The Fate of Lifshits Tails in Magnetic Fields

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Abstract

We investigate the integrated density of states of the Schrödinger operator in the Euclidean plane with a perpendicular constant magnetic field and a random potential. For a Poisson random potential with a non-negative algebraically decaying single-impurity potential we prove that the leading asymptotic behaviour for small energies is always given by the corresponding classical result in contrast to the case of vanishing magnetic field. We also show that the integrated density of states of the operator restricted to the eigenspace of any Landau level exhibits the same behaviour. For the lowest Landau level, this is in sharp contrast to the case of a Poisson random potential with a delta-function impurity potential.

Key words: Random Schrödinger operators, magnetic fields, Lifshits tails

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Version of February, 22 1995
To appear in Journal of Statistical Physics

1 Introduction

Random Schrödinger operators are differential operators on $\mathrm{L}^2(\mathbb{R}^d)$ formally given by $-\frac{1}{2}\nabla^2 + V_\omega$, where V_ω is an ergodic (or metrically transitive) random scalar potential. Here ∇ denotes the nabla operator in the d-dimensional Euclidean space $\mathbb{R}^d$ and $\mathrm{L}^2(\mathbb{R}^d)$ is the Hilbert space of Lebesgue square-integrable complex-valued functions on $\mathbb{R}^d$. These operators have been thoroughly investigated by physicists as well as mathematicians. See [38], [11] or [23] for reviews of basic concepts and rigorous results. For a more physical point of view see [43], [31]. The generalization of random Schrödinger operators to non-zero constant magnetic field are operators of the form

$$H(V_\omega) := \frac{1}{2}(i\nabla + A)^2 + V_\omega, \tag{1.1}$$

where $A : \mathbb{R}^d \to \mathbb{R}^d$ is a non-random vector potential such that the magnetic-field tensor (B_{jk}) given by $B_{jk} := \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}$ is constant. The two-dimensional version of (1.1) is widely believed to serve as a minimal model for the integer quantum Hall effect [48], [21], [2] and has therefore been intensively investigated by physicists, see for instance [1], [26], [48], [21].

Only recently rigorous studies of the spectral properties of random operators of the form (1.1) have appeared [34], [35], [8], [49], [50], [12], [16], [17], [10], but see also the related earlier work [19]. The self-averaging property of the integrated density of states has been established under fairly general conditions and various of its asymptotic properties have been considered. Also the existence of localized states has been proven [12], [16], [17].

In this note we are concerned with the Schrödinger operator (1.1) in two dimensions with a Poisson random potential determined by a non-negative single-site potential U, decaying algebraically at infinity. That is,

$$V_\omega(x) = \sum_j U(x - p_\omega(j)), \tag{1.2}$$

where $p_\omega(j)$ are random points in the Euclidean plane $\mathbb{R}^2$ distributed in accordance with Poisson's law. We will show how rigorous versions of results in [6] and [7] can be used to find the leading asymptotic behaviour of the integrated density of states as the energy approaches the infimum of the spectrum from above.

In the zero-field case the asymptotic behaviour changes its character depending on the decay of the single-site potential at infinity. For slow decay it is governed solely by the potential energy [37], that is, by classical effects. For rapid decay the kinetic energy also becomes relevant leading to genuine quantum effects. The form of the latter asymptotic behaviour has been discovered by Lifshits [29], [30], [31]. Convincing arguments for the validity of Lifshits' conjecture were given by Friedberg and Luttinger [18], [32]. Its rigorous proof [15], [36], [37] relies on Donsker's and Varadhan's involved large-deviation results [15], [14, Section 4.3] about the long-time asymptotics of certain Wiener integrals.

For the case of non-zero constant magnetic field we will show that the low-energy tail is always given by the classical result irrespective of the decay of the single-site potential. Basically, this is due to the fact that in this case the ground state of the unperturbed Schrödinger operator $H(0)$ consists of square-integrable functions.

The distance between successive eigenvalues of $H(0)$, commonly referred to as Landau levels, and the degeneracy per area of each eigenvalue increase linearly with the strength of the magnetic field. For a fixed concentration of non-interacting electrons and sufficiently strong field it is therefore physically reasonable to investigate only the restriction $E_0 H(V_\omega) E_0$ to the eigenspace $E_0 \mathrm{L}^2(\mathbb{R}^2)$ of the lowest Landau level instead of the full Schrödinger operator $H(V_\omega)$. For a rigorous discussion of this point see [34], [8], [10]. Remarkably, Wegner [51] succeeded in calculating the corresponding restricted integrated density of states for the case of a delta-correlated Gaussian random potential. A rigorous version of this derivation is given in [34]. Wegner's result was quickly extended to general delta-correlated random potentials by Brézin, Gross and Itzykson [5], including Poisson potentials, the single-site potential of which being a Dirac delta function. The calculation in [5] relies on the resummation of a suitable representation of an averaged Neumann series. A non-perturbative derivation was given by Klein and Perez [25].

We will show that the restriction to any Landau level does not alter the leading asymptotic behaviour of the integrated density of states at the lower spectral boundary for algebraically decaying single-site potentials. This behaviour is in sharp contrast to that for delta-correlated Poisson potentials, where it can happen that the integrated density of states is not continuous at the infimum of the spectrum.

Although we will only discuss the two-dimensional case, our result generalizes to even dimensions and a non-degenerate magnetic-field tensor, that is, to the case where the magnetic-field tensor (B_{jk}) is constant and has full rank.

2 Statement of the Result

We will assume throughout that the random potential V_ω is the **Poisson random potential** [38, Example 1.15(d)] with concentration $\varrho > 0$ and **non-negative single-site potential** $U : \mathbb{R}^2 \to [0, \infty]$, where U decays algebraically and integrably at infinity, that is

$$U \geq 0, \quad \lim_{|x| \to \infty} |x|^\alpha U(x) = \mu, \quad 0 < \mu < \infty, \ \alpha > 2. \tag{2.1}$$

In addition, for technical reasons we cannot allow for too strong local singularities. Therefore we will assume besides (2.1) either

$$U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) \tag{2.2}$$

or, more restrictively,

$$U \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2). \tag{2.3}$$

Remarks 2.1

i) Either of the conditions ensure that V_ω is an ergodic measurable non-negative random potential, where ergodicity is meant with respect to the group of spatial translations in $\mathbb{R}^2$.

ii) We note that (2.1) together with (2.2) implies

$$\|U\|_p < \infty \quad \text{for all } 1 \leq p \leq 2, \tag{2.4}$$

while (2.1) together with (2.3) implies

$$\|U\|_p \leq (\|U\|_1)^{1/p}\,(\|U\|_\infty)^{(p-1)/p} < \infty \quad \text{for all } 1 \leq p \leq \infty. \tag{2.5}$$

Here as usual

$$\|f\|_p := \begin{cases} \left(\displaystyle\int |f(y)|^p \,\mathrm{d}y\right)^{1/p} & \text{for} \quad p < \infty \\ \operatorname*{ess\,sup}_{y \in \mathbb{R}^2} |f(y)| & \phantom{\text{for}} \quad p = \infty \end{cases} \tag{2.6}$$

denotes the norm of $f \in \mathrm{L}^p(\mathbb{R}^2)$.

