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The Brjuno functions and their regularity properties

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ABSTRACT

We show that various possible versions of the Brjuno function, based on different kinds of continued fraction developments, are all equivalent and we study their regularity (L^p, BMO and Hölder) properties, through a systematic analysis of the functional equation which they fulfill.

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Abstract

We show that various possible versions of the Brjuno function, based on different kinds of continued fraction developments, are all equivalent and we study their regularity (L^p, BMO and Hölder) properties, through a systematic analysis of the functional equation which they fulfill.

0. Introduction

When an irrational rotation is analytically perturbed, it is a natural question to ask whether or not there exists a neighborhood of the fixed point where the dynamics looks like the unperturbed case. More precisely, does there exist a local holomorphic coordinate for which the perturbed transformation is expressed as an ordinary rotation ? When the rotation number satisfies the Brjuno condition [Br], such a coordinate exists in a domain called a Siegel disk. The Brjuno function tells more : it gives an estimate of minus the logarithm of the size of the Siegel disks as a function of the rotation number [Yo]. A similar result also holds in the simpler case of certain complex area–preserving maps [Ma, Da]. In this work, we first analyze the relation between the Brjuno function and the various kind of continued fractions. We establish the functional equation fulfilled by the Brjuno function, and show that its solution requires the inversion of an operator T. We estimate the value of the spectral radius of the operator T for all L^p norms, and also for the BMO (Bounded Mean Oscillation) space. The Brjuno function is obtained as the action of $(1-T)^{-1}$ on a logarithmic function which belongs to the BMO–space. Since the spectral radius of T is smaller than one, the Brjuno function is also in this space. Noticing that the adjoint of T is nothing else than the Ruelle-Frobenius-Perron operator associated to the dynamical system which generates the continued fraction, the identification of the space adapted to T seems to us promising for the study of dynamical properties. Finally, the action of T on continuous functions is described with respct to Hölder-continuity properties. We show that regular perturbations (at least $C^{1/2}$) of the logarithmic term do

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modify the solution only by a $C^{1/2}$ contribution, so that the most singular part remains unchanged. We anticipate that this result might be much more general : it might happen that the geometric renormalisation for holomorphic dynamical systems produces only C^1 perturbations to the renormalisation equation, and the most singular part of minus the logarithm of the size of the stability domains as function of the rotation number could be universally (that is modulo $C^{1/2}$) described by the Brjuno function. This is in agreement with the numerical results of [Ma]. A further motivation for the present study, and particularly the BMO-space results, is the problem of building a complex analytic extension of the Brjuno function. This will be the suject of a subsequent paper.

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1. On a family of continued fraction transformations

Let $\alpha \in [1/2, 1]$ and let $x \in \mathbb{R}$. We define

$$[x]_\alpha = \min\{p \in \mathbb{Z} \mid x < \alpha + p\} \ , \tag{1.1}$$

that is

$$[x]_\alpha = p \text{ iff } \quad \alpha - 1 + p \le x < \alpha + p \ .$$

Note that

$$[x]_\alpha = [x - \alpha + 1] \ ,$$

where $[\ ] = [\ ]_1$ denotes the usual integer part of a real number. We will consider the iteration of

$$A_\alpha : (0, \alpha) \mapsto [0, \alpha] \tag{1.2}$$

defined by

$$A_\alpha(x) = \left| \frac{1}{x} - \left[\frac{1}{x} \right]_\alpha \right| \ , \tag{1.3}$$

with branches

$$\begin{aligned} A_\alpha(x) =& \frac{1}{x} - k \quad \text{for} \quad \frac{1}{k+\alpha} < x \le \frac{1}{k} \\ A_\alpha(x) =& k - \frac{1}{x} \quad \text{for} \quad \frac{1}{k} < x \le \frac{1}{k+\alpha-1} \end{aligned} \ .$$

Let

$$G = \frac{\sqrt{5}+1}{2} \ , \quad \Gamma = \sqrt{2} + 1$$

and
$$g = G^{-1} = \frac{\sqrt{5}-1}{2}\ , \quad \gamma = \Gamma^{-1} = \sqrt{2}-1\ .$$

Theorem 1.1. *The dynamical system defined by the iteration of (1.3) preserves an absolutely continuous (w.r.t. Lebesgue) probability measure m_α with density $c_\alpha \rho_\alpha(x) : m_\alpha(dx) = c_\alpha\rho_\alpha(x)dx$. The density is given by :*

(i) if $g \le \alpha \le 1$, then
$$0 \le 1-\alpha \le A_\alpha(\alpha) = \alpha^{-1} - 1 \le \alpha\ ,$$
and
$$\begin{aligned} c_\alpha &= \frac{1}{\log(1+\alpha)}\ ,\\ \rho_\alpha(x) &= \frac{1}{1+x}\chi_{(\alpha^{-1}-1,\alpha)}(x) + \frac{1}{2+x}\chi_{(1-\alpha,\alpha^{-1}-1]}(x) \\ &+ \left(\frac{1}{2+x}+\frac{1}{2-x}\right)\chi_{(0,1-\alpha]}(x)\ , \end{aligned} \tag{1.4}$$

(ii) if $1-\gamma \le \alpha < g$, then
$$0 \le A_\alpha(\alpha) = 2-\alpha^{-1} < 1-\alpha \le A_\alpha(\alpha-1) = (1-\alpha)^{-1} - 2 < \alpha\ ,$$
and
$$\begin{aligned} c_\alpha &= \frac{1}{\log G}\\ \rho_\alpha(x) &= \frac{1}{G+x}\chi_{((1-\alpha)^{-1}-2,\alpha)}(x) + \frac{1}{2+x}\chi_{(1-\alpha,(1-\alpha)^{-1}-2]}(x) \\ &+ \left(\frac{1}{x+2}+\frac{1}{G+1-x}\right)\chi_{(2-\alpha^{-1},1-\alpha]}(x) \\ &+ \left(\frac{1}{x+2}+\frac{1}{2-x}\right)\chi_{(0,2-\alpha^{-1}]}(x)\ , \end{aligned} \tag{1.5}$$

(iii) if $\frac{1}{2} \le \alpha < 1-\gamma$, then
$$0 \le A_\alpha(\alpha) = 2-\alpha^{-1} \le A_\alpha(\alpha-1) = (1-\alpha)^{-1} - 2 < 1-\alpha \le \alpha\ ,$$
and
$$\begin{aligned} c_\alpha &= \frac{1}{\log G}\ ,\\ \rho_\alpha(x) &= \frac{1}{G+x}\chi_{(1-\alpha,\alpha)}(x) + \left(\frac{1}{G+x}+\frac{1}{G+1-x}\right)\chi_{((1-\alpha)^{-1}-2,1-\alpha]}(x) \\ &+ \left(\frac{1}{x+2}+\frac{1}{G+1-x}\right)\chi_{(2-\alpha^{-1},(1-\alpha)^{-1}-2]}(x) \\ &+ \left(\frac{1}{x+2}+\frac{1}{2-x}\right)\chi_{(0,2-\alpha^{-1}]}(x)\ , \end{aligned} \tag{1.6}$$

where $\chi_{(a,b)}(x)$ denotes the characteristic function of the interval (a,b).

The theorem can be proven by direct calculations, which are very tedious for general values of α. It is much easier to derive it from similar results of Nakada [Na], who considers the map $\tilde{A}_\alpha = \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right|\right]_\alpha$, instead of $A_\alpha = |\tilde{A}_\alpha|$. Note that if $\alpha = 1$, one recovers Gauss' result : $c_1\rho_1(x) = \dfrac{1}{(1+x)\log 2}$, and if $\alpha = 1/2$, one finds $c_{1/2}\rho_{1/2}(x) = \dfrac{1}{\log G}\left(\dfrac{1}{G+x} + \dfrac{1}{G+1-x}\right)$.

Remark 1.2. From the above theorem, it is obvious that the density of the measure m_α is bounded above and below, so that m_α and the Lebesgue measure are equivalent. More precisely, bounding each term of the type $(b \pm x)^{-1}$ by its values at 0 or 1, one easily shows that there exists a constant c_0 such that for any $\alpha \in [1/2, 1]$,

$$c_0^{-1} \le c_\alpha\rho_\alpha \le c_0 \ . \tag{1.7}$$

One finds

$$c_0 = 2/\log G = 4.156... \ , \tag{1.8}$$

which can be reduced to $1/\log G$ when $\alpha = 1/2$.

Remark 1.3. It is well known that the Kolmogorov-Sinai entropy of these maps is given by

$$h(\alpha) = -2\int_0^\alpha c_\alpha\rho_\alpha(x)\log x\,dx \ .$$

If $\alpha = 1$ one obtains

$$h(1) = \frac{\pi^2}{6\log 2} \ ,$$

for $\alpha = 1/2$ one obtains,

$$h(1/2) = \frac{\pi^2}{6\log G} \ ,$$

a result already given by Rieger [Ri]. For general α, we find that $c_\alpha h(\alpha)$ does not depend on α, thus

$$h(\alpha) = \begin{cases} \dfrac{\pi^2}{6\log(1+\alpha)} & \text{if } g \le \alpha \le 1 \ , \\ \dfrac{\pi^2}{6\log G} & \text{if } 1/2 \le \alpha \le g \ , \end{cases}$$

as already shown by Nakada [Na]. Using the terminology of the thermodynamical formalism for dynamical systems, these results show the existence of a phase transition at $\alpha = g$.

To each $x \in \mathbb{R}\setminus\mathbb{Q}$ we associate a continued fraction expansion by iterating A_α as follows. Let

$$\begin{aligned} x_0 &= |x - [x]_\alpha| \\ a_0 &= [x]_\alpha \end{aligned} \tag{1.9}$$

then one obviously has

$$x_0 = a_0 + \varepsilon_0 x_0 \quad \text{where} \quad \varepsilon_0 = \begin{cases} +1 \text{ iff } x \geq [x]_\alpha \\ -1 \text{ otherwise} \end{cases} . \tag{1.10}$$

We now define inductively for all $n \geq 0$

$$\begin{aligned} x_{n+1} &= A_\alpha(x_n) \\ a_{n+1} &= \left[\frac{1}{x_n}\right]_\alpha \geq 1 \end{aligned} \tag{1.11}$$

thus

$$x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1} \quad \text{where} \quad \varepsilon_{n+1} = \begin{cases} +1 \text{ iff } x_n^{-1} \geq a_{n+1} \\ -1 \text{ otherwise} \end{cases} . \tag{1.12}$$

Therefore we have

$$x = a_0 + \varepsilon_0 x_0 = a_0 + \frac{\varepsilon_0}{a_1 + \varepsilon_1 x_1} = \ldots = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \ddots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n x_n}}} \tag{1.13}$$

and we will write

$$x = [(a_0, \varepsilon_0), (a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n), \ldots] . \tag{1.14}$$

Note that for $\alpha = 1$ we recover the standard continued fraction expansion defined through the iteration of Gauss' map $x \mapsto x^{-1}$ mod 1, and all $\varepsilon_n = +1$. When $\alpha = 1/2$ one has the so-called nearest integer continued fraction, and $a_n \geq 2$ for all $n \geq 1$.

