ derivatives of ϕ_j , $g_\mu\big|_{t_j=0}$ and $\frac{\partial g_\mu}{\partial t_j}\Big|_{t_j=0}$ in the way that

$$e^{-\gamma[j] \cdot t[j]} \tilde{g}_{\mu,j} \in H^{\lambda_\mu - 3, \frac{\lambda_\mu - 3}{2}[j]}\big(\Omega' \cap \{t_j = 0\}\big) .$$

At last, $\tilde{\phi}_i = \frac{\partial \phi_i}{\partial t_j}\Big|_{t_j=0} \in H^{l-4}(G)$. There are more boundary conditions (being compatible in the view of General Compatibility Condition 2.2.8) in (2.2.27) than is needed. We have to choose one of them to satisfy the following

Condition 2.2.10 *(or* **Inductive polyparabolicity condition** *).*
We say that the problem (2.2.1) - (2.2.3) is inductively polyparabolic *for* $t_j = \alpha_j = 0$ *if for both* $j = 1$ *and 2 the corresponding system (2.2.26) - (2.2.28) (after the sufficient choice of the boundary condition) is parabolic.*

Theorem 2.2.11 *Suppose the problem (2.2.1) - (2.2.3) is* inductively polyparabolic *and* $l > 6$ *additionally. Then under the conditions of Theorem 2.2.9 the solution* $u(x,t)$ *to the problem (2.2.1) - (2.2.3) is unique and satisfies the following two-sided a priori estimate*

$$C_1 \, \|e^{-\gamma \cdot t} u\|_{l,\frac{l}{2}} \;\leq\; \|e^{-\gamma \cdot t} f\|_{\lambda,\frac{\lambda}{2}} \;+$$

$$+ \sum_{\mu=1}^{2} \|e^{-\gamma \cdot t} g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;+\; \sum_{j=1}^{2} \|e^{-\gamma[j]\cdot t[j]} \phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j]}$$

$$\leq\; C_2 \, \|e^{-\gamma \cdot t} u\|_{l,\frac{l}{2}} \;, \tag{2.2.29}$$

where the norms are taken over Ω_+ , Ω'_+ *and* $G \times \mathbb{R}_+$ *respectively and the constants are independent of functions.*

Proof. As usual, we prove only the left one of the inequalities (2.2.29), which implies uniqueness. The right is a trivial consequence of inequalities (2.1.18) - (2.1.22) (see Remark 2.1.4).

The main problem here is to substitute the function $u_0(x,t)$ by a function $u_1(x,t)$ such that $e^{-\gamma \cdot t} u_1(x,t) \in H^{l,\frac{l}{2}}(\Omega_+)$, in the way that we can estimate its norm through the norms of functions f, g_μ, ϕ_j (unfortunately, we are not able to do it with the norm of $u_0(x,t)$ itself).

We prove the estimate (2.2.29) by reduction to the one-time case. If we have only one time, then our statement is contained in Theorem 10.1 from (Agranovich M.S., Vishik M.I. (1964)). Thus, we suppose the estimate is true for one time and the solution to (2.2.1) - (2.2.3) is unique in this case.

We wish to estimate the initial values of the derivatives

$$\phi_{j,\rho} \;:=\; \frac{\partial^\rho u}{\partial t_j^\rho}\Big|_{t_j=0} \qquad \text{for} \;\; \rho = 1, ..., \frac{l}{2} - 1$$

through the norms of f, g_μ, ϕ_j .

In this way for $w = \frac{\partial u}{\partial t_j}\Big|_{t_j=0}$ we obtain the system (2.2.26) - (2.2.28) which is parabolic with one time, and thus, by the above theorem from (Agranovich M.S., Vishik M.I. (1964)) this system possesses the unique solution $w(x,t[j])$, and

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} w\|_{l-3,\frac{l-3}{2}[j]} \;\le\; \|e^{-\gamma[j]\cdot t[j]} \tilde{f}\|_{\lambda-1,\frac{\lambda-1}{2}[j]} \; +$$

$$+ \sum_{\mu=1}^{2} \|e^{-\gamma[j]\cdot t[j]} \tilde{g}_{\mu,j}\|'_{\lambda_\mu-3,\frac{\lambda_\mu-3}{2}[j]} \; + \; \sum_{i=1}^{2} \|\tilde{\phi}_i\|_{l-4} \; . \qquad (2.2.30)$$

In the view of the structure of functions $\tilde{f}, \tilde{g}_{\mu,j}, \tilde{\phi}_i$ and the inequalities (2.1.18) - (2.1.22) (see Remark 2.1.4) we can overwrite (2.2.30) without waves (and with different C_1) :

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} w\|_{l-3,\frac{l-3}{2}[j]} \;\le\; \|e^{-\gamma\cdot t} f\|_{\lambda,\frac{\lambda}{2}} \; +$$

$$+ \sum_{\mu=1}^{2} \|e^{-\gamma\cdot t} g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \; + \; \sum_{j=1}^{2} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\nu_j}(x,t[j])\|_{l-1,\frac{l-1}{2}[j]} \; . \qquad (2.2.31)$$

For the functions $\phi_{j,\rho}$ with $\rho = 2, ..., \frac{l}{2} - 1$ we obtain the same systems (2.2.26) - (2.2.28) but with different right-hand sides. Thus, they can be uniquely defined too and we are in position to overwrite (2.2.31) for them:

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\rho}\|_{l-\rho-1,\frac{l-\rho-1}{2}[j]} \;\le\; \|e^{-\gamma\cdot t} f\|_{\lambda,\frac{\lambda}{2}} \; +$$

$$+ \sum_{\mu=1}^{2} \|e^{-\gamma\cdot t} g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \; + \; \sum_{j=1}^{2} \|e^{-\gamma[j]\cdot t[j]} \phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j]} \; . \qquad (2.2.32)$$

Now, remembering that $v = u - u_0$ satisfies (2.2.23) - (2.2.25) and belongs to $\mathbf{P}^{l,\frac{l}{2}}\big(e^{-\gamma\cdot t}\big)$, we see that $\frac{\partial^\rho v}{\partial t_j^\rho}\Big|_{t_j=0} = 0$ for $\rho = 0, ..., \frac{l}{2} - 1$, and thus $\frac{\partial^\rho u_0}{\partial t_j^\rho}\Big|_{t_j=0} = \phi_{j,\rho}$ for $\rho = 0, ..., \frac{l}{2} - 1$. Further, in the view of Remark 2.1.4 4), there exists $u_1(x,t)$ such that

$$e^{-\gamma\cdot t} u_1(x,t) \;\in\; H^{l,\frac{l}{2}}\big(\Omega_+\big) \; , \qquad (2.2.33)$$

$$\frac{\partial^\rho u_1}{\partial t_j^\rho}\Big|_{t_j=0} \; = \; \phi_{j,\rho} \quad \text{for} \;\; \rho = 0, ..., \frac{l}{2} - 1 \;, \tag{2.2.34}$$

and

$$\|e^{-\gamma \cdot t} u_1(x,t)\|_{l, \frac{l}{2}} \; \le \; C \sum_{j=1}^{2} \sum_{\rho=0}^{\frac{l}{2}-1} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\rho}\|_{l-\rho-1, \frac{l-\rho-1}{2}[j]} \;. \tag{2.2.35}$$

As u_0 also satisfies (2.2.33) - (2.2.34), we have $u_1 - u_0 \in \mathbf{P}^{l,\frac{l}{2}}\big(e^{-\gamma \cdot t}\big)$ and thus

$$A[u_1 - u_0] \; = \; Au_1 \; - \; Au_0 \; \in \; \mathbf{P}^{\lambda,\frac{\lambda}{2}}\big(e^{-\gamma \cdot t}\big) \;,$$

and

$$B_\mu[u_1 - u_0]\big|_{\Omega'} \in \; \mathbf{P'}^{\,\lambda_\mu, \frac{\lambda_\mu}{2}}\big(e^{-\gamma \cdot t}\big) \;, \quad \mu = 1, 2 \,.$$

It means that we can substitute the function u_0 by u_1 without violating Compatibility Condition 2.2.8, but now we gain the estimate

$$C_1 \, \|e^{-\gamma \cdot t} u_1\|_{l,\frac{l}{2}} \; \le \; \|e^{-\gamma \cdot t} f\|_{\lambda, \frac{\lambda}{2}} \; +$$

$$+ \; \sum_{\mu=1}^{2} \|e^{-\gamma \cdot t} g_\mu\|'_{\lambda_\mu, \frac{\lambda_\mu}{2}} \; + \; \sum_{j=1}^{2} \|e^{-\gamma[j]\cdot t[j]} \phi_j\|_{l-1, \frac{l-1}{2}[j]} \;. \tag{2.2.36}$$

Now let $u(x,t)$ be an arbitrary solution to the problem (2.2.1) - (2.2.3) from the previous theorem. Computing its derivatives $\frac{\partial^\nu u}{\partial t_j^\nu}\Big|_{t_j=0}$ for $\nu = 2, ..., l/2{-}1$, in the same way, as we computed $\phi_{j,\rho}$ for $\rho = 2, ..., l/2 - 1$, we see in the force of inductive polyparabolicity assumption that $\frac{\partial^\nu (u-u_1)}{\partial t_j^\nu}\Big|_{t_j=0} = 0$, for $\nu = 0, ..., \frac{l}{2} - 1$.