Let $\mathbb{E}$ denote the expectation with respect to the random potential. By (2.4) or (2.5) one has the finiteness

$$\mathbb{E}\Big[(V_\omega(x))^2\Big] = \varrho \int (U(y))^2 \,\mathrm{d}y + \left(\varrho \int U(y)\,\mathrm{d}y\right)^2 < \infty \tag{2.7}$$

of the potential's second moment at any point $x \in \mathbb{R}^2$. This implies that the potential V_ω is almost surely locally square-integrable and, moreover, that the mapping

$$x \mapsto \int_{|x-y|\leq 1} (V_\omega(y))^2 \,\mathrm{d}y \tag{2.8}$$

is almost surely polynomially bounded as $|x| \to \infty$, confer [23, Proof of Theorem 5.1].

By (2.5) one has the finiteness

$$\begin{aligned} \mathbb{E}\Big[(V_\omega(x))^k\Big] &\leq k!\,(\|U\|_\infty)^k \left(\mathbb{E}\Big[\mathrm{e}^{V_\omega(x)/\|U\|_\infty}\Big] - 1\right) \\ &= k!\,(\|U\|_\infty)^k \left(\exp\Big\{\varrho \int \left(\mathrm{e}^{U(y)/\|U\|_\infty} - 1\right)\mathrm{d}y\Big\} - 1\right) \\ &\leq k!\,(\|U\|_\infty)^k \left(\exp\Big\{\varrho\,\frac{\|U\|_1}{\|U\|_\infty}\,(\mathrm{e}-1)\Big\} - 1\right) \end{aligned} \tag{2.9}$$

of the k-th moment for all positive integers $k \in \mathbb{N}$.

iii) The Laplace characteristic functional of V_ω

$$\mathbb{E}\Big[\mathrm{e}^{-\int J(x)V_\omega(x)\,\mathrm{d}x}\Big] = \exp\Big\{-\varrho \int \left(1 - \mathrm{e}^{-\int J(x-y)\,U(y)\,\mathrm{d}y}\right)\mathrm{d}x\Big\} \tag{2.10}$$

is well-defined for all non-negative $J \in \mathcal{S}(\mathbb{R}^2)$, confer for example [38, Proposition 1.16]. Here we have introduced $\mathcal{S}(\mathbb{R}^2)$ for the Schwartz space of rapidly decreasing arbitrarily-often-differentiable complex-valued functions on $\mathbb{R}^2$.

iv) One can look upon V_ω as given by

$$V_\omega(x) = \int U(x-y)\,\mathrm{d}m_\omega(y), \tag{2.11}$$

where m_ω is the Poisson random measure on $\mathbb{R}^2$ with concentration ϱ. Since m_ω is purely atomic, V_ω can be interpreted physically as the potential generated by uniformly distributed non-interacting impurities, the influence of each impurity being described by the same repulsive potential U, see (1.2).

In the sequel we suppose $B > 0$ and the vector potential

$$A(x) := \frac{B}{2}\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \quad x =: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.12}$$

given in the symmetric gauge. Then the **Schrödinger operator** (1.1) is almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ provided (2.1) and (2.2) hold. This follows from [13, Theorem 1.15] and the fact that $V_\omega \psi \in \mathrm{L}^2(\mathbb{R}^2)$ for all $\psi \in \mathcal{S}(\mathbb{R}^2)$ almost surely because of Remark 2.1.*ii)*. On account of gauge equivalence [28], with the choice (2.12) there is no loss of generality for the description of a constant magnetic field.

The spectral resolution of the unperturbed part dates back to Landau [27] and is given by

$$H(0) = \sum_{n=0}^{\infty} \varepsilon_n \, E_n, \quad \varepsilon_n := \left(n + \frac{1}{2}\right) B, \tag{2.13}$$

where the eigenvalues ε_n are called Landau levels and the corresponding infinite-dimensional eigenprojectors E_n are integral operators with kernels given by

$$E_n(x,y) := \frac{B}{2\pi} \, \mathrm{e}^{\frac{iB}{2}(x_1 y_2 - x_2 y_1) - \frac{B}{4}(x-y)^2} \, \mathrm{L}_n\!\left(\frac{B}{2}(x-y)^2\right). \tag{2.14}$$

Here L_n is the n-th Laguerre polynomial [20, Section 8.97].

The explicit formula for the integral kernel of E_n shows that $E_n(x, \bullet) \in \mathcal{S}(\mathbb{R}^2)$. Since $\mathcal{S}(\mathbb{R}^2)$ is stable under convolution, this implies that $\mathcal{S}(\mathbb{R}^2) \cap E_n \mathrm{L}^2(\mathbb{R}^2) \subset E_n \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$. Furthermore, $E_n \mathcal{S}(\mathbb{R}^2)$ is dense in $E_n \mathrm{L}^2(\mathbb{R}^2)$, because $\mathcal{S}(\mathbb{R}^2)$ is dense in $\mathrm{L}^2(\mathbb{R}^2)$. Taking finally into account, that the mapping (2.8) is almost surely polynomially bounded, we conclude that the Schrödinger operator restricted to the eigenspace of the n-th Landau level, in symbols $E_n H(V_\omega) E_n = \varepsilon_n E_n + E_n V_\omega E_n$, and all its natural powers $(E_n H(V_\omega) E_n)^k$, $k \in \mathbb{N}$, are almost surely well-defined non-negative operators from $\mathcal{S}(\mathbb{R}^2)$ to $\mathrm{L}^2(\mathbb{R}^2)$.

