Field theory and KAM tori

G. Gallavotti,[*] G. Gentile,[#], V. Mastropietro[&]

Abstract: The parametric equations of KAM tori for a l degrees of freedom quasi integrable system, are shown to be one point Schwinger functions of a suitable euclidean quantum field theory on the l dimensional torus. KAM theorem is equivalent to a ultraviolet stability theorem. A renormalization group treatment of the field theory leads to a resummation of the formal pertubation series and to an expansion in terms of l^2 new parameters forming a $l \times l$ matrix σ_ε (identified as a family of renormalization constants). The matrix σ_ε is an analytic function of the coupling ε at small ε: the breakdown of the tori at large ε is speculated to be related to the crossing by σ_ε of a "critical" surface at a value $\varepsilon = \varepsilon_c$ where the function σ_ε is still finite. A mechanism for the possible universality of the singularities of parametric equations for the invariant tori, in their parameter dependence as well as in the $\varepsilon_c - \varepsilon$ dependence, is proposed.

1. Introduction

We consider l rotators with inertia moments J, angular momenta $\vec{A} = (A_1, \ldots, A_l) \in \mathbf{R}^l$, and angular positions $\vec{\alpha} = (\alpha_1, \ldots, \alpha_l) \in \mathbf{T}^l$. Their motion will be described by the hamiltonian

$$
\begin{aligned}
H &= \frac{1}{2} J^{-1} \vec{A} \cdot \vec{A} + \varepsilon\, f(\vec{\alpha}, \vec{A}) \,, \qquad \vec{A} \in \mathbf{R}^l, \vec{\alpha} \in \mathbf{T}^l \,, \\
f &= \sum_{|\vec{\nu}| \le N} f_{\vec{\nu}}(\vec{A})\, e^{i\vec{\nu}\cdot\vec{\alpha}} \,, \qquad f_{\vec{\nu}}(\vec{A}) = f_{-\vec{\nu}}(\vec{A}) \,,
\end{aligned}
\tag{1.1}
$$

with $f_{\vec{\nu}}(\vec{A})$ a polynomial in $\vec{A}$.[1] Let $\vec{\omega}_0 = J^{-1}\vec{A}_0$ be a rotation vector, "angular velocity vector", verifying for $C_0, \tau > 0$ suitably chosen the *diophantine property*

$$
C_0 |\vec{\omega}_0 \cdot \vec{\nu}| > |\vec{\nu}|^{-\tau} \,, \qquad \vec{0} \ne \vec{\nu} \in \mathbf{Z}^l \,. \tag{1.2}
$$

The KAM theorem states the existence of a one parameter family $\varepsilon \to \mathcal{T}_\varepsilon$ of tori with parametric equations

$$
\vec{A} = \vec{A}_0 + \vec{H}(\vec{\psi}) \,, \qquad \vec{\alpha} = \vec{\psi} + \vec{h}(\vec{\psi}) \,, \quad \vec{\psi} \in \mathbf{T}^l \,, \tag{1.3}
$$

---

[*] Dipartimento di Fisica, I^a Università di Roma, P.le Moro 2, 00185 Roma, Italia; partially supported by Rutgers U. and CNR-GNFM.

[#] IHES, 91440 Bures s/Yvette, France.

[&] Dipartimento di Matematica, II^a Università di Roma, 00133 Roma, Italia; supported from CNR-GNFM.

[1] Analyticity of f in a domain $W(\vec{A}_0, \rho) = \{\vec{A} \in \mathbf{R}^l : |\vec{A} - \vec{A}_0|/|\vec{A}_0| < \rho\}$ would make the matter more complicate only slightly.

where $\vec H(\vec\psi)$ and $\vec h(\vec\psi)$ are analytic functions of ε, ψ_j, $j=1,\ldots,l$, divisible by ε, defined for $|\varepsilon|, |\mathrm{Im}\,\psi_j|$ small enough. Such tori are uniquely determined by the requirements:

$$\begin{aligned}&(a)\qquad \vec\psi\to\vec\psi+\vec\omega_0 t \text{ solves the equations of motion },\\ &(b)\qquad H(\vec\psi) \text{ is even in } \vec\psi\,,\\ &(c)\qquad h(\vec\psi) \text{ is odd in } \vec\psi\,,\end{aligned}\tag{1.4}$$

Consider the four (formally) gaussian vector fields $\vec\Phi\equiv(\vec{\mathcal H}^\sigma,\vec h^\sigma)$, $\sigma=\pm$, defined on the torus $\mathbf{T}^l$, and with propagators[2]

$$\begin{aligned}\langle h^+_{\vec\psi,j}h^-_{\vec\psi',j'}\rangle=&\delta_{j,j'}\sum_{\vec\nu}\frac{e^{i(\vec\psi-\vec\psi')\cdot\vec\nu}}{(i\vec\omega_0\cdot\vec\nu+\Lambda^{-1})^2}\equiv\delta_{j,j'}\,S^2(\vec\psi-\vec\psi')\,,\\ \langle h^{2+}_{\vec\psi,j}\mathcal H^-_{\vec\psi',j'}\rangle=&\langle\mathcal H^+_{\vec\psi,j}h^-_{\vec\psi',j'}\rangle=\delta_{j,j'}\sum_{\vec\nu}\frac{e^{i(\vec\psi-\vec\psi')\cdot\vec\nu}}{(i\vec\omega_0\cdot\vec\nu+\Lambda^{-1})}\equiv\delta_{j,j'}\,S^1(\vec\psi-\vec\psi')\,,\end{aligned}\tag{1.5}$$

where Λ is a *ultraviolet cut off.*[3] The other propagators are taken to be zero. The physical dimensions of the field $\vec h^+$, $\vec h^-$, $\vec{\mathcal H}^+$, $\vec{\mathcal H}^-$ are respectively $[1],[\omega^{-2}],[\omega],[\omega^{-1}]$ in terms of the dimension $[\omega]$ of $\vec\omega_0$. We sall also set $\vec\Phi^{1\pm}\equiv\mathcal H^\pm$ and $\vec\Phi^{2\pm}\equiv h^\pm$.

