ALGEBRO-GEOMETRIC QUASI-PERIODIC FINITE-GAP SOLUTIONS OF THE TODA AND KAC-VAN MOERBEKE HIERARCHIES

W. BULLA, F. GESZTESY, H. HOLDEN, AND G. TESCHL

Dedicated to the memory of Henrik H. Martens (1927–1993)

Abstract. Combining algebro-geometric methods and factorization techniques for finite difference expressions we provide a complete and self-contained treatment of all real-valued quasi-periodic finite-gap solutions of both the Toda and Kac-van Moerbeke hierarchies.

In order to obtain our principal new result, the algebro-geometric finite-gap solutions of the Kac-van Moerbeke hierarchy, we employ particular commutation methods in connection with Miura-type transformations which enable us to transfer whole classes of solutions (such as finite-gap solutions) from the Toda hierarchy to its modified counterpart, the Kac-van Moerbeke hierarchy, and vice versa.

Contents

1. Introduction

The primary goal of this paper is to construct all real-valued algebro-geometric quasi-periodic finite-gap solutions of the Kac-van Moerbeke (KM) hierarchy of nonlinear evolution equations.

While there exists a direct method to construct the finite-gap solutions of the KM hierarchy, we shall use an alternative route that exploits the close connection between the Toda and KM hierarchies and characterizes the KM hierarchy as the modified Toda hierarchy in precisely the same manner that connects the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) hierarchies or more generally, the Gel'fand-Dickey (GD) hierarchy and its modified counterpart, the Drinfeld-Sokolov (DS) hierarchy. The deep connection between these hierarchies of nonlinear evolution equations and their modified counterparts is based on Miura-type transformations which in turn rely on factorization techniques of the associated Lax differential, respectively difference expressions, as will be indicated below. (Alternatively, one can use the discrete analog of the formal pseudo differential calculus in connection with the GD and DS hierarchy as in [60], Ch. IV.) Accordingly, our approach consists of three main parts:
(i) A thorough treatment of the Toda hierarchy.
(ii) The algebro-geometric approach to completely integrable nonlinear evolution equations.
(iii) A transfer of classes of solutions of the Toda hierarchy to that of the Kac-van Moerbeke hierarchy and vice versa.
Our major accomplishments in this paper may be summarized as follows:
(α) Construction of an alternative approach to the Toda hierarchy, modeled after Al'ber [6], Jacobi [47], McKean [63], and Mumford [73], Sect. III a).1, particularly suited to derive its algebro-geometric quasi-periodic finite-gap solutions. Derivation of an intimate connection of this approach with spectral properties of the corresponding Lax operator.
(β) A complete presentation of the algebro-geometric approach to the Toda hierarchy which goes beyond results in the literature and leads, in particular, to an alternative theta function representation of $b(n,t)$ (in Flaschka's variables [35], cf. (6.66)).
(γ) A complete derivation of all real-valued algebro-geometric quasi-periodic finite-gap solutions of the KM hierarchy, our principal new result.

Before we describe the content of each section, and hence (α)–(γ) in some detail, we shall comment on items (i)–(iii) a bit further.

The (Abelian) Toda lattice (TL) in its original variables reads

$$\frac{d^2}{dt^2}Q(n,t) = \exp[Q(n-1,t) - Q(n,t)] - \exp[Q(n,t) - Q(n+1,t)], \quad (n,t) \in \mathbb{Z} \times \mathbb{R} \tag{1.1}$$

and similarly, the original Kac-van Moerbeke system, also called the Volterra system, in physical variables is of the type

$$\frac{d}{dt}R(n,t) = \frac{1}{2}\{\exp[-R(n-1,t)] - \exp[-R(n+1,t)]\}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}. \tag{1.2}$$