As for the essential self-adjointness of $E_n H(V_\omega) E_n$ on $\mathcal{S}(\mathbb{R}^2)$, we were not able to prove it under the assumptions (2.1) and (2.2). However, under the stronger assumptions (2.1) and (2.3) a proof can be taylored along the lines given in the proof of Theorem 2.1 in [16]. By appealing to Nelson's analytic-vector theorem [41, Theorem X.39] and to the fact that $\mathcal{S}(\mathbb{R}^2)$ contains a countable dense subset it suffices to check that

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathbb{E}\left[\left(\left\| (E_n V_\omega E_n)^k \, \psi \right\|_2 \right)^2 \right] \right)^{1/2} < \infty \tag{2.15}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^2)$ and some $t > 0$. We start by observing that there is a constant $c_n(B)$ such that

$$|E_n(x,y)| \le c_n(B) \, \frac{B}{8\pi} \, \mathrm{e}^{-\frac{B}{8}(x-y)^2} \tag{2.16}$$

for all $x, y \in \mathbb{R}^2$. Moreover, by an iterated version of Hölder's inequality and translation invariance one has

$$\mathbb{E}\left[\prod_{j=1}^{2k} V_\omega(x^{(j)}) \right] \le \mathbb{E}\left[(V_\omega(0))^{2k} \right] \tag{2.17}$$

for all $x^{(1)}, \ldots, x^{(2k)} \in \mathbb{R}^2$. Copying now the steps between Equations (2.15) and (2.20) in [16] our estimates (2.16), (2.17) and (2.9) yield for all $k \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}\left[\left(\left\|(E_n V_\omega E_n)^k \psi\right\|_2\right)^2\right] \;\leq\; & (2k)!\; (c_n(B)\, \|U\|_\infty)^{2k} \\ & \times \frac{B c_n(B)\, (\|\psi\|_1)^2}{8\pi(2k+1)} \left(\exp\left\{\varrho\, \frac{\|U\|_1}{\|U\|_\infty}\, (\mathrm{e}-1)\right\} - 1\right). \end{aligned} \tag{2.18}$$

As a consequence, (2.15) is valid for all $0 \leq t < (2\, c_n(B)\, \|U\|_\infty)^{-1}$.

In the following we are allowed and will understand the unrestricted operators $H(V_\omega)$ and the restricted operators $E_n H(V_\omega) E_n$ as being almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ under the imposed assumptions (2.1), (2.2) and (2.1), (2.3), repectively.

Let $\mathbb{R} \ni \lambda \mapsto P_\lambda(X) = \Theta(\lambda - X)$ denote the projection-valued measure for the self-adjoint operator X. Here Θ is Heaviside's unit-step function: $\Theta(a) = 0$ for $a < 0$ and $\Theta(a) = 1$ for $a \geq 0$. Our objects of study, the **integrated density of states** N and the ***n*-th restricted integrated density of states** R_n, are defined by

$$N(\lambda) := \mathbb{E}[P_\lambda(H(V_\omega))(x,x)] \tag{2.19}$$

and

$$R_n(\lambda) := \mathbb{E}[(E_n\, P_\lambda(E_n H(V_\omega) E_n)\, E_n)(x,x)]\,, \tag{2.20}$$

respectively.

Remarks 2.2

i) The integral kernel $(x,y) \mapsto P_\lambda(H(V_\omega))(x,y)$ of the spectral projection $P_\lambda(H(V_\omega))$ almost surely exists and is jointly continuous in (x,y). This can be seen from [9, Section 6] using Remark 2.1.*ii)*. For vanishing vector potential this result is standard [47, Theorem B.7.1]. The above continuity assures that (2.19) is well-defined. Since V_ω is translation invariant, the right-hand side of (2.19) is independent of $x \in \mathbb{R}^2$.

ii) In [49, Proposition 3.2], [10] it is shown that the more familiar and physically reasonable way of defining the integrated density of states by means of a macroscopic limit yields (2.19). This amounts to first restricting the Schrödinger operator $H(V_\omega)$ to a finite region – a square with Dirichlet boundary conditions, say. Then one defines the integrated density of states of the finite system to be the number of eigenvalues below λ divided by the region's area. Finally, one proves that this quantity becomes non-random in the macroscopic limit, which is usually summarized as the self-averaging property of the integrated density of states.

iii) The Cauchy-Schwarz inequality, $E_n^2 = E_n$ and $E_n(x,x) = B/(2\pi)$ yield for all $\psi \in \mathrm{L}^2(\mathbb{R}^2)$

$$\|E_n \psi\|_\infty \leq \left(\frac{B}{2\pi}\, \langle \psi, \psi \rangle\right)^{1/2} = \left(\frac{B}{2\pi}\right)^{1/2} \|\psi\|_2, \tag{2.21}$$

where, as usual, $\langle \bullet, \bullet \rangle$ symbolizes the standard scalar product for $\mathrm{L}^2(\mathbb{R}^2)$. Therefore, the restriction $E_n X E_n$ of any bounded self-adjoint operator X on

$L^2(\mathbb{R}^2)$ to the eigenspace of the n-th Landau level is a Carleman integral operator [47, Corollary A.1.2]. Since $E_n(x,y)$ is smoothing and a projection, the integral kernel $(x,y) \mapsto (E_n X E_n)(x,y)$ is jointly continuous in (x,y). Hence (2.20) is well-defined, too. Furthermore, the right-hand side of (2.20) is independent of $x \in \mathbb{R}^2$ due to the translation invariance of V_ω, see the Appendix for details.

iv) On physical grounds R_0 should be a reasonable approximation to N for strong magnetic fields. This is given a precise meaning in [34, Proposition 1], [8, Theorem 5] and [10]. For the significance of R_n for general n see [7], [8].

v) In the reasoning in the above Remarks *i)* and *iii)* we have swept measurability questions with respect to ω under the rug as we will do in the sequel. These problems, however, can be fixed by the methods of [24] or [11, Sections V.1, V.3].

Our aim is to identify the asymptotics of $N(\lambda)$ and $R_n(\lambda)$ as λ approaches the lower spectral boundary, that is, $\lambda \downarrow \inf \operatorname{spec}(H(V_\omega))$ or $\lambda \downarrow \inf \operatorname{spec}(E_n H(V_\omega) E_n)$, respectively. By employing the so-called magnetic translations [52], [25], see Equation (A.17) below, standard arguments [24], [23] show that both $H(V_\omega)$ and $E_n H(V_\omega) E_n$ are ergodic families of operators [49], [50], [12], [16], [10]. Therefore, their spectra are non-random quantities, see [49, Theorem 2.1], [24, Theorem 1] or [23, Theorem 4.3.1]. It will turn out that $\inf \operatorname{spec}(H(V_\omega)) = \varepsilon_0$ and $\inf \operatorname{spec}(E_n H(V_\omega) E_n) = \varepsilon_n$ as expected. Our result is the following.

