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Quantized chaotic dynamics and non-commutative KS entropy

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Abstract. We study the quantization of a classically chaotic dynamics, the Anosov dynamics of "cat maps" on a two dimensional torus. This dynamics is implemented as a discrete group of automorphisms of a von Neumann algebra of functions on a quantized torus. We compute the non-commutative generalization of the Kolmogorov-Sinai entropy, namely the Connes-Størmer entropy, of the generator of this group, and find that its value is equal to the classical value. This can be interpreted as a sign of stability of chaotic behavior in a dynamical system under quantization.

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I. Introduction

I.A. One of the characteristic features of chaos in classical dynamics is the positivity of the Kolmogorov-Sinai (KS) entropy. The KS entropy is a natural measure of mixing in phase space resulting from the time evolution of a dynamical system. Indeed, one can adopt the positivity of the KS entropy as a convenient way of defining chaos in a classical dynamical system. Through Pesin's theorem, this is related to another characteristic feature of chaotic evolution, namely the positivity of Lyapunov exponents.

The focus of the emerging field of "quantum chaology" [B2], [HT], [N], [V2], is the study of quantum dynamics arising from quantization of classically chaotic systems. Much emphasis is put on understanding the semiclassical approximation to the actual quantum dynamics, and it is, in fact, a somewhat controversial issue whether "quantum chaos" exists beyond this approximation.

In this paper we propose that a natural quantity to exhibit quantum chaos in a class of quantized dynamics is the positivity of the Connes-Størmer (CS) entropy. The CS entropy is defined in the context of von Neumann algebras, and is a natural extension of the KS entropy to the non-commutative context. We focus our attention on the quantized Anosov dynamics, and show that its CS entropy is positive and, in fact, equal to the classical value.

I.B. We begin by recalling the definition of the (classical) KS entropy. Let M be the phase space on which a probability measure ν and ν-preserving automorphism $\varphi : M \to M$ are defined. The latter means that φ is a measurable bijective function such that for all measurable sets $\mathcal{O}$, $\nu(\varphi(\mathcal{O})) = \nu(\mathcal{O})$. Let $\mathcal{A} = \{A_j\}$, $1 \leq j \leq$, be a finite partition of M into measurable and pairwise disjoint (up to measure zero) subsets. The entropy of this partition is defined by

$$H(\mathcal{A}) = \sum_j \eta(\nu(A_j)), \tag{I.1}$$

where the function η is given by

$$\eta(t) = -t \log t, \qquad 0 \leq t \leq 1. \tag{I.2}$$

Clearly, H is invariant under φ,

$$H(\varphi(\mathcal{A})) = H(\mathcal{A}), \tag{I.3}$$

where $\varphi(\mathcal{A}) = \{\varphi(A_1), \ldots, \varphi(A_n)\}$. Now, given two such partitions, $\mathcal{A}$ and $\mathcal{B}$, we form a finer partition $\mathcal{A} \vee \mathcal{B}$ by taking the intersections of the elements of $\mathcal{A}$ with the elements of $\mathcal{B}$. The entropy is subadditive with respect to the operation $\vee$,

$$H(\mathcal{A} \vee \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B}). \tag{I.4}$$

This and (I.3) imply that the limit

$$H(\mathcal{A}, \varphi) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{A} \vee \varphi(\mathcal{A}) \vee \ldots \vee \varphi^{n-1}(\mathcal{A})) \tag{I.5}$$

exists. The KS entropy of φ is defined as the supremum of $H(\mathcal{A}, \varphi)$ over all possible choices of the finite partition $\mathcal{A}$,

$$h_{KS}(\varphi) = \sup_{\mathcal{A}} H(\mathcal{A}, \varphi). \tag{I.6}$$

This definition does not lend itself to explicit computations. However, the fundamental theorem of Kolmogorov and Sinai [CFS] states that, in fact, $h_{KS}(\varphi)$ can be computed from a single partition, provided that it is sufficiently generic. More precisely, $h_{KS}(\varphi) = H(\mathcal{A}, \varphi)$, if $\mathcal{A}$ is a partition such that the sets $\mathcal{A} \vee \varphi(\mathcal{A}) \vee \ldots \vee \varphi^{k-1}(\mathcal{A})$, $k \in \mathbb{N}$, generate the σ-algebra of Borel sets on M.

We will explain in Section IV how Connes and Størmer generalized the theory outlined above to the non-commutative case.

I.C. The KS entropy has been computed for a large class of dynamical systems. One of the most famous such systems is that of Anosov dynamics on a torus. Let us briefly review the basic definitions. For a more complete presentation and a variety of results we refer the reader to [A], [AW], and [CFS]. We consider an element $\gamma \in SL(2, \mathbb{Z})$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{I.7}$$

with $|\mathrm{tr}(\gamma)| > 2$. Such a matrix has two positive eigenvalues $\mu_1 > 1$, and $\mu_2 = 1/\mu_1 < 1$. The action of γ on the plane $\mathbb{R}^2$ is given as usual by $(x_1, x_2) \to (y_1, y_2)$ with

$$\begin{aligned} y_1 &= ax_1 + bx_2, \\ y_2 &= cx_1 + dx_2. \end{aligned} \tag{I.8}$$

For later reference, we rewrite (I.8) in terms of the complex variable $z = (x_1 + ix_2)/\sqrt{2}$ as $z \to w$, with

$$w = \overline{\alpha} z + \beta \overline{z}, \tag{I.9}$$

where the complex parameters α and β are given by

$$\begin{aligned} \alpha &= (a + d + i(b - c))/2, \\ \beta &= (a - d + i(b + c))/2, \end{aligned} \tag{I.10}$$

and satisfy $|\alpha|^2 - |\beta|^2 = 1$. The transformation (I.8) is area preserving. Since the coefficients in (I.8) are integer, γ also defines an area preserving automorphism of the torus

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which we will denote by the same symbol γ. The group $\{\gamma^n\}_{n\in\mathbb{Z}}$ of automorphisms of $\mathbb{T}^2$ is called the Anosov dynamics on the torus (in fact, this is a special case of Anosov dynamics sometimes called the "cat dynamics").

The definitions above are of course meaningful without assuming that $|\mathrm{tr}(\gamma)| > 2$. The resulting dynamical systems are non-chaotic, and, as such, less relevant to the subject of this paper.

It turns out that for the Anosov dynamics,

$$h_{KS}(\gamma) = \log \mu_1, \tag{I.11}$$

where μ_1 is the larger of the eigenvalues of γ. A beautiful proof of this result in the context of symbolic dynamics is presented in [AW]. If $|\mathrm{tr}(\gamma)| \leq 2$, then $h_{KS}(\gamma) = 0$, showing that the corresponding dynamics is indeed non-chaotic.

I.D. One of the central concepts of this paper is that of quantization of a dynamical system. Without getting involved with technicalities we would like to emphasize several points which will explain the particular conceptual framework which we chose to work with.