We denote by $P(d\Phi)$ the formal functional integral with respect to the above gaussian process, and consider the field theory with $\vec\Phi$ as free field and *action*

$$\begin{aligned}V(\Phi)=&-\varepsilon\int_{\mathbf{T}^l}d\vec\psi\,J^{-1}h^-_{\vec\psi}\cdot\partial_{\vec\psi}f\big(\vec\psi+h^+_{\vec\psi},\vec A+J\mathcal H^+_{\vec\psi}\big)\\ &-\varepsilon\int_{\mathbf{T}^l}d\vec\psi\,\mathcal H^-_{\vec\psi}\cdot\partial_{\vec A}f\big(\vec\psi+h^+_{\vec\psi},\vec A+J\mathcal H^+_{\vec\psi}\big)+\Lambda^{-1}\vec a(\varepsilon)\cdot\int_{\mathbf{T}^l}d\vec\psi\,h^-_{\vec\psi}\end{aligned}\tag{1.6}$$

where $\vec a$ will be called *counterterm*, and its physical dimensions are $[\vec\omega]$.

It is easy to check that the Schwinger functions

$$S_n(\vec\psi_1,s_1,\sigma_1;\ldots;\vec\psi_n,s_n,\sigma_n)=\frac{\int P(d\Phi)\,e^{-V(\Phi)}\,\Phi^{s_1\sigma_1}_{\vec\psi_1}\ldots\Phi^{s_n\sigma_n}_{\vec\psi_n}}{\int P(d\Phi)\,e^{-V(\Phi)}}\tag{1.7}$$

of the non polynomial formal[4] action Eq. (1.6) are well defined if the one point Schwinger functions

$$\vec h(\vec\psi)\equiv S_1(\vec\psi,2,+)=\frac{\int P(d\Phi)\,e^{-V(\Phi)}\,h^+_{\vec\psi}}{\int P(d\Phi)\,e^{-V(\Phi)}},\qquad \vec H(\vec\psi)\equiv S_1(\vec\psi,1,+)=J\frac{\int P(d\Phi)\,e^{-V(\Phi)}\,\mathcal H^+_{\vec\psi}}{\int P(d\Phi)\,e^{-V(\Phi)}}\tag{1.8}$$

are well defined. The reason is simply that the structure of the free field and that of the action imply that all the Feynman graphs of the theory must be either trees or families of disconnected trees. The renormalization constant $\vec a(\varepsilon)$ will be fixed by requiring that the average of $\vec h$ vanishes. As in field theory one could fix $\vec a$ equivalently by requiring that the average of $\vec h$ has a prefixed value: it is only important that $\vec h$ is well defined when $\Lambda\to\infty$ and the value $\vec 0$ for its average has no special meaning, except that it is a convenient normalization which, as we shall see, makes use of the symmetry of the problem inherited by the fact that f has a cosine Fourier series and this simplifies some considerations.

---

[2] *i.e.* linear functionals on the space of complex felds on $\mathbf{T}^l$ such that the moments are evaluated by using the Wick rule.

[3] Because $\vec\omega_0\cdot\vec\nu$, $\vec\nu\neq\vec 0$, can become small only for $|\vec\nu|$ large.

[4] Because the Φ's are complex and f is a trigonometric polynomial.

The case in which f is $\vec{A}$-independent has been studied in [G3], where it has been shown that in the limit $\Lambda\to\infty$ the one point Schwinger functions are precisely the functions $\vec{h}$ and $\vec{H}$ defined by the KAM theorem, provided the counterterms $\vec{a}$ are chosen $\vec{0}$. In [G3] the $\vec{a}$ does not appear (as it is $\vec{0}$ for symmetry reasons) so that the analysis is considerably simpler and no cut off Λ is necessary. The necessity of $\vec{a}\neq\vec{0}$ arises only if f is $\vec{A}$-dependent (and it is related to the twist condition that becomes necessary in such a case: note that in [G3] the twist condition was not required; furthermore, as a consequence, only one field, namely $\vec{h}_{\vec{\psi}}$, was used).

In this paper we study the more general case in which the action Eq. (1.6) depends also on $\vec{A}$. If the ultraviolet cut off Λ is finite the perturbative expansion for the Schwinger functions is convergent for ε suitably small and for any choice of the counterterms. However, in the limit $\Lambda\to\infty$ the series is convergent for a *unique* choice of the counterterm $\vec{a}(\varepsilon)$. This is what happens generically in quantum field theory, in which the pertubative series for Schwinger functions converge only if a unique choice of the counterterm is made (see for instance the case of ϕ^4, [G1]). *Moreover the choice of the counterterms which makes the perturbative series finite in the limit $\Lambda\to\infty$ is such that $\vec{h}$, $\vec{H}$ in Eq. (1.8) coincide with the corresponding quantities in the KAM theorem.*

2. The Schwinger functions expansion.

The latter statement can proved by writing recursively the one point Schwinger function to order n, $H^{(n)}_{\vec{\nu},j}$ and $h^{(n)}_{\vec{\nu},j}$ and comparing it with a similar recursive construction of the Lindstedt series for the KAM functions $\vec{H},\vec{h}$.

The exponentials in Eq. (1.7) are expanded in powers of V and the P integrals of the resulting products of fields are evaluated using the Wick rule leading to the familiar Feynman diagrams: the special form of V immediately implies that the diagrams have no loops, *i.e.* they are tree diagrams.

The diagrams will be described later: here it is sufficient to remark that even without using the diagram representation the evaluation of the integrals immediately leads to the following recursive relations between the coefficients of the power series (in ε) expansion of the functions $\vec{H},\vec{h}$ in Eq. (1.8), *i.e.* the one field Schwinger functions of the theory described by Eq. (1.5), Eq. (1.6):

$$H^{(k)}_{\vec{\nu},j}=S^1_{\vec{\nu}}\Big\{\sum\nolimits^*(-i\vec{\nu}_0)_j\sum_{p,q\ge0}\frac{1}{p!q!}\prod_{s=1}^{p}\big(i\vec{\nu}_0\cdot\vec{h}^{(k_s)}_{\vec{\nu}_s}\big)\prod_{i=1}^{q}\big(\vec{H}^{(k'_i)}_{\vec{\nu}'_i}\cdot\partial_{\vec{A}}\big)\,f_{\vec{\nu}_0}(\vec{A})\Big|_{\vec{A}=\vec{A}_0}\Big\}+Ja^{(k)}_j\delta_{\vec{\nu},\vec{0}}\ ,\tag{2.1}$$

and:

$$\begin{aligned}h^{(k)}_{\vec{\nu},j}=&\,S^2_{\vec{\nu}}\Big\{\sum\nolimits^*(-iJ^{-1}\vec{\nu}_0)_j\sum_{p,q\ge0}\frac{1}{p!q!}\prod_{s=1}^{p}\big(i\vec{\nu}_0\cdot\vec{h}^{(k_s)}_{\vec{\nu}_s}\big)\prod_{i=1}^{q}\big(\vec{H}^{(k'_i)}_{\vec{\nu}'_i}\cdot\partial_{\vec{A}}\big)\,f_{\vec{\nu}_0}(\vec{A})\Big|_{\vec{A}=\vec{A}_0}\Big\}\\&+\Lambda a^{(k)}_j\delta_{\vec{\nu},\vec{0}}+S^1_{\vec{\nu}}\Big\{\sum\nolimits^*\sum_{p,q\ge0}\frac{1}{p!q!}\prod_{s=1}^{p}\big(i\vec{\nu}_0\cdot\vec{h}^{(k_s)}_{\vec{\nu}_s}\big)\prod_{i=1}^{q}\big(\vec{H}^{(k'_i)}_{\vec{\nu}'_i}\cdot\partial_{\vec{A}}\big)\partial_{\vec{A}_j}f_{\vec{\nu}_0}(\vec{A})\Big|_{\vec{A}=\vec{A}_0}\Big\}\ ,\end{aligned}\tag{2.2}$$

where the $\sum^*$ denotes sum over the integers $k_s,k'_i\ge1$ and over the integers $\vec{\nu}_0$, $\vec{\nu}_s$, $\vec{\nu}'_i$, with

$$\sum_{s=1}^{p}k_s+\sum_{i=1}^{q}k'_i=k-1,\qquad \vec{\nu}_0+\sum_{s=1}^{p}\vec{\nu}_s+\sum_{i=1}^{q}\vec{\nu}'_i=\vec{\nu}\ .\tag{2.3}$$

The integer vectors $\vec{\nu}_s,\vec{\nu}'_i,\vec{\nu}_0,\vec{\nu}$ may be $\vec{0}$.

For $\vec{\nu}=\vec{0}$, from the above relations we obtain

$$H^{(k)}_{\vec{0},j}=\Lambda X^{(k)}_j+Ja^{(k)}_j,\qquad h^{(k)}_{\vec{0},j}=J^{-1}\Lambda[\Lambda X^{(k)}_j+Ja^{(k)}_j]+\Lambda Y^{(k)}_j\ ,\tag{2.4}$$

where $X_j^{(k)}$ e $Y_j^{(k)}$ are read from Eq. (2.1) and Eq. (2.2) for $\vec\nu=\vec 0$. The condition that $\vec h_{\vec 0}^{(k)}=\vec 0$ determines, recursively, $a_j^{(k)}$ and implies $\vec H_0^{(k)}=-J\vec Y^{(k)}$.

The first order calculation yields

$$\begin{aligned}\vec H_{\vec\nu}^{(1)}&=S_{\vec\nu}^1(-i\vec\nu)\,f_{\vec\nu}+J\vec a^{(1)}\delta_{\vec\nu,\vec 0}\ ,\\ \vec h_{\vec\nu}^{(1)}&=J^{-1}S_{\vec\nu}^2(-i\vec\nu)\,f_{\vec\nu}+S_{\vec\nu}^1\vec a^{(1)}\delta_{\vec\nu,\vec 0}+S_{\vec\nu}^1\partial_{\vec A}f_{\vec\nu}\ ,\end{aligned}\tag{2.5}$$

and the limit as $\Lambda\to+\infty$ is well defined if $\vec a^{(1)}=J^{-1}\vec H_{\vec 0}^{(1)}=-\partial_{\vec A}f_{\vec 0}(\vec A_0)$, and it is

$$\begin{aligned}&\vec H_{\vec\nu}^{(1)}=\frac{(-i\vec\nu)\,f_{\vec\nu}(\vec A_0)}{i\vec\omega\cdot\vec\nu}\ ,\qquad \vec h_{\vec\nu}^{(1)}=\frac{(-iJ^{-1}\vec\nu)\,f_{\vec\nu}(\vec A_0)}{(i\vec\omega\cdot\vec\nu)^2}+\frac{\partial_{\vec A}f_{\vec\nu}(\vec A_0)}{i\vec\omega\cdot\vec\nu}\ ,\qquad \vec\nu\neq\vec 0\ ,\\ &\vec H_{\vec 0}^{(1)}=-J\partial_{\vec A}f_{\vec 0}(\vec A_0)\ ,\qquad \vec h_{\vec 0}^{(1)}=\vec 0\ ,\qquad \text{if }J\vec a^{(1)}=H_{\vec 0}^{(1)}\ ,\end{aligned}\tag{2.6}$$

with $\vec h_{\vec 0}^{(1)}=\vec 0$ and the functions $\vec H$ and $\vec h$ respectively even and odd in $\vec\nu$, (as in [GM]).

Then, if we want that the expressions in Eq. (2.1), Eq. (2.2) are well defined when $\Lambda\to\infty$, we proceed inductively by supposing that by suitably fixing $\vec a^{(k)}$ the functions $\vec H^{(k)}$ and $\vec h^{(k)}$ have a well defined limit as $\Lambda\to+\infty$ and become, respectively, even and odd in $\vec\nu$ when the limit is taken. We assume this to be true for $k'\le k-1$: we see that this implies $X_j^{(k)}=0$ in the first equation, and the choice $a_j^{(k)}=-Y_j^{(k)}$ makes the parity and finiteness requests to be fulfilled to order k.

3. The Lindsted series.

The classical construction of the formal series representation for the functions $\vec H,\vec h$ in Eq. (1.3) defining parametrically the KAM torus starts from the Hamilton equations of motion for Eq. (1.1). One imposes that by replacing $\vec\psi$ with $\vec\psi+\vec\omega_0 t$ in Eq. (1.3) one gets an exact solution to the equations of motion. The following equations are obtained:

$$\begin{aligned}\vec\omega_0\cdot\vec\partial_{\vec\psi}\,\vec H(\vec\psi)&=-\,\varepsilon\,\partial_{\vec\psi}f\left(\vec\psi+\vec h(\vec\psi),\vec A_0+\vec H(\vec\psi)\right)\ ,\\ \vec\omega_0\cdot\vec\partial_{\vec\psi}\,\vec h(\vec\psi)&=J^{-1}\vec H(\vec\psi)+\varepsilon\,\partial_{\vec A}f\left(\vec\psi+\vec h(\vec\psi),\vec A_0+\vec H(\vec\psi)\right)\ .\end{aligned}\tag{3.1}$$