Theorem 2.3

Under the assumptions (2.1) and (2.2) one has for all $B > 0$

$$\lim_{\lambda \downarrow 0} \lambda^{2/(\alpha-2)} \ln N(\varepsilon_0 + \lambda) = -C(\alpha, \mu, \varrho) \tag{2.22}$$

for the integrated density of states N *and, similarly, under the assumptions (2.1) and (2.3)*

$$\lim_{\lambda \downarrow 0} \lambda^{2/(\alpha-2)} \ln R_n(\varepsilon_n + \lambda) = -C(\alpha, \mu, \varrho) \tag{2.23}$$

for the n*-th restricted integrated density of states* R_n*,* $n \in \mathbb{N} \cup \{0\}$*.*
Here we have set

$$C(\alpha, \mu, \varrho) := \frac{1}{2}(\alpha - 2)\, \mu^{2/(\alpha-2)} \left(\frac{2\pi\varrho}{\alpha}\, \Gamma\!\left(\frac{\alpha-2}{\alpha}\right)\right)^{\alpha/(\alpha-2)} > 0, \tag{2.24}$$

where Γ *denotes Euler's gamma function.*

Remarks 2.4

i) The result for the unrestricted integrated density of states N should be compared with the case $B = 0$. The asymptotic decay at the lower spectral boundary coincides with the behaviour for $B = 0$, if $(d =)\, 2 < \alpha < 4\, (= d + 2)$, that is, for slowly decaying single-site potential, see [37] or [38, Corollary 9.14], but it differs in the case $\alpha > 4\,(= d+2)$, see [36], [37] or [38, Theorem 10.2]. This is plausible from the Rayleigh-Ritz-like variational principle due to Luttinger [32] and Pastur [37], see also Equation (17.3) and Chapter 21 in [31]. For $B = 0$ and $\alpha > 4$ the optimal wavefunction becomes too sharply localized so that the unperturbed (kinetic) energy begins to play a significant rôle. For $B > 0$ the contribution of the unperturbed energy relative to the ground-state energy ε_0 can always be kept zero by choosing one of the square-integrable ground-state wavefunctions of $H(0)$.

ii) The lowest restricted integrated density of states R_0 for a delta-correlated Poisson potential, corresponding to $U(x) = \nu\delta(x)$, $\nu > 0$, is known exactly [5], [25], not only in the low-energy tail. If the mean number $2\pi\varrho/B$ of impurities over the spatial extent πl^2 of the ground-state wavefunctions of $H(0)$,

$$l^2 := \frac{1}{E_0(y,y)} \int E_0(y,x)\,(x-y)^2\, E_0(x,y)\,\mathrm{d}x = \frac{2}{B}, \tag{2.25}$$

is smaller than one, R_0 exhibits a jump at ε_0, the height of which being proportional to $1 - 2\pi\varrho/B$. This is plausible, because in the case $2\pi\varrho/B < 1$ one can imagine that, effectively, only a fraction of the ground-state wavefunctions is affected by the impurities [5]. This reasoning does not seem applicable to algebraically decaying impurity potentials, in accordance with our result that R_n is continuous at ε_n.

iii) For the reader's convenience, we present the generalizations of (2.22) and (2.23) to even space dimensions $d \geq 2$

$$\begin{aligned} &\lim_{\lambda\downarrow 0} \lambda^{d/(\alpha-d)} \ln N(\varepsilon_0+\lambda) = \lim_{\lambda\downarrow 0} \lambda^{d/(\alpha-d)} \ln R_n(\varepsilon_n+\lambda) \\ = \quad & -\frac{\alpha-d}{d}\,\mu^{d/(\alpha-d)} \left(\frac{\varrho}{\alpha}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,\Gamma\!\left(\frac{\alpha-d}{\alpha}\right)\right)^{\alpha/(\alpha-d)}. \end{aligned} \tag{2.26}$$

Here it is assumed that $\alpha > d$ and that the magnetic-field tensor (B_{jk}) is constant and has full rank.

iv) It would be interesting to compute subleading corrections to (2.26) as Luttinger and Waxler [33] did for zero magnetic field and, of course, $d < \alpha < d+2$. In contrast to the leading term given by (2.26) we expect these corrections to depend on the field.

3 Proof

For the proof of Theorem 2.3 we follow the strategy in [37]. Instead of $N(\lambda)$ we investigate its Laplace transform and use a Tauberian argument [38, Theorem 9.7]. Since $H(V_\omega)$ is bounded below by ε_0, this argument shows that (2.22) is equivalent to

$$\lim_{t\to\infty} t^{-2/\alpha} \ln \widetilde{N}(t) = -\pi\varrho\mu^{2/\alpha}\Gamma\!\left(\frac{\alpha-2}{\alpha}\right) = -\varrho\mu^{2/\alpha}\int\left(1-\mathrm{e}^{-|x|^{-\alpha}}\right)\mathrm{d}x, \tag{3.1}$$

where

$$\widetilde{N}(t) := \int \mathrm{e}^{-t\lambda}\,\mathrm{d}N(\varepsilon_0+\lambda) = \mathrm{e}^{t\varepsilon_0}\int \mathrm{e}^{-t\lambda}\,\mathrm{d}N(\lambda), \quad t>0, \tag{3.2}$$

is the shifted Laplace transform of N. Analogously we define

$$\widetilde{R}_n(t) := \int \mathrm{e}^{-t\lambda}\,\mathrm{d}R_n(\varepsilon_n+\lambda) = \mathrm{e}^{t\varepsilon_n}\int \mathrm{e}^{-t\lambda}\,\mathrm{d}R_n(\lambda), \quad t>0, \tag{3.3}$$

to be the shifted Laplace transform of R_n. Then (2.23) is equivalent to (3.1) with $\widetilde{R}_n$ replacing $\widetilde{N}$.

To establish the leading asymptotic behaviour of $\widetilde{N}(t)$ and $\widetilde{R}_n(t)$ for large t we use the following

Basic Inequalities 3.1

Let

$$\phi_n(\bullet) := \left(\frac{2\pi}{B}\right)^{1/2} E_n(\bullet,0). \tag{3.4}$$

Then

$$\frac{1}{2\pi t}\mathbb{E}\Big[\mathrm{e}^{-t\langle\phi_0,V_\omega\phi_0\rangle}\Big] \le \widetilde{N}(t), \tag{3.5}$$

$$\widetilde{N}(t) \le \frac{B\,\mathrm{e}^{t\varepsilon_0}}{4\pi\sinh(t\varepsilon_0)}\mathbb{E}\Big[\mathrm{e}^{-tV_\omega(0)}\Big], \tag{3.6}$$

and

$$\frac{B}{2\pi}\mathbb{E}\Big[\mathrm{e}^{-t\langle\phi_n,V_\omega\phi_n\rangle}\Big] \le \widetilde{R}_n(t), \tag{3.7}$$

$$\widetilde{R}_n(t) \le \frac{B}{2\pi}\mathbb{E}\Big[\mathrm{e}^{-tV_\omega(0)}\Big]. \tag{3.8}$$

Remarks 3.2

i) One can infer from (2.14) that $\|\phi_n\|_2 = 1$ and $\phi_n \in \mathcal{S}(\mathbb{R}^2)$. Therefore, the left-hand sides of (3.5) and (3.7) are well-defined, because almost surely $V_\omega \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and the mapping (2.8) is polynomially bounded.