The nth-convergent is defined by

$$\frac{p_n}{q_n} = [(a_0, \varepsilon_0), (a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n)] = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \ddots + \cfrac{\varepsilon_{n-1}}{a_n}}} . \tag{1.15}$$

and it is immediate to check that the numerators p_n and denominators q_n are recursively determined by

$$p_{-1} = q_{-2} = 1 \;\; , \quad p_{-2} = q_{-1} = 0 \;\; , \tag{1.16}$$

and for all $n \geq 0$

$$\begin{aligned} p_n &= a_n p_{n-1} + \varepsilon_{n-1} p_{n-2} \; , \\ q_n &= a_n q_{n-1} + \varepsilon_{n-1} q_{n-2} \; . \end{aligned} \tag{1.17}$$

Moreover

$$x = \frac{p_n + p_{n-1}\varepsilon_n x_n}{q_n + q_{n-1}\varepsilon_n x_n} \tag{1.18}$$

$$x_n = -\varepsilon_n \frac{q_n x - p_n}{q_{n-1}x - p_{n-1}} \tag{1.19}$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \varepsilon_0 \ldots \varepsilon_{n-1} \quad . \tag{1.20}$$

Let

$$\beta_n = \Pi_{i=0}^n x_i = (-1)^n \varepsilon_0 \ldots \varepsilon_n (q_n x - p_n) \quad \text{for } n \geq 0, \quad \text{and } \beta_{-1} = 1 \quad . \tag{1.21}$$

Then

$$\begin{aligned} x_n &= \frac{\beta_n}{\beta_{n-1}} \\ \beta_{n-2} &= a_n \beta_{n-1} + \varepsilon_n \beta_n \end{aligned} \tag{1.22}$$

From the definitions given one easily proves by induction the following proposition

Proposition 1.4. *Given $\alpha \in [1/2, 1]$, for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and for all $n \geq 1$ one has*

(i) $q_{n+1} > q_n > 0$;

(ii) $p_n > 0$ *when* $x > 0$ *and* $p_n < 0$ *when* $x < 0$;

(iii) $|q_n x - p_n| = \dfrac{1}{q_{n+1} + \varepsilon_{n+1} q_n x_{n+1}}$, *so that* $\dfrac{1}{1+\alpha} < \beta_n q_{n+1} < \dfrac{1}{\alpha}$;

(iv) *if* $\alpha > g$, $\beta_n \leq \alpha g^n$;

(v) *if* $\alpha \leq g$, $\beta_n \leq \alpha \gamma^n$.

Proof. One gets parts (i) and (ii) by recursion using (1.17), in fact it is obvious only when $\alpha = 1$. When $\alpha \neq 1$, one could alternatively use Lemma 1.8 below. Part (iii) is easily obtained from (1.18).

The proof of (iv) is also easy : either $x_k \leq g$ for all $k = 0, \ldots, n$, or $x_k > g$ for some k. Then $x_{k+1} = x_k^{-1} - 1$ and $x_{k+1} < g^{-1} - 1 = g$, thus $x_k x_{k+1} = 1 - x_k < 1 - g = g^2$. In the sequence $\beta_n = x_0 \cdots x_n$ one then isolates the pairs x_k, x_{k+1} such that $x_k > g$ (since for each pair $x_k x_{k+1} < g^2$). The other terms in β_n are all smaller or equal to g, except for $x_n < \alpha$.

The proof of (v) is more complicated : the result is obvious if among $x_0, x_1, \ldots, x_n$, there are less than two of them taking values greater than γ. Otherwise, let x_k and x_{k+p+1} be two successive values greater than γ. Note that $x_k > \gamma$ implies $x_{k+1} = |2 - x_k^{-1}| < \gamma$, therefore we must have $p \geq 1$. Now statement (v) is an immediate consequence of the following assertion which we will then prove :

if $x_k > \gamma$, $p \geq 1$, $x_{k+1}, \ldots, x_{k+p} < \gamma$ *and* $x_{k+p+1} > \gamma$ *then* $\Pi_{i=k}^{k+p} x_i < \gamma^{p+1}$.

We divide the proof into some different cases.

(1) If $\gamma < x_k \leq 1/2$, then $x_{k+1} = x_k^{-1} - 2$ and $x_k x_{k+1} = 1 - 2x_k < 1 - 2\gamma = \gamma^2$, and the assertion holds. By the way this closes the proof in the case $\alpha = 1/2$.

(2) Let $x_k > 1/2$, thus $x_{k+1} = 2 - x_k^{-1} < g^2$. Now observe that the image of $[1/3, g^2]$ is $[0, g^2]$. If $x_{k+1} \le 1/3$ we let $m = 1$, and if $x_{k+1} > 1/3$ we let $m \ge 2$ such that

$$x_{k+1}, \ldots, x_{k+m-1} \in (1/3, g^2]\ , \quad x_{k+m} \in [0, 1/3]\ .$$

Note that $p \ge m \ge 1$.

(2.1) If $m \ge 4$, then $x_k x_{k+1} \cdots x_{k+m} \le \frac{1}{3} g^{2m-1}$, since $x_k < g$, $x_{k+1}, \ldots, x_{k+m-1} \le g^2$ and $x_{k+m} \le 1/3$. A numerical exercise shows that for $m \ge 4$, $\frac{1}{3} g^{2m-1} < \gamma^{m+1}$, and the assertion follows. We must now consider the three cases left : $m = 1$, $m = 2$ and $m = 3$.

(2.2) $m = 1$. If $x_k < 1 - \gamma$, then $x_k x_{k+1} = 2x_k - 1 < \gamma^2$; otherwise $x_k \ge 1 - \gamma$, and $x_{k+1} \ge 1 - \sqrt{2}/2$ so that $2/7 < x_{k+1} \le 1/3$ and $x_{k+2} = x_{k+1}^{-1} - 3$, $x_{k+1} x_{k+2} = 1 - 3x_{k+1}$ and finally $x_k x_{k+1} x_{k+2} = 3 - 5x_k < 5\gamma - 2 = \gamma^3$, which shows the assertion.

(2.3) $m = 2$, then $x_{k+2} = 3 - x_{k+1}^{-1}$ and $x_k x_{k+1} x_{k+2} = 5x_k - 3$. If this is smaller than γ^3, then the assertion follows. If not, which is equivalent to assume $x_k > \sqrt{2} - 4/5$, then $x_{k+1} > (48 - 25\sqrt{2})/34$ and $x_{k+2} > 0.3111\ldots$, so that $x_{k+2} \in [2/7, 1/3]$. Thus $x_{k+3} = x_{k+2}^{-1} - 3$ and $x_k x_{k+1} x_{k+2} x_{k+3} = 8 - 13x_k < 8 - 13(\sqrt{2} - 4/5)$. A numerical exercise shows that this last number is smaller than γ^4, which completes the assertion in this case.

(2.4) $m = 3$, then assume $x_k x_{k+1} x_{k+2} x_{k+3} = 13x_k - 8 > \gamma^4$. This would be equivalent to $x_k > (25 - 12\sqrt{2})/13$. However, from the definition of m, one gets $x_{k+3} \le 1/3$ which implies $x_k \le 21/34$, and the two inequalities on x_k are contradictory. Thus $x_k x_{k+1} x_{k+2} x_{k+3} \le \gamma^4$, and the proof of the assertion is completed. □

Remark 1.5. From *(iii)* and *(iv)* one gets if $\alpha > g$, $q_n \ge \dfrac{1}{\alpha(1+\alpha)} G^{n-1}$, and similarly, from *(iii)* and *(v)* if $\alpha \le g$, $q_n \ge \dfrac{1}{\alpha(1+\alpha)} \Gamma^{n-1}$.

Remark 1.6. From *(iii)* one gets

$$\frac{1}{2q_n q_{n+1}} < \frac{1}{q_n(q_n + q_{n+1})} \le \frac{1}{q_n(\alpha q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \tag{1.23}$$

if $\varepsilon_{n+1} = +1$, whereas

$$\frac{1}{q_n q_{n+1}} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n(q_{n+1} - (1-\alpha)q_n)} < \frac{1}{\alpha q_n^2} \tag{1.24}$$

if $\varepsilon_{n+1} = -1$. Note also that assertions *(iv)* and *(v)* remain valid for $x \in \mathbb{Q}$, with the convention that $\beta_n = 0$ as soon as one of the x_k , $k \le n$, vanish (in which case the x_k with larger order are undefined).

Remark 1.7. Using the estimates of the Remark 1.5, $q_k \geq \max(1, G^{k-1}/2)$, and the elementary inequality $\log q_k \leq (2/e) q_k^{1/2}$, there exists two positive constants c_1 and c_2 such that

$$\sum_{k=0}^{\infty} \frac{\log q_k}{q_k} \leq c_1 = \frac{2}{e}\left(3 + \frac{\sqrt{2}}{G - \sqrt{G}}\right) = 5.214... , \tag{1.25}$$

$$\sum_{k=0}^{\infty} \frac{\log 2}{q_k} \leq c_2 = 5 \log 2 = 3.465... , \tag{1.26}$$

for all $\alpha \in [1/2, 1]$ and for all $x \in (0, \alpha)$.

Let P_n/Q_n denote the n-th convergent to x according to the standard continued fraction expansions (i.e. obtained by the iteration of the Gauss map A_1). Following the method of [Bo], we shall now relate the n-th convergents of the α–continued fractions to P_n/Q_n. In fact the following result is obtained through a repeated use of the identity :

$$A - \frac{1}{B + x} = A - 1 + \cfrac{1}{1 + \cfrac{1}{B - 1 + x}} .$$

Lemma 1.8. *For fixed $x \in \mathbb{R} \setminus \mathbb{Q}$, let $k^\alpha : \mathbb{N} \to \mathbb{N}$ be the arithmetic function inductively defined by $k^\alpha(-1) = -1$ and*

$$k^\alpha(n+1) = \begin{cases} k^\alpha(n) + 1 & \text{if } \varepsilon_{n+1} = +1, \\ k^\alpha(n) + 2 & \text{if } \varepsilon_{n+1} = -1, \end{cases}$$

where ε_{n+1} is defined as in (1.10) and (1.12). Then k^α is strictly increasing and for all $n \in \mathbb{N}$

$$\frac{p_n}{q_n} = \frac{P_{k^\alpha(n)}}{Q_{k^\alpha(n)}} .$$

Moreover, when $k^\alpha(n+1) = k^\alpha(n) + 2$, we have for the denominators of the convergent of Gauss'continued fraction $Q_{k^\alpha(n+1)} = Q_{k^\alpha(n)+2} = Q_{k^\alpha(n)+1} + Q_{k^\alpha(n)}$.

In the remainder of this Section, we collect a few technical facts concerning the nearest integer continued fraction which will be systematically used in the various proofs of Sections 3 and 4. The reader who is mainly interested in the results can skip the following Lemmas.

Let $A = A_{1/2}$. We say that x and x' belong to the same branch of A^n, when x_k and x'_k belong to the same branch of A for $0 \leq k \leq n-1$. Then the coefficients a_k, ε_k, of the expansion of x and the coefficients a'_k, ε'_k of the expansion of x' do coincide for $0 \leq k \leq n$.

We now define an integer $n(x, x')$ which represents the number of iteration steps needed to separate the orbits of x x'.

Definition 1.9. *Let x, x' be two distinct irrationals in $(0,1/2)$. The* splitting order *$n(x,x')$ is the greatest integer m such that x, x' belong to the same branch or to two adjacent branches of A^m. We define also the integer $\delta(x,x')$ such that $m = n(x,x') - \delta(x,x')$ is the greatest integer such that x, x' belong to the same branch of A^m.*

We shall see in the sequel that $\delta(x,x')$ is equal to 0, 1, or 2. Indeed there are four possible situations, provided we also include the cases where x and x' are permuted (for brevity we will write n and δ for $n(x,x')$ and $\delta(x,x')$) :

(A) x and x' belong to the same branch of A^n. Then $\delta = 0$.