To make easier the comparison with the euclidean field theory of §2 we can introduce a cut off parameter Λ and consider the regularized equations

$$\begin{aligned}(\Lambda^{-1}+\vec\omega_0\cdot\partial_{\vec\psi})\,\vec H(\vec\psi)&=-\,\varepsilon\,\partial_{\vec\psi}f\left(\vec\psi+\vec h(\vec\psi),\vec A_0+\vec H(\vec\psi)\right)\ ,\\ (\Lambda^{-1}+\vec\omega_0\cdot\vec\partial_{\vec\psi})\,\vec h(\vec\psi)&=J^{-1}\vec H(\vec\psi)+\varepsilon\,\partial_{\vec A}f\left(\vec\psi+\vec h(\vec\psi),\vec A_0+\vec H(\vec\psi)\right)\ .\end{aligned}\tag{3.2}$$

We can solve Eq. (3.2) by a perturbation expansion, by writing $\vec H=\sum_{k=1}^\infty\varepsilon^k\vec H^{(k)}$ and $\vec h=\sum_{k=1}^\infty\varepsilon^k\vec h^{(k)}$. If one requires $\vec h_{\vec 0}^{(k)}=\vec 0$ then it follows immediately that the recursive construction of $\vec H^{(k)},\vec h^{(k)}$ is possible and in fact it clearly coincides with Eq. (2.1)÷Eq. (2.6). The existence of such formal series is known (if $\Lambda=+\infty$) as the *Lindstedt theorem*: and it goes back to Poincaré who extended to all orders the original proofs of Lindstedt and Newcomb.

The convergence radius of the Lindstedt series (hence of the euclidean field theory of §2) is uniform in Λ. For $\Lambda=+\infty$ this is the KAM theorem; a proof based on bounds on the coefficients $\vec H^{(k)},\vec h^{(k)}$ is due to Eliasson, [E]. It was recently "simplified" in various papers [G2], [GG], [GM], see also [CF] for a very similar approach. The proof in [G2], [GM] can be easily extended to cover the case $\Lambda<+\infty$.

Hence the theory is *uniform* in the ultraviolet cut off Λ (of course the convergence at fixed $\Lambda<\infty$ is quite trivial; the uniformity as $\Lambda\to\infty$, on the other hand, is equivalent to KAM).

4. The renormalization group and resonance resummation.

The KAM theory, thus, permits us to give a meaning to the non regularized field theory with action Eq. (1.6), a somewhat surprising fact. Therefore it is interesting to investigate in more detail the structure of the perturbation theory.

As already pointed out the model is, from the point of view of field theory, somewhat deceiving as its Feynman diagrams have no loops. Nevertheless the model is clearly non trivial and it requires a delicate analysis of a family of cancellations that make the ultraviolet stability possible at all.

With the choice of the counterterm $\vec{a}(\varepsilon)$ las in §2 the Feynman rules for the model can be summarized as follows. Consider k oriented lines, labeled from 1 to k: the final extreme v' of the lines will be called the *root* and the other extreme v will be a *vertex*. The lines, denoted $v'\leftarrow v$ are arranged on a plane by attaching in all possible ways the vertices of some segments to the roots of others, to form a connected tree.

In this way only one root r remains unmatched and it will be called the root of the graph whose lines will be called *branches* and whose vertices other than the root will be called *nodes*.

Each node v is given a *mode* label $\vec{\nu}_v$ which is one of the Fourier mode $\vec{\nu}$ such that $f_{\vec{\nu}}\neq 0$ (see Eq. (1.1)). We define the *momentum* flowing on the branch going from v to v' as $\vec{\nu}(v)=\sum_{w\le v}\vec{\nu}_w$. Furthermore each branch is regarded as composed by two halves each carrying a label H or h (so there are four possibilities per branch).

Trees that can be superposed modulo the action of the group of transformations generated by the permutation of the branches emerging from a node will be identified.

To each tree we associate a *value* obtained by assigning to a branch $v'\leftarrow v$ the following quantities, if $\vec{\nu}(v)\neq\vec{0}$,

a factor	$\dfrac{-i\vec{\nu}_{v'}\cdot iJ^{-1}\vec{\nu}_v}{(i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1})^2}$	$h\leftarrow h$
an operator	$\dfrac{i\vec{\nu}_{v'}\cdot\partial_{\vec{A}_v}}{i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1}}$	$h\leftarrow H$
an operator	$\dfrac{-\partial_{\vec{A}_{v'}}\cdot i\vec{\nu}_v}{i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1}}$	$H\leftarrow h$
just	0	$H\leftarrow H$

for all the branches distinct from the one containing the root: here the symbol to the right distiguishes the four type of labels that can be on the line $v'\leftarrow v$ (the arrow tells which is the right label and which is the left one). To the root branch we associate, instead, the following quantities, if $\vec{\nu}(v)\neq\vec{0}$,

a vector	$\dfrac{-iJ^{-1}\vec{\nu}_v}{(i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1})^2}$	$h\leftarrow h$
an operator	$\dfrac{\partial_{\vec{A}_v}}{i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1}}$	$h\leftarrow H$
a vector	$\dfrac{-i\vec{\nu}_v}{i\vec{\omega}_0\cdot\vec{\nu}(v)+\Lambda^{-1}}$	$H\leftarrow h$
just	0	$H\leftarrow H$

To each branch with $\vec{\nu}(v)=0$ which is not the root branch we associate a factor $-J\partial_{\vec{A}_v}\cdot\partial_{\vec{A}_{v'}}$, if $H \leftarrow h$, and 0 otherwise, while to the root branch we associate a factor $-J\partial_{\vec{A}_v}$, if $H \leftarrow h$, and 0 otherwise.

We multiply all the above operators (the factors are regarded as multiplication operators) and apply the resulting operator to the function $\prod_v f_{\vec{\nu}_v}(\vec{A}_v)$, evaluating the result at the point $\vec{A}_v \equiv \vec{A}_0$. This defines the Feynman rules: the $\vec{H}^{(k)}_{\vec{\nu}}$ and $\vec{h}^{(k)}_{\vec{\nu}}$ are given by $k!^{-1}$ times the sum of all the values of all the k branches trees with total momentum $\vec{\nu}$. In the limit $\Lambda \to \infty$, the above expressions are all well defined: this is easily checked. The expansion was developed in [G2], [GM] and it coincides essentially with the one used in [E] (and [CF]).