Quantization of a dynamical system has two components: kinematic and dynamic. The kinematic component of quantization involves the construction of a suitable quantized phase space of the system. This quantized phase space is given in terms of a non-commutative algebra $\mathfrak{A}_\hbar$ of observables. In the language of non-commutative geometry, $\mathfrak{A}_\hbar$ is an algebra of functions on the quantized phase space. Very much like in the classical situation, where (depending on the problem) one might be interested in the study of the algebra of continuous functions, smooth functions, compactly supported smooth functions, measurable functions, etc., specific choices of the composition of $\mathfrak{A}_\hbar$ can be made. This may result in imposing the structure of a $\mathbb{C}^*$-algebra, a von Neumann algebra or some suitably defined locally convex algebra, on the algebra of observables.

The dynamic component of quantization consists in defining a time evolution on the quantized phase space. A natural way of doing this is to find a suitable one parameter group of automorphisms of $\mathfrak{A}_\hbar$, where the parameter (discrete or continuous) has the meaning of time. Recall that an automorphism of an algebra $\mathfrak{R}$ is a linear one-to-one map Φ of $\mathfrak{R}$ onto itself such that $\Phi(ab) = \Phi(a)\Phi(b)$, for all $a, b \in \mathfrak{R}$. If $\mathfrak{R}$ is an algebra with involution, it is also required that $\Phi(a^*) = \Phi(a)^*$.

The "suitability" of the choices made, namely that of the algebra $\mathfrak{A}_\hbar$ and of the time evolution, is settled by the correspondence principle. This amounts to showing that limits of the quantized objects, as $\hbar \to 0$, yield the corresponding classical objects. Quantization is a highly non-unique procedure, and the correspondence principle is the only physical principle allowing one to decide whether a particular procedure is correct. To our taste, the most satisfying mathematical framework for quantization is that of "strict deformation quantization" proposed in [R1].

I.E. Quantization of the Anosov dynamics on the torus has been discussed before by a number of authors. The original reference is [HB], where a scheme is proposed using a group of unitary matrices on a finite dimensional Hilbert space. The generator of this group was determined from (i) the observation that the generating function of (I.8) is quadratic, and (ii) the assumption that, in the quadratic case, the semiclassical expressions are exact. This quantized dynamics was further studied in [K1,2], [MO], and [BDG], where a variety of beautiful number theoretic results were derived.

Our approach is slightly different, even though equivalent in the sense specified at the end of previous subsection. It is based on an infinite dimensional Hilbert space. The infinite dimensionality of the Hilbert space is due to the occurrence of Θ-vacua (to use the language of quantum gauge field theory), which in turn is a consequence of the fact that the phase space of the Anosov system, namely the torus, is not simply connected. We study a non-abelian algebra, known as the algebra of functions on a quantized torus [R2], and identify a group of automorphisms of this algebra as the quantized Anosov dynamics.

I.F. The paper is organized as follows. In Section II, we define the quantized linear dynamics on the plane. This will be the starting point for the construction of quantized Anosov dynamics. In Section III, we review the construction of the quantized torus, and show that the Anosov dynamics on the torus defines a group of automorphisms of the quantized torus. This group is the quantized Anosov dynamics on the torus. Section IV has a review character. We explain the properties and construction of the CS entropy. In Section V, we compute the CS entropy of the quantized Anosov dynamics on the torus.

II. Quantized linear dynamics on the plane

II.A. Of the many representations of quantum mechanics we choose the Bargmann representation (see e.g. [F]), as in this representation wave functions are defined on the phase space of the system. It can also be generalized to phase spaces other than flat spaces [B1], which should be important for future extensions of the results of this paper. In the Bargmann representation, the Hilbert space of states $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ consists of entire functions on $\mathbb{C}$ which are square integrable with respect to the probability measure $d\mu_\hbar(z) = (\pi\hbar)^{-1}\exp\{-|z|^2/\hbar\}d^2z$. This Hilbert space has two remarkable properties: (i) it has a reproducing kernel, namely the function $\exp\{\overline{w}z/\hbar\} \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ satisfies the equation

$$\int_{\mathbb{C}} \exp\{\overline{w}z/\hbar\}\phi(w)d\mu_\hbar(w) = \phi(z), \tag{II.1}$$

for all $\phi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$, and (ii) it carries a unitary projective representation of the group of translations of $\mathbb{C}$ given by

$$U(\zeta)\phi(z) = \exp\left\{\frac{1}{\hbar}(\overline{\zeta}z - |\zeta|^2/2)\right\}\phi(z-\zeta), \quad \zeta \in \mathbb{C}. \tag{II.2}$$

For future reference, we note that

$$U(\zeta)U(\xi) = e^{i\mathrm{Im}(\overline{\zeta}\xi)/\hbar}U(\zeta+\xi). \tag{II.3}$$

The algebra of observables (or functions on the quantized plane) can be defined as an algebra generated by Toeplitz operators. A Toeplitz operator $T_\hbar(f)$ with symbol f (where f is a measurable function on $\mathbb{C}$) is defined by

$$T_\hbar(f)\phi(z) = \int_{\mathbb{C}} e^{z\overline{w}/\hbar}f(w)\phi(w)d\mu_\hbar(w). \tag{II.4}$$

Various restrictions on the class of symbols f may be imposed, leading to various algebras of operators on $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$. Since the quantized plane is not the main concern of this paper, we ignore this issue, and refer the interested reader to e.g. [BC1,2] for precise statements. For our needs, it is only important that all bounded continuous functions are included in the class of symbols.

The Toeplitz operator with the symbol $f(z) = z$ is denoted by $A^\dagger$, and the Toeplitz operator with $f(z) = \overline{z}$ is denoted by A. These are the creation and annihilation operators obeying the usual commutation relation

$$[A, A^\dagger] = \hbar. \tag{II.5}$$

In fact, the quantization map $f \to T_\hbar(f)$ can be regarded as the anti-Wick ordering prescription, i.e., in the quantized expressions, all the annihilation operators are placed to the left of the creation operators.

II.B. To a γ as defined in the Introduction we assign the following Bogolubov transformation $(A^\dagger, A) \to (B^\dagger, B)$,

$$\begin{aligned} B^\dagger &= \overline{\alpha}A^\dagger + \beta A, \\ B &= \alpha A + \overline{\beta}A^\dagger. \end{aligned} \tag{II.6}$$

We will now show that this transformation is unitarily implementable, i.e. $B^\dagger = FA^\dagger F^{-1}$, and determine such a unitary F explicitly.

First, we note that the ground state $\omega_\gamma(z)$ for B satisfies the differential equation:

$$\hbar\alpha\omega_\gamma'(z) + \overline{\beta}\omega_\gamma(z) = 0, \tag{II.7}$$

and so

$$\omega_\gamma(z) = |\alpha|^{-1/2} \exp\{-\frac{\overline{\beta} z^2}{2\hbar\alpha}\}, \tag{II.8}$$

where the normalizing constant has been chosen so that $||\omega_\gamma|| = 1$. We require that F maps the function identically equal 1 (the ground state for A) to ω_γ. Then, using the Hausdorff-Baker-Campbell formula,

$$\begin{aligned}
F\exp\{\overline{w}z/\hbar\} &= F\exp\{\overline{w}A^\dagger/\hbar\}F^{-1}\omega_\gamma(z)\\
&= \exp\{\overline{w}(\overline{\alpha}A^\dagger + \beta A)/\hbar\}\omega_\gamma(z)\\
&= \exp\{(\overline{w\alpha}z + \overline{\alpha}\beta\overline{w}^2/2)/\hbar\}\exp\{\overline{w}\beta\frac{d}{dz}\}\omega_\gamma(z)\\
&= \exp\{(\overline{w\alpha}z + \overline{\alpha}\beta\overline{w}^2/2)/\hbar\}\omega_\gamma(z+\overline{w}\beta)\\
&= |\alpha|^{-1/2}\exp\{(\overline{w}z + \beta\overline{w}^2/2 - \overline{\beta}z^2/2)/\hbar\alpha\}.
\end{aligned}$$