(B) x and x' belong to the same branch of A^{n-1}, and there exists $k \geq 3$ such that

$$\frac{2}{2k+1} < x_{n-1} < \frac{1}{k} < x'_{n-1} < \frac{2}{2k-1} \; . \tag{1.27}$$

Since x and x' belong to the same branch of A^{n-1} and to adjacent branches of A^n, in this case $\delta = 1$.

(C) x and x' belong to the same branch of A^{n-2} and there exists $k \geq 3$ such that

$$\frac{5}{5k-2} < x_{n-2} < \frac{2}{2k-1} < x'_{n-2} < \frac{5}{5k-3} \; . \tag{1.28}$$

In this case, both x_n and x'_n belong to $[2/5, 1/2]$. x and x' belong to adjacent branches of A^{n-1} as well as of A^n, and $\delta = 2$.

(D) x and x' belong to the same branch of A^{n-1} and there exists $k \geq 3$ such that

$$\frac{1}{k} < x_{n-1} < \frac{2}{2k-1} < x'_{n-1} < \frac{1}{k-1} \; . \tag{1.29}$$

In this case $\delta = 1$, as in case (B) above, but one must add the condition that one of the numbers x_n, x'_n (at least) does not belong to $[2/5, 1/2]$, otherwise, one is in case (C).

From the above definitions, for all $l \leq n - \delta$ one has

$$\begin{aligned} a_l = a'_l \; , \;\; \varepsilon_l = \varepsilon'_l \; , \;\; p_l = p'_l \; , \;\; q_l = q'_l \; , \\ |\beta_l(x) - \beta_l(x')| = q_l |x - x'| \; , \\ |x - x'| = |x_l - x'_l| \beta_{l-1}(x) \beta_{l-1}(x') \; . \end{aligned} \tag{1.30}$$

In the case (B) one has

$$a_n = a'_n = k \; , \; p_n = p'_n \; , \;\; q_n = q'_n \; , \;\; \varepsilon_n = +1 \; , \; \varepsilon'_n = -1 \; . \tag{1.31}$$

Let

$$x'' = \frac{p_n}{q_n} \in (x, x') \; . \tag{1.32}$$

Then one has ${x''}_{n-1} = k^{-1}$, $\beta_{n-1}(x'') = q_n^{-1}$, ${x''}_n = 0$ and

$$\begin{aligned} |x - x''| &= q_n^{-1}\beta_{n-1}(x)x_n = q_n^{-1}\beta_n(x) \\ |x' - x''| &= q_n^{-1}\beta_{n-1}(x')x'_n = q_n^{-1}\beta_n(x') \ . \end{aligned} \tag{1.33}$$

In the case (C) one has

$$\begin{aligned} &a_{n-1} = k \ , \ \ \varepsilon_{n-1} = -1 \ , \ \ a_n = 2 \ , \ \ \varepsilon_n = +1 \\ &a'_{n-1} = k-1 \ , \ \ \varepsilon'_{n-1} = +1 \ , \ \ a'_n = 2 \ , \ \ \varepsilon'_n = +1 \\ &q_{n-1} = q'_{n-1} + q_{n-2} \ , \ \ q_n = q'_n = q_{n-1} + q'_{n-1} \\ &p_{n-1} = p'_{n-1} + p_{n-2} \ , \ \ p_n = p'_n = p_{n-1} + p'_{n-1} \ . \end{aligned} \tag{1.34}$$

Let

$$x'' = \frac{p_n}{q_n} = \frac{p_{n-1} + p'_{n-1}}{q_{n-1} + q'_{n-1}} \in (x, x') \ . \tag{1.35}$$

Then one has ${x''}_{n-2} = 2/(2k-1)$, $\beta_{n-2}(x'') = 2q_n^{-1}$, ${x''}_{n-1} = 1/2$, $\beta_{n-1}(x'') = q_n^{-1}$, ${x''}_n = 0$ and

$$\begin{aligned} |x - x''| &= q_n^{-1}\beta_{n-1}(x)x_n = q_n^{-1}\beta_n(x) \\ |x' - x''| &= q_n^{-1}\beta_{n-1}(x')x'_n = q_n^{-1}\beta_n(x') \ . \end{aligned} \tag{1.36}$$

In the case (D) one has

$$\begin{cases} a_n = k \ , \ \varepsilon_n = -1 \\ a'_n = k-1 \ , \ \varepsilon'_n = +1 \end{cases} \qquad \begin{cases} q_n = q'_n + q_{n-1} \\ p_n = p'_n + p_{n-1} \ . \end{cases} \tag{1.37}$$

Let

$$x'' = \frac{p_n + p'_n}{q_n + q'_n} \in (x, x') \ . \tag{1.38}$$

Then one has ${x''}_{n-1} = 2/(2k-1)$, ${x''}_n = 1/2$, $\beta_{n-1}(x'') = 2(q_n + q'_n)^{-1}$ and

$$\begin{aligned} |x - x''| &= 2(q_n + q'_n)^{-1}\beta_{n-1}(x)\left(\frac{1}{2} - x_n\right) \\ |x' - x''| &= 2(q_n + q'_n)^{-1}\beta_{n-1}(x')\left(\frac{1}{2} - x'_n\right) \end{aligned} \tag{1.39}$$

We recall that in the case (D) one also has

$$\max\left(\frac{1}{2} - x_n, \frac{1}{2} - x'_n\right) \geq \frac{1}{10} \ . \tag{1.40}$$

We give now to lemmas which relate the separation between two numbers and their splitting orders.

Lemma 1.10 *There exists a positive constant c_3 independent on x and x', such that for all $l < n = n(x,x')$, we have*

$$c_3^{-1}\beta_l(x') < \beta_l(x) < c_3\beta_l(x') \,. \tag{1.41}$$

Indeed one can take $c_3 = 9/2$.

Proof. For $l < n-\delta$, this is just a consequence of (1.30) and Proposition 1.4 *(iii)*, and the constant obtained in this case is 3. When $\delta = 1$, one gets from (1.27) or (1.29) $2/3 \le x_{n-1}/x'_{n-1} \le 3/2$, which leads to a constant $9/2$. When $\delta = 2$, one gets from (1.28), $12/13 \le x_{n-2}/x'_{n-2} \le 13/12$, and $4/5 \le x_{n-1}/x'_{n-1} \le 5/4$, so that we also get for the constant $65/16 < 9/2$. □

Lemma 1.11 *There exists a positive constant $c_4 > 0$ such that for all $x, x' \in (0,1/2)$, and $n \ge n(x,x')$, one has*

$$\max(\beta_n(x), \beta_n(x')) \le c_4|x-x'|^{1/2} \,. \tag{1.42}$$

Indeed one can take $c_4 = 9\sqrt{15}/2 = 17.42....$

Proof. In the case (D) one has $|x-x'| = |x-x''| + |x''-x'|$, so that

$$|x-x'| \ge \frac{1}{5}(q_n+q'_n)^{-1}\inf(\beta_{n-1}(x), \beta_{n-1}(x')) \ge \frac{1}{15}q_n^{-2} \ge \frac{1}{60}\beta_{n-1}^2(x) \,, \tag{1.43}$$

since $q_n = q'_n + q_{n-1} > q'_n$, and $(2/3)q_n^{-1} \le \beta_{n-1}(x) \le 2q_n^{-1}$. The previous Lemma then shows that $|x-x'| \ge (c_3^2/60)\beta_{n-1}^2(x')$. Since $\beta_n \le (1/2)\beta_{n-1}$, the constant c_4 is at most equal to $c_3\sqrt{15} = 9\sqrt{15}/2$.

In the cases (B) and (C) one has

$$|x-x'| = q_n^{-1}(\beta_n(x)+\beta_n(x')) \ge q_n^{-1}\max(\beta_n(x), \beta_n(x')) \,. \tag{1.44}$$

However, $q_n^{-1} \ge (1/2)\beta_{n-1}(x) \ge (1/2c_3)\beta_{n-1}(x') = (1/9)\beta_{n-1}(x')$, and therefore we get $c_4 \ge 3/\sqrt{2}$.

Finally in the case (A) one has $|x_n^{-1} - {x'}_n^{-1}| \ge 1$ since x, x' do not belong to two adjacent branches of A^{n+1}, from which follows that $|x_n - x'_n| \ge |x_n||x'_n|$. Suppose $x_n > x'_n$ (the other case resulting by symmetry), then if $x'_n \ge x_n/2$, we get $|x_n - x'_n| \ge |x_n|^2/2 >$, and if $x'_n < x_n/2$, we get $|x_n - x'_n| = x_n(1 - x'_n/x_n) > x_n/2 > x_n^2/2$. Therefore

$$|x_n - x'_n| \ge \frac{1}{2}[\max(x_n^2, x_n'^2)] \tag{1.45}$$

and

$$|x-x'| = \beta_{n-1}(x)\beta_{n-1}(x')|x_n - x'_n| \ge \frac{\beta_{n-1}(x)\beta_{n-1}(x')}{2}[\max(x_n^2, x_n'^2)]$$

and using the previous Lemma,

$$|x-x'| \ge \frac{\max(\beta_{n-1}^2(x), \beta_{n-1}^2(x'))}{2c_3}[\max(x_n^2, x_n'^2)] \tag{1.46}$$

so that one gets $c_4 \ge 3$ in this case. □

Lemma 1.12 *Let J the interval of definition of one single branch of A^m, and $|J|$ its length. One of its end-points is equal to p_m/q_m . We have*

$$\frac{1}{3q_m^2} \le |J| \le \frac{1}{q_m^2}\ , \qquad \text{and for } x \in J\ , \qquad \frac{1}{4}q_m^2 \le \left|\frac{dA^m(x)}{dx}\right| \le \frac{9}{4}q_m^2\ , \tag{1.47}$$

so that

$$\frac{1}{12} \le |J|\frac{dA^m(x)}{dx} \le \frac{9}{4}\ . \tag{1.48}$$

Proof. The end-points of J are obtained in setting $x_m = 0$ or $x_m = \pm 1/2$ in (1.18), that is p_m/q_m and $(2p_m \pm p_{m-1})/(2q_m \pm q_{m-1})$ repectively. Therefore we have $|J|^{-1} = q_m(2q_m \pm q_{m-1})$. On the other hand, one gets dx/dx_m from (1.18), so that its inverse $|dA^m/dx| = (q_m \pm q_{m-1}x_m)^2$, and the Lemma follows easily. □

Lemma 1.13 *Let $J$and J' be the intervals of definition of two adjacent branches of A^m, with respective lengths $|J|$ and $|J'|$, then there exists a constant c_5 (which can be taken $c_5 = 12$), such that*

$$c_5^{-1} \le \frac{|J|}{|J'|} \le c_5\ . \tag{1.49}$$

Proof. Let x be the common end-point. If $A^m(x) = 0$, then one has $x = p_m/q_m$, and the other end-points are $(2p_m \pm p_{m-1})/(2q_m \pm q_{m-1})$. Therefore $q_m = q'_m$, and $|J|/|J'| = (2q_m \pm q_{m-1})/(2q_m \pm q_{m-1}) \le 3$. If $A^m(x) = 1/2$, then one of the two intervals has the form $[p_m/q_m, (2p_m \pm p_{m-1})/(2q_m \pm q_{m-1})]$ and the same holds for the other, but with p_m and q_m replaced by p'_m and q'_m respectively. For the same reasons as in case (C) above (see Equation (1.34)), we have $p_m = p'_m \pm p_{m-1}$, $q_m = q'_m \pm q_{m-1}$, $p_{m-1} = p'_{m-1}$, $q_{m-1} = q'_{m-1}$, so that $q_m/q'_m \le 3$, and the length ratio $|J|/|J'| = q_n(2q_n \pm q_{n-1})/q'_n(2q'_n \pm q'_{n-1}) \le 3q_n^2/{q'}_n^2 \le 12$. Note that in both cases we have

$$\frac{1}{3} \le \frac{q_m}{q'_m} \le 3\ . \tag{1.50}$$

□

2. The Brjuno functions

Following Yoccoz [Yo] we define a (generalized) Brjuno function :

Definition 2.1 *The α-Brjuno function $B_\alpha : \mathbb{R}\setminus\mathbb{Q} \to \overline{\mathbb{R}}$ is defined by the formula*

$$B_\alpha(x) = -\sum_{i=0}^{\infty} \beta_{i-1}(x)\log x_i \tag{2.1}$$

where the x_n follow $x_0 = x$ by repeated iterations of A_α, as defined in (1.10) and (1.11), and the β_n' are given by (1.21). We have posed $\beta_{-1} = 1$.