Note that, in [GM], each time a line λ carries a vanishing momentum, all the subtrees of fixed order k_1 having λ as first branch are summed together to give, by construction, the value of the counterterm $\vec{a}^{(k_1)}$. Such a contribution is called *fruit* in [GM], and a line of a fruitful tree can have vanishing momentum only if it comes out from a fruit. Obviously the two ways to arrange the sums over the trees are equivalent, and give the same result, *once the sums are extended to all the possible trees.*

The scaling properties of the propagators (when $\Lambda = +\infty$) suggest decomposing them into components relative to various *scales.*

Let χ_1, χ be two smooth functions such that:
(1) $\chi_1(x) \equiv 0$ if $|x| < 1$ and $\chi_1(x) \equiv 1$ for $|x| \geq 1$.
(2) $\chi(x) \equiv 0$ for $|x| < \frac{1}{2}$ or for $|x| \geq 1$, and 1 otherwise.
(3) $1 \equiv \chi_1(x) + \sum_{n=-\infty}^{0} \chi(2^n x)$

Then we can write:

$$S^a_{\vec{\nu}} \equiv \frac{1}{(i\vec{\omega}_0 \cdot \vec{\nu})^a} = \frac{\chi_1(\vec{\omega}_0 \cdot \vec{\nu})}{(i\vec{\omega}_0 \cdot \vec{\nu})^a} + \sum_{n=-\infty}^{0} \frac{\chi(2^{-n}\vec{\omega}_0 \cdot \vec{\nu})}{(i\vec{\omega}_0 \cdot \vec{\nu})^a}\,, \qquad a = 1, 2, \tag{4.1}$$

and correspondingly we can break each Feynamn graph into a sum of many terms by developing the sums in Eq. (4.1). This can be simply represented by assigning to each branch λ an extra label n_λ and multiplying the factor associated to such a line times $\chi(2^{-n_\lambda}\vec{\omega}_0 \cdot \vec{\nu})$: the value of $\vec{H}^{(k)}_{\vec{\nu}}, \vec{h}^{(k)}_{\vec{\nu}}$ will be the sum over all possible new graphs which once deprived of the new scale labels would become "old" graphs contributing to $\vec{H}^{(k)}_{\vec{\nu}}, \vec{h}^{(k)}_{\vec{\nu}}$ respectively.

The branches of the new graphs are naturally collected into connected *clusters* "of fixed scale": a cluster of scale n ($n = 1, 0, -2, \ldots$) consists in a maximal connected set of branches with scale label $\geq n$, containing at least one line of scale n. By definition each cluster is again a tree graph. The lines which are not contained in a cluster, but have an extreme inside the clusters will be called the external lines of the cluster: if the extreme inside the resonance is the root, they will be *incoming,* while if the extreme is the node they will be *outgoing.* There can be at most one outgoing line per cluster.

The clusters are, by definition, hierarchically ordered and therefore they form a tree with respect to the partial ordering generated by the inclusion relation between clusters.

Examining the convergence of the perturbation series it becomes clear that if one considers the sum of the contributions to $\vec{H}^{(k)}, \vec{h}^{(k)}$ by all the graphs that *do not contain clusters with just one incoming and one outgoing branch which, furthermore, have the same momentum* $\vec{\nu}$, then the series so generated converge for ε small, [E], [FT].

Therefore the clusters of the latter type (with one incoming and one outgoing equal momentum branches) are called *resonances* and the KAM theory can be interpreted as an analysis of the reason why the resonances do not destroy the analyticity in ε at ε small, *i.e.* of the cancellations that make the resonances give a contribution *much smaller* than one could fear.

One can imagine to consider a graph and replace each resonance together its external lines with a new simple line, which will be called *dressed line.* We collect togheter all the graphs which become identical after such an operation.

We consider here for simplicity only the case in which f is $\vec{A}$ independent; the discussion of the more general case, $f = f(\vec{\alpha}, \vec{A})$, can be carried out in the same way and it is only notationally more involved, so that, for simplicity's sake, we relegate it into Appendix A2. If we multiply each graph value by the appropriate power of ε (equal to the number of branches of the graph) we see that the values of $\vec{H}$ and $\vec{h}$ can be computed by considering all the graphs without resonances and by adding resonant clusters to each of their lines. This simply means that a line factor of scale n has to be modified as:

$$\chi(2^{-n}\vec{\omega}_0\cdot\vec{\nu}(v))\frac{(-i\vec{\nu}_{v'}\cdot iJ^{-1}\vec{\nu}_v)}{(i\vec{\omega}_0\cdot\vec{\nu}(v))^2} \;\to\; \frac{\chi(2^{-n}\vec{\omega}_0\cdot\vec{\nu}(v))}{(i\vec{\omega}_0\cdot\vec{\nu}(v))^2}(-i\vec{\nu}_{v'}\cdot[(1-\sigma_{n,\varepsilon}(\vec{\omega}_0\cdot\vec{\nu}(v))]^{-1}iJ^{-1}\vec{\nu}_v) \quad (4.2)$$

where $\sigma_{n,\varepsilon}(\vec{\omega}_0\cdot\vec{\nu})$ is a suitable function representing the sum of all the possible insertions of a resonant cluster on the line $v'\leftarrow v$. The function $\sigma_{n,\varepsilon}(\vec{\omega}_0\cdot\vec{\nu})\equiv\sigma_{n,\varepsilon}(2^n x)$ does not vanish only for x in the interval $[\frac{1}{2},1]$.

The following result is an immediate consequence of the results in [G2], [GM2].

Theorem. *The matrix $\sigma_{n,\varepsilon}(2^n x)$ is analytic in ε for ε small, independently on n and there is a constant R such that $||\sigma_{n,\varepsilon}(2^n x)||<R|\varepsilon|$.*
Furthermore the limit:

$$\lim_{n\to-\infty}\sigma_{n,\varepsilon}(2^n x)=\sigma_\varepsilon \quad (4.3)$$

exists and is a x–independent function of ε, analytic for ε small enough and divisible by ε.

The second part of the above theorem is discussed in Appendix A1. The first part is proven in [GM2] in a version in which the χ functions are not characteristic functions as above, but are smoothed versions at least two times differentiable. However one can easily take them to be as above: this implies that when they are differentiated their derivatives have to be interpreted as combinations of delta functions. But one checks that most of of such terms cancel with each other with some obvious exceptions which can be easily bounded. The possibility of using characteristic functions in the decomposition Eq. (4.1) can also be seen from [G2], where the decomposition is done as above. The constant matrix σ_ε will be called the *resonance form factor*.

It is natural to consider the two parameters series $\vec{H}^*(\vec{\psi},\varepsilon,\sigma),\vec{h}^*(\vec{\psi},\varepsilon,\sigma)$ obtained from the resonance resummed series by replacing $\sigma_{n,\varepsilon}$ by a *new, independent* parameter σ. Then the above theorem and the results of [G2],[GG],[GM] imply that the functions $\vec{H}^*,\vec{h}^*$ are analytic both in ε and σ near the origin.