It implies $u{-}u_1 \in \mathbf{P}^{l,\frac{l}{2}}\big(e^{-\gamma \cdot t}\big)$ and thus, the estimate (2.2.13) from Theorem 2.2.6 is true for it:

$$C \, \|e^{-\gamma \cdot t}(u - u_1)\|_{l,\frac{l}{2}} \; \le \; \|e^{-\gamma \cdot t}(f - f_1)\|_{\lambda, \frac{\lambda}{2}} \; + \; \sum_{\mu=1}^{2} \|e^{-\gamma \cdot t}(g_\mu - g_{\mu,1})\|'_{\lambda_\mu, \frac{\lambda_\mu}{2}} \;, \tag{2.2.37}$$

where $f_1 = Au_1$ in Ω_+ and $g_{\mu,1} = B_\mu u_1$ on Ω'_+ . As , evidently,

$$\|e^{-\gamma \cdot t} f_1\|_{\lambda, \frac{\lambda}{2}} \; + \; \sum_{\mu=1}^{2} \|e^{-\gamma \cdot t} g_{\mu,1}\|'_{\lambda_\mu, \frac{\lambda_\mu}{2}} \; \le \; C \, \|e^{-\gamma \cdot t} u_1\|_{l, \frac{l}{2}} \;,$$

it follows that

$$C\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2}} \;\le\; \|e^{-\gamma\cdot t}f\|_{\lambda,\frac{\lambda}{2}} + \sum_{\mu=1}^{2}\|e^{-\gamma\cdot t}g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} + \|e^{-\gamma\cdot t}u_1\|_{l,\frac{l}{2}}\,.$$

Applying (2.2.36) we obtain the desired estimate (2.2.29). ▷

We now consider an example of inductively polyparabolic problems.

Example 2.2.12 *Let* $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ *be as in (2.2.1) and suppose that together with Dirichlet boundary conditions in* x *and Goursat initial conditions in* t *it forms the polyparabolic problem*

$$\begin{aligned} A\big(x, D_x, \frac{\partial}{\partial t}\big)u \;&=\; f\;,\\ u\big|_\Gamma \;=\; h_0(t_1,t_2,x')\;,\quad \frac{\partial u}{\partial\vec{n}}\Big|_\Gamma \;&=\; h_1(t_1,t_2,x')\;,\\ u\big|_{t_1=0} \;=\; \phi_1(t_2,x)\;,\quad u\big|_{t_2=0} \;&=\; \phi_2(t_1,x)\;. \end{aligned} \tag{2.2.38}$$

Then the respective systems (2.2.26) - (2.2.28) with the chosen Dirichlet boundary conditions are

$$\begin{aligned} \Big(\frac{\partial}{\partial t_2} - B\Big)w_1 \;&=\; C\frac{\partial\phi_1}{\partial t_2} - E\phi_1 + f_1 \;:=\; \tilde f_1\;,\\ w_1\big|_\Gamma \;=\; h_{0,t_1}(0,t_2,x')\;,\qquad w_1\big|_{t_2=0} \;&=\; \phi_{2,t_1}(0,x)\;, \end{aligned}$$

and

$$\begin{aligned} \Big(\frac{\partial}{\partial t_1} - A\Big)w_2 \;&=\; B\frac{\partial\phi_2}{\partial t_1} - E\phi_2 + f_2 \;:=\; \tilde f_2\;,\\ w_2\big|_\Gamma \;=\; h_{0,t_2}(t_1,0,x')\;,\qquad w_2\big|_{t_1=0} \;&=\; \phi_{1,t_2}(0,x)\;, \end{aligned}$$

where $w_1 = \frac{\partial u}{\partial t_1}\big|_{t_1=0}$, $w_2 = \frac{\partial u}{\partial t_2}\big|_{t_2=0}$, $f_1 = f(0,t_2,x)$, $f_2 = f(t_1,0,x)$ *and* z_{t_1} *means* $\frac{\partial z}{\partial t_1}$,

If A *and* B *are elliptic operators these systems are parabolic, and thus, the initial problem (2.2.38) is inductively polyparabolic and possesses the unique solution due to Theorem 2.2.11.*

Remark 2.2.13 *While proving the last theorem we found out that for inductively polyparabolic problem the initial values of derivatives* $\frac{\partial^\kappa u}{\partial t_j^\kappa}$ *for* $\kappa = 0,...,\frac{l}{2}-1$ *can be defined from initial conditions and equation (2.2.1), i.e. we do not need the function* $u(x,t)$ *itself. Moreover, if* u_0 *is the function from General Compatibility Condition 2.2.8, then it satisfies the same initial conditions as* u *and, thus, we obtain the following Compatibility Criterion: for* $\phi_{j,\rho}$ *with* $\rho = 0,...,\frac{l}{2}-1$ *being built from initial conditions (2.2.3) and equation (2.2.1), we construct according to Remark 2.1.4, item 4) their extension* u_0 *such that* $e^{-\gamma\cdot t}u_0 \in H^{l,\frac{l}{2}}(\Omega_+)$. *Then our Criterion consists of the execution of (2.2.21), (2.2.22).*

Though Example 2.2.3 gives us a nonpolyparabolic problem in the sense of Definition 2.2.1, it also shows that Inductive Parabolicity Condition is sufficient, but not necessary and systems (2.2.26) - (2.2.28) in this case turn into Cauchy problems for ordinary differential equations with constant coefficients.

Example 2.2.14 *Consider the equation*

$$\frac{\partial^2 u}{\partial t_1 \partial t_2} \; + \; a(x,D)u \; = \; f$$

(cf. Friedman A., Littman W. (1963)), where $a(x,D)$ *is an elliptic operator of order 4 with boundary and initial conditions*

$$u\big|_{\Gamma} \; = \; h_0(t_1,t_2,x') \; , \quad \frac{\partial u}{\partial x_n}\Big|_{\Gamma} \; = \; h_1(t_1,t_2,x') \; ,$$

$$u\big|_{t_1=0} \; = \; \phi_1(t_2,x) \; , \quad u\big|_{t_2=0} \; = \; \phi_2(t_1,x) \; .$$

Here we obtain the respective systems (2.2.26) - (2.2.28) in the form

$$\frac{\partial w_1}{\partial t_2} \; = \; f_1 - a(x,D)\phi_1 \; , \qquad w_1\big|_{t_2=0} \; = \; \phi_{2,t_1}(0,x) \; ,$$

and

$$\frac{\partial w_2}{\partial t_1} \; = \; f_2 - a(x,D)\phi_2 \; , \qquad w_2\big|_{t_1=0} \; = \; \phi_{1,t_2}(0,x) \; .$$

They are the Cauchy problems for simplest ordinary differential equations with respect to w_1 *and* w_2 *, i.e.* w_1 *and* w_2 *can be estimated through the functions* f *,* h_0 *,* h_1 *,* ϕ_1 *and* ϕ_2 *(possibly with increased* γ *in appropriate* $\mathbf{P}^{l,\frac{l}{2}}(e^{-\gamma\cdot t})$ *) via inequalities (2.1.18) - (2.1.22).*

In general, we observe that in all the cases where (2.2.26) - (2.2.28) are Cauchy problems for ordinary differential equations with constant coefficients, we obtain the desired uniqueness in our functional classes automatically. It is not so in the variable coefficients case. For example, if we obtain for w_1 or w_2 first order linear equations of the form $\frac{\partial w}{\partial t_2} + a(x,t)w \; = \; f(x,t)$ (this is exactly the case of Brish - Yurchuk equation (0.1.3), (0.1.4)), then we have to demand the function $a(x,t_1,t_2)$ to be bounded from below in t_2 uniformly in (x,t_1) , otherwise the solution does not fall into $\mathbf{P}^{l,\frac{l}{2}}(e^{-\gamma\cdot t})$ even with increased γ .