Using the fact that $\exp\{\overline{w}z/\hbar\}$ is the reproducing kernel for the measure $d\mu_\hbar$ we thus find that the action of F on $\phi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ is given by

$$\begin{aligned}
F\phi(z) &= |\alpha|^{-1/2}\exp\{-\frac{\overline{\beta}z^2}{2\hbar\alpha}\}\int_{\mathbb{C}}\exp\{\frac{\overline{w}z}{\hbar\alpha} + \frac{\beta\overline{w}^2}{2\hbar\alpha}\}\phi(w)d\mu_\hbar(w)\\
&= T_\hbar(\omega_\gamma)S_{\gamma^{-1}}T_\hbar(\omega_{\gamma^{-1}})^*\phi(z),
\end{aligned} \tag{II.9}$$

where S_γ is defined by $S_\gamma\phi(z) = |\alpha|^{1/2}\phi(z/\alpha)$. It is straightforward to verify that the inverse of F is given by

$$\begin{aligned}
F^{-1}\phi(z) &= |\alpha|^{-1/2}\exp\{\frac{\overline{\beta}z^2}{2\hbar\overline{\alpha}}\}\int_{\mathbb{C}}\exp\{\frac{\overline{w}z}{\hbar\overline{\alpha}} - \frac{\beta\overline{w}^2}{2\hbar\overline{\alpha}}\}\phi(w)d\mu_\hbar(w)\\
&= T_\hbar(\omega_{\gamma^{-1}})S_\gamma T_\hbar(\omega_\gamma)^*\phi(z),
\end{aligned} \tag{II.10}$$

and that F is unitary. Let us summarize the calculations above in the following theorem.

Theorem II.1. *There exists a unique unitary operator F satisfying $FA^\dagger F^{-1} = B^\dagger$, and $F1 = \omega_\gamma$. This operator and its inverse are given by equations (II.9) and (II.10).*

The group $\{F^n\}_{n\in\mathbb{Z}}$ of unitary operators on $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ is called the evolution group for the linear dynamics on the plane. The corresponding group of automorphisms of the algebra of observables is generated by $a \to FaF^{-1}$.

II.C. There is a simple relation between F and the unitary operators $U(\zeta)$ defined in (II.2).

Theorem II.2. *The conjugation of* $U(\zeta)$ *by* F *is equal to* $U(\gamma^{-1}\zeta)$*,*

$$FU(\zeta)F^{-1} = U(\alpha\zeta - \beta\overline{\zeta}). \tag{II.11}$$

Proof. The proof is a straightforward computation. Using (II.9) and (II.10) we find that

$$\begin{aligned} FU(\zeta)F^{-1}\phi(z) = |\alpha|^{-1} \exp\big\{ -\frac{1}{2\hbar}(|\zeta|^2 + \overline{\beta}z^2/\alpha - \overline{\beta}\zeta^2/\overline{\alpha})\big\} \\ \times \int_{\mathbb{C}} \exp\big\{\frac{1}{\hbar}\big(\frac{\beta\overline{w}^2}{2\alpha} + \frac{\overline{\beta}w^2}{2\overline{\alpha}} + \frac{z\overline{w}}{\alpha} + \frac{(\overline{\alpha}\overline{\zeta} - \overline{\beta}\zeta + \overline{v})w}{\alpha} + \frac{\overline{v}\zeta}{\alpha} + \frac{\beta\overline{v}^2}{2\overline{\alpha}}\big)\big\} \\ \times \phi(v)d\mu_\hbar(w)d\mu_\hbar(v). \end{aligned}$$

Evaluating the w-integral and using $|\alpha|^2 - |\beta|^2 = 1$ yields

$$\begin{aligned} FU(\zeta)F^{-1}\phi(z) = \exp\big\{\big(z\overline{(\alpha\zeta - \beta\overline{\zeta})} - |\alpha\zeta - \beta\overline{\zeta}|^2/2\big)/\hbar\big\} \\ \times \int_{\mathbb{C}} \exp\big\{(z - \alpha\zeta + \beta\overline{\zeta})\overline{v}/\hbar\big\}\phi(v)d\mu_\hbar(v), \end{aligned}$$

which by means of (II.1) is equal to

$$\exp\big\{z\overline{(\alpha\zeta - \beta\overline{\zeta})}/\hbar - |\alpha\zeta - \beta\overline{\zeta}|^2/2\hbar\big\}\phi(z - (\alpha\zeta - \beta\overline{\zeta})) = U(\gamma^{-1}\zeta)\phi(z),$$

as claimed. □

III. Quantized Anosov dynamics on the torus

III.A. Having defined the quantized linear dynamics on the plane we now proceed to constructing the quantized Anosov dynamics on the torus. As explained e.g. in [KL], one can regard the quantized torus as a suitably defined quotient of the quantized plane by the group $\mathbb{Z}^2$. Namely, we define the algebra of of observables on the quantized torus to be the algebra of all Toeplitz operators with continuous $\mathbb{Z}^2$-invariant symbols. Such symbols can be written as Fourier series, and so the algebra of observables is generated by $T_\hbar(f_1)$ and $T_\hbar(f_2)$, where $f_k(x_1, x_2) = \exp\{2\pi i x_k\}$. However, writing $ix_1 = i(z + \overline{z})/\sqrt{2}$, $ix_2 = (z - \overline{z})/\sqrt{2}$, we verify easily that

$$\begin{aligned} T(f_1) = e^{-\pi^2\hbar}U(-i\pi\hbar\sqrt{2}), \\ T(f_2) = e^{-\pi^2\hbar}U(\pi\hbar\sqrt{2}). \end{aligned} \tag{III.1}$$

It is thus natural to set

$$\begin{aligned} U &= U(-i\hbar\pi\sqrt{2}), \\ V &= U(\hbar\pi\sqrt{2}), \end{aligned} \tag{III.2}$$

and regard the operators U and V as generators of the algebra of functions on the quantized torus. Commutation relation (II.3) implies that they obey the following set of relations:

$$\begin{aligned} UU^* &= U^*U = I, \\ VV^* &= V^*V = I, \\ UV &= e^{i\lambda}VU, \end{aligned} \tag{III.3}$$

where for convenience we set $\lambda = 4\pi^2\hbar$. The algebra generated by U and V with the relations above has been studied extensively by both physicists and mathematicians, and we refer the reader to [R2] for an overview and extensive list of references. In particular, it has been established that "smooth elements" in this algebra obey a strong version of the correspondence principle [R1].

