A **q**-deformation of the Gauss distribution

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Abstract

The q-deformed commutation relation $aa^* - qa^*a = \mathbb{1}$ for the harmonic oscillator is considered with $q \in [-1, 1]$. An explicit representation generalizing the Bargmann representation of analytic functions on the complex plane is constructed. In this representation the distribution of $a + a^*$ in the vacuum state is explicitly calculated. This distribution is to be regarded as the natural q-deformation of the Gaussian.

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1 Introduction and notation

In classical probability the Gauss distribution plays a central role. According to the central limit theorem it is the distribution that the standardized sum of n classically independent identically distributed random variables converges to as n tends to infinity. In recent years the question has drawn attention, what distributions are obtained in this limit, if one replaces the classical commutative notion of independence by some other type. Anti-commutative independence, as occuring in Fermi noise [Mey93] and free independence as occurs in large random matrices [Maa92, Spe90, VDN92] have now been studied. In the former case the Gauss distribution is replaced by the measure $(\delta_1 + \delta_{-1})/2$ and in the latter by Wigner's semicircle distribution, which on its support $[-2, 2]$ has the density $x \mapsto \sqrt{4 - x^2}/2\pi$. It is clear, however, that these two cases exhaust the possibilities by no means. In 1991, Bożejko and Speicher [BS91, BS92] introduced a deformation of Brownian motion by a parameter $q \in [-1, 1]$ which is governed by classical independence for $q = 1$, anti-commutative independence for $q = -1$, and free independence for $q = 0$. Their construction was based on a q-deformation, $\Gamma_q(\mathfrak{H})$, of the Fock space over a Hilbert space $\mathfrak{H}$. Their central random variables are given by operators of the form

$$a(f) + a(f)^*, \qquad f \in \mathfrak{H},$$

where $a(f)$ and $a(f)^*$ are the annihilation and creation operators associated to f satisfying the q-deformed commutation relation

$$a(f)a(g)^* - qa(g)^*a(f) = \langle f, g\rangle \mathbb{1}, \qquad f, g \in \mathfrak{H}.$$

Algebraic aspects of these commutation relations have been studied in [JSW91]. Another interpolation between boson and fermion Brownian motion is described in [Sch91]. There are good reasons to regard the probability distribution ν_q of these operators $a(f) + a(f)^*$ in the vacuum state as the natural q-deformation of the Gaussian distribution (cf. [Spe90]). By combinatoric means, Bożejko and Speicher using recursion relations for orthogonal polynomials in $a(f) + a(f)^*$, calculated this q-Gaussian distribution ν_q. For $\|f\| = 1$, it is supported by the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$, on which it is given by

$$\nu_q(dx) = \frac{1}{\pi}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta}|^2\, dx, \tag{1}$$

where $\theta \in [0, \pi]$ is such that $x\sqrt{1-q} = 2\cos\theta$.
It is the purpose of this paper to understand this distribution in analytic terms, as the probability distribution of a non-commutative random variable $a + a^*$,

where a is a bounded operator on some Hilbert space satisfying

$$aa^* - qa^*a = \mathbb{1}, \tag{2}$$

for some $q \in [-1, 1)$. The calculation is inspired by the free case, $q = 0$, where a and a^* turn out to be the left and the right shift on $l^2(\mathbb{N})$. In this case a and a^* can be quite nicely represented as operators on the Hardy class $\mathcal{H}^2$ of all analytic functions on the unit disc with L^2 limits towards the boundary (for instance [RR94]) via the equivalence $l^2(\mathbb{N}) \to \mathcal{H}^2$ given by

$$(\xi_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} \xi_n z^n, \qquad |z| < 1.$$

Under this equivalence a^* and a change into multiplication by z and the operator

$$(af)(z) := \frac{f(z) - f(0)}{z}.$$

The probability distribution in the vector state 1 of the operator $a + a^*$ is now calculated by diagonalization: first find improper eigenvectors $z \mapsto K(z, x)$ of $a + a^*$ by solving

$$(a + a^*)K(\cdot, x) = xK(\cdot, x), \qquad x \in [-2, 2],$$

which yields

$$K(z, x) = \frac{1}{(z - v)(z - \overline{v})} = \frac{1}{\overline{v} - v}\left(\frac{1}{z - v} - \frac{1}{z - \overline{v}}\right),$$

where $v = \exp i\theta$ with $\theta \in [0, \pi]$ such that $x = 2\cos\theta$. Then introduce an operator $W\colon L^2([-2, 2], \nu_0) \to \mathcal{H}^2$ by

$$(Wf)(z) = \int_{-2}^{2} K(z, x)f(x)\nu_0(dx),$$

choosing ν_0 in such a way that W becomes unitary. Once this can be done, ν_0 is the probability distribution of $a + a^*$, since $W^*1 = 1$ and $(a + a^*)^n W^*1 = x^n$:

$$\langle 1, (a + a^*)^n 1\rangle = \int_{-2}^{2} x^n \, \nu_0(dx).$$

The measure ν_0 can be found explicitly by the calculation

$$\begin{aligned}(W^*f)(x) &= \lim_{\eta\uparrow 1} \oint_{|z|=1} \overline{K(\eta z, x)} f(z)\,\lambda(dz) \\ &= \lim_{\eta\uparrow 1} \frac{1}{v-\overline{v}} \oint_{|z|=1} \left(\frac{1}{\eta z^{-1} - \overline{v}} - \frac{1}{\eta z^{-1} - v} \right) f(z)\,\frac{dz}{2\pi i z} \\ &= \lim_{\eta\uparrow 1} \frac{1}{v-\overline{v}} \oint_{|z|=1} \left(\frac{v}{z-\eta v} - \frac{\overline{v}}{z-\eta\overline{v}} \right) f(z)\,\frac{dz}{2\pi i} \\ &= \lim_{\eta\uparrow 1} \frac{vf(\eta v) - \overline{v} f(\eta\overline{v})}{v-\overline{v}} = \frac{vf(v) - \overline{v}f(\overline{v})}{v-\overline{v}},\end{aligned}$$

where λ denotes the normalized Lebesgue measure on the unit circle. The map W^* is made unitary by letting ν_0 compensate a factor $v - \overline{v} \propto \sin\theta$, another one being compensated by the change of variable $x \mapsto \theta$. So let

$$\nu_0(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx = \frac{2}{\pi}\sin^2\theta\,d\theta.$$