ii) The bound (3.6) is a generalization of the classical upper bound for $\widetilde{N}$ [37], [38, Theorem 9.6] to non-zero magnetic fields. The essential ingredient for its derivation is the Golden-Thompson inequality. The bounds (3.7) as well as (3.8) stem from the Jensen-Peierls inequality. The inequality (3.5) is a variant of an inequality due to Berezin, Lieb and Luttinger, which in turn follows from the Jensen-Peierls inequality. In [37] and [38, Theorem 9.5] a lower bound similar to (3.5) is proven. It has the advantage of holding for more general families of random operators but allows for functions ϕ_0 with compact support only. For our proof it is essential, however, to allow for ϕ_0 as given in (3.4) with unbounded support.

iii) Non-rigorous derivations of the above bounds can be found in [6] and [7].

iv) A rigorous derivation of (3.6) for the case of a Gaussian random potential with a Gaussian covariance function can be read off from Equations (2.3), (2.8) and (2.10) in [34].

v) The above bounds are to some extent special cases of those presented in [10] for more general than Poisson random potentials. Our lines of reasoning are closely related to those in [10] but are considerably simplified by the fact that $V_\omega \geq 0$ almost surely. We have banished the proofs of the Basic Inequalities into the Appendix because on the one hand they follow the plan of [6] and [7] and on the other hand there are some unwieldy technicalities involved in supplying the missing rigour.

The Basic Inequalities provide us with asymptotically coinciding upper and lower bounds for the shifted Laplace transforms $\widetilde{N}$ and $\widetilde{R}_n$.

Upper bound

The inequality (3.8) and Remark 2.1.*iii)* imply

$$\widetilde{R}_n(t) \leq \frac{B}{2\pi} \exp\Big\{ -\varrho \int \Big(1 - \mathrm{e}^{-tU(x)}\Big)\, \mathrm{d}x \Big\}. \tag{3.9}$$

Therefore we have

$$\limsup_{t\to\infty} t^{-2/\alpha} \ln \widetilde{R}_n(t) \leq -\varrho \liminf_{t\to\infty} t^{-2/\alpha} \int \Big(1 - \mathrm{e}^{-tU(x)}\Big)\, \mathrm{d}x. \tag{3.10}$$

On the right-hand side of (3.10) we substitute $x \mapsto (\mu t)^{1/\alpha} x$ and use Fatou's lemma to interchange the limit and the integration. Since

$$\lim_{t\to\infty} t\, U\Big((\mu t)^{1/\alpha} x\Big) = |x|^{-\alpha}, \quad x \neq 0, \tag{3.11}$$

by (2.1), we conclude

$$\limsup_{t\to\infty} t^{-2/\alpha} \ln \widetilde{R}_n(t) \leq -\varrho \mu^{2/\alpha} \int \Big(1 - \mathrm{e}^{-|x|^{-\alpha}}\Big)\, \mathrm{d}x. \tag{3.12}$$

Note that this is also true with $\widetilde{N}$ replacing $\widetilde{R}_n$, because the different prefactors in the upper bounds (3.6) and (3.8) coincide asymptotically. ■

Lower bound

By means of Remark 2.1.*iii)* the inequality (3.7) reads more explicitly

$$\frac{B}{2\pi} \exp\Big\{ -\varrho \int \Big(1 - \mathrm{e}^{-t\int |\phi_n(x-y)|^2 U(y)\, \mathrm{d}y}\Big)\, \mathrm{d}x \Big\} \leq \widetilde{R}_n(t). \tag{3.13}$$

With the help of the substitution $x \mapsto (\mu t)^{1/\alpha} x$ this yields

$$-\varrho\mu^{2/\alpha} \limsup_{t\to\infty} \int \Big(1 - \mathrm{e}^{-\int \delta_t(x-y)\, t\, U\left((\mu t)^{1/\alpha} y\right)\, \mathrm{d}y}\Big)\, \mathrm{d}x \leq \liminf_{t\to\infty} t^{-2/\alpha} \ln \widetilde{R}_n(t), \tag{3.14}$$

where we have introduced the one-parameter family $\{\delta_t\}_{t>0} \subset \mathcal{S}(\mathrm{I\!R}^2)$ of probability densities on $\mathrm{I\!R}^2$

$$x \mapsto \delta_t(x) := \frac{B\,(\mu t)^{2/\alpha}}{2\pi}\, \mathrm{e}^{-B\,(\mu t)^{2/\alpha} x^2/2} \Big[\mathrm{L}_n\Big(B\,(\mu t)^{2/\alpha} x^2/2\Big)\Big]^2. \tag{3.15}$$

The Fourier representation

$$\delta_t(x) = \frac{1}{(2\pi)^2} \int \mathrm{e}^{ikx}\, \mathrm{e}^{-k^2/\left(2B\,(\mu t)^{2/\alpha}\right)} \left[\mathrm{L}_n\!\left(k^2/(2B\,(\mu t)^{2/\alpha})\right)\right]^2 \mathrm{d}k, \tag{3.16}$$

which may be verified with the help of [20, Equation 7.377], shows that δ_t approximates Dirac's delta function as $t \to \infty$. Therefore one can check

$$\limsup_{t\to\infty} \int \delta_t(x-y)\, t\, U\!\left((\mu t)^{1/\alpha} y\right)\, \mathrm{d}y \le |x|^{-\alpha}, \quad x \neq 0, \tag{3.17}$$

using (3.11) and the fact that U is both integrable and square-integrable. According to Fatou's lemma this suffices for the validity of

$$-\varrho\mu^{2/\alpha} \int \left(1 - \mathrm{e}^{-|x|^{-\alpha}}\right) \mathrm{d}x \le \liminf_{t\to\infty} t^{-2/\alpha} \ln \widetilde{R}_n(t). \tag{3.18}$$

To obtain the same estimate with $\widetilde{N}$ replacing $\widetilde{R}_n$ one only has to specialize to $n = 0$ and to note that the differing prefactors in (3.5) and (3.7) become both irrelevant on the logarithmic scale. ■

Appendix Proofs of the Basic Inequalities

The proofs of the bounds (3.5) and (3.6) for $\widetilde{N}$ rely on an approximation formula, which will be presented first.

