Normal forms of vector fields satisfying certain geometric conditions

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Abstract

The paper describes the simplified polynomial resonant normal forms of vector fields satisfying some geometric conditions (namely, preserving volume or symplectic forms or contact structures) with respect to C^k smooth changes of variables preserving the same structure.

Introduction

In the last years much attention has been paid to the theory of finitely smooth normal forms of vector fields in the neighbourhood of a singular point (see, for example, [1], [2]). The problem of reducing to normal form can be divided into three steps. At the first stage, one brings finite jets of the vector field to the *resonant* normal form by the aid of polynomial changes of variables. The second step deals with "cutting the tail", *i.e.*, reducing to the *polynomial* (resonant) normal form. Finally, one uses finitely smooth transformations to *simplify* the resonant normal form, *i.e.*, to eliminate as many monomial terms as possible. A great many of results obtained in these research directions are summed up in the book [2].

The recent work [3] is devoted to (the second step of) normalization of vector fields satisfying some geometric conditions (namely, preserving volume or symplectic forms or contact structures) by a smooth change of variables preserving the same structure. The method of proof used in [3] is based on the *deformation method* of singularity theory (R.Tom, J.Mather). As the authors emphasize, this method is ideally suited to discussing conjugacy problems in which a geometric structure is preserved. The deformation method reduces non-local non-linear problems to linear ones, namely, to solving *cohomology equations* (for the corresponding *hamiltonians*).

This paper is concerned with finitely smooth normal forms of volume preserving, symplectic and contact vector fields in the vicinity of a *hyperbolic* singular point. We make essential use of the techniques developed in [3] as well as of the results presented in [2,4]. Similar results hold for diffeomorphisms.

We would like to warn the reader that in this paper (as well as in the book [2]) the term *resonance* refers to the *real parts* of the eigenvalues. This enables us to obtain normal form which smoothly depend on the vector fields. That is why one may hope that these normal forms are of interest to bifurcation theory.

1 Geometric structures and differential forms

Let $\mathcal{M}$ be a contractible manifold of dimension $\dim \mathcal{M} = d$ (for example, a convex neighbourhood of the origin in $\mathbb{R}^n$). Let p be an integer, $0 \le p \le d$, and γ be a differential p-form. The form γ is said to be *non-degenerate* if the mapping $\xi \to i_\xi(\gamma)$ which to every vector field $\xi : \mathcal{M} \to T\mathcal{M}$ puts in correspondence the $(p-1)$-form $i_\xi(\gamma)$ is an isomorphism; that is, for each $x \in \mathcal{M}$ the mapping $T_x\mathcal{M} \to \Lambda^{p-1}(T_x^*\mathcal{M})$ defined by $v \mapsto i_v(\gamma(x))$ is an isomorphism of the tangent space $T_x\mathcal{M}$ onto the space of all skew-symmetric $(p-1)$-linear forms on $T_x\mathcal{M}$. This is possible only when

$$d = \dim(T_x\mathcal{M}) = \dim(\Lambda^{p-1}(T_x^*\mathcal{M})) = \binom{d}{p-1},$$

i.e., whenever $p = 2$ or $p = d$.

If γ is a non-degenerate p-form, then each differential $(p-1)$-form ω determines a unique vector field ξ such that $i_\xi(\gamma) = \omega$ (and *vice versa*).

If ω is a *closed* $(p-1)$-form (*i.e.*, $d\omega = 0$) then by Poincaré's lemma there exists a $(p-2)$-form F such that

$$i_\xi(\gamma) \equiv \omega = dF.$$

In this case the form F is called the *hamiltonian* of the vector field ξ. If, in addition, γ is a closed p-form, then the form $\omega = i_\xi(\gamma)$ is closed iff $L_\xi\gamma = 0$. The last equality means that the phase flow of ξ *respects* the p-form γ.

We shall consider three particular cases of geometric structures defined by differential forms.

Let μ be a *non-degenerate d-form* (*volume form*) on $\mathcal{M}$ (note that each volume form is closed). A vector field is called *volume preserving* (more exactly, preserving the volume form μ) if its (local) phase flow respects the form μ, *i.e.*, $L_\xi\mu = 0$. In this case, there exists a $(d-2)$-form F such that $i_\xi\mu = dF$. The hamiltonian F is determined up to adding a closed form. Each $(d-2)$-form F determines a μ-preserving vector field ξ.

Let $\mathcal{M}$ be a contractible manifold of dimension $2d$ provided with the *symplectic structure*, *i.e.*, with a *closed non-degenerate 2-form* ω. The vector field ξ is said to be *symplectic* if $L_\xi\omega = 0$. The *hamiltonian* F, $i_\xi(\omega) = dF$, is a function determined up to a constant. Each function may serve as a hamiltonian of some symplectic vector field.

Now let $\mathcal{M}$ be a contractible manifold of dimension $2d+1$ and θ be a differential 1-form on $\mathcal{M}$ such that $\theta \wedge (d\theta)^d$ is a volume form. The *contact structure* on $\mathcal{M}$ is defined to be the codimension 1 subbundle of the tangent bundle $T\mathcal{M}$ determined by $\theta = 0$. Note that if $\lambda : \mathcal{M} \to \mathbb{R}$ is a non-vanishing function then the form $\lambda\theta$ gives the same contact structure. The vector field ξ is said to be a *contact vector field* if its flow preserves the contact structure, *i.e.*, there exists a function $\mu_\xi : \mathcal{M} \to \mathbb{R}$ such that

$$L_\xi\theta = \mu_\xi\theta.$$

If ξ is a contact vector field, then the function $F = i_\xi\theta \equiv \theta(\xi)$ is called the *contact hamiltonian.* Every function $F : \mathcal{M} \to \mathbb{R}$ determines a contact vector field ξ. The correspondence $\xi \to F$, $F = \theta(\xi)$, is one-to-one.