Then for all $f \in \mathcal{H}^2$,

$$\begin{aligned}\|W^*f\|^2 &= \frac{2}{\pi}\int_0^\pi |(W^*f)(2\cos\theta)|^2 \sin^2\theta\,d\theta \\ &= \frac{2}{\pi}\int_0^\pi \left| \frac{e^{i\theta}f(e^{i\theta}) - e^{-i\theta}f(e^{-i\theta})}{2\sin\theta} \right|^2 \sin^2\theta\,d\theta \\ &= \frac{1}{2}\oint_{|z|=1} |zf(z) - \overline{z}f(\overline{z})|^2\,\lambda(dz) \\ &= \oint_{|z|=1} |f(z)|^2\,\lambda(dz) = \|f\|^2,\end{aligned}$$

since $z \mapsto zf(z)$ and $z \mapsto \overline{z}f(\overline{z})$ are orthogonal functions on the unit circle. Thus Wigner's semicircle law, ν_0, arises naturally as a normalization of improper eigenvectors. Our program is to do the same for the deformed measure ν_q.
This paper has the following structure. In section 2 we show the essential uniqueness of the bounded representation of the q-commutation relation (2). Subsequently we find a measure μ_q, $q \in [0,1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: μ_q is concentrated on a family of concentric circles, the largest of which has radius $1/\sqrt{1-q}$. Our representation space will be $\mathcal{H}^2(\mathfrak{D}_q, \mu_q)$, the completion of the analytic functions on $\mathfrak{D}_q = \{\, z \in \mathbb{C} \mid |z|^2 < 1/(1-q) \,\}$ with respect to the inner product defined by μ_q. In this space the annihilation operator a is represented by a q-difference operator D_q. As q tends to 1, μ_q will tend to the Gauss measure on $\mathbb{C}$, and D_q becomes differentiation. Thus one obtains Bargmann's representation of the

harmonic oscillator [Bar61]. It should be noted that there is no measure μ_q for $q < 0$. So far we essentially reproduce the work of Arik and Coon [AC76].
In section 3 we perform the diagonalization of $a + a^* = D_q + Z$ by constructing a unitary operator W like the one above. The q-Gaussian distribution ν_q is found naturally for $q \in [0, 1)$.
We shall make abundant use of the language of q-calculus, which is over a century old (cf. [All80, AI84, GR90, Jac10, KS94, Koo94]). We recall some basic notations here.
The natural number n has the following q-deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \qquad \text{with } [0]_q := 0.$$

Occasionally we shall write $[\infty]_q$ for the limit of these numbers: $1/(1-q)$. The q-factorials and q-binomial coefficients are defined naturally as:

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \qquad \text{with } [0]_q! := 1;$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j) \qquad \text{with } (a;q)_0 := 1,$$

is the q-shifted factorial, the q-analogue of the Pochhammer symbol. A product of these q-shifted factorials $(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n$ is denoted as $(a_1, a_2, \ldots, a_r; q)_n$. In terms of these products the deformed hypergeometric series can be defined as:

$${}_r\varphi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots b_s \end{matrix}; q, z\right] := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n,$$

where $\binom{n}{2} := n(n-1)/2$. For $q \in (-1, 1)$ this series converges absolutely for all $z \in \mathbb{C}$ if $r < s + 1$ and for $z \in \mathfrak{D}_0$ if $r = s + 1$, if $r > s + 1$ it diverges for all $z \in \mathbb{C} \setminus \{0\}$. We note two interesting deformations of Newton's binomial theorem:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-zq^{-n})^k = {}_1\varphi_0\left[\begin{matrix} q^{-n} \\ \text{—} \end{matrix}; q, z\right] = (zq^{-n}; q)_n \overset{q\uparrow 1}{=} (1-z)^n, \tag{3}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2} z^k}{(zq;q)_k} = {}_1\varphi_1\left[\begin{matrix} q^{-n} \\ zq \end{matrix}; q, zq^{n+1}\right] = \frac{1}{(zq;q)_n} \overset{q\uparrow 1}{=} \frac{1}{(1-z)^n}. \tag{4}$$

The proof of (3) and (4) is simple and can be found in [GR90, KS94, Koo94]. Finally, by $\sigma(x)$ we shall denote the spectrum of a bounded operator x.

2 A representation of the q-commutation relations

Proposition 2.1 *For $q \in (-1,1) \setminus \{ 0 \}$ relation (2) admits, up to unitary equivalence, a unique non trivial bounded irreducible representation given on the canonical basis $\{ e_n \mid n \in \mathbb{N} \}$ of $l^2(\mathbb{N})$ by:*

i) $a^* e_n = e_{n+1}$

ii) $a e_n = [n]_q e_{n-1}$

iii) $\langle e_n, e_m \rangle = \delta_{n,m} [n]_q!$

For $q = 0$ relation (2) reduces to $aa^* = \mathbb{1}$ and this obviously admits more then one representation: any isometry a^* suffices. By a representation with $q = 0$, we shall simply mean one satisfying i, ii and iii.