2.3 The case of a finite polycylinder

Once again we consider the mixed problem (2.2.1) - (2.2.3):

$$A\big(x, D_x, \frac{\partial}{\partial t}\big)\, u(x,t)$$

$$= 3D \ \frac{\partial^2 u}{\partial t_1 \partial t_2} - B(x, D_x)\frac{\partial u}{\partial t_1} - C(x, D_x)\frac{\partial u}{\partial t_2} + E(x, D_x)u \ = 3D \ f(x,t) \quad (2.3.1)$$

$$B_\mu\big(x, D_x, \frac{\partial}{\partial t}\big)u(x,t)\Big|_{x=3Dx'} = 3D \ g_\mu(x',t) \ , \quad \text{for } \mu = 3D1, 2 \, , \qquad (2.3.2)$$

$$u\big|_{t_1=3D0} = 3D\phi_1(x, t_2) \qquad u\big|_{t_2=3D0} = 3D\phi_2(x, t_1) \ . \qquad (2.3.3)$$

But now we consider equation (2.3.1) in the domain $\Omega_T \ = 3D\, G \times \prod_{j=3D1}^{2}(0, T_j)$, where $T_j < +\infty$ and $T = 3D(T_1, T_2)$, with boundary conditions (2.3.2) on its side surface $\Omega'_T \ = 3D\, \Gamma \times \prod_{j=3D1}^{2}(0, T_j)$ and the initial conditions (2.3.3) on polycylinders $\Omega_T[j] = 3DG \times (0, T_\rho)$, where $j, \rho = 3D1, 2$ and $\rho \neq j$. The functions in (2.3.1) - (2.3.3) will be considered in the following spaces

$$u(x,t) \ \in \ H^{l,\frac{l}{2}}\big(\Omega_T\big) \ , \qquad (2.3.4)$$

$$f(x,t) \ \in \ H^{\lambda,\frac{\lambda}{2}}\big(\Omega_T\big) \ \text{ for } \ \lambda \ = 3D\, l - 4 \ , \qquad (2.3.5)$$

$$g_\mu(x',t) \ \in \ H^{\lambda_\mu,\frac{\lambda_\mu}{2}}\big(\Omega'_T\big) \ \text{ for } \ \lambda_\mu \ = 3D\, l - k_\mu - 1/2, \qquad (2.3.6)$$

where $\mu = 3D1, 2$, and

$$\phi_{j,\nu_j}(x, t[j]) \ \in \ H^{l-1,\frac{l-1}{2}[j]}\big(\Omega_T[j]\big) \ . \qquad (2.3.7)$$

Here k_μ are the orders of operators B_μ , $\frac{l}{2}[j] \ = 3D\, l/2$,

$$l, \frac{l}{2} \ \in \ \mathbb{Z} \ \text{ and } \ l \ \geq \ \max(4, k_\mu) \ . \qquad (2.3.8)$$

We denote by $\tilde{\Omega}_T$, $\tilde{\Omega}'_T$, $\tilde{\Omega}_T[j]$ the corresponding domains, where the times t_i for $i \ = 3D1, 2$ are extended to the sets of negative numbers, i.e. $t_i \in (-\infty \, , T_i)$. Then we are in position to overwrite Compatibility Conditions 2.2.7, 2.2.8 in the following form:

Condition 2.3.1 *(Condition of Compatibility with zero).*
Suppose $\phi_j(x, t[j]) \ \equiv \ 0$ *for* $j \ = 3D\, 1, 2$. *We demand that after the zero extension of functions* $f(x,t)$ *and* $g_\mu(x',t)$ *into the domain where at least one of* t_j *is negative (i.e.* $\mathbb{R}^2 \setminus (\mathbb{R}_+)^2$), *the following inclusions are satisfied:*

$$f(x,t) \ \in \ H^{\lambda,\frac{\lambda}{2}}\big(\tilde{\Omega}_T\big) \ , \qquad g_\mu(x',t) \ \in \ H^{\lambda_\mu,\frac{\lambda_\mu}{2}}\big(\tilde{\Omega}'_T\big) \ ,$$

where $\mu = 3D1, 2$.

The above Condition means that the zero extension of the functions f and g_μ does not disturb their smoothness. If $\phi_j \neq 0$ we pose the following *General Compatibility Condition:*

Condition 2.3.2 *There exists a function* $u_0(x,t)$ *such that the following relations take place:*

$$u_0(x,t) \in H^{l,\frac{l}{2}}(\Omega_T) , \quad u_0\big|_{t_j=3D0} = 3D\ \phi_j(x,t[j])$$

for $j = 3D1,2$, *where* ϕ_j *are the functions from (2.3.3) and satisfy (2.3.7) for* $j = 3D1,2$. *Moreover, if we put*

$$f_0(x,t) = 3D\ Au_0 \ \text{ in } \ \Omega_+ \ \text{ and } \ g_{\mu 0}(x',t) = 3D\ B_\mu u_0\big|_{\Omega'_+}$$

for $\mu = 3D1,2$, *then after zero extension into* $\mathbb{R}^2 \setminus (\mathbb{R}_+)^2$, *the functions* $f - f_0$ *and* $g_\mu - g_{\mu 0}$ *for* $\mu = 3D1,2$, *being the new right-hand sides of (2.3.1) - (2.3.3) are compatible with zero:*

$$f - f_0 \in H^{\lambda,\frac{\lambda}{2}}(\tilde{\Omega}_T) , \qquad g_\mu - g_{\mu_0} \in H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\tilde{\Omega}'_T) .$$

Theorem 2.3.3 *Let (2.3.1)-(2.3.3) be a polyparabolic problem and* $l \in \mathbb{Z}$ *be an integer satisfying (2.3.8). Then for arbitrary right-hand sides* f, g_μ, ϕ_j *satisfying the relations (2.3.5) - (2.3.7) and General Compatibility Condition, there exists a solution* $u(x,t)$ *to the problem (2.3.1) - (2.3.3) such that*

$$u(x,t) \in H^{l,\frac{l}{2}}(\Omega_T) .$$

If, moreover, the problem (2.3.1)-(2.3.3) is inductively polyparabolic and $l > 6$, *then the above solution* $u(x,t)$ *is unique and satisfies the following two-sided a priori estimate*

$$C_1 \|u\|_{l,\frac{l}{2}} \le \|f\|_{\lambda,\frac{\lambda}{2}} + \sum_{\mu=3D1}^{2} \|g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} +$$

$$+ \sum_{j=3D1}^{2} \|\phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j]} \le C_2 \|u\|_{l,\frac{l}{2}} , \tag{2.3.9}$$

where the norms are taken over Ω_T , Ω'_T *and* $\Omega_T[j]$ *respectively, the constants are independent of functions.*

Proof. 1). We prove the existence first. Starting with a stationary problem to be obtained from (2.3.1)-(2.3.3) (with zero initial conditions) by means of Laplace transformation, we fix a vector $\gamma \in (\mathbb{R}_+)^2$ such that the boundary value problem

$$A(x,D_x,p)\,U(x,p) = 3D\ F(x,p) , \quad x \in G , \tag{2.3.10}$$

$$B_\mu(x, D_x, p)U(x,p)\big|_{x=3Dx'} = 3D\ G_\mu(x',p)\ , \quad x' \in \Gamma\ , \tag{2.3.11}$$

where $\mu = 3D1, 2$, is uniquely solvable. Now, in the view of Remark 2.1.4, section 5), the functions $e^{-\gamma\cdot t} f$, $e^{-\gamma\cdot t} g_\mu$, $e^{-\gamma\cdot t} u_0$, together with f, g_μ, u_0 belonging to $H^{\lambda,\frac{\lambda}{2}}(\Omega_T)$, $H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega'_T)$, $H^{l,\frac{l}{2}}(\Omega_T)$ respectively, may be continued to the domains with $t_j > T_j$. Thus, we obtain the functions $e^{-\gamma\cdot t} f^*$, $e^{-\gamma\cdot t} g^*_\mu$, $e^{-\gamma\cdot t} u^*_0$ belonging to the spaces $H^{\lambda,\frac{\lambda}{2}}(\Omega_+)$, $H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega'_+)$, $H^{l,\frac{l}{2}}(\Omega_+)$ respectively, where $\Omega_+ = 3DG \times (\mathbb{R}_+)^2$, $\Omega'_+ = 3D\Gamma \times (\mathbb{R}_+)^2$.