Let μ be volume form on $\mathbb{R}^d$, ω be a symplectic form on $\mathbb{R}^{2d}$ and θ be a contact form on $\mathbb{R}^{2d+1}$. According to a theorem due to Darboux, the following statements are true:

1. There exists a system of coordinates $u = (u_1, \ldots, u_d)$ on $\mathbb{R}^d$ such that

$$\mu(u) = du_1 \wedge \ldots \wedge du_d. \tag{1.1}$$

2. There exist coordinates $(u, v) = (u_1, \ldots, u_d, v_1, \ldots, v_d)$ on $\mathbb{R}^{2d}$ such that

$$\omega(u, v) = du_1 \wedge dv_1 + \ldots + du_d \wedge dv_d. \tag{1.2}$$

3. One can find coordinates $(u, v, w) = (u_1, \ldots, u_d, v_1, \ldots, v_d, w)$ on $\mathbb{R}^{2d+1}$ so that

$$\theta(u, v, w) = u_1 dv_1 + \ldots + u_d dv_d + dw. \tag{1.3}$$

In what follows, the forms (1.1)–(1.3) will be called *standard.* The changes of variables which respect these forms are called *canonical.*

2 Transition formulas

Let $u \in \mathbb{R}^d$, $1 \le i < j \le d$. In order to simplify notation, we shall write

$$\begin{aligned}
&\widehat{du_i} = du_1 \wedge \ldots \wedge du_{i-1} \wedge du_{i+1} \wedge \ldots \wedge du_d, \\
&\widehat{du_i} \wedge \widehat{du_j} = du_1 \wedge \ldots \wedge du_{i-1} \wedge du_{i+1} \wedge \ldots \wedge du_{j-1} \wedge du_{j+1} \wedge \ldots \wedge du_d.
\end{aligned}$$

For example, each $(d-1)$-form γ can be written as

$$\gamma(u) = \sum_{s=1}^{d} \gamma^s(u) \widehat{du}_s.$$

Similarly, if F is a $(d-2)$-form, then

$$F(u) = \sum_{p=1}^{d-1} \sum_{q=p+1}^{d} F^{pq}(u) \widehat{du}_p \wedge \widehat{du}_q$$

Let ξ be a vector field which preserves the volume form μ. With respect to canonical coordinates, we can choose a hamiltonian F so that

$$F^{pq}(u) = (-1)^{p+q} \int_0^1 [\xi^q(ut)u_p - \xi^p(ut)u_q] t^{d-2} dt \tag{2.1}$$

$$(p = 1, \ldots, d-1; \quad q = p+1, \ldots, d).$$

Conversely, every $(d-2)$-form F serves as a hamiltonian of the following volume preserving vector field:

$$\xi^p(u) = \sum_{q=1}^{p-1}(-1)^{p+q}\frac{\partial F^{qp}(u)}{\partial u_q} - \sum_{q=p+1}^{d}(-1)^{p+q}\frac{\partial F^{pq}(u)}{\partial u_q} \quad (p=1,\dots,d). \tag{2.2}$$

The relationship between symplectic vector fields $(\xi,\eta) = (\xi^1,\dots,\xi^d,\eta^1,\dots,\eta^d)$ on $\mathbb{R}^{2d}$ and the corresponding hamiltonians F can be expressed as follows:

$$F(u,v) = \sum_{p=1}^{d}\int_0^1 [\xi^p(ut,vt)v_p - \eta^p(ut,vt)v_p]dt; \tag{2.3}$$

$$\xi^p(u,v) = \frac{\partial F(u,v)}{\partial v_p}, \quad \eta^p(u,v) = -\frac{\partial F(u,v)}{\partial u_p} \quad (p=1,\dots,d). \tag{2.4}$$

Similarly, for contact vector fields (ξ,η,ζ) on $\mathbb{R}^{2d+1}$ and contact hamiltonians $F(u,v,w)$ one has

$$F(u,v,w) = \sum_{p=1}^{d} u_p\eta^p(u,v,w) + \zeta(u,v,w); \tag{2.5}$$

$$\begin{aligned} \xi^p(u,v,w) &= -\partial F/\partial v_p + u_p\partial F/\partial w, \\ \eta^p(u,v,w) &= \partial F/\partial u_p \quad (p=1,\dots,d), \\ \zeta(u,v,w) &= F(u,v,w) - \sum_{p=1}^{d} u_p\partial F/\partial u_p \end{aligned} \tag{2.6}$$

Moreover,

$$\mu_{(\xi,\eta,\zeta)}(u,v,w) = \partial F(u,v,w)/\partial w.$$

3 Normalization of linear parts of vector fields

We shall say that the hyperbolic linear vector field $\dot{x} = Ax \quad (x \in \mathbb{R}^d)$ is reduced to the *block-diagonal normal form* if:

(1) $A = \mathrm{diag}\{A_1,\dots,A_n\}$;

(2) A_i is a linear operator all of whose eigenvalues have one and the same real part θ_i $(i = 1,\dots,n)$;

(3) all the diagonal elements of the matrix A_i are equal to θ_i;

(4) if $i \neq j$, then $\theta_i \neq \theta_j$.

Let ξ be a vector field on $\mathbb{R}^d$ and the origin $0 \in \mathbb{R}^d$ be a hyperbolic singular point. Assume that ξ preserves the volume form μ. There exists a coordinate system $u = (u_1,\dots,u_d)$ on $\mathbb{R}^d$ which reduces μ to the standard form (1.1). Denote $B = D\xi(0)$. It is well-known that one can find a real linear change of variables $y = Su$, $\det S = 1$, which brings the operator B to the Jordan real normal form. Clearly, the vector field $\dot{y} = Ay$, where $A = SBS^{-1}$, takes the block-diagonal normal form. Because $\det S = 1$ the volume form is still the standard one. Observe that $\mathrm{div}A = \sum_{i=1}^{n} \theta_i \cdot \dim A_i = 0$.