Proof: Consider a bounded operator a on a Hilbert space $\mathfrak{H}$ that defines a non trivial irreducible representation of (2). Proposition 1.1.8 in [Sak71] states that $\sigma(xy) \cup \{ 0 \} = \sigma(yx) \cup \{ 0 \}$ for all x, y in some unital algebra over $\mathbb{C}$. In the case at hand this implies that $\sigma(a^*a) \cup \{ 0 \} = \sigma(aa^*) \cup \{ 0 \}$. Define a linear invertible mapping $\varphi : \mathbb{R} \to \mathbb{R} : x \mapsto 1 + qx$ for all $x \in \mathbb{R}$, then (2) can be written as $\varphi(a^*a) = aa^*$. By the spectral mapping theorem it follows that

$$\varphi(\sigma(a^*a)) \cup \{ 0 \} = \sigma(a^*a) \cup \{ 0 \} \tag{5}$$
$$\sigma(aa^*) \cup \{ 0 \} = \varphi^{-1}(\sigma(aa^*)) \cup \{ 0 \}. \tag{6}$$

Because a^*a is a positive operator, we have $\sigma(a^*a) \subset [0, \infty)$. From the definition of φ it is clear that $1/(1-q)$ is a fixed point, however it can not be an eigenvalue of a^*a because of irreducibility. Suppose $\lambda \in \sigma(a^*a) \cap (1/(1-q), \infty)$ then by (5) we have $\varphi^{-n}(\lambda) \in \sigma(a^*a)$, whereas

$$\lim_{n \to \infty} \varphi^{-n}(\lambda) = \lim_{n \to \infty} \frac{\lambda - 1}{q^n} = \infty$$

in contradiction with the boundedness of a^*a. It follows that there must be $\lambda \in \sigma(a^*a) \cap (0, 1/(1-q))$. Then because of (6) there must be some $n \in \mathbb{N}$ such that $\varphi^{-n}(\lambda) = 0$, therefore $\lambda = \varphi^n(0) = [n]_q$. We conclude that $\sigma(a^*a) = \{ [n]_q \mid n \in \mathbb{N} \}$ whereas because of (5) $\sigma(aa^*) = \sigma(a^*a) \setminus \{ 0 \}$. Note that $\lambda = 1/(1-q)$ is in $\sigma(a^*a)$ since this is a closed set.
Because 0 is an isolated point of $\sigma(a^*a)$ we can choose $e_0 \in \ker(a)$. Define $e_n := (a^*)^n e_0$, then i is satisfied and from (2) one easily obtains $\langle e_n, e_m \rangle = 0$ for $n \neq m$. Furthermore $ae_{n+1} = aa^*e_n = \varphi(a^*a)e_n = [n+1]_q e_n$ which yields ii.

To prove iii note that $\|e_{n+1}\|^2 = \langle e_n, aa^* e_n \rangle = [n+1]_q \|e_n\|^2$ and if we require $\|e_0\| = 1$ we find $\|e_n\|^2 = [n]_q!$. By irreducibility the vectors $\{\, e_n \mid n \in \mathbb{N} \,\}$ span the Hilbert space $\mathfrak{H}$. $\square$

For $q \in [-1, 1)$ and analytic $f : \mathbb{C} \to \mathbb{C}$ define operators Z and D_q as:

$$(Z\, f)\,(z) := z f(z)$$

$$(D_q\, f)\,(z) := \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases}$$

The operator D_q has the following properties:

i) $\lim_{q \uparrow 1} (D_q\, f)\,(z) = f'(z)$,

ii) $D_q(z^n) = [n]_q z^{n-1}$,

iii) $D_q\,(f(z) g(z)) = (D_q\, f)\,(z) g(z) + f(qz)\,(D_q\, g)\,(z)$,

iv) $D_q \left(\dfrac{f(z)}{g(z)} \right) = \dfrac{(D_q\, f)\,(z) g(z) - f(z)\,(D_q\, g)\,(z)}{g(z) g(qz)}$.

Note that iii is a q-analogue of the product rule and iv is a q-analogue of the quotient rule.

Lemma 2.2 *There exists a unique meromorphic function, $f : \mathbb{C} \to \mathbb{C}$, with $f(0) = 1$ such that $(D_q\, f)\,(z) = f(z)$. It is given by*

$$f(z) = \prod_{k=0}^{\infty} (1 - (1-q) q^k z)^{-1} = \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!},$$

where the series has radius of convergence $1/(1-q)$.

Proof: Writing out the difference equation gives $f(qz) = (1 - (1-q)z) f(z)$. Iterate this and use the fact that $\lim_{n \to \infty} f(q^n z) = f(0)$ to find the desired product formula for $f(z)$.
It is easy to check that the summation formula for $f(z)$ satisfies the difference equation with $f(0) = 1$. Convergence can be checked using the ratio test. $\square$

¿From now on we will refer to the function f defined in lemma 2.2 as $\exp_q(z)$.

To define what is called q-integration consider the equation $(D_q F)(z) = f(z)$ for some continuous f. This gives $F(x) - F(qx) = (1-q)xf(x)$ which, assuming $\lim_{n\to\infty} F(q^n x) = F(0)$, yields the following definition for the q-integral:

$$\int_0^a f(x)\, d_q x := F(a) - F(0) = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k, \qquad a \in \mathbb{R}^+.$$

A good definition for the q-gamma function is now given by:

$$\Gamma_q(x) := \int_0^{[\infty]_q} \frac{t^{x-1}}{\exp_q(qt)}\, d_q t = \frac{(q;q)_\infty}{(1-q)^n} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q;q)_k}.$$

The following classical lemma ([GR90], [Jac10], [KS94] and [Koo94]) is given here with proof because it illustrates some techniques needed later on.