Remark 2.2. It is useful to extend the above definition $x \in \mathbb{Q}$, by setting $B_\alpha(x) = +\infty$, or $\exp(-B_\alpha(x)) = 0$. The Brjuno function defined in [Yo] corresponds to $B_{1/2}$, the one defined by the nearest integer continued fraction map $A_{1/2}$.

Proposition 2.3. *Given $\alpha \in [1/2, 1]$ one has*

(i) $B_\alpha(x) = B_\alpha(x+1)$ *for all* $x \in \mathbb{R}$ *;*

(ii) For all $x \in (0, \alpha)$

$$B_\alpha(x) = -\log x + xB_\alpha\left(\frac{1}{x}\right) \ ; \tag{2.2}$$

(iii) if $x \in [\alpha - 1, 0)$ *then* $B_\alpha(-x) = B_\alpha(x)$ *;*

(iv) there exists a constant $C_1 > 0$ *(independent of α) such that for all* $x \in \mathbb{R}\setminus\mathbb{Q}$ *one has*

$$\left| B_\alpha(x) - \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} \right| \le C_1 \tag{2.3}$$

where $\{q_j\}_{j\ge 0}$ denotes the sequence of the denominators of the convergents to x of the α-continued fraction expansion.

Proof. Given $x \in \mathbb{R}\setminus\mathbb{Q}$, the sequences $(x_i)_{i\ge0}$ and $(\beta_i)_{i\ge0}$ associated to x and $x+1$ are the same, which proves (i). The same is true for x and $-x$ if $x \in (\alpha-1, 0)$, which proves (iii).

If $x \in (0, \alpha)$, let $y = 1/x$ and denote by y_i, $a_i(y)$, $\beta_i(y)$, and x_i, $a_i(x)$, $\beta_i(x)$ the sequences (1.11) and (1.21) associated to y and to x respectively. From (1.9) and (1.10) it follows that $x_0 = x$, $a_0(y) = a_1(x)$, $y_0 = x_1$ and by induction for all $n \ge 0$ $y_n = x_{n+1}$ and $\beta_n(y) = (\beta_{n+1}(x))/x$. Thus

$$\begin{aligned} B_\alpha(y) &= -\sum_{i=0}^{\infty}\beta_{i-1}(y)\log y_i = -\log y_0 - \sum_{i=1}^{\infty}\frac{1}{x}\beta_i(x)\log x_{i+1} \\ &= -\frac{1}{x}\sum_{i=1}^{\infty}\beta_{i-1}(x)\log x_i = \frac{1}{x}[B_\alpha(x) + \log x] \ , \end{aligned}$$

which proves (ii).

To prove (iv) we first remark that (1.21) implies

$$q_i\beta_{i-1}+\varepsilon_i q_{i-1}\beta_i=1$$

for all $i\geq 0$. Then

$$\begin{aligned}-B_\alpha(x)+\sum_{i=0}^\infty\frac{\log q_{i+1}}{q_i}&=\sum_{i=0}^\infty\beta_{i-1}\log\frac{\beta_i}{\beta_{i-1}}+\sum_{i=0}^\infty\left(\beta_{i-1}+\varepsilon_i\frac{q_{i-1}}{q_i}\beta_i\right)\log q_{i+1}\\&=\sum_{i=0}^\infty\beta_{i-1}\log\beta_i q_{i+1}-\sum_{i=0}^\infty\beta_{i-1}\log\beta_{i-1}+\sum_{i=0}^\infty\varepsilon_i\frac{q_{i-1}}{q_i}\beta_i\log q_{i+1}\ ,\end{aligned}$$

but by (1.21), Proposition 1.4 *(iii)*, and the estimates of Remark 1.7.,

$$\begin{aligned}\left|\sum_{i=0}^\infty\beta_{i-1}\log\beta_i q_{i+1}\right|&\leq 2\sum_{i=0}^\infty\frac{\log 2}{q_i}\leq 2c_2\ ,\\\left|\sum_{i=0}^\infty\beta_{i-1}\log\beta_{i-1}\right|&\leq 2\sum_{i=0}^\infty\frac{\log 2+\log q_i}{q_i}\leq 2(c_1+c_2)\ ,\\\left|\sum_{i=0}^\infty\varepsilon_i\frac{q_{i-1}}{q_i}\beta_i\log q_{i+1}\right|&\leq 2\sum_{i=0}^\infty\frac{\log q_{i+1}}{q_{i+1}}\leq 2c_1\ ,\end{aligned}$$

from which it follows that

$$\left|B_\alpha(x)-\sum_{i=0}^\infty\frac{\log q_{i+1}}{q_i}\right|\leq C_1=4(c_1+c_2)\ .$$

□

By means of Lemma 1.8 one can prove the following

Proposition 2.4. *There exists a positive constant $C_2>0$ such that for all $\alpha\in[1/2,1]$ and for all $x\in\mathbb{R}\setminus\mathbb{Q}$ one has*

$$\left|B_\alpha(x)-\sum_{j=0}^\infty\frac{\log Q_{j+1}}{Q_j}\right|\leq C_2 \tag{2.4}$$

Proof. Thanks to *(iv)*, Proposition 2.3, it suffices to compare $\sum_{j=0}^\infty(1/q_j)\log q_{j+1}$ with $\sum_{j=0}^\infty(1/Q_j)\log Q_{j+1}$. By Lemma 2.3, one has $q_j=Q_{k(j)}$ for all j, where for brevity we write $k(j)$ for $k^\alpha(j)$. Thus

$$\sum_{j=0}^\infty\frac{\log q_{j+1}}{q_j}=\sum_{k(j+1)=k(j)+1}\frac{\log Q_{k(j+1)}}{Q_{k(j)}}+\sum_{k(j+1)=k(j)+2}\frac{\log Q_{k(j+1)}}{Q_{k(j)}}\ .$$

Using the fact that $Q_{k(j+1)} = Q_{k(j)+2} = Q_{k(j)+1} + Q_{k(j)}$ we have

$$\frac{\log Q_{k+2}}{Q_k} = \frac{\log(Q_{k+1}+Q_k)}{Q_k} = \frac{\log Q_{k+1}}{Q_k} + \frac{\log\left(1+Q_k/Q_{k+1}\right)}{Q_k}$$

but

$$0 \le \frac{\log\left(1+Q_k/Q_{k+1}\right)}{Q_k} \le \frac{\log 2}{Q_k} \ ,$$

By applying the estimates of Remark 1.7 one gets the result :

$$\left|\sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} - \sum_{j=0}^{\infty} \frac{\log Q_{j+1}}{Q_j}\right| \le 2c_2 + c_1 \ .$$

and one gets $C_2 = 6c_2 + 5c_1$. □

Remark 2.5. The *Brjuno numbers* [Br] are usually defined by the *Brjuno condition*

$$\sum_{i=0}^{\infty} \frac{\log Q_{i+1}}{Q_i} < +\infty \ .$$

Proposition 2.4 shows that the α-Brjuno functions B_α are finite at x if and only if x is a Brjuno number and that all the generalized Brjuno functions differ one from the other for a L^∞ function.

On the other hand, the advantage of the functions B_α with respect to the Brjuno condition is that they verify a nice functional equation (2.2) under the action of the modular group $\mathrm{SL}\,(2,\mathbb{Z})$.

Another important characterization of the generalized Brjuno functions comes from their "uniqueness", as it is an immediate consequence of Theorem 2.6 below.

Let us consider the operator

$$(T_\nu f)(x) = x^\nu f\left(\frac{1}{x}\right) \ , \tag{2.5}$$

if $x \in (0,\alpha)$, where $\nu \ge 0$, defined for the moment on measurable functions of $\mathbb{R}$ which verify

$$f(x) = f(x+1) \text{ for almost every } x \in \mathbb{R} \ , \quad f(-x) = f(x) \ \text{ for a.e. } x \in (0, 1-\alpha) \ . \tag{2.6}$$

It is understood that the function $T_\nu f$ is completed outside $(0,\alpha)$ by imposing on $T_\nu f$ the same parity and periodicity conditions which are expressed for f in (2.6).

The functional equation for the α-Brjuno function can be written in the form

$$[(1-T_1)B_\alpha](x) = -\log x \ , \tag{2.7}$$

for all $x \in (0,\alpha)$, complemented with the periodicity and symmetry conditions (2.6). This suggest to study the operator T_ν on the Banach spaces

$$X_{\alpha,p} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ verifies (2.6)} \ , \quad f \in L^p((0,\alpha), dm_\alpha(x)), \} \tag{2.8}$$

endowed with the norm of $L^p((0,\alpha), dm_\alpha(x))$, where

$$dm_\alpha(x) = c_\alpha \rho_\alpha(x)\, dx \tag{2.9}$$

is the invariant measure defined in Section 1, so that

$$||f||_{\alpha,p} = \left(\int_0^\alpha |f(x)|^p dm_\alpha(x) \right)^{1/p} , \tag{2.10}$$

as α varies in $(1/2, 1)$ and $p \in [1,\infty]$. Note that if $p < p'$ one has the obvious inclusion $X_{\alpha,p'} \subset X_{\alpha,p}$ and let

$$X_\alpha = \cap_{p \geq 1} X_{\alpha,p} \ . \tag{2.11}$$

If $(1 - T_1)$ is invertible in the considered space, then (2.7) has a unique solution for B_α, provided that the right hand side also belongs to the space, which is easy to check. The invertibility property is given by the following theorem, which states in particular that the spectral radius of T_1 is strictly smaller than 1.