Now let ξ be a symplectic vector field on $\mathbb{R}^{2d}$, with the origin being a hyperbolic singular point. Choose coordinates $(u_1, \ldots, u_d, v_1, \ldots, v_d)$ so that the symplectic form ω takes the form (1.2). Denote $B = D\xi(0)$. It is known that $B = IC$, where C is a symplectic linear operator and

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

According to Williamson's theorem (see [6]), there exists a canonical linear transformation which brings $B = IC$ to the form

$$\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

where A is expressed in the block-diagonal normal form and all its eigenvalues have positive real parts.

Let ξ be a contact vector field on $\mathbb{R}^{2d+1}$ and the origin be a hyperbolic singular point of ξ. Suppose (u, v, w) are canonical coordinates, *i.e.*, the contact form θ is given by (1.3). Denote $B = D\xi(0)$, then we have

$$B(u, v, w) = \begin{pmatrix} Au \\ -Av + \gamma v \\ \gamma w \end{pmatrix},$$

where A is a hyperbolic $d \times d$-matrix and γ is a non-zero number. Now "symplectize" the linear part of the vector field ξ by introducing coordinates (z, v, w, λ) on $\mathbb{R}^{2d+2}$ so that $z = u\lambda,\ \lambda \in \mathbb{R}\backslash\{0\}$, and letting

$$\dot{z} = Az - \gamma z, \quad \dot{v} = -Av + \gamma v, \quad \dot{w} = \gamma w, \quad \dot{\lambda} = -\gamma\lambda.$$

This linear symplectic vector field can be reduced to the block-diagonal normal form by some canonical linear transformation which leaves invariant the subspace $w = 0,\ \lambda = 0$. Because this transformation commutes with the natural action of the multiplicative group of non-zero real numbers, it induces a canonical contact transformation on $\mathbb{R}^{2d+1}$ which brings the linear part of the contact vector field ξ to the block-diagonal normal form.

4 Deformation method

The deformation method is a powerful tool for solving various conjugacy problems [1,3]. Let us outline this very useful approach (in the context of vector fields).

Let ξ and η be vector fields defined on a manifold $\mathcal{M}$. For $\varepsilon \in \mathbb{R}$ set $\xi_\varepsilon = \varepsilon\eta + (1 - \varepsilon)\xi$. Define the vector field Ξ on $\mathcal{M} \times \mathbb{R}$ by $\dot{x} = \xi_\varepsilon(x)$, $\dot{\varepsilon} = 0$. Suppose we have found a vector field Φ on $\mathcal{M} \times \mathbb{R}$ of the form $\dot{x} = \varphi_\varepsilon(x)$, $\dot{\varepsilon} = 1$, which commutes with Φ, i.e.

$$[\Phi, \Xi] = 0. \tag{4.1}$$

Let $F(x, \varepsilon, t)$ $(x \in \mathcal{M},\ \varepsilon \in \mathbb{R},\ t \in \mathbb{R})$ denote the phase flow of Φ. Put $h(x) = \mathrm{pr}_1 \circ F(x, 0, 1)$. It is easy to show that $(h)_*(\xi) \equiv Th \circ \xi \circ h^{-1} = \eta$, *i.e.*, ξ and η are conjugate by h. Observe that (4.1) is equivalent to

$$D\varphi_\varepsilon \cdot \xi_\varepsilon(x) = D\xi_\varepsilon(x) \cdot \varphi_\varepsilon + \eta(x) - \xi(x) \quad (x \in \mathcal{M}), \tag{4.2}$$

which is called the *cohomology equation.* The corresponding characteristic system is

$$\dot{y} = D\xi_\varepsilon(x) \cdot y + \eta(x) - \xi(x), \;\; \dot{x} = \xi_\varepsilon(x) \quad (x, y \in \mathcal{M};\; \varepsilon \in \mathbb{R}). \tag{4.3}$$

The solutions of (4.2) are nothing else than invariant sections $y = \varphi_\varepsilon(x)$ of (4.3). Thus, the deformation method reduces the conjugacy problem to finding a smooth invariant section $y = \varphi_\varepsilon(x)$ of the affine extension (4.3) and then to integrating the system $\dot{x} = \varphi_\varepsilon(x)$, $\dot{\varepsilon} = 1$.

Now let us turn to the case where the vector fields ξ and η satisfy a certain geometric condition (for example, they are symplectic). It is naturally to ask whether one can choose the conjugation map to be canonical (in the corresponding sense). In order to solve this more delicate conjugacy problem, we need to use the relationships between vector fields and differential forms (see section 2), as suggested in [3].

Let γ be a closed non-degenerate k-form and ξ and η be vector fields which preserve γ. It is not hard to verify that

$$i_{[\xi_\varepsilon, \varphi_\varepsilon]}(\gamma) = d(i_{\xi_\varepsilon}(i_{\varphi_\varepsilon}(\gamma))),$$

therefore (4.2) implies that

$$d(i_{\xi_\varepsilon}(i_{\varphi_\varepsilon}(\gamma))) = i_{\eta-\xi}(\gamma).$$