Consequently, we obtain a polyparabolic problem over Ω_+ with the right-hand sides f^*, g^*_μ, ϕ_j and u^*_0 providing the execution of Compatibility Condition 2.2.8. Applying Theorem 2.2.9 we conclude that there exists a solution u^* to the problem and $e^{-\gamma\cdot t} u^* \in H^{l,\frac{l}{2}}(\Omega_+)$. Then $u(x,t) = 3D\, u^*(x,t)\big|_{\Omega_T}$ is the solution to the initial problem (2.3.1)-(2.3.3) and $u \in H^{l,\frac{l}{2}}(\Omega_T)$.

2). To prove the uniqueness we need some additional facts about spaces $\mathbf{P}^{a,\frac{a}{2}}(e^{-\gamma\cdot t})$ ($a \in \mathbb{Z}_+$ for Ω_+ and $a = 3D[a] + 1/2 > 0$ for Ω'_+).

Proposition 2.3.4 *Let* $u \in \mathbf{P}^{a,\frac{a}{2}}(e^{-\gamma\cdot t})$ *, then* $u = 3D0$ *while* $t_j < T_j$ *for* $j = 3D1$ *or 2 if and only if* $U = 3DLu$ *can be written in the form* $U = 3De^{-p_j T_j} U_1$ *, where* $U_1 \in \mathbf{E}_{a,\frac{a}{2}}(\gamma)$.

Proposition 2.3.5 *Consider the problem (2.3.1)-(2.3.3) in* Ω_+ *and suppose* $\phi_j = 3D0$ *,* $f \in \mathbf{P}^{\lambda,\frac{\lambda}{2}}(e^{-\gamma\cdot t})$ *,* $g_\mu \in \mathbf{P}^{\lambda_\mu,\frac{\lambda_\mu}{2}}(e^{-\gamma\cdot t})$ *, where* $\gamma \in (\mathbb{R}_+)^2$ *is such a vector, that this problem has a unique solution* $u \in \mathbf{P}^{l,\frac{l}{2}}(e^{-\gamma\cdot t})$.

Then, if $f, g_\mu = 3D0$ *while* $t_j < T_j$ *for both* $j = 3D1$ *and 2, we obtain that* $u = 3D0$ *for* $t \in \prod_{j=3D1=C1}^{2}[0, T_j)$.

We now return to the proof of Theorem 2.3.3.

In order to prove the uniqueness of the solution to the problem (2.3.1)-(2.3.3), we suppose that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ to the problem with the same right-hand sides f, g_μ, ϕ_j . Reviewing the proof of Theorem 2.2.11 we see that for $t_j = 3D0$

$$\frac{\partial^\nu u_1}{\partial t_j^\nu} = 3D\ \frac{\partial^\nu u_2}{\partial t_j^\nu} \quad \text{while} \quad \nu = 3D0, ..., \frac{l}{2} - 1\ . \tag{2.3.12}$$

Thus, an extension u^* of the function $u = 3Du_1 - u_2$ into Ω_+ (see Remark 2.1.4, sect.5)), such that $e^{-\gamma\cdot t} u^* \in H^{l,\frac{l}{2}}(\Omega_+)$, indeed belongs to $\mathbf{P}^{l,\frac{l}{2}}(e^{-\gamma\cdot t})$ (see (2.3.12)). We put

$$f^* = 3D\ A\big(x, D_x, \frac{\partial}{\partial t}\big)\, u^* \quad \text{in} \quad \Omega_+ \tag{2.3.13}$$

and

$$g^*_\mu = 3D\ B_\mu\big(x, D_x, \frac{\partial}{\partial t}\big) u^*\big|_{\Omega'_+}\ . \tag{2.3.14}$$

The functions f^* and g^*_μ belong to the spaces $\mathbf{P}^{\lambda,\frac{\lambda}{2}}\left(e^{-\gamma\cdot t}\right)$ and $\mathbf{P}^{\lambda_\mu,\frac{\lambda_\mu}{2}}\left(e^{-\gamma\cdot t}\right)$ respectively and are equal to zero for $t_j < T_j$ where $j = 3D1$ and 2.

Hence, it follows from Proposition 2.3.5 that $u^* = 3D0$ for $t \in \prod_{j=3D1}^{2}[0,T_j)$, that is, $u_1 = 3Du_2$ in Ω_T and the uniqueness follows.

3). The last fact we need to prove is the first inequality in the estimate (2.3.9). Let $u \in H^{l,\frac{l}{2}}(\Omega_T)$ be the solution to the problem (2.3.1)-(2.3.3) and let u^* be its extension to Ω_+ such that $e^{-\gamma\cdot t}u^* \in H^{l,\frac{l}{2}}(\Omega_+)$.

Let $Au^* = 3D\,f^*$ in Ω_+ and $B_\mu u^*\big|_{\Omega'_+} = 3D\,g^*_\mu$, $\mu = 3D1,2$. The functions f^*, g^*_μ coinside with f, g_μ for $t \in \prod_{j=3D1}^{2}[0,T_j)$ and

$$e^{-\gamma\cdot t}f^* \in H^{\lambda,\frac{\lambda}{2}}(\Omega_+)\ ,\quad e^{-\gamma\cdot t}g^*_\mu \in H^{\lambda_\mu,\frac{\lambda_\mu}{2}}(\Omega'_+)\ .$$

We now construct extensions f^{**} and g^{**}_μ of f and g_μ in the view of Remark 2.1.4, sect.5) and in the way that

$$\left\|e^{-\gamma\cdot t}f^{**}\right\|_{\lambda,\frac{\lambda}{2},\Omega_+} \le C_1\left\|e^{-\gamma\cdot t}f\right\|_{\lambda,\frac{\lambda}{2},\Omega_T} \le C_2\|f\|_{\lambda,\frac{\lambda}{2},\Omega_T}\,, \tag{2.3.15}$$

and

$$\left\|e^{-\gamma\cdot t}g^{**}_\mu\right\|_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'_+} \le C_1\left\|e^{-\gamma\cdot t}g_\mu\right\|_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'_T} \le C_2\|g_\mu\|_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'_T}\,. \tag{2.3.16}$$

We denote by u^{**} the solution to the problem (2.3.1)-(2.3.3) with the right-hand sides f^{**}, g^{**}_μ ϕ_j. As $f^* = 3Df^{**} = 3Df$ and $g^*_\mu = 3Dg^{**}_\mu = 3Dg_\mu$ on $\prod_{j=3D1}^{2}[0,T_j)$ we are able to apply Proposition 2.3.5 to the function $u^* - u^{**}$ and thus, obtain that $u = 3Du^* = 3Du^{**}$ for $t \in \prod_{j=3D1}^{2}[0,T_j)$.

Applying Theorem 2.2.11 to u^{**} we have

$$C\,\|e^{-\gamma\cdot t}u^{**}\|_{l,\frac{l}{2},\Omega_+} \le \|e^{-\gamma\cdot t}f^{**}\|_{\lambda,\frac{\lambda}{2},\Omega_+} +$$

$$+\sum_{\mu=3D1}^{2}\|e^{-\gamma\cdot t}g^{**}_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'_+} + \sum_{j=3D1}^{2}\|e^{-\gamma[j]\cdot t[j]}\phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j],G}\,.$$

Finally, applying (2.3.15), (2.3.16) and obvious inequalities

$$\|u\|_{l,\frac{l}{2},\Omega_T} \le C_1\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2},\Omega_T} \le C_2\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2},\Omega_+}\,,$$

we obtain the desired estimate

$$C\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2}} \;\leq\; \|e^{-\gamma\cdot t}f\|_{\lambda,\frac{\lambda}{2}} \;+$$

$$+\;\sum_{\mu=3D1}^{2}\|e^{-\gamma\cdot t}g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;+\; \sum_{j=3D1}^{2}\|e^{-\gamma[j]\cdot t[j]}\phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j]}\;.$$

$\triangleright$

Corollary 2.3.6 *The following statement is true in the view of Proposition 2.3.5 and Theorem 2.3.3.*

Suppose $\phi_j \equiv 0$ *for* $j = 3D1,2$ *. Then for arbitrary* f, g_μ *, being compatible with zero, there exists a unique compatible with zero solution* $u \in H^{l,\frac{l}{2}}(\Omega_T)$ *to the problem (2.3.1)-(2.3.3) satisfying (2.3.9) with* $\phi_j \equiv 0$ *.*

In the rest of this section we show how to substitute the norms in (2.3.9) by equivalent ones in the way to obtain the constants C_1 and C_2 independent of $T = 3D(T_1, T_2)$ with fixed $T_j \;:\; 0 \leq T_j \;\leq \text{const}$.