III.B. For our purposes, we consider the von Neumann algebra $\mathfrak{A}_\hbar$, generated by U and V. Recall [D] that an algebra of bounded operators $\mathfrak{R}$ on a Hilbert space $\mathcal{H}$ is called a von Neumann algebra, if (i) it is closed under taking the hermitian conjugate, and (ii) it is equal to its bicommutant, $\mathfrak{R} = \mathfrak{R}''$. Here $\mathfrak{R}'' = (\mathfrak{R}')'$, where the commutant $\mathcal{S}'$ of a set of operators $\mathcal{S}$ on $\mathcal{H}$ is defined as the set of all bounded operators on $\mathcal{H}$ which commute with all the elements of $\mathcal{S}$. The von Neumann algebra generated by a set $\mathcal{S}$ is defined as the smallest von Neumann algebra containing $\mathcal{S}$. If $\mathcal{S}$ is closed under taking the hermitian adjoint, this turns out to be $\mathcal{S}''$. In other words, $\mathfrak{A}_\hbar = \{U, U^*, V, V^*\}''$. In fact, $\mathfrak{A}_\hbar$ is isomorphic to the universal enveloping von Neumann algebra generated by U and V which obey the relations (III.3). This means, in particular, that (III.3) are the only relations between U and V. One can think of the elements of $\mathfrak{A}_\hbar$ as bounded (but not necessarily continuous) functions on the quantized torus.

Technically, $\mathfrak{A}_\hbar$ is a von Neumann algebra of type II_1. We will not reproduce here the precise definition of a type II_1 von Neumann algebra (see e.g. [D]). One should just keep in mind that a typical example of such an algebra is the space $L^\infty(M)$ of essentially bounded functions on a compact space M with a Borel probability measure $d\nu$. Characteristic for type II_1 von Neumann algebras is the existence of a faithful normal trace τ. A trace on a von Neumann algebra is a non-commutative extension of an integral, and again we refer the reader to [D] for the precise definitions. For the abelian example, such a trace is given by

$$\tau(f) = \int_M f \, d\nu. \tag{III.4}$$

On the algebra $\mathfrak{A}_\hbar$, a faithful normal trace is determined by

$$\tau_\hbar\Big(\sum_{j,k} \alpha_{jk} U^j V^k\Big) = \alpha_{00}. \tag{III.5}$$

III.C. Let us now derive the transformation rules for U and V under conjugation by the operator F. Using Theorem II.2 and (II.3) we obtain

$$\begin{aligned} FUF^{-1} &= U(-i\hbar\pi\sqrt{2}(\overline{\alpha}+\beta)) \\ &= U(-i\hbar\pi(a+ic)\sqrt{2}) \\ &= e^{i\hbar\pi^2 ac}U(-i\hbar\pi\sqrt{2}a)U(\hbar\pi\sqrt{2}c) \\ &= e^{i\lambda ac/2}U^a V^c, \end{aligned}$$

and likewise

$$FVF^{-1} = e^{i\lambda bd/2}U^b V^d.$$

These expressions define an automorphism $\Gamma_\hbar$ of $\mathfrak{A}_\hbar$. We call the group $\{\Gamma_\hbar^n\}_{n\in\mathbb{Z}}$ of automorphisms generated by $\Gamma_\hbar$ the quantized Anosov dynamics on the torus.

Theorem III.1. *The transformation*

$$\begin{aligned} \Gamma_\hbar(U) &= e^{i\lambda ac/2}U^a V^c, \\ \Gamma_\hbar(V) &= e^{i\lambda bd/2}U^b V^d. \end{aligned} \tag{III.6}$$

defines an automorphism of $\mathfrak{A}_\hbar$. *The trace* $\tau_\hbar$ *is invariant under* $\Gamma_\hbar$, *i.e.* $\tau_\hbar(\Gamma_\hbar(a)) = \tau_\hbar(a)$.

Proof. We need to show that $\Gamma_\hbar(U)$ and $\Gamma_\hbar(V)$ form a new set of generators. Using (III.3) we compute:

$$\Gamma_\hbar(U)^* = e^{-i\lambda ac/2}V^{-c}U^{-a} = \left(e^{i\lambda ac/2}U^a V^c\right)^{-1} = \Gamma_\hbar(U)^{-1}.$$

Likewise, $\Gamma_\hbar(V)^* = \Gamma_\hbar(V)^{-1}$. Furthermore,

$$\begin{aligned} \Gamma_\hbar(U)\Gamma_\hbar(V) &= e^{i\lambda(ac+bd)/2}U^a V^c U^b V^d \\ &= e^{i\lambda(ac+bd)/2-i\lambda bc}U^b U^a V^c V^d \\ &= e^{i\lambda(ac+bd)/2+i\lambda(ad-bc)}U^b V^d U^a V^c \\ &= e^{i\lambda}\Gamma_\hbar(V)\Gamma_\hbar(U). \end{aligned}$$

We also note that the inverse of $\Gamma_\hbar$ is given by

$$\begin{aligned} \Gamma_\hbar^{-1}(U) &= e^{-i\lambda cd/2}U^d V^{-c}, \\ \Gamma_\hbar^{-1}(V) &= e^{-i\lambda ab/2}U^{-b}V^a. \end{aligned} \tag{III.7}$$

Finally, the $\Gamma_\hbar$-invariance of $\tau_\hbar$ is an immediate consequence of (III.5). $\square$

III.D. At this point it is not quite clear that $\Gamma_\hbar$ is indeed a quantization of the classical map γ, i.e. that its classical limit $\hbar \to 0$ indeed yields γ. The goal of this subsection is to show that it is so. A related analysis can be found in [DGI]. We let $||\cdot||_\hbar$ denote the operator norm on the Hilbert space $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$.

Theorem III.2. *Let f be a continuous $\mathbb{Z}^2$-invariant function on $\mathbb{C}$. Then:*

$$||FT_\hbar(f)F^{-1} - T_\hbar(f\circ\gamma)||_\hbar \to 0, \quad \text{as} \quad \hbar \to 0. \tag{III.8}$$

Proof. Let $\epsilon > 0$. We are going to show that for all sufficiently small $\hbar$,

$$||FT_\hbar(f)F^{-1} - T_\hbar(f\circ\gamma)||_\hbar \leq \epsilon. \tag{III.9}$$

We proceed in steps.
Step 1. By the Stone-Weierstraß theorem, there is a trigonometric polynomial P such that

$$||f - P||_\infty \leq \epsilon/3,$$

where $||f||_\infty = \sup_z |f(z)|$ is the usual sup-norm. Since the operator norm of a Toeplitz operator does not exceed the sup-norm of its symbol, $||T_\hbar(f)||_\hbar \leq ||f||_\infty$, this yields the following inequality:

$$||T_\hbar(f) - T_\hbar(P)||_\hbar \leq \epsilon/3. \tag{III.10}$$

Step 2. A trigonometric polynomial $P(z)$ is a linear combination of terms of the form $\exp(\overline{w}z - \overline{z}w)$. In terms of the creation and annihilation operators, for the corresponding Toeplitz operator we have:

$$T_\hbar(e^{\bar{w}z-\overline{z}w}) = e^{-wA}e^{\overline{w}A^\dagger}.$$

Conjugating the above equation by F yields:

$$\begin{aligned} FT_\hbar(e^{\overline{w}z-\overline{z}w})F^{-1} &= Fe^{-wA}F^{-1}Fe^{\overline{w}A^\dagger}F^{-1} \\ &= e^{-w(\alpha A+\overline{\beta}A^\dagger)}e^{\overline{w}(\overline{\alpha}A^\dagger+\beta A)} \\ &= e^{-h(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2)/2}e^{-\alpha wA}e^{-\overline{\beta}wA^\dagger}e^{\beta\overline{w}A}e^{\overline{\alpha w}A^\dagger}, \end{aligned}$$

where we have used the Hausdorff-Baker-Campbell formula as in the derivation of (II.9). Commuting the third and the fourth terms gives further:

$$\begin{aligned} FT_\hbar(e^{\overline{w}z-\overline{z}w})F^{-1} &= e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}e^{-w\alpha A+\overline{w}\beta A}e^{\overline{w\alpha}A^\dagger-w\overline{\beta}A^\dagger} \\ &= e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{-w\alpha\overline{z}+\overline{w}\beta\overline{z}+\overline{w\alpha}z-w\overline{\beta}z}) \\ &= e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{\overline{w}(\overline{\alpha}z+\beta\overline{z})-w(\alpha\overline{z}+\overline{\beta}z)}) \\ &= e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{\overline{w}\gamma(z)-\overline{\gamma(z)}w}). \end{aligned}$$