Let F and G be the hamiltonians of the vector fields ξ and η. Denote by H_ε the hamiltonian of the vector field φ_ε we are seeking. Then

$$d(i_{\xi_\varepsilon}(dH_\varepsilon)) = dG - dF.$$

Taking into account the equality

$$L_{\xi_\varepsilon} H_\varepsilon = i_\varepsilon(dH_\varepsilon) + d i_{\xi_\varepsilon}(H_\varepsilon)$$

we get

$$dL_{\xi_\varepsilon} H_\varepsilon = d i_{\xi_\varepsilon}(dH_\varepsilon),$$

whence

$$d(L_{\xi_\varepsilon} H_\varepsilon - G + F) = 0.$$

It suffices to find the hamiltonian H_ε from the equation

$$L_{\xi_\varepsilon} H_\varepsilon = G - F, \tag{4.3}$$

which is called the *cohomology equation for hamiltonians.*

Let θ be a contact differential 1-form. Because

$$i_{[\xi_\varepsilon, \varphi_\varepsilon]}(\theta) = L_\xi \theta(\varphi_\varepsilon) - \mu_{\xi_\varepsilon} \theta(\varphi_\varepsilon), \quad L_{\xi_\varepsilon} \theta = \mu_{\xi_\varepsilon} \theta,$$

the equation (4.2) writes

$$L_{\xi_\varepsilon} H_\varepsilon - \mu_{\xi_\varepsilon} H_\varepsilon = G - F, \tag{4.4}$$

where F and G are the contact hamiltonians of the given contact vector fields ξ and η, and H_ε is the contact hamiltonian of the unknown vector field φ_ε.

Thus, in order to prove that two vector fields which preserve the volume form (the symplectic form, the contact structure) are conjugate with one another by a diffeomorphism respecting the corresponding structure, we have to solve the cohomology equation (4.3) or (4.4), then find the corresponding vector field φ_ε and integrate the system $\dot{x} = \varphi_\varepsilon(x)$, $\dot{\varepsilon} = 1$.

5 Cohomology equations for hamiltonians

Let us write out the cohomology equations for hamiltonians expressed in the canonical coordinates (see (1.1)–(1.3)) as well as the corresponding characteristic systems.

a) The volume preserving case

$$
\begin{aligned}
&\sum_{t=1}^{d} \xi_\varepsilon^t \frac{\partial H_\varepsilon^{pq}}{\partial u_t} + \sum_{t=1}^{d} \frac{\partial \xi_\varepsilon^t}{\partial u_t} H_\varepsilon^{pq} - \left(\frac{\partial \xi_\varepsilon^p}{\partial u_p} + \frac{\partial \xi_\varepsilon^q}{\partial u_q}\right) H_\varepsilon^{pq} \\
&+ \sum_{t=1}^{p-1} [(-1)^{q+t} \frac{\partial \xi_\varepsilon^q}{\partial u_t} H_\varepsilon^{tp} + (-1)^{p+t-1} \frac{\partial \xi_\varepsilon^p}{\partial u_t} H_\varepsilon^{tq}] \\
&+ \sum_{t=p+1}^{q-1} [(-1)^{q+t-1} \frac{\partial \xi_\varepsilon^q}{\partial u_t} H_\varepsilon^{pt} + (-1)^{p+t-1} \frac{\partial \xi_\varepsilon^p}{\partial u_t} H_\varepsilon^{tq}] \\
&+ \sum_{t=q+1}^{d} [(-1)^{q+t-1} \frac{\partial \xi_\varepsilon^q}{\partial u_t} H_\varepsilon^{pt} + (-1)^{p+t} \frac{\partial \xi_\varepsilon^p}{\partial u_t} H_\varepsilon^{qt}] \\
&\equiv \sum_{t=1}^{d} \xi_\varepsilon^t \frac{\partial H_\varepsilon^{pq}}{\partial u_t} + \mathcal{A}_{pq} \cdot H_\varepsilon = G_{pq} - F_{pq} \\
&(p = 1, \ldots, d-1; \quad q = p+1, \ldots, d);
\end{aligned}
\tag{5.1}
$$

$$
\begin{gathered}
\dot{H}_\varepsilon^{pq} = -\mathcal{A}_{pq}(u) \cdot H_\varepsilon + G^{pq}(u) - F^{pq}(u), \ \dot{u} = \xi_\varepsilon(u) \\
(1 \le p < q \le d; \ u \in \mathbb{R}^d, \ H_\varepsilon^{pq} \in \mathbb{R}).
\end{gathered}
\tag{5.2}
$$

Here $H_\varepsilon = \{H_\varepsilon^{pq} : 1 \le p < q \le d\}$ and, for each $u \in \mathbb{R}^d$, $\mathcal{A}_{pq}(u) \in L(\mathbb{R}^{d(d-1)/2}, \mathbb{R})$.

b) The symplectic case

$$
\sum_{t=1}^{d} \left(\xi_\varepsilon^t \frac{\partial H_\varepsilon}{\partial u_t} + \eta_\varepsilon^t \frac{\partial H_\varepsilon}{\partial v_t}\right) = G - F;
\tag{5.3}
$$

$$
\dot{H}_\varepsilon = G(u,v) - F(u,v), \quad \dot{u} = \xi_\varepsilon(u,v), \quad \dot{v} = \eta_\varepsilon(u,v).
\tag{5.4}
$$

c) The contact case

$$
\sum_{t=1}^{d} \left(\xi_\varepsilon^t \frac{\partial H_\varepsilon}{\partial u_t} + \eta_\varepsilon^t \frac{\partial H_\varepsilon}{\partial v_t}\right) + \xi_\varepsilon \frac{\partial H_\varepsilon}{\partial w} - \mu_{\xi_\varepsilon} H_\varepsilon = G - F;
\tag{5.5}
$$

$$
\begin{gathered}
\dot{H}_\varepsilon = \mu_{\xi_\varepsilon} \cdot H_\varepsilon + G(u,v,w) - F(u,v,w), \\
\dot{u} = \xi_\varepsilon(u,v,w), \quad \dot{v} = \eta_\varepsilon(u,v,w), \quad \dot{w} = \zeta_\varepsilon(u,v,w).
\end{gathered}
\tag{5.6}
$$

6 Introducing integrated variables

Before proceeding to the normalization of jets of vector fields and their hamiltonians we have to introduce integrated canonical "coordinates". As it was shown in section 3, without loss of generality we may assume that the linear parts of vector fields are reduced to the block-diagonal normal form (with respect to the canonical coordinates).