Lemma 2.3 $\forall n \in \mathbb{N}$: $\Gamma_q(n+1) = [n]_q!$

Proof: Note that

$$\int_0^{[\infty]_q} D_q\left(\frac{t^n}{\exp_q(t)}\right) d_q t = \begin{cases} \left.\dfrac{t^n}{\exp_q(t)}\right|_0^{[\infty]_q} = -\delta_{0,n} \\ [n]_q \displaystyle\int_0^{[\infty]_q} \frac{t^{n-1}}{\exp_q(qt)}\, d_q t - \int_0^{[\infty]_q} \frac{t^n}{\exp_q(qt)}\, d_q t \end{cases}$$

where we have used the q-quotient rule. Putting $n = 0$ to find that $\Gamma_q(1) = 1$ and put $n > 0$ to find that $\Gamma_q(n+1) = [n]_q \Gamma_q(n)$. The statement follows by induction. □

It is clear that in the limit $q \uparrow 1$ we recover Riemannian integration and the usual gamma function.
Next we show that the operators D_q and Z give a bounded representation of (2) as described in proposition 2.1.

Lemma 2.4 *The operators D_q and Z satisfy $D_q Z - q Z D_q = \mathbb{1}$.*

Proof: Use the q-product rule. □

With respect to the measure

$$\mu_q(dz) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \lambda_{r_k}(dz), \quad 0 \le q < 1 \text{ and } r_k = \frac{q^{k/2}}{\sqrt{1-q}}, \tag{7}$$

where λ_{r_k} is the normalized Lebesgue measure on the circle with radius r_k, we define the inner product $\langle f, g\rangle_{\mu_q} := \int_{\mathbb{C}} \overline{f(z)} g(z)\, \mu_q(dz)$ for all $f, g \in \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$. Note that μ_0 is the normalized Lebsgue measure on the unit circle and that, in the limit $q \uparrow 1$, μ_q tends to the Gauss measure on the complex plane.

Proposition 2.5 *The identifications $a = D_q$ and $a^* = Z$ determine a representation of* (2) *on* $\mathcal{H}^2(\mathfrak{D}_q, \mu_q)$.

In particular, with $e_n := z^n$, i, ii and iii of proposition 2.1 are satisfied, and therefore ${D_q}^* = Z$.

Proof: Properties i and ii of proposition 2.1 are already verified, so we prove property iii here:

$$\begin{aligned}
\langle z^n, z^m \rangle_{\mu_q} &= \int_{\mathbb{C}} \overline{z}^n z^m \, \mu_q(dz) \\
&= (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \int_{\mathbb{C}} \overline{z}^n z^m \, \lambda_{r_k}(dz) \\
&= \frac{(q;q)_\infty}{2\pi} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \int_0^{2\pi} r_k^{n+m} e^{i(m-n)\varphi} \, d\varphi \\
&= \delta_{n,m} \frac{(q;q)_\infty}{(1-q)^n} \sum_{k=0}^{\infty} \frac{q^{nk}}{(q;q)_k} \\
&= \delta_{n,m} \Gamma_q(n+1) = \delta_{n,m} [n]_q!
\end{aligned}$$

□

This means that D_q and Z give a bounded representation of the q-commutation relation on the Hilbert space $\mathcal{H}^2(\mathfrak{D}_q, \mu_q)$.

3 Probability distribution in the ground state

In this section we construct an operator $W\colon L^2(\mathbb{R}, \nu_q) \to \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$, for $q \in [0,1)$, which diagonalizes $D_q + Z$ in the sense that:

$$D_q + Z = W X W^{-1}, \qquad \text{with } W1 = 1,$$

where X denotes the operator of multiplication by the coordinate x in $L^2(\mathbb{R}, \nu_q)$. As a consequence ν_q can be viewed as the probability distribution of $D_q + Z$ in the vector state $1 \in \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$ since $D_q + Z$ is bounded and for all $n \in \mathbb{N}$:

$$\langle 1, (D_q + Z)^n 1 \rangle_{\mu_q} = \left\langle 1, W X^n W^{-1} 1 \right\rangle_{\mu_q} = \langle 1, X^n 1 \rangle_{\nu_q} = \int_{\mathbb{R}} x^n \, \nu_q(dx).$$

It turns out that for W to be unitary we need ν_q as follows

$$\nu_q(dx) = \frac{1}{\pi} \sqrt{1-q} \sin\theta (q, qv^2, qv^{-2}; q)_\infty \, dx$$

where $v = \exp i\theta$, $v + \overline{v} = x\sqrt{1-q}$ and $\operatorname{supp} \nu_q = [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. In the limit $q \uparrow 1$ this measure yields the Gaussian measure on $\mathbb{R}$, whereas for $q \downarrow -1$ ν_q gives a δ-distribution concentrated on $x = -1$ and $x = 1$. For $q = 0$ we recover the Wigner distribution. This means that although the Bargmann space measure μ_q only exists for $q \in [0, 1)$, the resulting measure ν_q seems to be valid for all $q \in (-1, 1)$.