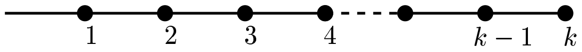
In fact it is clear that the functions $\vec{H}^*,\vec{h}^*$ depend only on the variables $\eta=\varepsilon(1-\sigma_\varepsilon)^{-1}$. Thus the possibility arises that a singularity for $\vec{H}^*,\vec{h}^*$ is reached at a value ε_c of ε where σ_ε is *still finite*. It seems natural, to us, to think that the singularities of the functions $\vec{h},\vec{H}$ as $\varepsilon\to\varepsilon_c$ are the same as those of $\vec{h}^*,\vec{H}^*$. If so the breakdown of the torus can be studied by using for it a much simpler perturbation representation, *i.e.* a representation in which no resonance appears in the graphs representing the $\vec{H}^*,\vec{h}^*$.

5. Heuristic discussion of a possible universality mechanism for the brakdown of the tori.

The scalar quantity σ_ε plays the role of a stability indicator and it would be nice to see some independent physical interpretation of it. A numerical study of the function σ_ε appears highly desirable, as well as that of the functions $\vec{H}^*,\vec{h}^*$.

The possibility that the singularities of $\vec{H}^*,\vec{h}^*$, as functions both of ε and $\vec{\psi}$, have a *universal nature* becomes clear because the behaviour of the large order coefficients of $\vec{H}^*,\vec{h}^*$, as series in ε, is likely to be very mildly dependent on the actual values of the Fourier components $f_{\vec{\nu}}$. This can be seen to happen when only the contributions to the coefficients arising from simple classes of trees are taken into account.

The simplest class of graphs which does not give a trivial contribution, *i.e.* contribution which is an entire function of ε, to the invariant tori is given by the set of trees of the form (*linear chains*):



We consider the contribution to $\vec{h}^*(\vec{\psi}, \varepsilon, \sigma)$ due to the above trees. For simplicity we fix $l = 2$, $\vec{\omega} = (r, 1)$ with $r = \frac{\sqrt{5}-1}{2}$ = *golden section* and the perturbation as an even function of $\vec{\alpha}$ only as $f(\vec{\alpha}) = a \cos \alpha_1 + b \cos(\alpha_1 - \alpha_2)$ ("Escande Doveil pendulum").

Let us call "resonant line" the line ortogonal to $\vec{\omega}$, *i.e.* parallel to $(1, -r)$. Let (p_n, q_n) be the convergents for continued fraction for r (*i.e.* $p_1 = 1, p_2 = 1, p_3 = 2, \ldots$ = *Fibonacci sequence*, and $q_1 = 1, q_2 = 2, q_3 = 3, \ldots$ with $q_n = p_{n+1}$ and $p_{n+1} = p_n + p_{n-1}$, and we set $p_0 = q_{-1} = 0$ and $p_{-1} = q_0 = 1$).

Any integer $s \geq 1$ can be written:

$$s = q_n + \sigma_{n-2} q_{n-2} + \ldots + \sigma_1 q_1 \tag{5.1}$$

if $q_n \leq s < q_{n+1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n-2} = 0, 1$, with the constraint $\sigma_j \sigma_{j+1} = 0$, $j = 1, \ldots, n-3$. Let Λ_{q_n} be the family of self avoiding walks on the integer lattice $\mathbf{Z}^2$ starting at $(0,0)$, ending at $(q_n, -p_n)$ and contained in the strip $0 < x \leq q_n$, except for the left extreme points. Then a self avoiding walk joining $(0,0)$ to (s, s') with s given by Eq. (5.1) and $s' = p_n + \sigma_{n-2} p_{n-2} + \ldots + \sigma_1 p_1$ can be obtained by simply joining a path in Λ_{q_n}, one in $\Lambda_{q_{n-2}}$ if $\sigma_{n-2} = 1, \ldots,$ one in Λ_1 if $\sigma_1 = 1$. The latter self-avoiding walks will define the class Λ_s of walks. It is clear by the construction that the above class Λ_s of self avoiding walks contains many of the ones which have the largest products $\prod_j \frac{1}{(i\vec{\omega} \cdot \vec{\nu}(j))^2}$ of small divisors. Therefore we define:

$$\vec{Z}(\Lambda_{q_n}) = \sum_{\text{paths in } \Lambda_{q_n}} (-i\eta J^{-1} \vec{\nu}_1) \frac{f_{\vec{\nu}_1}}{(i\vec{\omega} \cdot \vec{\nu}(1))^2} \prod_{j=2}^{k} \frac{f_{\vec{\nu}_{v_j}} (\vec{\nu}_{j-1} \cdot \eta J^{-1} \vec{\nu}_j)}{(\vec{\omega} \cdot \vec{\nu}(j))^2} e^{i(q_n \psi_1 - p_n \psi_2)} \tag{5.2}$$

(with $\vec{\nu}(1) = (q_n, -p_n)$ which can be always realized with the vectors $\vec{\nu}_1 = (1, 0)$ and $\vec{\nu}_2 = (1, -1)$).

We expect:

$$\vec{Z}(\Lambda_{q_n}) = \vec{\zeta}\, \frac{C(\eta, f)^{q_n}}{q_n^{\delta}}\, e^{i(q_n \psi_1 - p_n \psi_2)} (1 + O(q_n^{-1})) \equiv \vec{\zeta} Z(\Lambda_{q_n}) = Z_n\, e^{i(q_n \psi_1 - p_n \psi_2)} \ , \tag{5.3}$$

where $\vec{\zeta}$ is a suitable unit vector, $C(\eta, f)$ is a suitable function of ηf and δ is a critical exponent characteristic of the golden section. Then the contribution to $\vec{h}^*$ due to the above classes of trees and paths can be computed approximately, by noting that, if $\varepsilon_n = (r p_n - q_n)$ and $Z(\Lambda_s)$ is defined as in Eq. (5.2), Eq. (5.3), with the sum being over the paths in Λ_s and $\vec{\nu}(1) = (s, -s')$,

$$\begin{aligned} Z(\Lambda_s) \simeq & Z(\Lambda_{q_n}) Z(\Lambda_{q_{n-2}})^{\sigma_{n-2}} \ldots Z(\Lambda_{q_1})^{\sigma_1} \ , \qquad s < q_{n+1} \ , \\ Z(\Lambda_{q_{n+1}}) \simeq & Z(\Lambda_{q_n}) Z(\Lambda_{q_{n-1}}) \Big(\frac{\varepsilon_{n-1}}{\varepsilon_{n+1}}\Big)^2 \ , \qquad s = q_{n+1} \ , \end{aligned} \tag{5.4}$$