Denote $d_i = \dim A_i$ $(i = 1, \dots, n)$. Then $d_1 + \dots + d_n = d$. Given a point $u = (u_1, \dots, u_d) \in \mathbb{R}^d$, denote $x_i = (u_{j_{i-1}+1}, \dots, u_{j_i})$, where $j_i = \sum_{p=1}^{i} d_p$ $(i = 1, \dots, n)$, $j_0 = 0$. We shall also use the notation $J_i = \{j_{i-1} + 1, \dots, j_i\}$.

Let $\omega \in \mathbb{Z}_+^d$ be some multi-index. Denote $\tau^i = \sum_{j \in J_i} \omega^j$ $(i = 1, \dots, n)$. Clearly, $\tau = (\tau^1, \dots, \tau^n) \in \mathbb{Z}_+^n$. Thus, we have defined an operator $S : \mathbb{Z}_+^d \to \mathbb{Z}_+^n$, $\tau = S(\omega)$. Every monomial $u^\omega \equiv u_1^{\omega^1} \cdot \ldots \cdot u_d^{\omega^d}$ can be written as $x^\tau = x_1^{\tau^1} \cdot \ldots \cdot x_n^{\tau^n}$, where $\tau = S(\omega)$.

Let $\bar{\xi}$ be a polynomial vector field of degree Q, *i.e.*,

$$\bar{\xi}_j(u) = A_j u + \sum_{|\omega|=2}^{Q} \bar{a}_\omega^j u^\omega \quad (j = 1, \dots, d). \tag{6.1}$$

In terms of the integrated variables, $\bar{\xi}$ can be rewritten as

$$\xi_i(x) = A_i x_i + \sum_{|\tau|=2}^{Q} a_\tau^i x^\tau \quad (i = 1, \dots, n), \tag{6.2}$$

where $\xi_i = \{\bar{\xi}_j : j \in J_i\}$, $a_\tau^i = \{\bar{a}_\omega^j : j \in J_i, \tau = S(\omega)\}$.

Assume that the vector field (6.1) preserves the standard volume from (1.1), then its hamiltonian F defined by (2.1) is a polynomial of degree $Q + 1$, *i.e.*,

$$\bar{F}^{pq}(u) = \sum_{|\omega|=2}^{Q+1} \bar{b}_\omega^{pq} u^\omega \quad (p = 1, \dots, d-1;\ q = p+1, \dots, d). \tag{6.3}$$

By using the integrated variables $x_1, \dots, x_n$, we get

$$F^{ij}(x) = \sum_{|\tau|=2}^{Q+1} b_\tau^{ij} x^\tau \quad (1 \le i \le j \le n), \tag{6.4}$$

where

$$F^{ij} = (\bar{F}^{pq} : p \in J_i, q \in J_j),\ b_\tau^{ij} = (\bar{b}_\omega^{pq} : p \in J_i, q \in J_j, \tau = S(\omega)), \quad (1 \le i < j \le n). \tag{6.5}$$

Here it is convenient to assume that $\bar{F}^{pq} = -\bar{F}^{qp}$ for $q, p \in J_i$, $q < p$.

If ξ is a symplectic polynomial vector field on $\mathbb{R}^{2d}$, then its hamiltonian expressed in the integrated variables becomes

$$F(x, y) = \sum_{|\alpha|+|\beta|=2}^{Q+1} b_{\alpha\beta} x^\alpha y^\beta, \tag{6.6}$$

where $\alpha, \beta \in \mathbb{Z}_+^n$. Similarly, for the contact hamiltonian we get

$$F(x, y, z) = \sum_{|\alpha+\beta+\gamma|=2}^{Q+1} b_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma + bz. \tag{6.7}$$

7 Normalization of jets of vector fields and hamiltonians

The polynomial vector field (6.2) is said to be *resonant* if the condition $\theta_i \neq \langle \tau, \theta \rangle \equiv \tau^1\theta_1 + \ldots + \tau^n\theta_n$ implies $a^i_\tau = 0$.

Assume that the vector field (6.2) preserves the standard volume form (1.1) and F is its hamiltonian defined by (2.1). It is easy to verify that (6.2) is resonant iff the condition $\theta_i + \theta_j \neq \langle \tau, \theta \rangle,\ i < j$, implies that $b^{ij}_\tau = 0$ (see (6.4)). Thus, it is natural to say that the hamiltonian (6.4) is *resonant* if it contains only terms $b^{ij}_\tau x^\tau$ with $\theta_i + \theta_j = \langle \tau, \theta \rangle$.

Similarly, the symplectic hamiltonian (6.6) is called *resonant* if $\langle \alpha, \theta \rangle \neq \langle \beta, \theta \rangle$ implies $b_{\alpha\beta} = 0$. Recall that in the symplectic case the real parts of the eigenvalues of A are equal to $\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n$.

In the contact case, the real parts of the eigenvalues are equal to $\theta_1, \ldots, \theta_n, -\theta_1 + \nu, \ldots, -\theta_n + \nu, \nu$. Therefore, the contact hamiltonian (6.7) is said to be *resonant* if the condition $\langle \alpha, \theta \rangle - \langle \beta, \theta \rangle + (\gamma + \beta_1 + \ldots + \beta_n)\nu \neq \nu$ implies $b_{\alpha\beta\gamma} = 0$.