Approximation A.1

Define for $\Omega \ge 0$

$$\hat{V}_{\omega,\Omega}(x) := V_\omega(x) + \frac{\Omega^2}{2}\, x^2. \tag{A.1}$$

Then for $t, \Omega > 0$ *the Euclidean propagator* $\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}$ *is almost surely of trace class and one has*

$$\widetilde{N}(t) = \lim_{\Omega\downarrow 0} \frac{\Omega^2 t}{2\pi}\, \mathrm{e}^{t\varepsilon_0}\, \mathbb{E}\!\left[\mathrm{Tr}\!\left\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\right\}\right]. \tag{A.2}$$

Proof

According to the Feynman-Kac-Itô formula, see for example [44, Theorem 15.5], $\hat{V}_{\omega,\Omega} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\hat{V}_{\omega,\Omega} \ge 0$ ensure that $\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}$ has the integral kernel

$$\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}(x,y) := \frac{1}{2\pi t}\, \mathrm{e}^{-(x-y)^2/(2t)} \int \mathrm{e}^{-S_t(\hat{V}_{\omega,\Omega}|b)}\, \mathrm{d}\mu_{0,x}^{t,y}(b), \tag{A.3}$$

where $\mu_{0,x}^{t,y}$ denotes the probability measure associated with the two-dimensional Brownian bridge from $b(0) = x$ to $b(t) = y$. Here the potentials' part of the Euclidean action is given by

$$S_t(\hat{V}_{\omega,\Omega}|b) := i \int_0^t A(b(s))\, \mathrm{d}b(s) + \int_0^t \hat{V}_{\omega,\Omega}(b(s))\, \mathrm{d}s. \tag{A.4}$$

Since the Brownian bridge is a continuous semimartingale [39, Example V.6.3], the Itô stochastic line integral in (A.4) is well-defined [39, Section II.4].

In [9, Section 6] it is proven that the right-hand side of (A.3) is jointly continuous in (t, x, y), $t > 0$, $x, y \in \mathbb{R}^2$, and we will get as a by-product in the proof of (3.6) below that $\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}$ is of trace class. Thus [22, Example X.1.18] we may write

$$\mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\} = \frac{1}{2\pi t}\int \mathrm{e}^{-S_t(\hat{V}_{\omega,\Omega}|b+x)}\,\mathrm{d}\mu_{0,0}^{t,0}(b)\,\mathrm{d}x, \tag{A.5}$$

where we have performed the rigid shift $b \mapsto b + x$. Due to the translation invariance of V_ω we get with the help of Itô's formula [39, Corollary to Theorem II.32]

$$\mathbb{E}\Big[\mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\}\Big] = \frac{1}{\Omega^2 t^2}\,\mathbb{E}\Big[\int \mathrm{e}^{-S_t(V_\omega|b)-\Omega^2 t\,\sigma_t^2(b)/2}\,\mathrm{d}\mu_{0,0}^{t,0}(b)\Big], \tag{A.6}$$

where

$$\sigma_t^2(b) := \int_0^t (b(s))^2\,\frac{\mathrm{d}s}{t} - \left(\int_0^t b(s)\frac{\mathrm{d}s}{t}\right)^2 \geq 0. \tag{A.7}$$

Since by $V_\omega \geq 0$ one has

$$\left|\mathrm{e}^{-S_t(V_\omega|b)-\Omega^2 t\,\sigma_t^2(b)/2}\right| \leq 1, \tag{A.8}$$

the theorem of dominated convergence is applicable and yields

$$\lim_{\Omega\downarrow 0}\frac{\Omega^2 t}{2\pi}\mathbb{E}\Big[\mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\}\Big] = \frac{1}{2\pi t}\,\mathbb{E}\Big[\int \mathrm{e}^{-S_t(V_\omega|b)}\,\mathrm{d}\mu_{0,0}^{t,0}(b)\Big]. \tag{A.9}$$

Employing again the Feynman-Kac-Itô formula, for $\Omega = 0$, the continuity of the involved integral kernels and the translation invariance of the random potential, one achieves by (2.19) and (3.2)

$$\frac{1}{2\pi t}\,\mathbb{E}\Big[\int \mathrm{e}^{-S_t(V_\omega|b)}\,\mathrm{d}\mu_{0,0}^{t,0}(b)\Big] = \mathrm{e}^{-t\varepsilon_0}\,\widetilde{N}(t), \tag{A.10}$$

which concludes the proof of (A.2). ■

Proof of (3.5)

Let

$$\psi_{p,q}(x) := \mathrm{e}^{ip\left(x+\frac{q}{2}\right)}\,\phi_0(x-q). \tag{A.11}$$

Then $\{\psi_{p,q}\}_{p,q\in\mathbb{R}^2}$ is nothing but the standard overcomplete family of coherent states associated with the Heisenberg-Weyl group generated from the ground state of a two-dimensional isotropic harmonic oscillator. Since $\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}$ is almost surely of trace class whenever $t, \Omega > 0$, we may write, see [3, Equation 1.13] or [46, Theorem A.1.2]

$$\frac{1}{(2\pi)^2}\int\Big\langle\psi_{p,q}, \mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\psi_{p,q}\Big\rangle\,\mathrm{d}p\,\mathrm{d}q = \mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\}. \tag{A.12}$$

Since $H(\hat{V}_{\omega,\Omega})$ is almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ and $\psi_{p,q} \in \mathcal{S}(\mathbb{R}^2)$, the Jensen-Peierls inequality [4], [45, Section 8(c)] implies

$$\mathrm{e}^{-t\left\langle\psi_{p,q}, H(\hat{V}_{\omega,\Omega})\psi_{p,q}\right\rangle} \leq \Big\langle\psi_{p,q}, \mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\psi_{p,q}\Big\rangle. \tag{A.13}$$

Due to the translation invariance of the random potential, we get by inserting (A.13) into (A.12) after some calculation

$$\frac{2\pi}{\Omega^2 t}\,\mathrm{e}^{-\Omega^2 t/B}\,\frac{\mathrm{e}^{-t\varepsilon_0}}{2\pi t}\,\mathbb{E}\Big[\mathrm{e}^{-t\langle\phi_0, V_\omega\phi_0\rangle}\Big] \leq \mathbb{E}\Big[\mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\}\Big]. \tag{A.14}$$

With the help of (A.2) this proves (3.5). ■

Proof of (3.6)

First note that for $\Omega > 0$ the operators $H(0)$, $\hat{V}_{\omega,\Omega}$ and $H(\hat{V}_{\omega,\Omega})$ are almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ and non-negative. Furthermore, $\mathrm{e}^{-tH(0)/2}\,\mathrm{e}^{-t\hat{V}_{\omega,\Omega}/2}$ is almost surely Hilbert-Schmidt, since [40, Theorem VI.23]

$$\int \left| \mathrm{e}^{-tH(0)/2}(x,y)\,\mathrm{e}^{-t\hat{V}_{\omega,\Omega}(y)/2} \right|^2 \mathrm{d}x\,\mathrm{d}y = \frac{B}{4\pi\sinh(t\varepsilon_0)} \int \mathrm{e}^{-t\hat{V}_{\omega,\Omega}(y)}\,\mathrm{d}y < \infty. \tag{A.15}$$