Suppose $u(x,t) \in H^{l,\frac{l}{2}}(\Omega_T)$. The norm

$$\langle u\rangle_{l,\frac{l}{2}} \;= 3D\;\langle u\rangle_{l,\frac{l}{2},\Omega_T} \;= 3D\,\inf\|v\|_{l,\frac{l}{2},\Omega_+}\;, \tag{2.3.17}$$

is equivalent to the initial one. Here the infimum is taken over all functions $v(x,t) \in H^{l,\frac{l}{2}}(\Omega_+)$ coinsiding with u almost everywhere in Ω_T . The norms $\langle\cdot\rangle'_{l,\frac{l}{2},\Omega_T}$ over Ω'_T are defined in the same way. The equivalence of the norms $\langle\cdot\rangle_{l,\frac{l}{2},\Omega_T}$ and $\|\cdot\|_{l,\frac{l}{2},\Omega_T}$ is rather trivial. Indeed, if $v \in H^{l,\frac{l}{2}}(\Omega_+)$ and $u = 3Dv$ on Ω_T , it is clear, that

$$\|u\|_{l,\frac{l}{2},\Omega_T} \;= 3D\;\|v\|_{l,\frac{l}{2},\Omega_T} \;\leq\; \|v\|_{l,\frac{l}{2},\Omega_+}\;,$$

and thus,

$$\|u\|_{l,\frac{l}{2}} \;\leq\; \langle u\rangle_{l,\frac{l}{2}}.$$

On the other hand, using section 5) of Remark 2.1.4, we obtain

$$\langle u\rangle_{l,\frac{l}{2}} \;\leq\; \|Lu\|_{l,\frac{l}{2},\Omega_+} \;\leq\; C\|u\|_{l,\frac{l}{2}}\;.$$

Proposition 2.3.7 *Under the conditions of Theorem 2.3.3 it is possible to overwrite inequalities (2.3.9) in terms of norms (2.3.17)*

$$C_1\,\langle u\rangle_{l,\frac{l}{2}} \;\leq\; \langle f\rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=3D1}^{2}\langle g_\mu\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;+$$

$$+\;\sum_{j=3D1}^{2}\|\phi_j(x,t[j])\|_{l-1,\frac{l-1}{2}[j]} \;\leq\; C_2\,\langle u\rangle_{l,\frac{l}{2}}\;, \tag{2.3.18}$$

where the constants C_1 *and* C_2 *are independent of* T *.*

2.4 Generalizations to the case of time - dependent operators

Suppose the operators A (i.e. B, C, E) and B_μ in (2.3.1) - (2.3.2) have coefficients being infinitely smooth functions in (x,t) in the closed *finite* polycylinder $\overline{\Omega_T}$, i.e.

$$A(t) \ \equiv \ A\big(x,t,D_x,\frac{\partial}{\partial t}\big) \ , \quad B_\mu(t) \ \equiv \ B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big) \tag{2.4.1}$$

for $\mu = 1,2$. The problem (2.3.1) - (2.3.3) with such operators is called polyparabolic in Ω_T if the problem to be obtained from it by means of freezing the values $t_j = \alpha_j$ in the coefficients of operators A and B_μ is polyparabolic in the sense of Definition 2.2.1 for all fixed α_j , $0 \le \alpha_j \le T_j$, in other words the operators

$$A\big(x,\alpha,D_x,q^{\vec{2}}\big) \ , \quad B_\mu\big(x,\alpha,D_x,q^{\vec{2}}\big) \ ,$$

where $q^{\vec{2}} \ = \ (q_1^2,q_2^2)$, define the elliptic boundary value problem with parameters, i.e. the problem satisfying in the cone $\big\{|\arg q_j| \ \le \ \frac{\pi}{4} \ , \ j = 1,2\big\}$ the Conditions 1.3.2, 1.3.3.

The problem (2.3.1) - (2.3.3) with operators (2.4.1) is called inductively polyparabolic in Ω_T , if for all fixed $\alpha \in \prod_{j=1}^2 [0,T_j)$ operators (2.4.1) form the inductively polyparabolic system in the sense of Condition 2.2.10 for $t = \alpha$ ($t_j = \alpha_j$ for $j = 1,2$).

The compatibility conditions do not differ from Conditions 2.3.1, 2.3.2.

Theorem 2.4.1 *Theorem 2.3.3 remains valid for operators* $A\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ *and* $B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ *forming the polyparabolic problem in* Ω_T *(resp. inductively polyparabolic problem in* Ω_T *for the validity of (2.3.9) with the consequent uniqueness).*

Proof. 1). As we need many polycylindrical domains, we introduce new special notations for them:

$$\Omega[\alpha,\beta] \ = \ G \times \prod_{j=1}^{2} (\alpha_j,\beta_j) \ , \quad \Omega'[\alpha,\beta] \ = \ \Gamma \times \prod_{j=1}^{2} (\alpha_j,\beta_j) \ ,$$

where $\alpha_j \ge -\infty$, $\beta_j \le +\infty$, and, for example, $\Omega_T = \Omega[0,T]$, $\Omega'_T = \Omega'[0,T]$.

We say that $u(x,t) \in H^{l,\frac{l}{2}}\big(\Omega[\alpha,\beta]\big)$ is compatible with zero for $t = \alpha$, if after zero extension into the domain where at least one of $t_j < \alpha_j$ we obtain the function $\tilde{u} \in H^{l,\frac{l}{2}}\big(\Omega[-\infty,\beta]\big)$.

We start the proof of the theorem with the uniqueness, that is, with the estimate (2.3.9). As usual, the second inequality in (2.3.9) is trivial and we have to prove only the first one in our case.

Suppose $0 \le \alpha_j \le T_j$ and $\alpha_j < \beta_j < T_j^1$, where $T_j < T_j^1 < T_j + \varepsilon$ for sufficiently small ε , and the coefficients of operators $A(t)$ and $B_\mu(t)$ are smoothly extended to $\Omega[0, T^1]$.

As follows from Proposition 2.3.7, for $u \in H^{l,\frac{l}{2}}\big(\Omega[\alpha, \beta]\big)$ with sufficiently large l , we have

$$C_1 \langle u \rangle_{l,\frac{l}{2}} \;\le\; \langle A(\alpha)u \rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=1}^{2} \langle B_\mu(\alpha)u \rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;+$$

$$+\; \sum_{j=1}^{2} \|u(x, t[j], \alpha_j)\|_{l-1,\frac{l-1}{2}[j]} \;\le\; C_2 \langle u \rangle_{l,\frac{l}{2}} \;, \tag{2.4.2}$$

where $A(\alpha) \;=\; A\big(x, \alpha, D_x, \frac{\partial}{\partial t}\big)$, $B_\mu(\alpha) \;=\; B_\mu\big(x, \alpha, D_x, \frac{\partial}{\partial t}\big)$, $u(x, t[j], \alpha_j) \;=\; u(x,t)\big|_{t_j=\alpha_j}$, the norms are taken over $\Omega[\alpha, \beta]$, $\Omega'[\alpha, \beta]$, and $\Omega\big[\alpha[j], \beta[j]\big]$, and the constants C_1 , C_2 are independent of α, β and u .

Indeed, only independence of β was proved in Proposition 2.3.7. Independence of α follows trivially if we examine the proofs of Theorems 2.2.6, 2.2.9, 2.2.11, 2.3.3, and Proposition 2.3.7 and observe that the estimates obatined there are independent of α (in the view of Propositions 2.3.4, 2.3.5).