Let us consider the vector field ξ,

$$\xi_i(x) = A_i x_i + \sum_{|\tau|=2}^{Q} a^i_\tau x^\tau + \xi_Q(x), \tag{7.1}$$

where $\xi_Q \in C^K,\ K \geq Q+1,\ D^p\xi_Q(0) = 0 \quad (p = 0, 1, \ldots, Q)$. Assume that (7.1) preserves the standard volume form. Clearly, the corresponding hamiltonian $F = (F^{ij})$ is of the form

$$F^{ij}(x) = \sum_{|\tau|=2}^{Q+1} b^{ij}_\tau x^\tau + F^{ij}_{Q+1}(x), \tag{7.2}$$

where $F_{Q+1} \in C^K,\ D^pF_{Q+1}(0) = 0 \quad (p = 0, 1, \ldots, Q+1)$.

Lemma 7.1. *Let $s \in \{1, \ldots, n\},\ t \in \{1, \ldots, n\},\ s \leq t$. Let $\sigma \in \mathbb{Z}^n_+$ be a multi-index such that $|\sigma| = Q+1$ and $\theta_s + \theta_t \neq \langle \sigma, \theta \rangle$. There exists a canonical change of variables $y = g(x)$ such that $D^p(g - \mathrm{id})(0) = 0$ $(p = 0, 1, \ldots, Q-1)$ and the hamiltonian $\tilde{F}$ of the vector field $\tilde{\xi} = g_*\xi$ takes the form*

$$\tilde{F}^{ij}(y) = \sum_{|\tau|=2}^{Q+1} \tilde{b}^{ij}_\tau y^\tau + \tilde{F}^{ij}_{Q+1}(y),$$

where $\tilde{b}^{ij}_\tau = b^{ij}_\tau$ whenever $(i, j, \tau) \neq (s, t, \sigma)$; $\tilde{b}^{st}_\sigma = 0,\ D^p\tilde{F}^{ij}_{Q+1}(0) = 0$ for $p = 0, 1, \ldots, Q+1$.

Proof. Let $\dot{x} = Ax$ be the linear part of ξ at the point $x = 0$. Let $\hat{f} = (\hat{f}^{ij})$ be the $(d-2)$-form defined by $\hat{f}^{ij} = 0$ if $(i, j) \neq (s, t)$, $\hat{f}^{st}(x) = b^{st}_\sigma x^\sigma$. Consider the equation

$$L_{Ax}G = \hat{f}. \tag{7.3}$$

Try to find the unknown $(d-2)$-form $G = (G^{ij})$ so that $G^{ij} = 0$ for $(i, j) \neq (s, t)$ and $G^{st}(x) = h^{st}_\sigma x^\sigma$. Note that (7.3) is a special case of (5.1). Hence, (7.3) reduces to

$$\begin{aligned}&\sum_{l=1}^{n} \sum_{r \in J_l} \sum_{m \in J_l} a_{rm} u_m \frac{\partial G^{pq}}{\partial u_r} + \sum_{r \in J_s} (-1)^{p+r-1} a_{pr} G^{rq} \\ &+ \textstyle\sum_{r \in J_t} (-1)^{q+r-1} a_{qr} G^{pr} = \hat{f}^{pq} \quad (p \in J_s, \quad q \in J_t).\end{aligned} \tag{7.4}$$

Let us show that the real parts of all the eigenvalues of the linear operator $L_{A,s,t} : G^{st} \mapsto L_{Ax}G^{st}$ are equal to the number $\langle \sigma, \theta \rangle - \theta_s - \theta_t$. To this end, approximate the (complexified) matrix A by a semi-simple matrix B. Then

$$L_{B,p,q}(h^{pq}_\omega u^\omega) = \langle \omega, \nu \rangle h^{pq}_\omega u^\omega - (\nu_p + \nu_q) h^{pq}_\omega u^\omega,$$

where $\nu_1, \dots, \nu_d$ are the eigenvalues of $B, p \in J_s, q \in J_t, \sigma = S(\omega)$. Taking into account that the spectrum of a matrix depends continuously on its entries and that A is a block-diagonal matrix we easily arrive to the conclusion that the spectrum of $L_{A,s,t}$ lies on the line $\mathrm{Re}\lambda = \langle \sigma, \theta \rangle - \theta_s - \theta_t$, as asserted above.

Because $\langle \sigma, \theta \rangle - \theta_s - \theta_t \neq 0$, we see that system (7.4) (as well as equation (7.3)) has a solution of the desired form.

Identify the jet $j_0^Q(\xi)$ with the Taylor's polynomial of ξ of degree Q. Because ξ preserves the standard volume form, the polynomial vector field $j_0^Q(\xi)$ is also volume preserving. Its hamiltonian, f, coincides with $j_0^{Q+1}(F)$. Let

$$\tilde{f}^{ij}(x) = \sum_{|\tau|=2}^{Q+1} \tilde{b}^{ij}_\tau x^\tau, \quad \tilde{f} = (\tilde{f}^{ij}), \quad \hat{f} = \tilde{f} - f.$$

Clearly, $\hat{f}^{ij}(x) = 0$ if $(i,j) \neq (s,t)$; $\hat{f}^{st} = b^{st}_\sigma x^\sigma$. Let G denote the solution of (7.3) and η be the corresponding vector field. Let $g^t : \mathbb{R}^d \to \mathbb{R}^d$ be the phase flow of η. Define $\xi_\varepsilon = g^\varepsilon_* \xi$. Let F_ε be the hamiltonian of ξ_ε (see (2.1)). Then G is nothing else than the solution of the cohomology equation

$$L_{\xi_\varepsilon} G = \partial F_\varepsilon / \partial \varepsilon. \tag{7.5}$$

Observe that the components of G are $O(\|x\|^{Q+1})$ as $\|x\| \to 0$. Therefore $j_0^{Q+1}(L_{\xi_\varepsilon} G) = L_{Ax}G$, whence $j_0^{Q+1}(\partial F_\varepsilon / \partial \varepsilon) = \hat{f}$. Thus, $j_0^{Q+1}(F_\varepsilon) = f + \varepsilon \hat{f}$, hence $j_0^{Q+1}(F_1) = \tilde{f}$. We conclude that $g^1 : \mathbb{R}^d \to \mathbb{R}^d$ is the desired canonical C^∞ smooth change of variables and $\tilde{F} = F_1$. The proof is complete.