where we can define, for consistency, $Z(\Lambda_{q_0}) = \varepsilon_0^{-2} = r^{-2}$ and $Z(\Lambda_{q_{-1}}) = (\varepsilon_0 / \varepsilon_{-1})^2 = r^2$. This means that the contribution to $\vec{h}^*$ can be written approximately, if $r_n = \frac{p_n}{q_n}$:

$$\begin{aligned} & \vec{\zeta} \sum_{n=1}^{\infty} \sum_{\sigma_1, \ldots, \sigma_{n-2} = 0,1} Z_n\, Z_1^{\sigma_1} T_{\sigma_1 \sigma_2}\, Z_2^{\sigma_2} T_{\sigma_2 \sigma_3} \ldots T_{\sigma_{n-3} \sigma_{n-2}} Z_{n-2}^{\sigma_{n-2}}\, e^{i(s\psi_1 - s'\psi_2)} \\ & \simeq \vec{\zeta} \sum_{n=1}^{\infty} Z_n e^{i(q_n \psi_1 - p_n \psi_2)}\, \mathrm{Tr}\, [\Theta_1 \Theta_2 \ldots \Theta_{n-2}] \ , \end{aligned} \tag{5.5}$$

where $T_{\sigma\sigma'}$ is the compatibility matrix defined to be $T_{11} = 0$, $T_{00} = T_{01} = T_{10} = 1$, and Θ_j, $j = 2, \ldots, n-2$, are defined as $(\Theta_j)_{\sigma\sigma'} = T_{\sigma\sigma'} Z_j^{\sigma'}$ and $(\Theta_1)_{\sigma\sigma'} = Z_1^{\sigma'}$.

If $T_{\sigma\sigma'}$ were $\equiv 1$ the trace would be simply $\prod_j (1 + C(\eta, f)^{q_j} q_j^{-\delta})$; so that, in the above approximation the series will become singular when $|C(\eta, f)| = 1$ and in that case $\mathrm{Tr}\,[\Theta_1 \Theta_2 \ldots \Theta_{n-2}]$ can probably be replaced by a constant, as far as the determination of the singularity in $\vec{\psi}$ is concerned (and perhaps in η or ε as well). Hence we find the following representation of the contribution to $\vec{h}^*$ that we are considering:

$$\vec{\zeta} \sum_{n=1}^{\infty} \frac{[C(\eta, f) e^{i(\psi_1 - r_n \psi_2)}]^{q_n}}{q_n^\delta} = \vec{\zeta} \sum_{n=1}^{\infty} \frac{[C(\eta, f) e^{i(\psi_1 - r\psi_2)}]^{q_n}}{q_n^\delta} e^{-i\psi_2 O(\frac{1}{q_n})} \,. \tag{5.6}$$

We expect that the singularities of $\vec{H}^*, \vec{h}^*$, as ε grows, are the same as those of $\vec{H}, \vec{h}$, and furthermore we expect the above considered contributions to the functions $\vec{H}^*, \vec{h}^*$ to be the *most singular*. Hence we interpret Eq. (5.6) as saying that we should expect $\vec{h}, \vec{H}$ to be, at the breakdown of the invariant torus which corresponds to $|C(\eta, f)| = 1$, singular as functions of ψ_1, ψ_2 and of η (hence of ε).

Furthermore the set $|C(\eta, f)| = 1$ is in the η-plane a natural boundary for the functions $\vec{h}^*, \vec{H}^*$ as functions of η and, if $\eta = \frac{\varepsilon}{1 - \sigma_\varepsilon}$ is smooth in ε, or at least Lipshitz continuous, as mentioned above, when $\varepsilon \to \varepsilon_c^-$, $C(\eta, f) \simeq (1 - \gamma(\varepsilon_c - \varepsilon))$ so that the singularity of $\vec{h}(\psi_1, 0)$ or $\vec{H}(\psi_1, 0)$ in ε, ψ_1 is described by the singularity of a single function $\xi(z)$, or $\xi'(z)$, of the single variable $z = e^{-\gamma(\varepsilon_c - \varepsilon)} e^{i\psi}$:

$$\xi(z) = \sum_{k=1}^{\infty} \frac{z^{q_k}}{q_k^\delta}, \qquad \text{or} \quad \xi'(z) = \sum_{k=1}^{\infty} \frac{z^{q_k}}{q_k^{\delta+1}} \tag{5.7}$$

which would mean that the *critical torus* has a Lipshitz continuous regularity with *any* exponent $\delta' < \delta$ in the ψ_1–variable and δ in the $\varepsilon - \varepsilon_c$ variable, [K].

For instance if we fix ψ_2, *e.g.* as $\psi_2 = 0$ (which can be regarded as a special Poincaré section of the invariant torus), then $\vec{h}^*$ is a $C^{\delta'}$ function and $\vec{H}^*$, which is obtained from $\vec{h}$ by applying the operator $\vec{\omega} \cdot \partial_{\vec{\psi}}$, is a $C^{1+\delta}$ function, [K]. Note that the structure of the operator $\vec{\omega} \cdot \partial_{\vec{\psi}}$ is such that when it is applied to $\vec{h}$ as in Eq. (5.6) it generates a *smoother function.* Therefore, based on the hypothesis that the singularity of $\vec{H}, \vec{h}$ and of $\vec{H}^*, \vec{h}^*$ are the same, see §4, and on the above heuristic discussion, the following conjecture emerges.

Conjecture. Consider the conjugacy to a pure rotation of the motion generated by the Poincaré map on a circle on the critical torus. There is $\delta > 0$ such that it is described by two functions $\vec{h}, \vec{H}$ and written as $\vec{\alpha} = (\psi, 0) + (h_1(\psi), h_2(\psi))$ and $\vec{A} = (H_1(\psi), H_2(\psi))$ with $\vec{h}$ Hölder continuous with exponent $\delta' < \delta$ and $\vec{H}$ of class $C^{1+\delta'}$. Furthermore the above conjugacy has a Hölder continuous regularity $\delta' < \delta$ in the $\varepsilon - \varepsilon_c$ variable.

The mechanism for universality in the breakdown of the invariant tori that we propose above is, in our opinion, a refined version of an important idea in [PV]: except that we have *not* made here the simplifying assumption of absence of resonances (*i.e.* we *allow* for non zero Fourier components of opposite wave label $\pm\vec{\nu}$, and find resummations that in some sense eliminate them).