Therefore, we can use the Golden-Thompson inequality in the version given in the Corollary to [42, Theorem XIII.103] to conclude that $\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}$ is almost surely of trace class and its trace is bounded from above by the squared Hilbert-Schmidt norm (A.15). Taking the expectation with respect to the random potential, this shows

$$\mathbb{E}\Big[\mathrm{Tr}\Big\{\mathrm{e}^{-tH(\hat{V}_{\omega,\Omega})}\Big\}\Big] \leq \frac{2\pi}{\Omega^2 t}\,\frac{B}{4\pi\sinh(t\varepsilon_0)}\,\mathbb{E}\Big[\mathrm{e}^{-tV_\omega(0)}\Big]. \tag{A.16}$$

Using (A.2) once again, (3.6) is proven. ■

As a preliminary to the proofs of (3.7) and (3.8) we show that the n-th restricted integrated density of states R_n is independent of $x \in \mathbb{R}^2$. This is achieved with the help of the unitary magnetic-translation operator W_x defined by

$$(W_x\,\psi)\,(y) := \mathrm{e}^{\frac{iB}{2}(y_1x_2 - y_2x_1)}\,\psi(y-x), \quad \psi \in \mathrm{L}^2(\mathbb{R}^2), \tag{A.17}$$

because on the one hand the eigenprojectors E_n are left invariant under its action, that is

$$W_x^\dagger\,E_n\,W_x = E_n, \tag{A.18}$$

and on the other hand the potential is simply shifted

$$W_x^\dagger\,V_\omega\,W_x = V_\omega \circ T_x, \quad T_x y := y + x. \tag{A.19}$$

Using these properties together with the translation invariance of the random potential, the right-hand side of (2.20) is seen to be independent of $x \in \mathbb{R}^2$. For a complete discussion of the magnetic-translation group associated with $H(0)$ see [52], [25].

Proof of (3.7)

The combination of the definition (2.20) for $x = 0$ with (3.3) and (3.4) gives a more explicit expression

$$\widetilde{R}_n(t) = \frac{B}{2\pi}\mathbb{E}\Big[\Big\langle \phi_n, \mathrm{e}^{-tE_nV_\omega E_n}\,\phi_n \Big\rangle\Big] \tag{A.20}$$

for the shifted Laplace transform of the n-th restricted integrated density of states. Since $\phi_n \in \mathcal{S}(\mathbb{R}^2)$ and $E_nV_\omega E_n$ is almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$, we may use the Jensen-Peierls inequality [4], [45, Section 8(c)] and the fact that $E_n\phi_n = \phi_n$ to estimate

$$\mathrm{e}^{-t\langle\phi_n, V_\omega\phi_n\rangle} \leq \Big\langle \phi_n, \mathrm{e}^{-tE_nV_\omega E_n}\,\phi_n \Big\rangle. \tag{A.21}$$

The insertion of (A.21) into (A.20) completes the proof of (3.7). ■

For the proof of (3.8) we will use again an approximation formula which we single out as

Approximation A.2

Define the centered Poisson potential, truncated outside a disk of radius $r > 0$ *about the origin,*

$$\check{V}_{\omega,r}(x) := \Theta(r - |x|)\,(V_\omega(x) - \mathbb{E}[V_\omega(0)])\,. \tag{A.22}$$

Then

$$\lim_{r\to\infty} \frac{1}{\pi r^2} \int_{|x|\le r} \mathbb{E}\Big[\Big(E_n\,\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}\,E_n\Big)\,(x,x)\Big]\,\mathrm{d}x = \mathrm{e}^{t\,\mathbb{E}[V_\omega(0)]}\,\widetilde{R}_n(t). \tag{A.23}$$

Proof

First we use for the magnetic-translation operator (A.17) the properties (A.18) and (A.19) with $\check{V}_{\omega,r}$ replacing V_ω, the definition (3.4) of ϕ_n and the translation invariance of V_ω to rewrite

$$\begin{aligned} &\frac{1}{\pi r^2} \int_{|x|\le r} \mathbb{E}\Big[\Big(E_n\,\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}\,E_n\Big)\,(x,x)\Big]\,\mathrm{d}x \\ =\;& \frac{B}{2\pi^2} \int_{|\zeta|\le 1} \mathbb{E}\Big[\Big\langle \phi_n, \mathrm{e}^{-tE_n\overline{V}_{\omega,r,\zeta}E_n}\phi_n\Big\rangle\Big]\,\mathrm{d}\zeta. \end{aligned} \tag{A.24}$$

Here we have introduced

$$\overline{V}_{\omega,r,\zeta}(x) := \Theta\Big(1 - \Big|\zeta - \frac{x}{r}\Big|\Big)\,(V_\omega(x) - \mathbb{E}[V_\omega(0)])\,. \tag{A.25}$$

As a next step we claim that for all $|\zeta| < 1$ and almost all ω

$$E_n\overline{V}_{\omega,r,\zeta}E_n \overset{r\to\infty}{\longrightarrow} E_n\,(V_\omega - \mathbb{E}[V_\omega(0)])\,E_n \tag{A.26}$$

in the strong resolvent sense. Since $E_n\overline{V}_{\omega,r,\zeta}E_n$ is almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ for all $r > 0$ and so is the right-hand side of (A.26), it is sufficient [40, Theorem VIII.25] to check that for all $\psi \in \mathcal{S}(\mathbb{R}^2)$ and almost all ω

$$\Big\| E_n\,\Big(\overline{V}_{\omega,r,\zeta} - V_\omega + \mathbb{E}[V_\omega(0)]\Big)\,E_n\psi\Big\|_2 \overset{r\to\infty}{\longrightarrow} 0. \tag{A.27}$$

Since E_n is bounded and maps $\mathcal{S}(\mathbb{R}^2)$ into itself, this follows from

$$\begin{aligned} &\int \Big|\Big(\overline{V}_{\omega,r,\zeta}(x) - V_\omega(x) + \mathbb{E}[V_\omega(0)]\Big)\,\psi(x)\Big|^2\,\mathrm{d}x \\ \le\;& \int_{|x|>r(1-|\zeta|)} |(V_\omega(x) - \mathbb{E}[V_\omega(0)])\,\psi(x)|^2\,\mathrm{d}x \overset{r\to\infty}{\longrightarrow} 0 \end{aligned} \tag{A.28}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^2)$, $|\zeta| < 1$. The last inequality is due to the estimate $\Theta\Big(\Big|\zeta - \frac{x}{r}\Big| - 1\Big) \le \Theta(|x| - r\,(1 - |\zeta|))$ and its right-hand side vanishes as $r \to \infty$, because the mapping (2.8) is almost surely polynomially bounded.