Analogous results hold for symplectic and contact vector fields.

Lemma 7.2. *Let F be a symplectic hamiltonian of the form*

$$F(x,y) = \sum_{|\alpha+\beta|=2}^{Q+1} b_{\alpha\beta} x^\alpha y^\beta + F_{Q+1}(x,y),$$

where $F_{Q+1} \in C^K$, $K \geq Q+1$ and $D^p F_{Q+1}(0) = 0$ $(p = 0, 1, \dots, Q+1)$. Let the pair $(\bar{\alpha}, \bar{\beta})$ be such that $|\bar{\alpha}| + |\bar{\beta}| = Q+1$, $\langle \bar{\alpha}, \theta \rangle \neq \langle \bar{\beta}, \theta \rangle$. Then there exists a C^∞ smooth canonical change of variables which brings the hamiltonian F to the form

$$\tilde{F}(x,y) = \sum_{|\alpha+\beta|=2}^{Q+1} \tilde{b}_{\alpha\beta} x^\alpha y^\beta + \tilde{F}_{Q+1}(x,y),$$

where $\tilde{b}_{\alpha\beta} = b_{\alpha\beta}$ whenever $(\alpha, \beta) \neq (\bar{\alpha}, \bar{\beta})$, and $\tilde{b}_{\bar{\alpha}\bar{\beta}} = 0$.

Lemma 7.3. *Let $F(x,y,z)$ be a contact hamiltonian of the form*

$$F(x,y,z) = \sum_{|\alpha+\beta+\gamma|=2}^{Q+1} b_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma + bz + F_{Q+1}(x,y,z),$$

where $F_{Q+1} \in C^K$, $K \geq Q+1$, $D^p F_{Q+1}(0) = 0$ $(p = 0,1,\ldots,Q+1)$. Let $(\bar\alpha, \bar\beta, \bar\gamma)$ be a triple of multi-indices such that $|\bar\alpha| + |\bar\beta| + |\bar\gamma| = Q+1$, and $\langle\bar\alpha, \theta\rangle + (\bar\gamma + |\bar\beta|)\nu \neq \langle\bar\beta, \theta\rangle + \nu$. Then there exists a C^∞ smooth canonical change of variables which reduces F to the form

$$\tilde F(x,y,z) = \sum_{|\alpha+\beta+\gamma|=2}^{Q+1} \tilde b_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma + bz + \tilde F_{Q+1}(x,y,z),$$

where $\tilde b_{\alpha\beta\gamma} = b_{\alpha\beta\gamma}$ if $(\alpha,\beta,\gamma) \neq (\bar\alpha,\bar\beta,\bar\gamma)$, and $\tilde b_{\bar\alpha\bar\beta\bar\gamma} = 0$.

8 Resonant polynomial normal forms of hamiltonians

Let ξ be a C^K smooth vector field, $\xi(0) = 0$, $A = D\xi(0)$ and $\theta_1, \ldots, \theta_n$ denote the real parts of the eigenvalues of A. Assume that $\theta_i \neq 0$ $(i = 1,\ldots,n)$, *i.e.*, the origin is a *hyperbolic* singular point of ξ. Denote

$$\mathrm{M} = \max\{\theta_i : i = 1,\ldots,n\}, \quad \mu = \min\{\theta_i : \theta_i > 0\},$$

$$\Lambda = \max\{-\theta_i : \theta_i < 0\}, \quad \lambda = \min\{-\theta_i : \theta_i < 0\}.$$

Lemma 8.1. *Let ξ preserve the standard volume form and F be its hamiltonian defined by* (2.1). *Let k be an integer satisfying the inequality*

$$K > \frac{2\Lambda}{\lambda} + \frac{2\mathrm{M}}{\mu} + (k+1)\left(\frac{\mathrm{M}}{\lambda} + \frac{\Lambda}{\mu}\right) + 2. \tag{8.1}$$

Then there exists a canonical transformation of class C^k defined in the vicinity of the origin which conjugates F and $j_0^K(F)$.

Proof. Identify the jet $j_0^{K-1}(\xi)$ with the corresponding Taylor's polynomial. Clearly, the polynomial vector field $j_0^{K-1}(\xi)$ respects the standard volume form. Recall that the hamiltonian F is a $(d-2)$-form of class C^K. It easily follows that $j_0^K(F)$ is the hamiltonian for $j_0^{K-1}(\xi)$. Consider the cohomology equation (5.1) and the related affine extension (5.2), where $\xi_\varepsilon = \varepsilon j_0^{K-1}(\xi) + (1-\varepsilon)\xi$, $G = j_0^K(F)$. Then $\mathcal{A} = (\mathcal{A}_{pq}(u))$ is a family of linear morphisms of class $C^{K-1} \subset C^{k+1}$ and $j_0^K(G - F) = 0$. It is easily seen from (5.1) that the spectrum of the matrix $\mathcal{A}(0)$ lies between the lines $\mathrm{Re}\nu = -2\Lambda$ and $\mathrm{Re}\nu = 2\mathrm{M}$. Now apply Theorem $2'$ from [4]. The proof is complete.

Lemma 8.2. *Let ξ be a symplectic vector field on $\mathbb{R}^{2d}$, F be its hamiltonian, k be an integer satisfying*

$$K > \frac{2\Lambda(k+1)}{\lambda} + 2. \tag{8.2}$$

Then there exists a (locally defined) canonical change of variables of class C^k which conjugates F and $j_0^K(F)$.

The proof is similar to that of Lemma 8.1. Observe that the spectrum of the linear part of ξ is symmetric relatively the imaginary axis and the first eigenvalue of (5.4) equals to 0.