Theorem 2.6. *T_ν is a linear bounded operator from $X_{\alpha,p}$ into itself for all $\nu > 0$, for all $\alpha \in [\frac{1}{2}, 1]$ and for all $p \in [1,\infty]$. Indeed its spectral radius on $X_{\alpha,p}$ satisfies*

$$r(T_\nu) \leq \begin{cases} g^\nu \ , & \text{if } \alpha > g \\ \gamma^\nu \ , & \text{if } \alpha \leq g. \end{cases} \tag{2.12}$$

Proof. It is a simple calculation. We observe first that

$$(T_\nu^n f)(x) = (\beta_{n-1}(x))^\nu f(x_n) = (\beta_{n-1}(x))^\nu (f \circ A_\alpha^n)(x) \ , \tag{2.13}$$

therefore

$$\begin{aligned} \int_0^\alpha |T_\nu^n f(x)|^p m_\alpha(x) dx &= \int_0^\alpha (\beta_{n-1}(x))^{\nu p} |f(A_\alpha^n(x))|^p dm_\alpha(x) \\ &\leq [\alpha \gamma_\alpha^{n-1}]^{\nu p} \int_0^\alpha |f(x)|^p dm_\alpha(x) \end{aligned} \tag{2.14}$$

where we have used Proposition 1.4 (iv) and (v) (therefore $\gamma_\alpha = g$ if $\alpha > g$, $\gamma_\alpha = \gamma$ if $\alpha \leq g$) and the invariance of the measure $dm_\alpha(x)$ w.r.t. A_α. From (2.14) it immediately follows that

$$||T_\nu^n f||_{\alpha,p} \leq [\alpha \gamma_\alpha^{n-1}]^\nu ||f||_{\alpha,p} \tag{2.15}$$

and one gets (2.12) by taking the $1/n$–th root of both sides. $\square$

The use of the invariant measure in (2.8) makes the evaluation of the spectral radius remarkably simple. We of course get the same result if we replace the measure in (2.8) by the Lebesgue measure which is equivalent (see Remark 1.2). In particular, the above Theorem implies that the spectral radius is also given by (2.12) for the operator T_ν in the spaces $L^p(\mathbb{T})$, naturally introduced using the periodicity property. It is more difficult to tell whether T_1 is itself contracting (see [MMY] for some results in this direction), we only mention here that in the case $\alpha = 1/2$, T_1 is a contraction for all Lebesgue L^p-norms on $[0,1/2]$.

3. The Brjuno function and the BMO space

In the previous Section, it has been shown that the Brjuno functions B_α belong to $L^p(\mathbb{T})$ and therefore to the intersection $\bigcap_{p=1}^{\infty} L^p(\mathbb{T})$. The purpose of this Section is to show a stronger result : the Brjuno functions B_α belong to BMO($\mathbb{T}$). We recall the definition and the main properties of BMO spaces in Appendix A.

In fact, we already know that all Brjuno functions B_α differ by L^∞ functions, and since $L^\infty \subset$ BMO, it will be enough to prove that B_α is in BMO($\mathbb{T}$) for a fixed value of α. In this section we fix $\alpha = 1/2$ and denote $A_{1/2}$ and $B_{1/2}$ simply by A and B respectively. We will also write

$$dm(x) = dm_{1/2}(x) = \frac{1}{\log G}\left(\frac{1}{G+x} + \frac{1}{G+1-x}\right) dx \ . \tag{3.1}$$

On the interval I, we define now the mean value f_I of f

$$f_I = \frac{1}{m_I}\int_I f(x)\, dm(x) \quad , \quad m_I = \int_I dm(x)\ , \tag{3.2}$$

and its quadratic oscillation $\mathcal{O}_I(f)$

$$\begin{aligned}\mathcal{O}_I(f) &= \left(\frac{1}{m_I}\int_I \big(f(x) - f_I\big)^2 dm(x)\right)^{1/2} \\ &= \left(\frac{1}{2m_I^2}\int_{I\times I} \big(f(s) - f(t)\big)^2 dm(s)\, dm(t)\right)^{\frac{1}{2}}\end{aligned} \tag{3.3}$$

We now consider the space $X_* \subset X_{1/2} = \cap_{p=1}^{\infty} X_{1/2,p}$ defined as :

$$X_* = \{f \in \mathrm{BMO}(\mathbb{R}) \mid f(x+1) = f(x)\ \forall x \in \mathbb{R}\ ,\ f(-x) = f(x)\ \forall x \in [0,1/2]\}\ , \tag{3.4}$$

with the norm

$$||f||_* = |f|_* + ||f||_2\ , \tag{3.5}$$

where $||f||_2 = ||f||_{L^2((0,1/2),dm)}$, and

$$|f|_* = \sup_{I\subset[0,1/2]} \mathcal{O}_I(f) \ . \tag{3.6}$$

Therefore we have

$$\begin{aligned} ||f||_* = & \sup_{I\subset[0,1/2]} \left(\frac{1}{2m_I^2}\int_{I\times I}\big(f(s)-f(t)\big)^2\, dm(s)\, dm(t)\right)^{\frac{1}{2}} \\ & + \left(\int_0^{1/2}(f(x))^2\, dm(x)\right)^{\frac{1}{2}} \ . \end{aligned} \tag{3.7}$$

Remark 3.1. Due to the equivalence between the measure m and the Lebesgue measure, one gets an equivalent norm in replacing $m(I)$ by the length $|I|$ in (3.7) We explain in the appendix why the norm we use here is equivalent to the usual BMO norm : it differs with the usual one [Gr,GCRF] in two respects, first, we use here the invariant measure dm instead of the Lebesgue measure, second, we use here a L^2-norm definition of BMO instead of the usual L^1-norm definition. The equivalence between the L^1 and L^2 definitions is a corollary of the John-Nirenberg Theorem which is far from obvious. Furthermore, the BMO-norm on $[0,1/2]$ is equivalent to the BMO-norm on $\mathbb{T}$ only for even functions, which is the case for $B_{1/2}$.

We shall now prove the following theorem,

Theorem 3.2. *The Brjuno function $B = B_{1/2}$ belongs to X_*, and therefore to BMO($\mathbb{T}$). For $1/2 \le \alpha \le 1$, the functions B_α also belong to BMO($\mathbb{T}$).*

The proof follows immediately from the following Theorem 3.3, which states in particular that for $\alpha = 1/2$, $1 - T_1$ is invertible in X_*. And it is easy to show by direct computations that, for $\alpha = 1/2$, the right hand side of (2.7), namely the even and periodic function equal to $\log x$ on $(0, 1/2]$, is in X_*. It results that $B = B_{1/2}$ is also in X_*, as well as all B_α for $1/2 \le \alpha \le 1$. Note that $B \not\subset L^\infty$ because the logarithmic function is unbounded.

Theorem 3.3. *In the case $\alpha = 1/2$, and for all $\nu > 0$, T_ν is a bounded linear operator from X_* to X_*. Indeed, its spectral radius in X_* is at most equal to $\gamma^\nu = (\sqrt{2}-1)^\nu$.*

Proof. In order to prove the theorem, we must estimate $\mathcal{O}_I(T_\nu^m f)$ for $m \ge 0$, $I \subset [0, 1/2]$. Let $I = [x, x']$, $n = n(x, x')$ the splitting order of x and x'. We divide the proof into three cases.

First case : one has $m > n$.

Let $\widehat{I}$ denote the union of the domains of the branches of A^m which meet the interval I. Then one has $I \subset \widehat{I}$, and from (1.49), $1 \le |\widehat{I}|/|I| \le (1 + 2c_5)$, since in this case I contains the full interval of at least one branch of A^m. Setting $g = T_\nu^m f$, one gets

$$\mathcal{O}_I^2(g) = \frac{1}{m_I}\int_I (g - g_I)^2\, dm \le \frac{1}{m_I}\int_I g^2\, dm \le c_0(1+2c_5)\frac{1}{|\widehat{I}|}\int_{\widehat{I}} g^2\, dm\ , \tag{3.8}$$

where we have used (1.7) and (1.49). For the domain J of any complete branch of A^m, we take $A^m(t)$ instead of t as the integration variable. Then, it follows from (1.48) that

$$\int_J (T_\nu^m f)^2\, dm \le ||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}\int_J (f\circ A^m)^2\, dm \le 12|J|c_0^2||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}||f||_2^2 \tag{3.9}$$

where $||\ ||_{\mathcal{C}}^0$ denotes the sup-norm on $[0,1/2]$. Therefore one gets

$$\int_{\widehat{I}} (T_\nu^m f)^2\, dm \le 12c_0^2||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}||f||_2^2|\widehat{I}|\ , \tag{3.10}$$

from which it follows that

$$\mathcal{O}_I(T_\nu^m f) \le 2c_0\sqrt{3c_0(1+2c_5)}||\beta_{m-1}||_{\mathcal{C}^0}^{\nu}||f||_2\ . \tag{3.11}$$

Second case : one has $m \le n - \delta$.

Then I is contained in the domain J of a single branch of A^m. Let $I_1 = A^m(I) \subset [0,1/2]$. We have

$$2m_I^2\mathcal{O}_I^2(T^m f) = \int_{I\times I}(T_\nu^m f(s) - T_\nu^m f(t))^2\, dm(s)\, dm(t) \le 2(M_1 + M_2)\ , \tag{3.12}$$

where

$$M_1 = \int_{I\times I}\beta_{m-1}^{2\nu}(s)(f(A^m s) - f(A^m t))^2\, dm(s)\, dm(t)\ , \tag{3.13}$$

$$M_2 = \int_{I\times I}(f(A^m t))^2(\beta_{m-1}^{\nu}(s) - \beta_{m-1}^{\nu}(t))^2\, dm(s)\, dm(t)\ . \tag{3.14}$$

Now, from the bound (1.47) on dA^m/dx, we deduce upper and lower bounds on the ratio $|I_1|/|I|$

$$\frac{q_m^2}{4} \le \frac{|I_1|}{|I|} \le \frac{9q_m^2}{4}\ , \quad \text{and,} \quad \left|\frac{dA^m(x)}{dx}\right|^{-1} \le \frac{4}{q_m^2} \le 9\frac{|I|}{|I_1|} \le 9c_0^2\frac{m_I}{m_{I_1}}\ . \tag{3.15}$$

Taking $A^m(s)$ and $A^m(t)$ as new integration variable, one gets

$$\begin{aligned} M_1 &\le c_0^4||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}\max_{x\in I_1}\left(\left|\frac{dA^m}{dx}\right|^{-2}\right)\int_{I_1\times I_1}(f(s') - f(t'))^2\, dm(s')\, dm(t') \\ &\le 81c_0^8||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}2m_I^2\mathcal{O}_{I_1}^2(f) = c_6^2||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu}m_I^2\mathcal{O}_{I_1}^2(f)\ , \end{aligned} \tag{3.16}$$

with $c_6 = 9\sqrt{2}c_0^4$. On the other hand, from (1.30) one gets

$$|\beta_{m-1}(s) - \beta_{m-1}(t)| \le q_{m-1}|I| \le |I|^{1/2} , \tag{3.17}$$

since from (1.47), one has $|I| \le |J| \le q_m^{-2} \le q_{m-1}^{-2}$. Then ,using the obvious inequality $|x^\nu - y^\nu| \le \nu \max(x^{\nu-1}, y^{\nu-1})|x-y|$, and Lemma 1.10, we get

$$\begin{aligned} M_2 &\le \nu^2 \int_{I\times I} (f(A^m t))^2 \max(\beta_{m-1}^{2\nu-2}(s), \beta_{m-1}^{2\nu-2}(t))(\beta_{m-1}(s) - \beta_{m-1}(t))^2\, dm(s)dm(t) \\ &\le \nu^2 |I| c_3^2 \int_{I\times I} (f(A^m t))^2 \beta_{m-1}^{2\nu-2}(t)\, dm(s)dm(t) \end{aligned} \tag{3.18}$$

$$M_2 \le \nu^2 |I| m_I c_0 c_3^2 ||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu} \int_I (f(A^m(t)))^2 \beta_{m-1}^{-2}(t)dt \le c_7^2 ||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu} m_I^2 ||f||_2^2 . \tag{3.19}$$

We have bounded β_{m-1} in the integral using Proposition 1.4 *(iii)* and taken $A^m(t)$ as new integration variable using (1.47), so that setting $c_7 = 3\nu c_0^{3/2} c_3$, one obtains

$$\begin{aligned} \mathcal{O}_I(T_\nu^m f) &\le ||\beta_{m-1}||_{\mathcal{C}^0}^{\nu} \left(c_6^2 \mathcal{O}_{I_1}^2(f) + c_7^2 ||f||_2^2\right)^{1/2} \\ &\le ||\beta_{m-1}||_{\mathcal{C}^0}^{\nu} \left(c_6 \mathcal{O}_{I_1}(f) + c_7 ||f||_2\right) . \end{aligned} \tag{3.20}$$

Third case : one has $n - \delta < m \le n$.