We can thus make the following estimate:

$$
\begin{aligned}
||FT_\hbar(e^{\overline{w}z-\overline{z}w})F^{-1} &- T_\hbar(e^{\overline{w}g(z)-\overline{g(z)}w})||_\hbar \\
&\leq |e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}-1|\,||T_\hbar(e^{\overline{w}g(z)-\overline{g(z)}w})||_\hbar \\
&\leq |e^{-\hbar(\alpha\overline{\beta}w^2+\overline{\alpha}\beta\overline{w}^2+2|\beta|^2|w|^2)/2}-1|.
\end{aligned}
$$

Clearly, the right hand side of the above inequality goes to zero, as $\hbar \to 0$. Since P is a linear combination of finitely many terms of the above form, we can find δ (depending on P) such that for $\hbar < \delta$ we have:

$$
||FT_\hbar(P)F^{-1} - T_\hbar(P\circ\gamma)||_\hbar \leq \epsilon/3. \tag{III.11}
$$

Step 3. We can now conclude the argument:

$$
\begin{aligned}
&||FT_\hbar(f)F^{-1} - T_\hbar(f\circ\gamma)||_\hbar \\
&\quad \leq ||FT_\hbar(f)F^{-1} - FT_\hbar(P)F^{-1}||_\hbar + ||FT_\hbar(P)F^{-1} - T_\hbar(P\circ\gamma)||_\hbar + ||T_\hbar(P\circ\gamma) - T_\hbar(f\circ\gamma)||_\hbar \\
&\quad \leq ||T_\hbar(f) - T_\hbar(P)||_\hbar + \epsilon/3 + ||P\circ\gamma - f\circ\gamma||_\infty \\
&\quad \leq 2||f-P||_\infty + \epsilon/3 \\
&\quad \leq \epsilon,
\end{aligned}
$$

where we have used (III.10) and (III.11). □

III.E. So far the value of Planck's constant has not been restricted in any way other than it should be a positive number. In particular, the von Neumann algebra $\mathfrak{A}_\hbar$ is a well defined object for all such $\hbar$. On the other hand, its structure depends crucially on whether $\lambda/2\pi$ is a rational number or not. It is well known that physics requires $\lambda/2\pi$ to be rational. The standard informal argument is that the volume of the phase space should be an integer multiple of the elementary cell volume $2\pi\hbar$. Hence

$$
\hbar = \frac{1}{2\pi N}, \qquad N \in \mathbb{N}, \tag{III.12}
$$

or

$$
\lambda = \frac{2\pi}{N}. \tag{III.13}
$$

Incidentally, this is precisely the integrality condition of geometric quantization which requires the symplectic form on the torus divided by 2π to define a deRham cohomology class with integer coefficients. Throughout the rest of this paper, we will be assuming that the condition above is satisfied. Trivial changes in our arguments show that the conclusions below hold for arbitrary positive rational $\lambda/2\pi$.

III.F. The von Neumann algebra $\mathfrak{A}_\hbar$ has a simple structure which is described in the theorem below. This theorem is well known, and the references to the original literature can be found in [R2]. Since the proof is not easy to extract from the original references (and for the sake of completeness), we include an elementary proof. We denote by $\mathcal{M}_N$ the (von Neumann) algebra of complex $N \times N$ matrices, while by $L^\infty(\mathbb{T}^2)$ we denote the space of all essentially bounded functions on the torus regarded as a von Neumann algebra on the Hilbert space $L^2(\mathbb{T}^2)$.

Theorem III.3. *We have the following isomorphism of von Neumann algebras*

$$\iota: \quad \mathfrak{A}_\hbar \to L^\infty(\mathbb{T}^2) \otimes \mathcal{M}_N. \tag{III.14}$$

Under this isomorphism, the trace $\tau_\hbar$ factorizes into a tensor product of traces,

$$\tau_\hbar \circ \iota^{-1} = \tau \otimes (1/N)\ \mathrm{tr}, \tag{III.15}$$

where τ is given by (III.4).

Proof. It is clear from the relations (III.3) that U^N and V^N are in the center of $\mathfrak{A}_\hbar$. Let us denote by $\mathfrak{Z}$ the von Neumann algebra generated by U^N and V^N. Obviously, $\mathfrak{Z}$ is isomorphic with $L^\infty(\mathbb{T}^2)$, with the isomorphism given by $U^N \to e^{2\pi i\theta_1}$ and $V^N \to e^{2\pi i\theta_2}$. Consider now the following (discontinuous) functions in $L^\infty(\mathbb{T}^2)$: $f_1(\theta) = e^{2\pi i\theta_1/N}$ and $f_2(\theta) = e^{2\pi i\theta_2/N}$, and let X_1 and X_2 be the corresponding elements of $\mathfrak{Z}$. Then the two elements $u = X_1^{-1}U$ and $v = X_2^{-1}V$ obey the following set of relations:

$$\begin{aligned} uu^* &= u^*u = I, \\ vv^* &= v^*v = I, \\ uv &= e^{i\lambda}vu, \\ u^N &= v^N = I. \end{aligned} \tag{III.16}$$

This algebra has the following realization. In the Hilbert space $\mathbb{C}^N$, choose an orthonormal basis $e_1, \ldots, e_N$, and set $ue_j = e^{i(j-1)\lambda}e_j, \quad ve_j = e_{j+1}$, where $e_{N+1} = e_1$. A short computation shows that the only matrices commuting with u and v are scalar multiples of the identity, and thus the von Neumann algebra generated by u and v can be identified with the full matrix algebra $\mathcal{M}_N$.