If one accepts that the above pendulum system has the same critical exponents for the golden mean torus in the standard map then it follows that $\delta = 0.7120834$ by the scaling argument on p.207 of [Ma].[1] The regularity of the two conjugators is in fact in that case not smoother than C^δ for the analogue of $\vec{h}$ and of $C^{1.9568}$ for the analogue of $\vec{H}$: hence the above conjecture is in agreement with the data and gives some independent reasons for the difference of about 1 between the regularity of

[1] Private communication of MacKay.

$\vec{h}$ and that of $\vec{H}$. Unfortunately an exact computation of the regularity of $\vec{H}$ does not seem to have bee attempted yet.[2]

Appendix A1. *The stability constant σ_ε.*

We fix n and we consider the contribution to $\sigma^{(k)}_{n,\varepsilon}(2^n x)$ arising from a k-th order term corresponding to a given Feynman graph: it will be given by the sum of products of factors whose dependence on the variable $2^n x$ is through terms of the form:

$$(\vec{\omega}_0 \cdot (\vec{\nu}^0_\lambda + 2^n x))^{-1} \ ,$$

where $\vec{\nu}^0_\lambda$ is the momentum of the branch λ inside the resonance, *i.e.* the sum of all the modes of the vertices preceding λ contained in the resonance. Then $|\vec{\nu}^0_\lambda| \le kN$ and by the diophantine property $|\vec{\omega}_0 \cdot \vec{\nu}^0_\lambda| > [C_0(kN)]^{-\tau}$ so that $n_\lambda > \tilde{n} = -\tau \log(kN) - \log C_0$, for all λ inside the resonance. Then, if k is fixed and $n \to -\infty$, the quantity $|\vec{\omega}_0 \cdot \vec{\nu}^0_\lambda|$ remains bounded from below because $|\vec{\omega}_0 \cdot \vec{\nu}_0| \ge 2^{\tilde{n}}$ while $2^n x \to 0$ and the x–dependence is only via quantities like $(\vec{\omega}_0 \cdot \vec{\nu}_\lambda + 2^n \sigma x)$, $\sigma = 0, 1$. Therefore the dependence on x disappears, and we have:

$$\lim_{n\to\infty} \sigma^{(k)}_{n,\varepsilon}(2^n x) = \sigma^{(k)}_\varepsilon \ .$$

On the other hand, as $\sigma^{(k)}_{n,\varepsilon}(2^n x)$ is a power series in ε uniformly convergent, see [GM], and we can pass to the limit under the sign of series and the theorem is proven.

Appendix A2. *Resonance form factors for an action dependent interaction*

In general the interaction potential depends also on the action variables. This yields that all the line factors introduced in §4 are possible, so that to the dressed lines we associate the following quantities

a factor	$\chi(2^{-n}\vec{\omega}_0 \cdot \vec{\nu}(v)) \dfrac{-i\vec{\nu}_{v'} \cdot [1 - \sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu}(v))]^{-1} i J^{-1} \vec{\nu}_v}{(i\vec{\omega}_0 \cdot \vec{\nu}(v) + \Lambda^{-1})^2}$	$h \leftarrow h$
an operator	$\chi(2^{-n}\vec{\omega}_0 \cdot \vec{\nu}(v)) \dfrac{i\vec{\nu}_{v'} \cdot [1 - \sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu}(v))]^{-1} \partial_{\vec{A}_v}}{i\vec{\omega}_0 \cdot \vec{\nu}(v) + \Lambda^{-1}}$	$h \leftarrow H$
an operator	$\chi(2^{-n}\vec{\omega}_0 \cdot \vec{\nu}(v)) \dfrac{-\partial_{\vec{A}_{v'}} \cdot [1 - \sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu}(v))]^{-1} i\vec{\nu}_{v'}}{i\vec{\omega}_0 \cdot \vec{\nu}(v) + \Lambda^{-1}}$	$H \leftarrow h$
just	0	$H \leftarrow H$

where n is the scale label of the line, and $\sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu})$, $s = 1, \ldots, 4$, will have a different form depending on the labels (H or h) attached to the half branches contributing to form, respectively, the outgoing and the incoming external lines of the resonant clusters whose values add to $\sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu})$. The analysis in [GM] applies to all kinds of resonance, so that a result analogous to the theorem of §4 holds for all the functions $\sigma^s_{n,\varepsilon}(\vec{\omega}_0 \cdot \vec{\nu})$, and four resonance form factors can be shown to be well defined and depending only on ε: the proof can be carried out exactly in the same way.

Acknowlegments: *We are indebted to G. Parisi for an early suggestion discussed in [G3] and to R. MacKay for explaining us the results on the standard map. This work is part of the research*

[2] Private communication of MacKay.

program of the European Network on: "Stability and Universality in Classical Mechanics", # ER-BCHRXCT940460.

References

[CF] Chierchia, L., Falcolini, C.: *A direct proof of a theorem by Kolmogorov in hamiltonian systems*, Annali della Scuola Normale Superiore di Pisa, **21**, 541–593, 1994.
[E] Eliasson, L.H.: *Absolutely convergent series expansions for quasi periodic motions*, report 2-88, Dept. of Mathematics, University of Stockholm, 1988.
[FT] Feldman, J., Trubowitz, E.: *Renormalization in classical mechanics and many body quantum field theory*, Journal d'Analyse Mathématique, **58**, 213-247, 1992.
[G1] Gallavotti, G.: *Renormalization Theory and Ultraviolet Stability for scalar Fields via Renormalization Group Methods*, Reviews of Modern Physics, **57**, 471–562, 1985.
[G2] Gallavotti, G.: *Twistless KAM Tori*, Communications in Mathematical Physics, **164**, 145–156, 1994.
[G3] Gallavotti, G.: *Perturbation Theory*, in "Mathematical physics towards the XXI century", p. 275–294, ed. R. Sen, A. Gersten, Ben Gurion University Press, Ber Sheva, 1994.
[GG] Gallavotti, G., Gentile, G.: *Majorant series for the KAM theorem*, in `mp_arc@math.utexas.edu`, #93-229, to appear in Ergodic Theory and Dynamical Systems.
[GM] Gentile, G., Mastropietro, V.: *Tree expansion and multiscale decomposition for KAM tori*, Roma 2, CARR–preprint 8/9, 1994.
[K] Katznelson, Y.: *An introduction to harmonic analysis*, Dover, 1976.
[Ma] MacKay R.: *Renormalization in area preserving maps*, World Scientific, London, 1993.