Thus one is in one of the cases (B), (C) or (D), discussed above after Definition 1.9 : the interval I is contained in the union of two adjacent branches of A^m and the point x'' is the common point of the two branches. Let $I^- = [x, x'']$, $I^+ = [x'', x']$ and we still define M_1 and M_2 as in Equations (3.13) and (3.14), so that we get (3.12) as a bound of the oscillation on I. We first bound M_2 as in the previous case, but now from (1.30),

$$\begin{aligned} |\beta_{m-1}(s) - \beta_{m-1}(t)| &\le |\beta_{m-1}(s) - \beta_{m-1}(x'')| + |\beta_{m-1}(x'') - \beta_{m-1}(t)| \\ &\le q_{m-1}|I_+| + q'_{m-1}|I_-| \\ &\le |I_+|^{1/2} + |I_-|^{1/2} \le \sqrt{2}|I|^{1/2} \end{aligned} \tag{3.21}$$

where the beforelast inequality is obtained as (3.17) above. Then

$$\int_I f(A^m(t))^2 \beta_{m-1}^{-2}(t)dt \le \left(\int_{I^-} + \int_{I^+}\right) f(A^m(t))^2 \beta_{m-1}^{-2}(t)dt \le 2c_7^2 ||f||_2^2 , \tag{3.22}$$

from which it follows that

$$M_2 \le 4c_7^2 ||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu} m_I^2 ||f||_2^2 . \tag{3.23}$$

On the other hand

$$M_1 \le c_0^2 ||\beta_{m-1}||_{\mathcal{C}^0}^{2\nu} \int_{I\times I} (f(A^m(s)) - f(A^m(t)))^2\, dsdt . \tag{3.24}$$

Let $I_1^+ = A^m(I^+)$, $I_1^- = A^m(I^-)$; one has, using (3.15),

$$\begin{aligned}\int_{I^\varepsilon\times I^\varepsilon}(f(A^m(s))-f(A^m(t)))^2 dsdt &\le 81\frac{|I^\varepsilon|^2}{|I_1^\varepsilon|^2}\int_{I_1^\varepsilon\times I_1^\varepsilon}(f(s')-f(t'))^2 ds'dt' \\ &\le 81 c_0^4 2 m_I^2 \mathcal{O}^2_{I_1^\varepsilon}(f)\ .\end{aligned} \tag{3.25}$$

Finally one gets

$$\int_{I^+\times I^-}(f(A^m(s))-f(A^m(t)))^2 dsdt \le 81\frac{|I^+||I^-|}{|I_1^+||I_1^-|}\int_{I_1^+\times I_1^-}(f(s')-f(t'))^2 ds'dt'\ . \tag{3.26}$$

Let $I_1 = I_1^+ \cup I_1^-$. From the discussion of cases (B), (C) and (D) made after Definition 1.9, it follows that I_1^+ and I_1^- have a common end-point which is either 0 or 1/2. Therefore, either $I_1^+ \subset I_1^-$ or $I_1^- \subset I_1^+$, then

$$\int_{I_1^+\times I_1^-}(f(s')-f(t'))^2 ds'dt' \le c_0^2 2 m_{I_1}^2 \mathcal{O}^2_{I_1}(f)\ . \tag{3.27}$$

However, as read from (3.15), $\frac{|I^+|}{|I_1^+|}q_m^2$, and $\frac{|I^-|}{|I_1^-|}{q'}_m^2$ both belong to the interval $[4/9, 4]$. Then, from (1.50), one gets

$$\frac{|I^+||I^-|}{|I_1^+||I_1^-|} \le c_8^2\left(\frac{|I|}{|I_1|}\right)^2\ , \tag{3.28}$$

with $c_8 = 2^2 3^4$, from which it follows that

$$\int_{I^+\times I^-}(f(A^m(s))-f(A^m(t)))^2 dsdt \le c_9^2 2 m_I^2 \mathcal{O}^2_{I_1}(f)\ . \tag{3.29}$$

with $c_9 = 81 c_8^2 c_0^4$. Putting all constants together, one therefore finds c_{10} and c_{11} such that

$$\begin{aligned} M_1 &\le c_{10}||\beta_{m-1}||^{2\nu}_{\mathcal{C}^0} m_I^2(\mathcal{O}^2_{I_1^+}(f)+\mathcal{O}^2_{I_1^-}(f)+\mathcal{O}^2_{I_1}(f))\ , \\ \mathcal{O}_I(T_\nu^m f) &\le c_{11}||\beta_{m-1}||^{\nu}_{\mathcal{C}^0}(\mathcal{O}_{I_1^+}(f)+\mathcal{O}_{I_1^-}(f)+\mathcal{O}_{I_1}(f)+||f||_2)\ .\end{aligned} \tag{3.30}$$

In the three cases we have considered, we get a bound for the oscillation of $T_\nu^m(f)$ as a linear combination with constant coefficients of the sup of the oscillation of f, the norm $||f||_2$, and *an overall factor* $||\beta_{m-1}||^\nu_{\mathcal{C}^0}$ *which contains the only dependance on* m. Since (12.5) gives the same result for the L^2-norm, we easily deduce that, for the norm (3.5), there exist a constant c_{12} independant of m and f, such that

$$||T_\nu^m f||_* \le c_{12}||\beta_{m-1}||^\nu_{\mathcal{C}^0}||f||_* \le (\gamma^\nu)^m\,(c_{12}/2\gamma)||f||_* \tag{3.31}$$

The first consequence is that T_ν is a bounded linear operator from X_* to X_*, the second is that its spectral radius $r(T_\nu) = \lim ||(T_\nu)^m||^{1/m}$ is bounded by γ^ν. □

The spectral radius computation is rather general, and does not require a tight adjustment for the constants : more specific results on the norm of T_ν itself depend on the peculiar norm taken. See [MMY] for some results of this kind. We observe that the computation of the spectral radius for the L^p-norm and the BMO-norm both come from the leading behaviour of β_{m-1}, as given by Theorem 1.4. The embedding of the BMO space in between the L^p spaces and L^∞ makes this result very natural. We expect now interesting consequences for the complex extension of the Brjuno functions.

4. The Brjuno function and its Hölder stability properties

The functional equation (1.29) for the Brjuno function for $\alpha = 1/2$ is

$$[(1-T_1)B_{1/2}](x) = -\log x \ , \tag{4.1}$$

for all $x \in (0,1/2)$, complemented with the condition that $B = B_{1/2}$ is even and periodic. In this Section we will suppose that the right hand side of this equation is pertubed, by an additional term f, which is less singular than the logarithmic function. We want to study the singular properties of the perturbed solution. Since the equation is linear, we only need to consider the action on f of T_1 and $(1-T_1)^{-1}$, which will be conveniently called the Brjuno operator $\mathbf{B}$. We will consider even and periodic functions f which are *continuous*. It is sufficient to know the value of f on $[0,1/2]$, so we assume $f \in \mathcal{C}^0_{[0,1/2]}$. One can check that $T_1 f$ (resp T_ν for $\nu > 0$) is also continuous provided we set $(T_1 f)(0) = 0$ (resp $(T_\nu f)(0) = 0$). We need now the usual Hölder's type semi-norms for continuous functions.

Definition 4.1. *Let $f \in \mathcal{C}^0_{[0,1/2]}$, Then we define the Hölder's η-norm as*

$$|f|_\eta = \sup_{0 \le x < y \le 1/2} \frac{|f(x)-f(y)|}{|x-y|^\eta} \ , \tag{4.2}$$

with $0 < \eta \le 1$. This is a seminorm since it vanishes on constant functions, so that we introduce the norm :

$$||f||_\eta = |f|_\eta + ||f||_{\mathcal{C}^0} \ , \tag{4.3}$$

where $||f||_{\mathcal{C}^0} = \max_{0\le x\le 1/2} |f(x)|$. We say that $f \in \mathcal{C}^\eta_{[0,1/2]}$, if $f \in \mathcal{C}^0_{[0,1/2]}$ and $|f|_\eta$ is finite.

We now have :

Theorem 4.2. *Let $f \in \mathcal{C}^\eta_{[0,1/2]}$, $0 < \eta \le 1$. Then :*

(1) $T_\nu^m f$ is Hölder continuous, of exponent $\overline{\eta} = \min(\eta, \nu/2)$;

(2) If $\nu > 2\eta$ (thus $\overline{\eta} = \eta$), T_ν is a bounded linear operator in $\mathcal{C}^\eta$, of spectral radius smaller or equal to $\gamma^{\nu-2\eta} < 1$. The operator $\mathbf{B}_\nu = (1 - T_\nu)^{-1}$ is defined in this space and fulfils $\mathbf{B}_\nu = \sum_{m=0}^\infty T_\nu^m$.

(3) If $\nu = 2\eta$ (thus $\overline{\eta} = \eta = \nu/2$), there exists a positive constant $c_{14} > 0$ such that

$$||T_\nu^m||_{\mathcal{C}^{\nu/2}(0,1/2)} \le c_{14} \ , \quad \textit{for all}\, m \ge 0 \ . \tag{4.4}$$

Proof. Let $x, x' \in [0, 1/2]$, $m \ge 0$, $\eta \in (0, 1]$, $\nu > 0$. We want to estimate

$$|T_\nu^m f(x) - T_\nu^m f(x')| \tag{4.6}$$

under the assumption that $f \in \mathcal{C}^\eta_{[0,1/2]}$. We let $n = n(x, x')$.

First case : $m > n$. From (2.13), one has

$$\begin{aligned} |T_\nu^m f(x) - T_\nu^m f(x')| &\le ||f||_{\mathcal{C}^0}(\beta^\nu_{m-1}(x) + \beta^\nu_{m-1}(x')) \\ &\le ||f||_{\mathcal{C}^0}(\beta^\nu_n(x)\beta^\nu_{m-n-2}(x_{n+1}) + \beta^\nu_{n-1}(x')\beta^\nu_{m-n-2}(x'_{n+1})) \\ &\le 2c_4^\nu ||f||_{\mathcal{C}^0}|x - x'|^{\nu/2}||\beta_{m-n-2}||^\nu_{\mathcal{C}^0} \ , \end{aligned} \tag{4.7}$$

where we have used Lemma 1.11.