LEMMA 3.1 *The measure $\nu_q(dx)$ is a probability measure for $q \in (-1, 1)$.*

PROOF: The triple product in ν_q can be rewritten as a Θ_1-function (cf. [Cha84, GR90]):

$$\nu_q(dx) = \frac{1}{2\pi} q^{-1/8} \sqrt{1-q}\, \Theta_1\!\left(\frac{\theta}{\pi}, \frac{1}{2\pi i} \log q\right) dx.$$

This yields:

$$\begin{aligned}\int_{\mathbb{R}} \nu_q(dx) &= \frac{2}{\pi} q^{-1/8} \int_0^\pi \sum_{k=0}^\infty (-1)^k q^{(2k+1)^2/8} \sin{(2k+1)\theta} \sin\theta \, d\theta \\ &= \frac{2}{\pi} \int_0^\pi \sin^2\theta \, d\theta = 1\end{aligned}$$

We note that this calculation actually works for every $q \in (-1, 1)$. □

Since $D_q + Z$ is self-adjoint, diagonalization can be done in much the same way as for a symmetric matrix; by using its (improper) eigenvectors. The eigenvalue equation

$$(D_q + Z)\, K(z, x) = x K(z, x) \tag{8}$$

normalized by the requirement that $K(0, x) = 1$, has a unique meromorphic solution $K(z, x)$ given by:

$$K(z, x) = (vz\sqrt{1-q}, v^{-1}z\sqrt{1-q}; q)_\infty^{-1},$$

again with $|x| \leq 2/\sqrt{1-q}$ and $v + \overline{v} = x\sqrt{1-q}$. This can be seen easily by iterating the difference equation (8) as was done for $\exp_q$ in the proof of lemma 2.2.

The poles of the function $z \mapsto K(z, x)$ are given by $z = v^{\pm 1}/q^k\sqrt{1-q}$ with $k \in \mathbb{N}$, two of which are on the edge of the disc $\mathfrak{D}_q$, whereas the remaining ones lie outside. This implies that the function $z \mapsto K(z, x)$ is not in $\mathcal{H}^2(\mathfrak{D}_q, \mu_q)$ although it is analytic on the interior of $\mathfrak{D}_q$. This is of course related to the fact

that $D_q + Z$ has a continuous spectrum admitting only improper eigenfunctions. We deal with this by introducing a bounded kernel $z \mapsto K_\eta(z,x)$ by $K_\eta(z,x) := K(\eta z, x)$ with $\eta \in (0,1)$, so $K(z,x)$ is recovered in the limit $\eta \uparrow 1$.
The kernel $K_\eta(z,x)$ will be used to define a compact operator W_η as follows:

$$W_\eta \colon L^2(\mathbb{R},\nu_q) \to \mathcal{H}^2(\mathfrak{D}_q,\mu_q) \colon (W_\eta f)(z) = \int_{\mathbb{R}} K_\eta(z,x) f(x)\,\nu_q(dx), \quad |z|<1, \tag{9}$$

for all $\eta \in (0,1)$. Note that W_η is a well-defined operator and that the right hand side of (9) also makes sense for $\eta = 1$.

Proposition 3.2 *The definition* (9) *defines a bounded operator $W = W_1$, and we have the strong operator limits:*

$$\lim_{\eta\uparrow 1} W_\eta = W \qquad \textit{and} \qquad \lim_{\eta\uparrow 1} W_\eta^* W_\eta = \mathbb{1}.$$

Consequently, W is an isometry.

In the proof it will be convenient to identify $L^2(\mathbb{R},\nu_q)$ with the space $L^2_{\mathrm{s}}(\partial\mathfrak{D}_0,\tilde{\nu}_q) = \{ f \in L^2(\partial\mathfrak{D}_0,\tilde{\nu}_q) \mid f(v) = f(\overline{v})\ \forall v \in \partial\mathfrak{D}_0 \}$, the symmetric functions on the unit circle, via the map:

$$J \colon L^2(\mathbb{R},\nu_q) \to L^2_{\mathrm{s}}(\partial\mathfrak{D}_0,\tilde{\nu}_q) \colon (Jf)(v) = f\Big(\frac{v+\overline{v}}{\sqrt{1-q}}\Big), \qquad |v|=1.$$

The map J becomes unitary if we transfer the measure ν_q to $\partial\mathfrak{D}_0$:

$$\tilde{\nu}_q(dv) = \nu_q(dx) = \frac{1}{2}|v-\overline{v}|^2 (q, qv^2, qv^{-2}; q)_\infty\, \lambda(dv),$$

where λ is the normalized Lebesgue measure on $\partial\mathfrak{D}_0$. Then W_η decomposes naturally as $W_\eta = \tilde{W}_\eta \circ J$, where

$$(\tilde{W}_\eta f)(z) = \oint_{\partial\mathfrak{D}_0} \tilde{K}_\eta(z,v) f(v)\,\tilde{\nu}_q(dv), \qquad |z|<1,$$

and $\tilde{K}_\eta(z,v) = K_\eta(z,x)$ if $v + \overline{v} = x\sqrt{1-q}$.
We start with a lemma connecting functions on $\partial\mathfrak{D}_0$ to functions on $\mathfrak{D}_0$. Let A denote the contraction from $L^2(\partial\mathfrak{D}_0,\lambda)$ to $\mathcal{H}^2(\mathfrak{D}_0,\lambda)$ given by:

$$A \colon \sum_{n\in\mathbb{Z}} a_n z^n \mapsto \sum_{n\in\mathbb{N}} a_n z^n,$$

which can also be written as

$$(Af)(z) = \frac{1}{2\pi i}\oint_{\partial\mathfrak{D}_0} \frac{f(v)}{v-z}\,dv, \qquad |z|<1,$$

Cauchy's formula for the analytic part of f.