In Flaschka's variables [35] for (1.1) and similarly for (1.2),

$$\begin{aligned} a(n,t) &= \frac{\epsilon(n)}{2}\exp\{[Q(n,t) - Q(n+1,t)]/2\}, \\ b(n,t) &= \dot{Q}(n,t)/2, \quad \epsilon(n) \in \{+1,-1\}, \end{aligned} \tag{1.3}$$

$$\rho(n,t) = \frac{\epsilon(n)}{2}\exp[-R(n,t)/2], \quad \epsilon(n) \in \{+1,-1\} \tag{1.4}$$

one can rewrite (1.1) and (1.2) in the form

$$\mathrm{TL}_0(a,b) = (\dot{a} - a(b - b^+), \dot{b} - 2[(a^-)^2 - a^2]) = 0 \tag{1.5}$$

and

$$\mathrm{KM}_0(\rho) = \dot{\rho} - \rho[(\rho^+)^2 - (\rho^-)^2] = 0, \tag{1.6}$$

the latter also known as Langmuir lattice. Here "$\cdot$" denotes d/dt and we employed the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$ and regarded all equations in the multiplication algebra of sequences. Moreover, introducing the shift operators

$$(S^\pm f)(n) = f(n \pm 1) = f^\pm(n), \quad n \in \mathbb{Z} \tag{1.7}$$

in $\ell^\infty(\mathbb{Z})$, the systems (1.5) and (1.6) are well-known to be equivalent to the Lax equations

$$\dot{L} - [P_2, L] = 0 \tag{1.8}$$

and

$$\dot{M} - [Q_2, M] = 0. \tag{1.9}$$

Here L and P_2 are the difference expressions

$$L = aS^+ + a^-S^- - b, \quad P_2 = aS^+ - a^-S^- \tag{1.10}$$

defined on $\ell^\infty(\mathbb{Z})$, and M and Q_2 the matrix-valued difference expressions

$$\begin{aligned} M &= \begin{pmatrix} 0 & \rho_o^- S^- + \rho_e \\ \rho_o S^+ + \rho_e & 0 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} \rho_e \rho_o S^+ - \rho_e^- \rho_o^- S^- & 0 \\ 0 & \rho_e^- \rho_o S^+ - \rho_e \rho_o - S^- \end{pmatrix} \end{aligned} \tag{1.11}$$

defined on $\ell^\infty(\mathbb{Z}) \otimes \mathbb{C}^2$, with ρ_e and ρ_o the "even" and "odd" parts of ρ, that is,

$$\rho_e(n,t) = \rho(2n,t), \quad \rho_o(n,t) = \rho(2n+1,t), \quad n \in \mathbb{Z}, \tag{1.12}$$

assuming a, b, $\rho \in \ell^\infty(\mathbb{Z})$.

Since the literature on the Toda lattice (even if one considers only the infinite lattice $\mathbb{Z}$, our main interest) is extensive, we will only refer to a few standard monographs such as [29], [76], [77], [84]. In the case of the Kac-van Moerbeke system we refer to [15], [49], [50], [53], [60], [61], [62], [71], [87] and the references therein.

While (1.5) and (1.6) describe the original Toda and Kac-van Moerbeke lattices, one can develop a systematic generalization to Lax pairs of the type (L, P_{2g+2}) and (M, Q_{2g+2}), where P_{2g+2} (Q_{2g+2}) are (matrix-valued) difference expressions of order $2g+2$ with certain polynomial coefficients in a, b (ρ). The associated Lax equations

$$\dot{L} - [P_{2g+2}, L] = 0 \tag{1.13}$$

and

$$\dot{M} - [Q_{2g+2}, M] = 0 \tag{1.14}$$

(cf. Section 2) are then equivalent to the TL_g and KM_g equations denoted by

$$\mathrm{TL}_g(a,b) = 0 \tag{1.15}$$

and

$$\mathrm{KM}_g(\rho) = 0. \tag{1.16}$$

Varying $g \in \mathbb{N}_0$ then yields the corresponding hierarchies of nonlinear evolution equations for (a,b) and ρ.