Second case : $m \le n - \delta$. One has

$$\begin{aligned} |T_\nu^m f(x) - T_\nu^m f(x')| \le (\beta_{m-1}(x))^\nu |f(A^m(x)) - f(A^m(x'))| \\ + |f(A^m(x'))||\beta^\nu_{m-1}(x) - \beta^\nu_{m-1}(x')| \ . \end{aligned} \tag{4.8}$$

From (1.47) and Proposition 1.4 *(iii)*, one has

$$|A^m(x) - A^m(x')| \le 9(\beta_{m-1}(x))^{-2}|x - x'| \ , \tag{4.9}$$

and, using (1.30), and (1.41)

$$\begin{aligned} |\beta^\nu_{m-1}(x) - \beta^\nu_{m-1}(x')| &\le \nu \max(\beta^{\nu-1}_{m-1}(x), \beta^{\nu-1}_{m-1}(x'))|\beta_{m-1}(x) - \beta_{m-1}(x')| \\ &\le \nu c_3^{\nu-1}(\beta_{m-1}(x))^{\nu-1} q_{m-1}|x - x'| \\ &\le \nu c_3^{\nu-1}(\beta_{m-1}(x))^{\nu-2}|x - x'| \ . \end{aligned} \tag{4.10}$$

Therefore

$$\begin{aligned} &|x - x'|^{-\eta}|T_\nu^m f(x) - T_\nu^m f(x')| \\ \le&(\beta_{m-1}(x))^{\nu-2\eta}\left(9^\eta|f|_\eta + \nu c_3^{\nu-1}||f||_{\mathcal{C}^0}(\beta_{m-1}(x))^{2\eta-2}|x - x'|^{1-\eta}\right) \ , \\ \le&(\beta_{m-1}(x))^{\nu-2\eta}\left(9^\eta|f|_\eta + \nu c_3^{\nu-1}||f||_{\mathcal{C}^0}(16/9)^{1-\eta}\right) = K_f(\beta_{m-1}(x))^{\nu-2\eta} \ , \end{aligned} \tag{4.11}$$

where we have used $|x - x'| \le |J|$, and (from 1.48) $\beta_{m-1}(x) \ge (3/4)q_m^{-1} \ge (3/4)|J|^{1/2}$, J being the domain of the branch of A^m which contains x and x'.

Third case : one has $n-\delta < m \le n$: one is then led to consider the cases (B), (C) or (D) defined above, after Definition 1.9. One then introduces the intermediate point x'' and one gets the same estimates of the second case. More precisely,

$$\begin{aligned} |T_\nu^m f(x) - T_\nu^m f(x')| &\le |T_\nu^m f(x) - T_\nu^m f(x'')| + |T_\nu^m f(x') - T_\nu^m f(x'')| \\ &\le 2K_f \max\left[(\beta_{m-1}(x))^{\nu-2\eta}|x-x''|^\eta, (\beta_{m-1}(x'))^{\nu-2\eta})|x'-x''|^\eta\right] \\ &\le 2K_f c_3^{\nu-2\eta}(\beta_{m-1}(x))^{\nu-2\eta}|x-x'|^\eta, \end{aligned} \tag{4.12}$$

where we have used (1.41).

One can summarize the possible cases, in view of Theorem 4.3, by stating that there exists always a constant c_{13} such that

$$|T_\nu^m f(x) - T_\nu^m f(x')| \le c_{13}||f||_\eta |\beta_{m-1}(x)|^{\nu-2\eta}|x-x'|^\eta \ . \tag{4.13}$$

The statements of the theorem result easily from the Equations (4.3), (4.7), (4.11), (4.12), and from the obvious inequality for the $\mathcal{C}_0$ norm

$$|T_\nu^m f(x)| \le ||f||_{\mathcal{C}^0} \sup[\beta_{m-1}^\nu(x)] \le \frac{1}{2}\gamma^{(m-1)\nu}||f||_{\mathcal{C}^0} \ . \tag{4.14}$$

□

Remark 4.3. It will be useful to notice here that an estimate similar to (4.13) holds for $|\varepsilon_m(x)T_\nu^m f(x) - \varepsilon_m(x')T_\nu^m f(x')|$ provided

$$f(0) = f(1/2) = 0 \ , \tag{4.15}$$

namely

$$|\varepsilon_m(x)T_\nu^m f(x) - \varepsilon_m(x')T_\nu^m f(x')| \le c_{13}||f||_\eta |\beta_{m-1}(x)|^{\nu-2\eta}|x-x'|^\eta \ . \tag{4.16}$$

In order to get (4.16), we follow the same argument as we used in the previous proof, and distinguish three cases. The proof is immediate in the first case where $m > n$, and the bound is obtained in the same way as (4.7). In the second case where $m \le n-\delta$, we also get the result as in (4.8), since we now have $\varepsilon_m(x) = \varepsilon_m(x')$. In the third case, where $n-\delta < m \le n$, one introduces once more the intermediate point x'', and one observes that either $x''_m = 0$, or $x''_m = 1/2$, according to the discussion which follows Definition 1.9. It follows that $T_\nu^m(x'') = \beta_{m-1}(x'')f(x''_m) = 0$, due to condition (4.15). Therefore $|\varepsilon_m(x)T_\nu^m f(x) - \varepsilon_m(x')T_\nu^m f(x')| \le |T_\nu^m f(x) - T_\nu^m f(x'')| + |T_\nu^m f(x') - T_\nu^m f(x'')|$, and the required estimate is obtained as in (4.12). This completes the proof of (4.16).

When $f \in \mathcal{C}^\eta_{[0,1/2]}$, with $\nu < 2\eta$, we easily deduce from the proof of the preceding theorem that $T_\nu f$ is in $\mathcal{C}^{\nu/2}$, and that $\mathbf{B}_\nu f$ is in $\mathcal{C}^\theta$, for any $\theta < \nu/2$. In fact, one has a slightly stronger result for $\mathbf{B}_\nu f$

Theorem 4.4. *Let $f \in \mathcal{C}^{\eta}_{[0,1/2]}$,*
(1) If $\nu < 2\eta$, then the function $\mathbf{B}_\nu f = \sum_{m\geq 0} T_\nu^m f$ is in $\mathcal{C}^{\nu/2}$.
(2) If $\nu = 2\eta$, then $\mathbf{B}_\nu f$ is in $\mathcal{C}^{\eta}$ for $0 \leq \eta < \nu/2$. Indeed it admits $x^{\nu/2}|\log x|$ as continuity modulus.

Proof. We have, using (4.13) and setting $n = n(x,x')$,

$$
\begin{aligned}
|\mathbf{B}_\nu f(x) - \mathbf{B}_\nu f(x')| &\leq \sum_0^n |T_\nu^m f(x) - T_\nu^m f(x')| + \sum_{n+1}^{\infty} |T_\nu^m f(x) - T_\nu^m f(x')| \\
&\leq c_{13}||f||_\eta |x-x'|^{\nu/2}\left(|x-x'|^{\eta-\nu/2}\left(\sum_0^n (\beta_{m-1}(x))^{\nu-2\eta}\right) + \sum_{n+1}^{\infty} ||\beta_{m-n-2}||^{\nu}_{\mathcal{C}^0}\right) .
\end{aligned}
\tag{4.17}
$$

From the bound on the β_n, one sees immediately that the second series in the previous inequality converges. Furthermore all x_k belong to $(0,1/2]$, and one has

$$
\sum_0^n (\beta_{m-1}(x))^{\nu-2\eta} \leq (\beta_{n-1}(x))^{\nu-2\eta}\sum_0^n \left(\frac{1}{2^{n-m}}\right)^{2\eta-\nu} = c_{15}(q_n(x))^{2\eta-\nu} ,
$$

with $c_{15} = (3/2)^{2\eta-\nu}\sum_0^n 2^{-(m(2\eta-\nu))}$, which is finite. Now, if x and x' belong to the same branch of A^n, let K be its interval of definition. If they belong to two adjacent branches, let K be the union of their intervals of definition. In both cases one has $|x-x'| \leq |K|$ and, using Lemmas 1.12 and 1.13, one easily sees that $|K|q_n^2(x)$ is bounded, above and below, so that finally, one can find c_{16} such that

$$
\text{for} \quad \nu < 2\eta , \qquad |\mathbf{B}_\nu f(x) - \mathbf{B}_\nu f(x')| \leq c_{16}||f||_\eta |x-x'|^{\nu/2} . \tag{4.18}
$$

When $\nu = 2\eta$, we get

$$
|\mathbf{B}_\nu f(x) - \mathbf{B}_\nu f(x')| \leq c_{17}(1+n(x,x'))||f||_\eta |x-x'|^{\nu/2} . \tag{4.19}
$$

However, x and x' belong to the same branch $|J|$ of A^{n-2}, so that, using (1.48), one gets $|x-x'| \leq |J| \leq q_{n-2}^{-2}$, and following remark 1.5, $|x-x'| \leq (3/4)\gamma^{n-3}$. We therefore have, up to some constant c, $\log|x-x'| \geq n\log\gamma + c$, which means that n is bounded : $n \leq (\log(|x-x'|^{-1}) - c)/\log\Gamma$. Therefore there exists a constant c_{18} such that

$$
\text{for} \quad \nu = 2\eta , \qquad |\mathbf{B}_\nu f(x) - \mathbf{B}_\nu f(x')| \leq c_{18}||f||_\eta \log(|x-x'|^{-1})|x-x'|^{\nu/2} . \tag{4.20}
$$

□

As a consequence of the above theorem, we observe that if we consider a perturbation of the functional equation (2.7) for the Brjuno function (in the case $\alpha = 1/2$), that is

$$\text{for} \quad 0 < x \le \frac{1}{2} \quad , \quad [(1 - T_1)B_f](x) = -\log x + f(x) \ ,$$

the function B_f differs from B by *1/2-Hölder continuous function* when f is analytic, or at least 1/2- Hölder continuous. Therefore the 'most singular' part of B_f does not depend on the perturbation when it is sufficiently regular. The present result could provide an explanation for the so-called 'modular smoothing' of critical functions observed in earlier works [BPV,MS].

The above result seems to give a special role to the Brjuno function $B = B_{1/2}$. However, the theorem below displays a somewhat surprising result, namely that the difference $B_1 - B_{1/2}$ is not only bounded, as we already know, but also 1/2-Hölder continuous. We first give a preparatory statement.