LEMMA 3.3 *The following holds for all $f \in L^2_{\mathrm{s}}(\partial\,\mathfrak{D}_0)$:*

$$\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{|v-\overline{v}|^2 f(v)}{(v-z)(\overline{v}-z)}\,\lambda(dv)=\frac{1}{z}A((v-\overline{v})f(v))(z),\qquad |z|<1.$$

PROOF: Note the partial fraction decomposition

$$\frac{1}{(v-z)(\overline{v}-z)}=\frac{1}{(\overline{v}-v)}\left(\frac{1}{v-z}-\frac{1}{\overline{v}-z}\right).$$

Therefore since $f(v)=f(\overline{v})$:

$$\begin{aligned}\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{|v-\overline{v}|^2 f(v)}{(v-z)(\overline{v}-z)}\,\lambda(dv)&=\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{(v-\overline{v})f(v)}{(v-z)}\,\lambda(dv)-\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{(v-\overline{v})f(v)}{(\overline{v}-z)}\,\lambda(dv)\\&=\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{(v-\overline{v})f(v)}{(v-z)}\,\lambda(dv)+\frac{1}{2}\oint_{\partial\,\mathfrak{D}_0}\frac{(\overline{v}-v)f(\overline{v})}{(\overline{v}-z)}\,\lambda(d\overline{v})\\&=\oint_{\partial\,\mathfrak{D}_0}\frac{(v-\overline{v})f(v)}{(v-z)}\,\lambda(dv)\\&=\oint_{\partial\,\mathfrak{D}_0}\frac{(v-\overline{v})f(v)}{(v-z)}\,\frac{dv}{2\pi i v}\\&=\frac{1}{z}A((v-\overline{v})f(v))(z).\end{aligned}$$

□

We now turn to the proof of proposition 3.2.

PROOF: Choose a smooth symmetric function f on $\partial\,\mathfrak{D}_0$. Then, for $|z|<1$,

$$\begin{aligned}(\tilde{W}f)(z)&=\oint_{\partial\,\mathfrak{D}_0}\tilde{K}(z,v)f(v)\,\tilde{\nu}_q(dv)\\&=(q;q)_\infty\oint_{\partial\,\mathfrak{D}_0}\frac{\tilde{K}_q(z,v)f(v)|\varphi(v)|^2|v-\overline{v}|^2}{(v-z\sqrt{1-q})(\overline{v}-z\sqrt{1-q})}\,\lambda(dv),\end{aligned}$$

where $\varphi(v):=(qv^2;q)_\infty/\sqrt{2}$. Now let $g_z\in L^2_{\mathrm{s}}(\partial\,\mathfrak{D}_0)$ denote the function

$$g_z(v)=(q;q)_\infty\tilde{K}_q(z,v)|\varphi(v)|^2,$$

uniformly bounded in z and v: $|g_z(v)|\le C$, with C a constant. Then by lemma 3.3,

$$(\tilde{W}f)(z)=\frac{1}{z\sqrt{1-q}}A((v-\overline{v})g_z\cdot f)(z\sqrt{1-q})$$

so that

$$\begin{aligned}\|\tilde{W}f\|^2 &= \int_{\mathbb{C}} |(\tilde{W}f)(z)|^2\,\mu_q(dz)\\ &\le \int_{\partial\mathfrak{D}_0} |(\tilde{W}f)(u/\sqrt{1-q})|^2\,\lambda(du)\\ &\le \frac{4C^2\|A\|\|f\|^2}{1-q},\end{aligned}$$

so $\tilde{W}$, and hence W, is a bounded operator. From the analyticity of $z \mapsto (\tilde{W}f)(z)$ and the fact that $(\tilde{W}_\eta f)(z) = (\tilde{W}f)(\eta z)$ it follows that for all $f \in L^2_{\mathrm{s}}(\partial\mathfrak{D}_0)$:

$$L^2\text{-}\lim_{\eta\uparrow 1}\tilde{W}_\eta f = \tilde{W}f,$$

and also $W_\eta \to W$ strongly.
To prove the second statement in proposition 3.2, let L_η denote the integral kernel of the operator $\tilde{W}^*_\eta\tilde{W}_\eta$:

$$\begin{aligned}L_\eta(v,w) &:= \int_{\mathbb{C}} \overline{\tilde{K}_\eta(z,v)}\tilde{K}_\eta(z,w)\,\mu_q(dz)\\ &= (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} L^{(k)}_\eta(v,w),\end{aligned} \tag{10}$$

where

$$L^{(k)}_\eta(v,w) = \oint_{|z|=r_k} \overline{\tilde{K}_\eta(z,v)}\tilde{K}_\eta(z,w)\,\lambda(dz).$$

$L^{(k)}_\eta$can be calculated using the residue theorem. The meromorphic extension of the restriction of the integrand $z \mapsto \overline{\tilde{K}_\eta(z,v)}\tilde{K}_\eta(z,w)$ to the disc $\{\,|z| \le r_k\,\}$,

$$\tilde{K}_\eta(z,w)\prod_{j=k}^{\infty}\frac{z^2}{(z-\eta v q^j\sqrt{1-q})(z-\eta\overline{v}q^j\sqrt{1-q})}, \qquad \text{for } |z| \le r_k,$$

has poles at $z = \eta q^n v^{\pm 1}/\sqrt{1-q}$ with $n \ge k$. Summation of the residues at these poles yields:

$$\begin{aligned}L^{(k)}_\eta(v,w) = \sum_{n=k}^{\infty}\Bigg(&\frac{v\tilde{K}_\eta\left(\frac{\eta q^n v}{\sqrt{1-q}},w\right)}{(v-v^{-1})}\prod_{\substack{j=k\\ j\neq n}}^{\infty}\frac{q^{2n}v^2}{(q^j-q^n)(q^j-q^n v^2)} +\\ &+\frac{v^{-1}\tilde{K}_\eta\left(\frac{\eta q^n v^{-1}}{\sqrt{1-q}},w\right)}{(v^{-1}-v)}\prod_{\substack{j=k\\ j\neq n}}^{\infty}\frac{q^{2n}v^{-2}}{(q^j-q^n)(q^j-q^n v^{-2})}\Bigg),\end{aligned}$$

where $\tilde{K}_\eta\Big(\frac{\eta q^n v}{\sqrt{1-q}}, w\Big)$ is given by:

$$\tilde{K}_\eta\Big(\frac{\eta q^n v}{\sqrt{1-q}}, w\Big) = \frac{(\eta^2 vw, \eta^2 vw^{-1}; q)_n}{(\eta^2 vw, \eta^2 vw^{-1}; q)_\infty}. \tag{11}$$