Now let ξ be a contact vector field on $\mathbb{R}^{2d+1}$, F be its contact hamiltonian and η be the symplectic vector field on $\mathbb{R}^{2d+2}$ that corresponds to ξ. Since $\xi(0) = 0$, we have $\eta(0) = 0$ and the origin is a hyperbolic singular point of η. Denote $A = D\eta(0)$. Define the numbers λ and Λ for A, as before.

Lemma 8.3. *If k satisfies the inequality* (8.2) *then there exists a canonical transformation of class C^k which conjugates F and $j_0^K(F)$ near the origin.*

Combining these results with the results presented in the preceding section, we obtain

Theorem 8.1. *Let ξ be a vector field of class C^K which preserves the (standard) volume form (symplectic form, contact structure) and the origin be a hyperbolic singular point of ξ. Let F be the corresponding local hamiltonian. If k satisfies the inequality* (8.1) *or, respectively,* (8.2), *then there exists a local canonical transformation of class C^k which reduces the hamiltonian F (and, consequently, the vector field ξ) to the polynomial resonant normal form.*

9 Simplified resonant normal forms

It is known (see, for example, [6,2]) that some resonant monomials x^τ can be eliminated by using C^k smooth changes of variables. Let us present the following sufficient condition that comprises all the known ones (see [2]).

Condition $\mathfrak{A}(k)$. The multi-index $\tau = (\tau^1, \ldots, \tau^n) \in \mathbb{Z}_+^n$ is said to satisfy the condition $\mathfrak{A}(k)$, $k \geq 1$, if there is a collection $\sigma = (\sigma_1, \ldots, \sigma_p)$ of n-vectors $\sigma_i = (\sigma_i^1, \ldots, \sigma_i^n)$ $(i = 1, \ldots, p)$ with non-negative components such that either

$$\langle \sigma_i, \theta \rangle \equiv \sum_{j=1}^{n} \sigma_i^j \theta_j = 1 \quad (i = 1, \ldots, p)$$

or

$$\langle \sigma_i, \theta \rangle = -1 \quad (i = 1, \ldots, p)$$

and the inequality

$$\langle \tau, \bar{u} \rangle > k \max \{\bar{u}_j : j = 1, \ldots, n\}$$

holds for every vertex $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ of the convex polyhedral domain $D = D(\sigma)$ determined by the inequalities

$$u_j \geq 0 \quad (j = 1, \ldots, n), \quad \langle \sigma_i, \theta \rangle \geq 1 \quad (i = 1, \ldots, p).$$

Let Q be an integer, $Q \geq 2$. The polynomial

$$P(x) = (\sum_{|\omega|=2}^{Q} P_\omega^1 x^\omega, \ldots, \sum_{|\omega|=2}^{Q} P_\omega^m x^\omega)$$

is called τ-*divisible* if $P^j_\omega \neq 0$ implies $\tau^1 \leq \omega^1, \ldots, \tau^n \leq \omega^n$.

Lemma 9.1. *Let E and F be finite-dimensional real vector spaces, $\dim E = d$; $\dot{x} = Ax$ be a hyperbolic linear vector field on E and $\dot{y} = By$ be a linear vector field on F. Let $\tau \in \mathbb{Z}^n_+$ satisfy condition $\mathfrak{A}(k)$ with respect to the vector field $\dot{x} = Ax$. Let $P : E \to F$ be a τ-divisible resonant polynomial. Let $Q_\varepsilon : E \times F \to F$ be a C^∞ smooth family of resonant polynomials, $Q_\varepsilon(x, 0) = 0$ $(x \in E)$, and $q_\varepsilon : E \to E$ be a C^∞ smooth family of resonant polynomials. Then the extension*

$$\dot{y} = By + Q_\varepsilon(x, y) + P(x), \quad \dot{x} = Ax + q_\varepsilon(x) \tag{9.1}$$

admits a C^k smooth invariant section $y = \varphi_\varepsilon(x)$ which C^∞ smoothly depends on the parameter ε.

The proof of this lemma is based on the techniques elaborated in [2], Chapter II. It uses the iterative method of introducing additional variables (see Lemma II.8.1) as well as Theorem II.5.23 adapted to the case of vector fields. The details are left to the interested reader.

Let us say that the hamiltonian F of the vector field ξ is reduced to the *simplified resonant polynomial k-normal form* if F is a polynomial which contains only such resonant terms x^τ that τ does not satisfy condition $\mathfrak{A}(k+1)$.

Theorem 9.1. *Let the vector field ξ and its hamiltonian F satisfy the hypotheses of Theorem 8.1, then F can be reduced to the simplified resonant polynomial k-normal form by a local canonical transformation of class C^k.*

Proof. Let F be the resonant polynomial hamiltonian of the vector field ξ. Represent F in the form $F = P + Q$, where P is a τ-divisible resonant polynomial and τ satisfies condition $\mathfrak{A}(k+1)$. Let $A = D\xi(0)$ and p be the polynomial vector field which corresponds to the hamiltonian P. Denote $F_\varepsilon = Q + \varepsilon P$, then $\xi_\varepsilon(x) = Ax + q(x) + \varepsilon p(x)$.

Let us prove that the cohomology equations (5.1), (5.3), (5.5) with $G - F = P$ have C^{k+1} smooth solutions. Consider the corresponding characteristic systems (5.2), (5.4) and (5.6). In each of the three cases, the characteristic system is of the form (9.1). According to Lemma 9.1, there exists a family of invariant sections $y = H_\varepsilon(x)$ of class C^{k+1}. The hamiltonian H_ε determines a change of variables of class C^k (see Section 4).

Applying successively these arguments to all multi-indices τ satisfying condition $\mathfrak{A}(k+1)$, we get the desired result.

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