2). We now wish to substutute $A(\alpha)$ and $B_\mu(\alpha)$ in (2.4.2) by $A(t)$ and $B_\mu(t)$. First, we prove it for a small "polycylinder shell" $\Omega[\alpha, \beta]$ with $\beta_j - \alpha_j \le \delta < \varepsilon$ and $u(x,t)$ being compatible with zero for $t_j = \alpha_j$, i.e. for $j = 1, 2$ the relation $u(x, t[j], \alpha_j) \;=\; 0$ holds.

For such functions we obtain in the view of (2.4.2) that

$$C \langle u \rangle_{l,\frac{l}{2}} \;\le\; \langle A(\alpha)u \rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=1}^{2} \langle B_\mu(\alpha)u \rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;,$$

and thus, in $\Omega[\alpha, \beta]$ we have

$$C \langle u \rangle_{l,\frac{l}{2}} \;\le\; \langle A(t)u \rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=1}^{2} \langle B_\mu(t)u \rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;+\; \mathcal{R}_{\alpha\beta}(u) \;, \tag{2.4.3}$$

where

$$\mathcal{R}_{\alpha\beta}(u) \;=\; \big\langle [A(\alpha) - A(t)]u \big\rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=1}^{2} \big\langle [B_\mu(\alpha) - B_\mu(t)]u \big\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;. \tag{2.4.4}$$

Writing the expressions for operators $A(\alpha) - A(t)$ and $B_\mu(\alpha) - B_\mu(t)$

$$A(\alpha) - A(t) \;=\; \sum_{|r|+2s \le 4} a_{rs}(x,t) D_x^r \partial_t^s \;, \tag{2.4.5}$$

$$B_\mu(\alpha) - B_\mu(t) \;=\; \sum_{|r|+2s\le k_\mu} b_{\mu rs}(x,t) D_x^r \partial_t^s \;, \tag{2.4.6}$$

we see that $a_{rs}(x,t)$, $b_{\mu rs}(x,t) \in C^\infty\big(\Omega[\alpha,T_1]\big)$ and are equal to zero for $t=\alpha$. In order to control their growth for large $|t|$, we shall change them for $t_j > \alpha_j + \delta$, where $j=1,2$, considering instead of a_{rs} and $b_{\mu rs}$ the functions $\tilde a_{rs} = a_{rs}\tilde\chi(t)$ and $\tilde b_{\mu rs} = b_{\mu rs}\tilde\chi(t)$, where $\tilde\chi(t) = \prod_{j=1}^2 \chi(t_j)$ and $\chi(t_j) \in C^\infty(\bar{\mathbb{R}}_+)$ is the cut-off function with the properties: $\chi(t_j)=1$ for $\alpha_j \le t_j \le \alpha_j+\delta$, $\chi(t_j)=0$ for $t_j \ge \alpha_j + 2\delta$ and $\big|\partial_{t_j}^{\nu_j}\chi(t_j)\big| \le C\delta^{1-\nu_j}$ for $\nu_j \ge 1$ (for the construction of such a function see Sect.28 in the book (Shubin M.A. (1978))). We put

$$\varepsilon(\delta) \;=\; \max_{\Omega[\alpha,\alpha+\delta]} \big\{ |D_x^N a_{rs}| \,,\; |D_x^N b_{\mu rs}| \big\} \;,\; \text{where } \; |N| \le N_1 \;,$$

and

$$M \;=\; \max_{\Omega[0,T^1]} \big\{ |D_x^N \partial_t^\nu a_{rs}| \,,\; |D_x^N \partial_t^\nu b_{\mu rs}| \big\} \;,\; \text{where } \; |N| + |\nu| \le N_1$$

for fixed $N_1 \in \mathbb{N}$, and observe that

$$|D_x^N \tilde a_{rs}| \,,\; |D_x^N \tilde b_{\mu rs}| \;\le\; \varepsilon(\delta) \tag{2.4.7}$$

for $|N| \le N_1$, and

$$|D_x^N \partial_t^\nu \tilde a_{rs}| \,,\; |D_x^N \partial_t^\nu \tilde b_{\mu rs}| \;\le\; CM\delta^{1-|\nu|} \tag{2.4.8}$$

for $|N|+|\nu| \le N_1$, $|\nu| \ge 1$, and $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Using the notation $u_{rs} = D_x^r \partial_t^s u$ we are in position to estimate the terms in (2.4.4):

$$\big\langle [A(\alpha) - A(t)]u \big\rangle_{\lambda,\frac{\lambda}{2},\Omega[\alpha,\beta]} \;\le\; \Big\| \sum_{|r|+2s\le 4} \tilde a_{rs} u_{rs} \Big\|_{\lambda,\frac{\lambda}{2},\Omega[\alpha,\infty]} \tag{2.4.9}$$

and

$$\big\langle [B_\mu(\alpha) - B_\mu(t)]u \big\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega[\alpha,\beta]} \;\le\; \Big\| \sum_{|r|+2s\le k_\mu} \tilde b_{\mu rs} u_{rs} \Big\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\alpha,\infty]} \;\le$$

$$\le\; C \sum_{|r|+2s\le k_\mu} \|\tilde b_{\mu rs} u_{rs}\|_{l-k_\mu,\frac{l-k_\mu}{2},\Omega'[\alpha,\infty]} \;, \tag{2.4.10}$$

where the constant C is resulting from the embedding theorem in $\Omega[\alpha,\infty]$ and thus, is independent of α,β and functions.

We now wish to estimate the right-hand sides of (2.4.9) and (2.4.10). Starting with (2.4.9) we see that

$$\|\tilde a_{rs} u_{rs}\|^2_{\lambda,\frac{\lambda}{2},\Omega[\alpha,\infty]} \;=\; \sum_{|\sigma|\le\lambda} \big\| D_x^\sigma [\tilde a_{rs} u_{rs}] \big\|^2_{0,0} + \sum_{\tau:\,2\tau\le\lambda} \big\| \partial_t^\tau [\tilde a_{rs} u_{rs}] \big\|^2_{0,0} \,, \tag{2.4.11}$$

where the norms in the right-hand side are taken over $\Omega[\alpha,\infty]$. Using (2.4.7) we obtain

$$\sum_{|\sigma|\le\lambda} \left\| D_x^\sigma[\tilde{a}_{rs}u_{rs}]\right\|_{0,0}^2 \;\le\; C\,[\varepsilon(\delta)]^2\|u_{rs}\|_{|\sigma|,0}^2 \;\le\; C\,|\varepsilon(\delta)|^2\|u\|_{l,\frac{l}{2}}^2 \;. \qquad (2.4.12)$$

Further, for $|\tau|\ge 1$

$$\left\|\partial_t^\tau[\tilde{a}_{rs}u_{rs}]\right\|_{0,0}^2 \;\le\; C \sum_{|\tau_1|+|\tau_2|=|\tau|} \left\|\partial_t^{\tau_1}\tilde{a}_{rs}\partial_t^{\tau_2}u_{rs}\right\|_{0,0}^2 \;.$$

The terms with $\tau_2 = 0$ are estimated as in (2.4.12). For $\tau_2 \ne 0$ we have $\partial_t^{\tau_1}\tilde{a}_{rs} = 0$ if $t_j > \alpha_j + 2\delta$ and according to (2.4.8)

$$\left\|\partial_t^{\tau_1}\tilde{a}_{rs}\partial_t^{\tau_2}u_{rs}\right\|_{0,0}^2 \;\le\; C^2M^2\delta^{2-2|\tau_1|}\int_G\int_{\alpha_1}^{\alpha_1+2\delta}\int_{\alpha_2}^{\alpha_2+2\delta}\left|\partial_t^{\tau_2}u_{rs}\right|^2 dxdt \;. \qquad (2.4.13)$$

We now remind that $u(x,t)$ is compatible with zero for $t_j = \alpha_j$ and use the following property of such functions: for a smooth function v on $\mathbb{R}$ which is compatible with zero for $t=\alpha\in\mathbb{R}$ we have $v(t) = \int_\alpha^t v'(\theta)\,d\theta$, and thus for $t\in[\alpha,\alpha+2\delta]$:

$$|v(t)| \;\le\; \left[\int_\alpha^{\alpha+2\delta}|v'(\theta)|\,d\theta\right]^2 \;\le\; 2\delta\int_\alpha^{\alpha+2\delta}|v'(\theta)|^2\,d\theta \;.$$

Integrating over t we obtain

$$\int_\alpha^{\alpha+2\delta}|v(t)|^2\,dt \;\le\; 4\delta^2\int_\alpha^{\alpha+2\delta}|v'(\theta)|^2\,d\theta \;.$$