The first product in the right hand side of $L_\eta^{(k)}$ can be rewritten as:

$$\prod_{j=k}^{n-1} \frac{q^{2(n-j)} v^2}{\big(1-q^{(n-j)}v^2\big)\big(1-q^{(n-j)}\big)} \prod_{j=n+1}^{\infty} \frac{1}{\big(1-q^{(j-n)}v^{-2}\big)\big(1-q^{(j-n)}\big)}$$
$$= \frac{q^{(n-k)(n-k+1)} v^{2(n-k)}}{(q, qv^2; q)_{n-k} (q, qv^{-2}; q)_\infty}.$$

Note that this result is also correct if $n = k$. A similar argument works for the second product in $L_\eta^{(k)}$. Substitute $L_\eta^{(k)}$ back into (10) and interchange the order of summation to find for L_η:

$$L_\eta(v, w) = (q;q)_\infty \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{q^{n-k}}{(q;q)_{n-k}} \left(\frac{v^{2k+1} q^{k(k+1)} \tilde{K}_\eta\Big(\frac{\eta q^n v}{\sqrt{1-q}}, w\Big)}{(v-v^{-1})(q, qv^2; q)_k (q, qv^{-2}; q)_\infty} + \right.$$
$$\left. + \frac{v^{-(2k+1)} q^{k(k+1)} \tilde{K}_\eta\Big(\frac{\eta q^n v^{-1}}{\sqrt{1-q}}, w\Big)}{(v^{-1}-v)(q, qv^{-2}; q)_k (q, qv^2; q)_\infty} \right). \tag{12}$$

The first double sum on the right hand side will be called Ω. We use the q-binomial theorem (4) to rewrite Ω as:

$$(v - v^{-1}) \frac{\Omega}{v} = \sum_{n=0}^{\infty} \frac{q^n \tilde{K}_\eta\Big(\frac{\eta q^n v}{\sqrt{1-q}}, w\Big)}{(q;q)_n (qv^{-2}; q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2} v^{2k}}{(qv^2; q)_k}$$
$$= \sum_{n=0}^{\infty} \frac{q^n \tilde{K}_\eta\Big(\frac{\eta q^n v}{\sqrt{1-q}}, w\Big)}{(q;q)_n (qv^{-2}; q)_\infty (qv^2; q)_n}, \tag{13}$$

and note that a similar result holds for the second double sum in (12). Put Ω back into (12) and use (11) to find for L_η;

$$L_\eta(v, w) = \frac{2}{|v - \overline{v}|^2 (q, qv^2, qv^{-2}; q)_\infty} \operatorname{Re}(\Lambda_\eta(v, w)),$$

where

$$\Lambda_\eta(v, w) = \frac{v(v - \overline{v})(q, qv^2; q)_\infty}{(\eta^2 vw, \eta^2 vw^{-1}; q)_\infty} \, {}_2\varphi_1\!\left[\begin{matrix} \eta^2 vw, \eta^2 vw^{-1} \\ qv^2 \end{matrix}; q, q\right].$$

We claim that $\mathrm{Re}(\Lambda_\eta(v,w))$, as a kernel, tends to the identity operator on $L^2_{\mathrm{s}}(\partial\mathfrak{D}_0)$ as $\eta\uparrow 1$. As a result we have for all $f\in L^2_{\mathrm{s}}(\partial\mathfrak{D}_0,\tilde\nu_q)$:

$$\begin{aligned}(\tilde W_\eta^*\tilde W_\eta f)(v) &= \int_{\partial\mathfrak{D}_0} L_\eta(v,w)f(w)\,\tilde\nu_q(dw)\\ &= \int_{\partial\mathfrak{D}_0} \frac{2|w-\overline{w}|^2(q,qw^2,qw^{-2};q)_\infty}{2|v-\overline{v}|^2(q,qv^2,qv^{-2};q)_\infty}\,\mathrm{Re}(\Lambda_\eta(v,w))\,f(w)\,\lambda(dw)\\ &\xrightarrow{\eta\uparrow 1} f(v),\end{aligned}$$

showing that $\tilde W_\eta^*\tilde W_\eta\to\mathbb{1}$ strongly, hence also $W_\eta^*W_\eta\to\mathbb{1}$ strongly. As a bonus the measure $\tilde\nu_q$ and therefore ν_q comes out naturally in the same way as the measure ν_0 came out in the free case, $q=0$, treated in the introduction.
It remains to prove our claim that that for all $f\in L^2_{\mathrm{s}}(\partial\mathfrak{D}_0)$:

$$L^2\text{-}\lim_{\eta\uparrow 1}\oint_{\partial\mathfrak{D}_0}\mathrm{Re}(\Lambda_\eta(\cdot,w))\,f(w)\,\lambda(dw)=f.$$

Rewrite $\mathrm{Re}(\Lambda_\eta(v,w))$ as follows:

$$\mathrm{Re}(\Lambda_\eta(v,w))=\hat\Lambda_\eta(v,w)\,\mathrm{Re}\left(\frac{v(v-\overline{v})}{(1-\eta^2vw)(1-\eta^2vw^{-1})}\right)$$

with $\hat\Lambda_\eta$ defined as:

$$\hat\Lambda_\eta(v,w):=\frac{(q,qv^2;q)_\infty}{(\eta^2qvw,\eta^2qvw^{-1};q)_\infty}\,{}_2\varphi_1\left[\begin{matrix}\eta^2vw,\eta^2vw^{-1}\\ qv^2\end{matrix};q,q\right]\xrightarrow{\eta\uparrow 1}1.$$