We have $U = X_1u$, $V = X_2v$, and the required isomorphism is given by

$$\iota(U) = f_1 \otimes u, \quad \iota(V) = f_2 \otimes v. \tag{III.17}$$

To prove (III.15), we note that

$$\big(\tau \otimes (1/N)\ \mathrm{tr}\big)(f_1^j f_2^k \otimes u^j v^k) = \int_0^1 \int_0^1 e^{2\pi i(j\theta_1 + k\theta_2)/N} d\theta_1 d\theta_2\ (1/N)\ \mathrm{tr}(u^j v^k). \tag{III.18}$$

Using the explicit realization of the operators u and v we see that $\mathrm{tr}(u^j v^k) = 0$, unless $k = pN$, $p \in \mathbb{Z}$. However, $\int_0^1 e^{2\pi i p\theta_2} d\theta_2 = 0$, for $p \neq 0$, and so (III.18) is zero for $k \neq 0$. Let $k = 0$, and $j = Np + q$, $0 \leq q \leq N-1$. If $q > 0$, then $\mathrm{tr}(u^j) = 0$. If $q = 0$, but $p \neq 0$, then $\int_0^1 e^{2\pi i p\theta_1} d\theta_1 = 0$. Consequently,

$$\big(\tau \otimes (1/N)\ \mathrm{tr}\big)\big(f_1^j f_2^k \otimes u^j v^k\big) = \delta_{j0}\delta_{k0} = \tau_\hbar(U^j V^k), \tag{III.19}$$

and the claim follows. $\square$

Let us parenthetically remark that the corresponding result for the $\mathbb{C}^*$-algebra of functions on a quantized torus involves a bundle of full matrix algebras over the torus rather than a tensor product.

III.G. It is now easy to see that, under the isomorphism above, the automorphism $\Gamma_\hbar$ becomes a tensor product of automorphisms of the factors in (III.17). A straightforward calculation yields the following theorem.

Theorem III.4. *We have*

$$\iota\Gamma_\hbar\iota^{-1} = \Psi \otimes \Phi_\hbar, \tag{III.20}$$

where Ψ is an automorphism of $L^\infty(\mathbb{T}^2)$ given by

$$\begin{aligned} \Psi(f_1)(\theta) &= e^{2\pi i(a\theta_1 + c\theta_2)}, \\ \Psi(f_2)(\theta) &= e^{2\pi i(b\theta_1 + d\theta_2)}, \end{aligned} \tag{III.21}$$

and where $\Phi_\hbar$ is an automorphism of $\mathcal{M}_N$ given by

$$\begin{aligned} \Phi_\hbar(u) &= e^{i\lambda ac/2} u^a v^c, \\ \Phi_\hbar(v) &= e^{i\lambda bd/2} u^b v^d. \end{aligned} \tag{III.22}$$

A natural way of thinking about the above decomposition of $\Gamma_\hbar$ is to regard Ψ the purely classical component of the dynamics, and $\Phi_\hbar$ its purely quantum component.

IV. Connes-Størmer entropy

IV.A. To motivate the construction of the CS entropy we first reformulate the definition of the classical KS entropy in purely algebraic (rather than measure theoretic) terms (see also [B3]). We assume that M is a compact phase space with a Borel probability measure $d\nu$ defined on it, and τ define the normal faithful trace on $L^\infty(M)$ given by (III.4). Given a partition $\mathcal{A}$ of M (defined as in the Introduction), we consider the finite dimensional subalgebra $\mathfrak{N} \subset L^\infty(M)$ which is generated by the characteristic functions χ_{A_j}. The operator of multiplication by χ_{A_j} is a projection operator and we denote it by p_j. Note that each projection p_j is minimal (i.e. is not a sum of two non-trivial projections in $\mathfrak{N}$), and $\sum_j p_j = I$. We define the entropy of the subalgebra $\mathfrak{N}$ to be

$$H(\mathfrak{N}) = \sum_j \eta(\tau(p_j)) = H(\mathcal{A}). \tag{IV.1}$$

For two such subalgebras, $\mathfrak{N}_1$ and $\mathfrak{N}_2$, we let $\mathfrak{N}_1 \vee \mathfrak{N}_2$ denote the (finite dimensional) subalgebra generated by $\mathfrak{N}_1$ and $\mathfrak{N}_2$.

Now, a measure preserving automorphism φ of M defines a τ-preserving automorphism Φ of $\mathfrak{R}$,

$$\Phi f(x) = f \circ \varphi(x). \tag{IV.2}$$

We set

$$H(\mathfrak{N}, \Phi) = \lim_{k\to\infty} \frac{1}{k} H(\mathfrak{N} \vee \Phi(\mathfrak{N}) \vee \ldots \vee \Phi^{k-1}(\mathfrak{N})) = H(\mathcal{A}, \varphi),$$

and define the entropy of the automorphism Φ as the supremum of this quantity over all possible choices of $\mathfrak{N}$ (this is, of course, equal to $h_{KS}(\varphi)$).

IV.B. The construction above of the entropy of a measure preserving automorphism was generalized to the non-commutative case by Connes and Størmer [CS] (in the von Neumann algebraic setup), and later by Connes, Narnhofer and Thirring [CNT] (in the $\mathbb{C}^*$-algebraic setup). We choose the original Connes-Størmer construction as it suits our needs best.

Let $\mathfrak{R}$ be a type II_1 von Neumann algebra, and let τ be a finite faithful normal trace on $\mathfrak{R}$. Consider a collection $\mathfrak{N}_1, \ldots, \mathfrak{N}_k$ of finite dimensional von Neumann subalgebras of $\mathfrak{R}$. The key difficulty to overcome here is the fact that $\mathfrak{N} \vee \mathfrak{P}$ may not be finite dimensional, even though $\mathfrak{N}$ and $\mathfrak{P}$ are. Connes and Størmer defined a function $H(\mathfrak{N}_1, \ldots, \mathfrak{N}_k)$ which replaces $H(\mathfrak{N}_1 \vee \ldots \vee \mathfrak{N}_k)$ but reduces to it in the commutative case. Specifically, this function satisfies the following properties:

(A) $H(\mathfrak{N}_1, \ldots, \mathfrak{N}_k) \leq H(\mathfrak{P}_1, \ldots, \mathfrak{P}_k)$, if $\mathfrak{N}_j \subset \mathfrak{P}_j$, for all $1 \leq j \leq k$;

(B) $H(\mathfrak{N}_1, \ldots, \mathfrak{N}_m, \mathfrak{N}_{m+1}, \ldots, \mathfrak{N}_n) \leq H(\mathfrak{N}_1, \ldots, \mathfrak{N}_m) + H(\mathfrak{N}_{m+1}, \ldots, \mathfrak{N}_n)$;

(C) if $\mathfrak{N}_1, \ldots, \mathfrak{N}_m \subset \mathfrak{N}$, then $H(\mathfrak{N}_1, \ldots, \mathfrak{N}_m, \mathfrak{N}_{m+1}, \ldots, \mathfrak{N}_n) \leq H(\mathfrak{N}, \mathfrak{N}_{m+1}, \ldots, \mathfrak{N}_n)$;

(D) if $\{p_\alpha\}$ is any family of minimal projections in $\mathfrak{N}$ such that $\sum_\alpha p_\alpha = I$, then $H(\mathfrak{N}) = \sum_\alpha \eta(\tau(p_\alpha))$;

(E) if $\mathfrak{N}_1, \dots, \mathfrak{N}_k$ are pairwise commuting then $H(\mathfrak{N}_1, \dots, \mathfrak{N}_k) = H((\mathfrak{N}_1 \cup \dots \cup \mathfrak{N}_k)'')$;

(F) if Φ is an automorphism of $\mathfrak{R}$ preserving the trace τ, then $H(\Phi(\mathfrak{N}_1), \dots, \Phi(\mathfrak{N}_k)) = H(\mathfrak{N}_1, \dots, \mathfrak{N}_k)$.