Applying this inequality to $\partial_t^{\tau_2}u_{rs}$ in (2.4.13), we get after integrating over x that

$$\int_G\int_{\alpha_1}^{\alpha_1+2\delta}\int_{\alpha_2}^{\alpha_2+2\delta}\left|\partial_t^{\tau_2}u_{rs}\right|^2 dxdt \;\le\; 2^{2|\tau_1|}\,\delta^{2|\tau_1|}\int_G\int_{\alpha_1}^{\alpha_1+2\delta}\int_{\alpha_2}^{\alpha_2+2\delta}\left|\partial_t^{\tau}u_{rs}\right|^2 dxdt \;\le$$

$$\le\; 2^{2|\tau_1|}\,\delta^{2|\tau_1|}\left\|\partial_t^\tau u_{rs}\right\|_{0,0,\Omega[\alpha,\infty]}^2 \,,$$

and thus

$$\left\|\partial_t^{\tau_1}\tilde{a}_{rs}\partial_t^{\tau_2}u_{rs}\right\|_{0,0}^2 \;\le\; CM^2\delta^2\left\|\partial_t^\tau u_{rs}\right\|_{0,0}^2 \;\le\; CM^2\delta^2\,\|u\|_{l,\frac{l}{2}}^2 \;. \qquad (2.4.14)$$

Applying (2.4.12) and (2.4.14) to (2.4.11) we obtain

$$\left\|\tilde{a}_{rs}u_{rs}\right\|^2_{\lambda,\frac{\lambda}{2}} \;\le\; C\left[(\varepsilon(\delta))^2+\delta^2\right]\|u\|^2_{l,\frac{l}{2}} \tag{2.4.15}$$

and, by analogy,

$$\left\|\tilde{b}_{\mu rs}u_{rs}\right\|^2_{\lambda_\mu,\frac{\lambda_\mu}{2}} \;\le\; C\left[(\varepsilon(\delta))^2+\delta^2\right]\|u\|^2_{l,\frac{l}{2}} \tag{2.4.16}$$

with the constants being independent of α,β and u . We now see from (2.4.4), (2.4.9), (2.4.10), (2.4.15) and (2.4.16), that

$$\mathcal{R}_{\alpha\beta}(u) \;\le\; \varepsilon_1(\delta)\,\|u\|^2_{l,\frac{l}{2},\Omega[\alpha,\infty]} \;\le\; 2\varepsilon_1(\delta)\,\langle u\rangle^2_{l,\frac{l}{2},\Omega[\alpha,\beta]} \tag{2.4.17}$$

where $\varepsilon_1(\delta)$ is independent of α,β and u and vanishes as $\delta\to 0$ $(\alpha_j\le\beta_j\le\alpha_j+\delta)$.

In this way for $\varepsilon_1(\delta)<\frac14$ we obtain that $\mathcal{R}_{\alpha\beta}(u)\le\frac12\langle u\rangle^2_{l,\frac{l}{2},\Omega[\alpha,\beta]}$, and in the view of (2.4.3) we get the desired inequality

$$C\,\langle u\rangle_{l,\frac{l}{2}} \;\le\; \langle A(t)u\rangle_{\lambda,\frac{\lambda}{2}} \;+\; \sum_{\mu=1}^{2}\langle B_\mu(t)u\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2}} \tag{2.4.18}$$

in $\Omega[\alpha,\alpha+\delta]$.

We now fix $\delta>0$ such that in the "shell" $\Omega[\alpha,\alpha+\delta]$ with $0\le\alpha_j\le\alpha_j+\delta\le T_j$ the inequality (2.4.18) takes place and suppose $r_j\delta<T_j\le(r_j+1)\delta$, where $r_j\in\mathbb{Z}$. We cut the domain $\Omega[0,T]$ into shells $\Omega[\vec{i}]:=\Omega[\delta\vec{i},\delta(\vec{i}+\vec{1})]$, where $\vec{i}=(i_1,i_2)$, $\vec{i}+\vec{1}=(i_1+1,i_2+1)$, where $0\le i_j\le r_j$ and for $i_j=r_j$ we put $i_j+1=T_j$. Thus, the last shell in each row may be smaller than the others. Also, we denote by $\Omega'[\vec{i}]$ the "side" surface of the shell $\Omega[\vec{i}]$ and by

$$G[\vec{i};0] \;=\; \cup_{j=1}^2 G[\vec{i};j;0]\ ,\ \text{and}\ \ G[\vec{i};1] \;=\; \cup_{j=1}^2 G[\vec{i};j;1]$$

its "lower" and "upper" bases, where

$$G[\vec{i};j;0] \;=\; \left\{(x,t)\in\Omega[\vec{i}]\;:\;t_j=i_j\delta\right\}$$

and

$$G[\vec{i};j;1] \;=\; \left\{(x,t)\in\Omega[\vec{i}]\;:\;t_j=(i_j+1)\delta\right\}\ .$$

As for the fixed shell the norms $\|\cdot\|$ and $\langle\cdot\rangle$ are equivalent then, in the view of (2.4.18), we obtain

$$C\,\|u\|_{l,\frac{l}{2},\Omega[\vec{i}]} \;\le\; \|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{i}]} \;+\; \sum_{\mu=1}^{2}\|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{i}]}$$

for functions $u \in H^{l,\frac{l}{2}}\left(\Omega[\vec{i}]\right)$ being compatible with zero on $G[\vec{i};0]$. Finally, for all $u \in H^{l,\frac{l}{2}}\left(\Omega[\vec{i}]\right)$ we obtain in the view of inductive polyparabolicity the two-sided a priori estimate

$$C\,\|u\|_{l,\frac{l}{2},\Omega[\vec{i}]} \;\leq\; \|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{i}]} \;+\; \sum_{\mu=1}^{2}\|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{i}]} \;+$$

$$+\;\sum_{j=1}^{2}\|u(x,t[j],i_j\delta)\|_{l-1,\frac{l-1}{2}[j],G[\vec{i};j;0]}\;, \tag{2.4.19}$$

where the constant C is independent of u and $i_j = 0,1,...,r_j$.

3). We now overcome to the proof of the estimate (2.3.9) in our case.

Let $u \in H^{l,\frac{l}{2}}\left(\Omega[0,T]\right)$, then $u \in H^{l,\frac{l}{2}}\left(\Omega[\vec{0}]\right)$ (here all $i_j = 0$ for all $j = 1,2$) and $u \in H^{l,\frac{l}{2}}\left(\Omega[(1,0,...,0)]\right)$, where $i_1 = 1$ and $i_2 = 0$. In these shells we have (2.4.19) with $\vec{i} = (0,0)$ and $\vec{i} = (1,0)$ respectively. In the view of Remark 2.1.4, item 3)

$$\|u(x,t[j],(i_j+1)\delta)\|_{l-1,\frac{l-1}{2}[j],G[\vec{i};j;1]} \;\leq\; C\,\|u\|_{l,\frac{l}{2},\Omega[\vec{i}]}\;. \tag{2.4.20}$$

¿From (2.4.20) and (2.4.19) with $\vec{i} = (0,0)$ and in the view of the obvious property of the sets G : $G[(1,0);1;0] = G[(0,0);1;1]$ we obtain that

$$C\,\|u(x,\vec{0}[1],\delta)\|_{l_\nu,1,\frac{l_\nu,1}{2}[1],G[\vec{0};1;1]} \;\leq\; \|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{0}]} \;+$$

$$\sum_{\mu=1}^{2}\|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{0}]} \;+\; \sum_{j=1}^{2}\|u(x,t[j],0)\|_{l-1,\frac{l-1}{2}[j],G[\vec{0};j;0]}\;.$$

Now, as obviously,

$$\|u\|_{l,\frac{l}{2},\Omega[\vec{0},\delta(2,0)]} \;\leq\; \|u\|_{l,\frac{l}{2},\Omega[\vec{0}]} \;+\; \|u\|_{l,\frac{l}{2},\Omega[(1,0)]}$$

and on the other hand, for $\vec{i} = \vec{0}$ and $(1,0)$ in polycylinders $\Omega[\vec{0},\delta(2,0)] = G \times [0,2\delta] \times [0,\delta]$ and $G[\vec{0}\cup(1,0);j;0] = \Omega[\vec{0}[j],\delta(2,0)[j]]$ we have

$$\|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{i}]} \;\leq\; \|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{0},\delta(2,0)]}\;,$$

$$\|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{i}]} \;\leq\; \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{0},\delta(2,0)]}$$

and

$$\|u(x,t[j],0)\|_{l-1,\frac{l-1}{2}[j],G[\vec{i};j;0]} \;\leq\; \|u(x,t[j],0)\|_{l-1,\frac{l-1}{2}[j],G[\vec{0}\cup(1,0);j;0]}\;.$$