The strong resolvent convergence (A.26) now implies [40, Theorem VIII.20] together with (A.20)

$$\lim_{r\to\infty} \mathbb{E}\Big[\Big\langle \phi_n, \mathrm{e}^{-tE_n\overline{V}_{\omega,r,\zeta}E_n}\,\phi_n\Big\rangle\Big] = \frac{2\pi}{B}\,\mathrm{e}^{t\,\mathbb{E}[V_\omega(0)]}\,\widetilde{R}_n(t). \tag{A.29}$$

From $\overline{V}_{\omega,r,\zeta}(x) \geq -\mathbb{E}[V_\omega(0)]$ one has

$$\left\langle \phi_n, \mathrm{e}^{-tE_n\overline{V}_{\omega,r,\zeta}E_n}\,\phi_n\right\rangle \leq e^{t\,\mathbb{E}[V_\omega(0)]} \tag{A.30}$$

almost surely. Therefore one can use the dominated-convergence theorem, (A.29) and (A.24) to obtain (A.23). ■

Proof of (3.8)

Let $\check{V}_{\omega,r}$ be as given in (A.22) and $0 < r < \infty$. We first note that $E_n\check{V}_{\omega,r}E_n$ is almost surely of trace class. This can be seen [40, Theorem VI.22(h)], for example, by noting that both $E_n\left|\check{V}_{\omega,r}\right|^{1/2}$ and $\mathrm{sgn}(\check{V}_{\omega,r})\left|\check{V}_{\omega,r}\right|^{1/2}E_n$ are almost surely Hilbert-Schmidt because [40, Theorem VI.23]

$$\int\left|E_n(x,y)\left|\check{V}_{\omega,r}(y)\right|^{1/2}\right|^2 \mathrm{d}x\,\mathrm{d}y = \frac{B}{2\pi}\int\left|\check{V}_{\omega,r}(y)\right|\mathrm{d}y < \infty. \tag{A.31}$$

Therefore

$$\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n} - 1 = E_n\check{V}_{\omega,r}E_n\sum_{k=1}^{\infty}\frac{(-t)^k}{k!}\left(E_n\check{V}_{\omega,r}E_n\right)^{k-1} \tag{A.32}$$

is of trace class, because it is the product of a trace-class operator and a norm-convergent sum [40, Theorem VI.19(c)]. Thus we have

$$\begin{aligned}&\int\mathbb{E}\Big[\Big(E_n\left(\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}-1\right)E_n\Big)(x,x)\Big]\,\mathrm{d}x\\ =\;&\mathbb{E}\Big[\mathrm{Tr}\Big\{E_n\left(\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}-1\right)E_n\Big\}\Big]\end{aligned} \tag{A.33}$$

due to the continuity of the integral kernel [22, Example X.1.18], see Remark 2.2.*iii)*.

Analogously to the arguments in the proof of (3.7) we may rewrite the integrand in the approximation formula (A.23) as a scalar product, use the Jensen-Peierls inequality and in a next step the Jensen inequality together with $\mathbb{E}\big[\check{V}_{\omega,r}(x)\big] = 0$ to show

$$\mathbb{E}\Big[\Big(E_n\left(\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}-1\right)E_n\Big)(x,x)\Big] \geq \frac{B}{2\pi}\,\mathbb{E}\Big[\mathrm{e}^{-\frac{2\pi}{B}t\,(E_n\check{V}_{\omega,r}E_n)(x,x)}-1\Big] \geq 0. \tag{A.34}$$

Thanks to (A.34), (A.33) and the approximation formula (A.23) can be combined to yield the preliminary estimate

$$\mathrm{e}^{t\,\mathbb{E}[V_\omega(0)]}\,\widetilde{R}_n(t) - \frac{B}{2\pi} \leq \limsup_{r\to\infty}\frac{1}{\pi r^2}\mathbb{E}\Big[\mathrm{Tr}\Big\{E_n\left(\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}-1\right)E_n\Big\}\Big]. \tag{A.35}$$

Finally, we use the Jensen-Peierls inequality in the version of Berezin [4], [45, Section 8(c)], which implies

$$\mathrm{Tr}\Big\{E_n\left(\mathrm{e}^{-tE_n\check{V}_{\omega,r}E_n}-1\right)E_n\Big\} \leq \mathrm{Tr}\Big\{E_n\left(\mathrm{e}^{-t\check{V}_{\omega,r}}-1\right)E_n\Big\}. \tag{A.36}$$

This is justified, since both $E_n\check{V}_{\omega,r}E_n$ and $\check{V}_{\omega,r}$ are almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$, $E_n\mathcal{S}(\mathbb{R}^2)$ is dense in $E_n\mathrm{L}^2(\mathbb{R}^2)$ and because the right-hand side is well-defined. The latter can be seen by an argument analogous to that at the beginning of this proof employing $\left(\mathrm{e}^{-t\check{V}_{\omega,r}}-1\right)\in \mathrm{L}^1(\mathbb{R}^2)$. Due to the continuity of the integral kernel of $E_n\left(\mathrm{e}^{-t\check{V}_{\omega,r}}-1\right)E_n$ we are allowed [22, Example X.1.18] to calculate

$$\begin{aligned}
&\mathbb{E}\Big[\mathrm{Tr}\Big\{E_n\left(\mathrm{e}^{-t\check{V}_{\omega,r}}-1\right)E_n\Big\}\Big] \\
=\;& \frac{B}{2\pi}\,\mathbb{E}\Big[\int\left(\mathrm{e}^{-t\check{V}_{\omega,r}(x)}-1\right)\mathrm{d}x\Big] \\
=\;& \frac{Br^2}{2}\left(\mathrm{e}^{t\,\mathbb{E}[V_\omega(0)]}\,\mathbb{E}\Big[\mathrm{e}^{-tV_\omega(0)}\Big]-1\right).
\end{aligned}\tag{A.37}$$

To finish the proof of (3.8), we only have to insert (A.36) into (A.37) and perform the limit with the help of (A.37). ■

Acknowledgments

We thank L. A. Pastur for stimulating comments. We are much indebted to W. Fischer and P. Müller for constructive criticism. K. B. would like to thank the members of the Mathematics Department of the Ruhr-Universität Bochum for their kind hospitality, made possible by financial support of the Sonderforschungsbereich 237: “Unordnung und große Fluktuationen”. D. H. would like to thank the “Graduiertenkolleg Geometrie und Mathematische Physik an der Ruhr-Universität Bochum” for financial support. Moreover, this work was partially supported by the Deutsche Forschungsgemeinschaft under Grant No. Le 330/5-1.

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