Now, if Φ is a τ-preserving automorphism of $\mathfrak{R}$, then properties (B) and (F) imply that that the limit

$$H(\mathfrak{N}, \Phi) = \lim_{k \to \infty} \frac{1}{k} H(\mathfrak{N}, \Phi(\mathfrak{N}), \dots, \Phi^{k-1}(\mathfrak{N})) \tag{IV.3}$$

exists. We define the CS entropy as the supremum of the above quantity over all possible choices of the finite dimensional algebra $\mathfrak{N}$,

$$h_{CS}(\Phi) = \sup_{\mathfrak{N},\ \dim \mathfrak{N} < \infty} H(\mathfrak{N}, \Phi). \tag{IV.4}$$

To be able to compute $h_{CS}(\Phi)$ we need a non-commutative version of the Kolmogorov-Sinai theorem. Such a theorem was proved in [CS] and is formulated as follows.

Theorem IV.1. *Let $\{\mathfrak{N}_k\}$ be an increasing sequence of finite dimensional von Neumann subalgebras of $\mathfrak{R}$ such that the weak closure $\left(\bigcup_k \mathfrak{N}_k\right)^-$ of $\bigcup_k \mathfrak{N}_k$ is $\mathfrak{R}$. Then*

$$h_{CS}(\Phi) = \lim_{k \to \infty} H(\mathfrak{N}_k, \Phi). \tag{IV.5}$$

Von Neumann algebras having the property assumed in the theorem above are called hyperfinite. This theorem was used in [CS] to compute the entropy of the non-commutative Bernoulli shift.

As expected, the CS entropy reduces to the KS entropy in the commutative case.

Theorem IV.2. *Let M be a compact space with a Borel probability measure $d\nu$ and let φ be a measure preserving automorphism of M. Consider the von Neumann algebra $\mathfrak{R} = L^\infty(M)$ with the trace τ given by (III.4), and the automorphism Φ of $\mathfrak{R}$ defined by (IV.2). Then $h_{CS}(\Phi) = h_{KS}(\varphi)$.*

IV.C. The actual definition of $H(\mathfrak{N}_1,\dots,\mathfrak{N}_k)$ will play a role in the next section and so we summarize it briefly.

We consider a von Neumann subalgebra $\mathfrak{N}\subset\mathfrak{R}$, and define the following inner product on $\mathfrak{N}$: $(x,y)=\tau(x^*y)$. The completion of $\mathfrak{N}$ in the norm induced by this inner product is a Hilbert space which we denote by $L^2(\mathfrak{N})$. Let $P_{\mathfrak{N}}:L^2(\mathfrak{R})\to L^2(\mathfrak{N})$ be the orthogonal projection on $L^2(\mathfrak{N})$ and let $E_{\mathfrak{N}}$ denote the restriction of $P_{\mathfrak{N}}$ to the dense subspace $\mathfrak{R}\subset L^2(\mathfrak{R})$. This is a non-commutative version of the conditional expectation operator.

Let now $\mathcal{S}_k$ be the set of all sequences of elements of $\mathfrak{R}$, $x=\{x_{\mathbf{i}}\}$, where $\mathbf{i}\in\mathbb{N}^k$, such that:

(a) $x_{\mathbf{i}}\geq 0$;
(b) all but finitely many $x_{\mathbf{i}}$ are zero;
(c) $\sum_{\mathbf{i}} x_{\mathbf{i}}=I$.

For $x\in\mathcal{S}_k$ and $1\leq l\leq k$ we set

$$x_j^l=\begin{cases} x_j, & \text{if } k=1;\\ \sum_{i_1\dots i_{l-1}i_{l+1}\dots i_k} x_{i_1\dots i_{l-1}ji_{l+1}\dots i_k}, & \text{if } k\geq 2.\end{cases} \tag{IV.6}$$

We define

$$H(\mathfrak{N}_1,\dots,\mathfrak{N}_k)=\sup_{x\in\mathcal{S}_k}\Big\{\sum_{\mathbf{i}\in\mathbb{N}^k}\eta(\tau(x_{\mathbf{i}}))-\sum_{l,j}\tau(\eta(E_{\mathfrak{N}_l}x_j^l))\Big\}. \tag{IV.7}$$

It now takes quite a lot of skill to establish the results stated above, and we refer the interested reader to [CS] for details.

V. Entropy of the quantized Anosov dynamics

V.A. We are now ready to compute the CS entropy of the quantized Anosov dynamics.

Theorem V.1. *The CS entropy of the quantized Anosov dynamics on the torus is equal to the classical value,*

$$h_{CS}(\Gamma_\hbar)=\log\mu_1. \tag{V.1}$$

Furthermore, if $|\mathrm{tr}(\gamma)|\leq 2$*, then* $h_{CS}(\Gamma_\hbar)=0$.

The proof of this theorem is based on the following technical result.

Lemma V.2. *Let* $\mathfrak{R}_1=L^\infty(M)$*, where* M *is a compact space with a Borel probability measure* $d\nu$ *and the natural faithful normal trace* $\tau_1(\cdot)=\int_M(\cdot)d\nu$*, let* $\mathfrak{R}_2$ *be a finite dimensional von Neumann algebra with a faithful trace* τ_2*, and let* Φ *and* Ψ *be trace preserving automorphisms of* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$*, respectively. Consider the hyperfinite type* II_1 *von Neumann algebra* $\mathfrak{R}=\mathfrak{R}_1\otimes\mathfrak{R}_2$ *with the faithful normal trace* $\tau=\tau_1\otimes\tau_2$*, and the* τ*-preserving automorphism* $\Gamma=\Phi\otimes\Psi$ *of* $\mathfrak{R}$*. Then* $h_{CS}(\Gamma)=h_{CS}(\Phi)$.

V.B. The proof of this lemma proceeds in steps.

Step 1. For any collection of finite dimensional subalgebras $\mathfrak{N}_1, \dots, \mathfrak{N}_k \subset \mathfrak{R}_1$,

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, \dots, \mathfrak{N}_k \otimes \mathfrak{R}_2) = H((\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2). \tag{V.2}$$

To prove this, note first that by property (C) of Section IV,

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, \dots, \mathfrak{N}_k \otimes \mathfrak{R}_2) \leq H((\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2), \tag{V.3}$$

as $\mathfrak{N}_j \otimes \mathfrak{R}_2 \subset (\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2$. To prove that

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, \dots, \mathfrak{N}_k \otimes \mathfrak{R}_2) \geq H((\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2), \tag{V.4}$$

we proceed as follows. Let $P_1^j, \dots, P_{n_j}^j$, $1 \leq j \leq n_j$, where $n_j = \dim \mathfrak{R}_j$, denote the minimal projections in $\mathfrak{N}_j$, and let $E_1, \dots, E_n$ be minimal projections in $\mathfrak{R}_2$ such that $\sum_j E_j = I$. We set

$$x_{i_0 i_1 \dots i_k} = E_{i_0} \otimes P_{i_1} \cdot \ldots \cdot P_{i_k} = (E_{i_0} \otimes P_{i_1}) \cdot \ldots \cdot (E_{i_0} \otimes P_{i_k}), \tag{V.5}$$

and observe that $\{x_{i_0 i_1 \dots i_k}\} \in S_{k+1}$ and it forms a system of minimal projections in $(\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2$. Since $\eta(x_j^l) = 0$, property (D) of Section IV implies that

$$\begin{aligned} H((\mathfrak{N}_1 \cup \ldots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2) &= \sum_{i_0 i_1 \dots i_k} \eta(\tau(x_{i_0 i_1 \dots i_k})) \\ &\leq H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, \dots, \mathfrak{N}_k \otimes \mathfrak{R}_2). \end{aligned}$$