And thus, we overcome to the desired inequality in the joined shell $[\vec{0}, \delta(2,0)]$ unifying the inequalities (2.4.19) for $\vec{i} = \vec{0}$ and $(1,0)$:

$$C\,\|u\|_{l,\frac{l}{2},\Omega[\vec{0},\delta(2,0)]} \;\leq\; \|A(t)u\|_{\lambda,\frac{\lambda}{2},\Omega[\vec{0},\delta(2,0,\dots,0)]} \;+$$

$$\sum_{\mu=1}^{2} \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2},\Omega'[\vec{0},\delta(2,0)]} \;+\; \sum_{j=1}^{2} \|u(x,t[j],0)\|_{l-1,\frac{l-1}{2}[j],\Omega[\vec{0}[j],\delta(2,0)[j]]} \;. \tag{2.4.21}$$

Continuing to move over $i_1 = 2, \dots, r_1$ we obtain (2.4.21) in the strip $\Omega[\vec{0},(T_1,0)]$. Afterwards we continue the process by adding a new strip $\Omega[\vec{0},(T_1,\delta)]$ (in the view of (2.4.20)) and combining the appropriate inequalities (2.4.21) for them into one. Then we combine the strips untill we obtain (2.4.21) in the whole polycylinder $\Omega[0,T]$. It means, that (2.3.9) is proved together with the uniqueness of the problem.

4). We start the proof of the existence again in a small shell $\Omega[\alpha,\beta]$. Denote by $\mathcal{A}(t)$ an operator which is mapping the functions $u \in H^{l,\frac{l}{2}}\big(\Omega[\alpha,\beta]\big)$ being compatible with zero for $t_j = \alpha_j$ into vectors

$$(f,g) \;=\; (f,g_1,g_2) \;=\; \big(A(t), B_1(t)\big|_{\Omega'}, B_2(t)\big|_{\Omega'}\big)$$

with elements being compatible with zero for $t_j = \alpha_j$ and lying in the space

$$\mathcal{H}^l\big(\Omega[\alpha,\beta]\big) \;:=\; H^{\lambda,\frac{\lambda}{2}}\big(\Omega[\alpha,\beta]\big) \;\times\; \prod_{\mu=1}^{2} H^{\lambda_\mu,\frac{\lambda_\mu}{2}}\big(\Omega'[\alpha,\beta]\big)\,.$$

Again we use the norms $\langle\cdot\rangle$. For the norm $\langle f,g\rangle$ of a vector $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ we take the sum of the norms $\langle\cdot\rangle$ of respective components.

Also, we note that the vectors which are compatible with zero for $t_j = \alpha_j$ form a closed subspace in $\mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$.

We now prove (cf. Sect.2 of this proof) that there exists δ such that in the shell $[\alpha,\beta]$ with $\beta_j - \alpha_j \leq \delta$ the operator $\mathcal{A}(t)$ possesses the right inverse $\mathcal{A}^{-1}(t)$ being defined on the subspace of vectors $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ which are compatible with zero for $t_j = \alpha_j$. The operator $\mathcal{A}^{-1}(t)$ maps this subspace boundedly into the subspace of functions $u \in H^{l,\frac{l}{2}}\big(\Omega[\alpha,\beta]\big)$ being also compatible with zero for $t_j = \alpha_j$.

Indeed, in the view of Corollary 2.3.6, in our shell exists a bounded inverse $\mathcal{A}^{-1}(\alpha)$ to the operator $\mathcal{A}(\alpha)$. This operator is defined on vectors $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ which are compatible with zero for $t_j = \alpha_j$, and maps them boundedly to functions $u \in H^{l,\frac{l}{2}}\big(\Omega[\alpha,\beta]\big)$ being compatible with zero for $t_j = \alpha_j$ too. Thus,

$$\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)(f,g) \;=\; (f,g) \;+\; (f^1,g^1)\;,$$

where

$$(f^1, g^1) \;=\; \big(\mathcal{A}(t) - \mathcal{A}(\alpha)\big)\mathcal{A}^{-1}(\alpha)(f, g) \;=\; \big(\mathcal{A}(t) - \mathcal{A}(\alpha)\big)u \;,$$

for $u = \mathcal{A}(\alpha)(f, g)$, but it follows from (2.4.2) that

$$\langle u\rangle_{l,\frac{l}{2}} \;\leq\; C\,\langle f, g\rangle_l,$$

where C is independent neither of α and β nor of the functions. Now, as follows from (2.4.4) and (2.4.17),

$$\langle f^1, g^1\rangle_l \;\leq\; \varepsilon(\delta)\langle u\rangle_{l,\frac{l}{2}},$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ equicontinuously in α . Thus,

$$\langle f^1, g^1\rangle_l \;\leq\; \frac{1}{2}\langle f, g\rangle_l$$

for sufficiently small shell $[\alpha, \beta]$. An inverse operator $\big[\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)\big]^{-1}$ is defined in such shells and we simply put

$$\mathcal{A}^{-1}(t) \;=\; \mathcal{A}^{-1}(\alpha)\big[\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)\big]^{-1} \;.$$

As in Section 3) of this proof we devide $\Omega[0, t]$ into shells $\Omega[\vec{i}]$. The measures of the shells are fixed and we may return to the initial norms $\|\cdot\|$ in every shell. The operator $\mathcal{A}^{-1}(t)$ provides in each $\Omega[\vec{i}]$ the solvability of the problem under the condition of compatibility with zero on its lower base, and we have to review the proof of Theorem 2.2.9 in this case in order to obtain the solvability of our problem in each $\Omega[\vec{i}]$ under nonhomogeneous initial conditions on the lower base, satisfying General Compatibility Condition.

Suppose now f, g_μ, ϕ_j belong to the corresponding spaces and satisfy General Compatibility Condition in the shell $\Omega[\vec{0}]$, and, hense, there exists a solution $u(x,t)$ to our problem, belonging to $H^{l,\frac{l}{2}}\big(\Omega[\vec{0}]\big)$.

Consider the next shell $\Omega[(1,0)]$. We take the corresponding values of the above-built in $\Omega[\vec{0}]$ solution and its derivatives while $t_1 = \delta$ and $t_2 = 0$ for the initial conditions. We have to check General Compatibility Condition in the shell $\Omega[(1,0)]$. Indeed, let u_0 be an extension of the above solution from $\Omega[\vec{0}]$ into $\Omega[0,(2\delta,\delta)]$ belonging to $H^{l,\frac{l}{2}}\big(\Omega[0,(2\delta,\delta)]\big)$. Then the functions f and $A(t)u_0$ belong to $H^{\lambda,\frac{\lambda}{2}}\big(\Omega[0,(2\delta,\delta)]\big)$ and coinside for $t \in \Omega[\vec{0}]$ $(t_1 < \delta$, $(t_2 < \delta$.

Thus, the difference $f - A(t)u_0$ is compatible with zero as $t_1 = \delta$ and $t_2 = 0$. In the same way $g_\mu - B_\mu(t)u_0\big|_{\Omega'}$ are compatible with zero. Hence, it is possible to construct a solution $u_1(x,t)$ in $\Omega[(1,0)]$. The difference $u_1 - u_0$ is compatible with zero on the common part of the boundaries of $\Omega[\vec{0}]$ and

$\Omega[(1,0)]$, thus u_1 together with the above-built in $\Omega[\vec{0}]$ solution u defines the joint solution belonging to $H^{l,\frac{l}{2}}\big(\Omega[0,(2\delta,\delta)]\big)$.

Moving this construction over the shells and strips as in Section 3) of this proof, we obtain the solution to the problem in the whole $\Omega[0,T]$. $\triangleright$

We conclude the paper with a few general remarks about possible generalizations and further results.

Remark 2.4.2 *A). The above results can be reformulated for operators acting on manifolds.*

B). It is possible to define polyparabolic problems both for operators of an arbitrary order and for an arbitrary number of "times".

C). It would be interesting to consider pseudodifferential boundary problems depending on a number of times. The one - time case was considered in the book (Grubb G. (1986)).

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