The limit for $\hat\Lambda_\eta$ being calculated using the q-Gauss sum II.8 in [GR90]. Let $g\in L^2_{\mathrm{s}}(\partial\mathfrak{D}_0)$ be such that we can write $g(w)=|w-\overline{w}|^2f(w)$ with $f\in L^2_{\mathrm{s}}(\partial\mathfrak{D}_0)$, then applying lemma 3.3 yields:

$$\begin{aligned}\oint_{\partial\mathfrak{D}_0}\mathrm{Re}(\Lambda_\eta(v,w))\,g(w)\,\lambda(dw) &= \frac{v(v-\overline{v})}{2}\oint_{\partial\mathfrak{D}_0}\frac{\hat\Lambda_\eta(v,w)f(w)|w-\overline{w}|^2}{(1-\eta^2vw)(1-\eta^2v\overline{w})}\,\lambda(dw)+\\ &\quad+\frac{\overline{v}(\overline{v}-v)}{2}\oint_{\partial\mathfrak{D}_0}\frac{\overline{\hat\Lambda_\eta(v,w)}f(\overline{w})|w-\overline{w}|^2}{(1-\eta^2\overline{vw})(1-\eta^2\overline{v}w)}\,\lambda(d\overline{w})\\ &=(v-\overline{v})A((w-\overline{w})\hat\Lambda_\eta(\cdot,w)f)(\eta^2v)+\\ &\quad+(\overline{v}-v)A((\overline{w}-w)\overline{\hat\Lambda_\eta(\cdot,w)}f)(\eta^2\overline{v})\\ &\xrightarrow{\eta\uparrow 1}|v-\overline{v}|^2f(v)=g(v),\end{aligned}$$

for almost all $v\in\partial\mathfrak{D}_0$.

¿From the strong convergence $W_\eta^* W_\eta \to \mathbb{1}$ we know that W is isometric since for all $f, g \in L^2(\mathbb{R}, \nu_q)$:

$$\langle Wf, Wg \rangle_{\mu_q} = \lim_{\eta \uparrow 1} \langle W_\eta f, W_\eta g \rangle_{\mu_q} = \lim_{\eta \uparrow 1} \left\langle f, W_\eta^* W_\eta g \right\rangle_{\nu_q} = \langle f, g \rangle_{\nu_q}.$$

□

THEOREM 3.4 *The map $W\colon L^2(\mathbb{R}, \nu_q) \to \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$ has the following properties:*

i) $W1 = 1$

ii) $(D_q + Z)W = WX$

iii) *W is unitary.*

PROOF: To prove property i note that for all $\eta \in (0,1)$ and $|x| < 2/\sqrt{1-q}$,

$$(W_\eta^* 1)(x) = \int_{\mathbb{C}} \overline{K_\eta(z,x)}\, \mu_q(dz) = \overline{\int_{\mathbb{C}} K_\eta(z,x)\, \mu_q(dz)} = \overline{K_\eta(0,x)} = 1.$$

Since $W_\eta^* \to W^*$, at least weakly, it follows that $W^* 1 = 1$. The operator WW^* is the range projection of W and since ν_q and μ_q are both probability measures:

$$\|W1 - 1\|_{\mu_q}^2 = \|(WW^* - \mathbb{1})1\|_{\mu_q}^2 = \langle 1, (WW^* - \mathbb{1})1 \rangle_{\mu_q} = \|1\|_{\nu_q}^2 - \|1\|_{\mu_q}^2 = 0.$$

Property ii follows from the definition of K:

$$\begin{aligned} ((D_q + Z)Wf)(z) &= \int_{\mathbb{R}} (D_q + Z) K(z,x) f(x)\, \nu_q(dx) \\ &= \int_{\mathbb{R}} x K(z,x) f(x)\, \nu_q(dx) \\ &= (WXf)(z). \end{aligned}$$

To prove iii it suffices to show that z^n lies in the range of W for all $n \in \mathbb{N}$. By i this is already the case for $n = 0$. Proceeding by induction, assume that $WW^* z^k = z^k$ for all $k \le n$. ¿From ii we have that $ZW = WX - D_q W$ and $W^* Z = XW^* - W^* D_q$. Hence

$$\begin{aligned} WW^* z^{n+1} &= W(XW^* - W^* D_q) z^n \\ &= (Z + D_q) WW^* z^n - WW^* [n]_q z^{n-1} \\ &= (Z + D_q) z^n - [n]_q z^{n-1} = Z z^n = z^{n+1}. \end{aligned}$$

So z^{n+1} is in $\operatorname{Ran} W$ as well. □

As a final result we calculate the operator W^* explicitly.

Lemma 3.5 *The operator W^* is, for all $f \in \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$, given by:*

$$(W^*f)(x) = \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \left(\frac{f\left(\frac{q^k}{\sqrt{1-q}} v\right)}{(qv^2;q)_k (v^{-2};q)_\infty} + \frac{f\left(\frac{q^k}{\sqrt{1-q}} v^{-1}\right)}{(qv^{-2};q)_k (v^2;q)_\infty} \right).$$

Proof: This is an immediate consequence of (13) with $K_\eta(z,y)$ replaced by an arbitrary function $f(z) \in \mathcal{H}^2(\mathfrak{D}_q, \mu_q)$. □

Finally note that we can obtain relation II.23 in [GR90] using lemma 3.5 and the fact that $W^*1 = 1$.

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