Step 2. If $\mathfrak{N}$ is a finite dimensional subalgebra of $\mathfrak{R}_1$, then

$$H(\mathfrak{N} \otimes \mathfrak{R}_2) = H(\mathfrak{N}) + H(\mathfrak{R}_2). \tag{V.6}$$

To prove this, note that for a projection $P \in \mathfrak{N}_1$ and a projection $E \in \mathfrak{R}_2$,

$$\eta(\tau(P \otimes E)) = \eta(\tau_1(P))\tau_2(E) + \tau_1(P)\eta(\tau_2(E)). \tag{V.7}$$

Denoting by $P_1, \dots, P_m$ and $E_1, \dots, E_n$ systems of minimal projections in $\mathfrak{N}$ and $\mathfrak{R}_2$, respectively, and using property (D), we obtain

$$\begin{aligned} H(\mathfrak{N} \otimes \mathfrak{R}_2) &= \sum_{j,k} \eta(\tau_1(P_j))\tau_2(E_k) + \tau_1(P_j)\eta(\tau_2(E_k)) \\ &= \sum_j \eta(\tau_1(P_j)) + \sum_k \eta(\tau_2(E_k)) \\ &= H(\mathfrak{N}) + H(\mathfrak{R}_2). \end{aligned}$$

Step 3. Choose now an increasing sequence $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ of finite dimensional subalgebras of $\mathfrak{R}_1$, such that $\left(\bigcup_n \mathfrak{P}_n\right)^- = \mathfrak{R}_1$. Then $\{\mathfrak{P}_n \otimes \mathfrak{R}_2\}_{n\in\mathbb{N}}$ forms an increasing sequence of finite dimensional subalgebras of $\mathfrak{R}_1 \otimes \mathfrak{R}_2$, and $\left(\bigcup_n \mathfrak{P}_n \otimes \mathfrak{R}_2\right)^- = \mathfrak{R}_1 \otimes \mathfrak{R}_2$. Therefore, by Theorem IV.1,

$$h_{CS}(\Phi \otimes \Psi) = \lim_{n\to\infty} H(\mathfrak{P}_n \otimes \mathfrak{R}_2, \Phi \otimes \Psi). \tag{V.8}$$

By Steps 1 and 2,

$$\begin{aligned}
H(\mathfrak{P}_n &\otimes \mathfrak{R}_2, \Phi(\mathfrak{P}_n) \otimes \Psi(\mathfrak{R}_2), \ldots, \Phi^{k-1}(\mathfrak{P}_n) \otimes \Psi^{k-1}(\mathfrak{R}_2)) \\
&= H(\mathfrak{P}_n \otimes \mathfrak{R}_2, \Phi(\mathfrak{P}_n) \otimes \mathfrak{R}_2, \ldots, \Phi^{k-1}(\mathfrak{P}_n) \otimes \mathfrak{R}_2) \\
&= H((\mathfrak{P}_n \cup \Phi(\mathfrak{P}_n) \cup \ldots \cup \Phi^{k-1}(\mathfrak{P}_n))'' \otimes \mathfrak{R}_2) \\
&= H((\mathfrak{P}_n \cup \Phi(\mathfrak{P}_n) \cup \ldots \cup \Phi^{k-1}(\mathfrak{P}_n))'') + H(\mathfrak{R}_2) \\
&= H(\mathfrak{P}_n, \Phi(\mathfrak{P}_n), \ldots, \Phi^{k-1}(\mathfrak{P}_n)) + H(\mathfrak{R}_2).
\end{aligned}$$

But $H(\mathfrak{R}_2)$ is a constant independent of k, and so

$$H(\mathfrak{R}_1 \otimes \mathfrak{R}_2, \Phi \otimes \Psi) = H(\mathfrak{R}_1, \Phi), \tag{V.9}$$

which proves the lemma. □

V.C. It is now easy to prove Theorem V.1. Indeed, according to Theorem III.3 and Theorem III.4, $\mathfrak{A}_\hbar$ and $\Gamma_\hbar$ have precisely the structure required by the above lemma. Hence, $h_{CS}(\Gamma_\hbar) = h_{CS}(\Phi)$. Theorem IV.2 implies that $h_{CS}(\Phi) = h_{KS}(\gamma)$, and our theorem follows from the results quoted in the Introduction. □

It is an interesting and difficult question, even if without physical significance, whether Theorem V.1 holds without the assumption that $\lambda/2\pi$ is rational. In that case, $\mathfrak{A}_\hbar$ is not isomorphic to a finite dimensional algebra tensored by an abelian algebra, and so Lemma V.2 cannot be applied. In the case of topological entropy, Voiculescu [V1] has recently shown that the entropy of the quantized dynamics does not exceed the classical value.

V.D. We conclude this section with a brief discussion of a dynamical system on a torus which is ergodic but is not chaotic. Consider the Kronecker map on the torus defined by

$$K: \ (x_1, x_2) \to (x_1 + \omega_1, x_2 + \omega_2). \tag{V.10}$$

This map is known to be ergodic if and only if the frequencies ω_1 and ω_2 are linearly independent over $\mathbb{Z}$. The Kronecker map is, however, not chaotic, as its KS entropy is easily found to be zero [CFS].

In terms of the complex variable z, the Kronecker map reads

$$K \ : z \to z + \omega,$$

with $\omega = (\omega_1 + i\omega_2)/\sqrt{2}$, and so to quantize it we need to find a unitary operator implementing the following Bogolubov transformation:

$$A^\dagger \to A^\dagger + \omega I \ .$$

As in the case of the Anosov map the unitary operator is uniquely (up to a phase) determined by the above condition. In fact, an easy consequence of (II.2) is that

$$U(-\omega)A^\dagger U(-\omega)^{-1} = A^\dagger + \omega I,$$

and so $U(-\omega)$ is the required unitary operator.

Let now $K_\hbar$ be the automorphism of the quantum torus given by by $K_\hbar(\cdot) = U(-\omega)(\cdot)U(-\omega)^{-1}$. Evaluated on the generators of $\mathfrak{A}_\hbar$, $K_\hbar$ is:

$$\begin{aligned} K_\hbar(U) &= e^{2i\pi\omega_1}U, \\ K_\hbar(V) &= e^{2i\pi\omega_2}V. \end{aligned} \tag{V.11}$$

Assume now that $\hbar = 1/2\pi N$, in which case Theorem III.3 is applicable. It is easy to see that $K_\hbar$ can be factorized, with the first factor given by the following automorphism of $L^\infty(\mathbb{T}^2)$:

$$f(\theta) \to f(\theta_1 + N\omega_1, \theta_2 + N\omega_2). \tag{V.12}$$

Hence, the CS entropy of $K_\hbar$ is equal to the KS entropy of (V.12) and is thus zero.

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