Proposition 4.5 *Let B_1^+ and B_1^- be the even and odd part of B_1. B_1^+ and B_1^- are periodic, so that they are determined by their values in $[0, 1/2]$. We have*

$$\text{for} \quad x \in [0, 1/2] \quad , \quad B_1^-(x) = \frac{x}{2} \log\left(\frac{1-x}{x}\right) \tag{4.21}$$

$$B_1^+(x) = x B_1^+\left(\frac{1}{x}\right) + g(x) - \log x \ , \tag{4.22}$$

with, still for $x \in [0, 1/2]$,

$$g(x) = -\frac{x}{2} \log\left(\frac{1-x}{x}\right) + x B_1^-\left(\frac{1}{x}\right) \ . \tag{4.23}$$

Proof. For $x \in [0, 1/2] \cap [\mathbb{R} \backslash \mathbb{Q}]$, we have $B_1(-x) = B_1(1-x)$, and $1 < (1-x)^{-1} < 2$, so that $B_1(-x) = -\log(1-x) + (1-x)B_1((1-x)^{-1} - 1)$. But

$$B_1\left(\frac{1}{1-x} - 1\right) = B_1\left(\frac{x}{1-x}\right) = \log\left(\frac{1-x}{x}\right) + \frac{x}{1-x} B_1\left(\frac{1}{x}\right) \ ,$$

since $0 < x/(1-x) < 1$. Therefore

$$B_1(-x) = -x\log(1-x) - (1-x)\log x + x B_1(x^{-1}) \ .$$

Since we also have

$$B_1(x) = -\log x + x B_1(x^{-1}) \ ,$$

we easily get (4.21) by subtraction. By addition, we get

$$B_1^+(x) = x B_1\left(\frac{1}{x}\right) - \log x - \frac{x}{2} \log\left(\frac{1-x}{x}\right) = x B_1\left(\frac{1}{x}\right) - \log x - B_1^-(x) \ , \tag{4.24}$$

which leads to (4.22) since we already know B_1^-. □

The odd part of $B_1 - B_{1/2}$ coincides with B_1^-, and (4.21) shows that B_1^- is in $\mathcal{C}^\eta$ for $0 \le \eta < 1$. The even part of $B_1 - B_{1/2}$, namely

$$\Delta(x) = B_1^+(x) - B_{1/2}(x) \tag{4.25}$$

satisfies $\Delta(x) = g(x) + x\Delta(x^{-1})$. From (4.23) and Theorem 4.2, one deduces that $g \in \mathcal{C}^\eta$, for $0 \le \eta < 1/2$. Then Theorem 4.4 tells that $\Delta \in \mathcal{C}^\eta$ for $0 \le \eta < 1/2$, and the same holds for the difference $B_1 - B_{1/2}$. It is easy to see that the continuity extends to $x = 0$ and $x = 1/2$, by setting $\Delta(0) = \Delta(1/2) = 0$. In fact, the following theorem gives a stronger result which includes the case $\eta = 1/2$.

Theorem 4.6. *The difference $B_1 - B_{1/2}$ may be extended from $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{R}$ as an 1/2-Hölder continuous periodic function with period one.*

Proof. From (4.24) and (4.25), we get for $0 < x < 1/2$,

$$\Delta(x) = x\Delta(x^{-1}) + xB_1^-(x^{-1}) - B_1^-(x) = x_0\Delta(x_1) + \varepsilon_1 x_0 B_1^-(x_1) - B_1^-(x_0)\ , \tag{4.26}$$

where we have used the first step of the continued fraction expansion of x. Solving (4.26) by iteration leads to the even and periodic functions Δ, Δ_1, Δ_2 defined for $0 < x < 1/2$ as :

$$\Delta(x) = \Delta_1(x) + \Delta_2(x)\ , \tag{4.27}$$

$$\Delta_1(x) = -\sum_{n=0}^{\infty} \beta_{n-1}(x) B_1^-(x_n)\ , \tag{4.28}$$

$$\Delta_2(x) = \sum_{n=1}^{\infty} \varepsilon_n(x)\beta_{n-1}(x) B_1^-(x_n)\ . \tag{4.29}$$

We deduce from Theorem 4.4 that $\Delta_1 = -(1 - T_1)^{-1} B_1^-$ is an even periodic function in $\mathcal{C}^{1/2}$, since B_1^- is sufficiently regular. We now observe that $B_1^-(0) = B_1^-(1/2) = 0$, therefore the evaluation of $|\Delta_2(x) - \Delta_2(x')|$ is made following the proof of Theorem 4.4, for $\nu = 1$, using (4.16) instead of (4.13). The conclusion is the same as Theorem 4.4, part (i), that is that also $\Delta_2 \in \mathcal{C}^{1/2}$. Now, Δ is the even part of $B_1 - B_{1/2}$, and since the odd part of $B_1 - B_{1/2}$ is nothing but B_1^-, we deduce that $B_1 - B_{1/2}$ is 1/2-Hölder-continuous. $\square$

Appendix : B. M. O. Norms

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We define the mean value f_I of f on the interval I as :

$$f_I = \frac{1}{|I|}\int_I f\,dx\ , \tag{A.1}$$

where $|I|$ is the length of the interval I. Then we define for any interval U

$$||f||_{*,U} = \sup_{I\subset U}\frac{1}{|I|}\int_I |f - f_I|\,dx\ . \tag{A.2}$$

We then say f belongs to the space $\mathrm{BMO}(U)$ if $||f||_{*,U} < \infty$, *i.e.* is finite. BMO is an abbreviation for 'bounded mean oscillation'. $||f||_{*,U}$ is a seminorm on $\mathrm{BMO}(U)$, since for any constant c, we have $||f+c||_{*,U} = ||f||_{*,U}$. In particular $||f||_{*,U} = 0$ if f is constant on U. This applies to $U = \mathbb{R}$ and leads to the space $\mathrm{BMO}(\mathbb{R})$, abbreviated as BMO and the seminorm $||f||_{*,\mathbb{R}}$ on $\mathrm{BMO}(\mathbb{R})$ will simply be written $||f||_*$. In fact this seminorm is a norm on the quotient space of function in $L^1_{\mathrm{loc}}(\mathbb{R})$ modulo the constant functions. With this norm, the quotient space is complete. We list now some more or less classical results and lemmas [Gr,GRCF].

Proposition A.1. *The space $L^\infty(U)$ is a subspace of* $\mathrm{BMO}(U)$*, and $||f||_{*,U} \le \inf_C ||f-c||_{\infty,U}$.*

Proposition A.2. *Let f in* $\mathrm{BMO}(U)$ *and let I be an interval. Then for any $\lambda > 0$, the Lebesgue measure of the set of points $t \in I$ such that $|f(t) - f_I| > \lambda$ is bounded by $K_1\exp(-K_2\lambda/||f||_{*,U})$.*

This is the John-Nirenberg Theorem theorem, see Garnett [Gr]. The proof there is for $U = \mathbb{R}$, but a careful reading show that it works for any U. The constants K_1 and K_2 do not depend on U, λ, and f. Roughly speaking, this theorem says that where f is unbounded, it behaves at most as as a logarithmic function.

Proposition A.3. *We have the following 'magic reverse Hölder's inequality' : let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and suppose that for some interval $U \subset \mathbb{R}$, the seminorm $||f||_{*,U}$ is finite, then for any bounded real $p \ge 1$, there exists a constant A_p such that*

$$\sup_{I\subset U}\left(\frac{1}{|I|}\int_I |f - f_I|^p\,dx\right)^{\frac{1}{p}} \le A_p ||f||_{*,U}\ . \tag{A.3}$$

In fact it is an easy corollary of the John-Nirenberg theorem, see Garnett [Gr]. The constant A_p *does not depend on* U, and may be shown to be smaller than pC with an explicit constant C. Note that the inequality does not work in the limit $p \to \infty$. $\square$

The preceding proposition shows that replacing the L^1 norm in the definition of the BMO norm $||f||_{*,U}$, by the analogous L^p norm (with p finite), leads to the same BMO space. More precisely using the usual L^p norm

$$||f||_{p,U} = \left(\int_U |f|^p \, dx \right)^{\frac{1}{p}} , \tag{A.4}$$

we define

$$||f||_{*,p,U} = \sup_{I \subset U} \left(\frac{1}{|I|} \int_I |f - f_I|^p \, dx \right)^{\frac{1}{p}} = \sup_{I \subset U} |I|^{-\frac{1}{p}} ||f - f_I||_{p,I} . \tag{A.5}$$

We then have

Proposition A.4. *The space* BMO(U), *is a subspace of* $L^p(U)$ *when* U *is a bounded interval.*

Thus, BMO(U) is a subspace of $\cap_{p=1}^{\infty} L^p(U)$, but *not* a subspace of $L^\infty(U)$. In fact, on BMO(U), we have a family of equivalent norms, as shown in the following proposition.

Proposition A.5. *On* BMO(U), *where* U *is a bounded interval, define for any real* $a > 0$, *and* $b > 0$, *and for any integer* $p \geq 1$ *(finite), the following family of norms*

$$N(f, a, b, p) = a||f||_{*,p,U} + b||f||_{p,U} , \tag{A.6}$$

then these norms are all equivalent for various a *and* b *and* p.

In BMO$(\mathbb{R})$, we will also define the seminorm $||f||_{*,\mathbb{T}}$ as follows

$$||f||_{*,\mathbb{T}} = \sup_{|I| \leq 1} \frac{1}{|I|} \int_I |f - f_I| \, dx , \tag{A.7}$$

the supremum being taken over intervals $I \subset \mathbb{R}$ with length less or equal to 1. The seminorm $||f||_{*,\mathbb{T}}$ is convenient for periodic functions with period 1. For any $f \in$ BMO$(\mathbb{R})$ we obviously have : $||f||_{*,[-(1/2),+(1/2)]} \leq ||f||_{*,\mathbb{T}} \leq ||f||_{*,\mathbb{R}}$. This observation will be useful if we now consider functions $f \in$ BMO$(\mathbb{R})$ *which are even and periodic with period 1* : for such functions, we also have the following result

Proposition A.6. *There exist constants* $K_3 > 1$, $K_4 > 1$, $K_5 > 1$ *such that for any* $f \in$ BMO$(\mathbb{R})$, *which is even and periodic with period 1, we have*

$$\begin{array}{ll} a) & ||f||_{*,\mathbb{R}} \leq K_3 ||f||_{*,\mathbb{T}} \\ b) & ||f||_{*,\mathbb{T}} \leq K_4 ||f||_{*,[0,1]} \\ b) & ||f||_{*,[0,1]} \leq K_5 ||f||_{*,[0,1/2]} \end{array} .$$

See [MMY] for a detailed proof. Note that parts b) and c) are not true if the periodic function f is not even. A non trivial, but immediate consequence of these results is the following corollary.

Corollary A.7. *Let f be a function defined in $[0,1/2]$, which belongs to* $\mathrm{BMO}([0,1/2])$. *The function g which is even and periodic with period 1, and which coincides with f on $[0,1/2]$ is in* $\mathrm{BMO}(\mathbb{R})$.

As indicated in Propositions A.5 and A.6, we have a wide choice among possible equivalent norms. The L^2-norm is especially useful, due to the following elementary identity :

$$\frac{1}{|I|}\int_I (f-f_I)^2\,ds = \frac{1}{2|I|^2}\int_{I\times I}(f(s)-f(t))^2\,dsdt\ . \tag{A.8}$$

Defining the oscillation of f in I with the L_2-norm, namely

$$\mathcal{O}_I(f) = \left(\frac{1}{2|I|^2}\int_{I\times I}(f(s)-f(t))^2\,dsdt\right)^{\frac{1}{2}}\ , \tag{A.9}$$

the norm $N(f,a,b,2)$ can be rewritten as

$$N(f,a,b,2) = a\sup_{I\subset U}\mathcal{O}_I(f) + b||f||_{2,U}\ , \tag{A.10}$$

In the above expression, (A.9) and (A.10), replacing the Lebesgue measure on U by any equivalent mesure, leads to an equivalent norm.

We now end this appendix by quoting Fefferman's theorem, which makes the link between analysis on the real line and harmonic or complex extension on the upper halfplane.

Proposition A.8 *The space* $\mathrm{BMO}(\mathbb{R})$ *is the dual space of the Hardy space H^1 on $\mathbb{R}$. If $f\in L^1_{\mathrm{loc}}(\mathbb{R})$, $f\in \mathrm{BMO}(\mathbb{R})$ if and only if there exist a constant c, and functions ϕ and ψ in L^∞, such that $f=c+\phi+H\psi$, where the Hilbert transform $H\psi$ is the harmonic conjugate of ψ. Furthermore, ϕ and ψ can be chosen such that $||\phi||_\infty \le C||f||_*$ and $||\psi||_\infty \le C||f||_*$, with C a constant.*

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