The special case of stationary TL_g and KM_g equations, characterized by $\dot{L} = 0$, $\dot{M} = 0$ in (1.13), (1.14), or equivalently, by commuting difference expressions of the type

$$[P_{2g+1}, L] = 0, \tag{1.17}$$

$$[Q_{2g+2}, M] = 0 \tag{1.18}$$

then yields a polynomial relationship between L and P_{2g+2}, respectively M and Q_{2g+2}. In fact, (1.17) and (1.18) imply the following analogs of the Burchnall-Chaundy polynomials familiar from the theory of commuting ordinary differential expressions [16], [17]

$$P_{2g+2}^2 = \prod_{m=0}^{2g+1} (L - E_m), \qquad \{E_m\}_{0 \leq m \leq 2g+1} \subset \mathbb{C}, \tag{1.19}$$

$$Q_{2g+2}^2 = \prod_{m=0}^{2g+1} (M^2 - e_m), \quad \{e_m\}_{0 \leq m \leq 2g+1} \subset \mathbb{C}. \tag{1.20}$$

In particular, (1.19) and (1.20) yield the following hyperelliptic curves

$$y^2 = \prod_{m=0}^{2g+1} (z - E_m), \tag{1.21}$$

$$y^2 = \prod_{m=0}^{2g+1} (w^2 - e_m), \tag{1.22}$$

the fundamental ingredients of the algebro-geometric approach for the TL and KM hierarchies. For spectral theoretic reasons (see, e.g., Theorems 4.2 and 8.2) algebro-geometric solutions (a, b) and ρ of (1.15) and (1.16) are called g-gap solutions following the conventional terminology.

If θ denotes the Riemann theta function associated with the curve (1.21) (and a fixed homology basis), the ultimate goal of the algebro-geometric approach is then a θ-function representation of the solutions (a, b) and ρ of the TL_r and KM_r equations,

$$\mathrm{TL}_r(a, b) = 0, \quad \mathrm{KM}_r(\rho) = 0, \quad r \in \mathbb{N}_0 \tag{1.23}$$

with g-gap initial conditions,

$$(a, b) = (a^{(0)}, b^{(0)}), \quad \rho = \rho^{(0)} \text{ at } t = t_0, \tag{1.24}$$

where $(a^{(0)}, b^{(0)})$ and $\rho^{(0)}$ are stationary solutions of the TL_g and KM_g equations, that is,

$$\begin{aligned} &\mathrm{TL}_g(a^{(0)}, b^{(0)}) = 0, \quad \dot{a}^{(0)} = \dot{b}^{(0)} = 0, \\ &\mathrm{KM}_g(\rho^{(0)}) = 0, \qquad \dot{\rho}^{(0)} = 0 \end{aligned} \tag{1.25}$$

for some fixed $g \in \mathbb{N}_0$.

Next, we illustrate the close connection between the Toda hierarchy and its modified counterpart, the KM hierarchy. Introducing the difference expressions

$$A = \rho_o S^+ + \rho_e, \quad A^* = \rho_o^- S^- + \rho_e \tag{1.26}$$

in $\ell^\infty(\mathbb{Z})$ one infers that

$$M = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \tag{1.27}$$

$$M^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix} = L_1 \oplus L_2 , \tag{1.28}$$

with

$$L_1 = A^*A, \quad L_2 = AA^*, \tag{1.29}$$

$$L_k = a_k S^+ + a_k^- S^- - b_k, \quad k = 1, 2, \tag{1.30}$$

$$a_1 = \rho_e \rho_o, \quad b_1 = -\rho_e^2 - (\rho_o^-)^2, \tag{1.31}$$

$$a_2 = \rho_e^+ \rho_o, \quad b_2 = -\rho_e^2 - \rho_o^2 \tag{1.32}$$

and

$$Q_{2g+2} = \begin{pmatrix} P_{1,2g+2} & 0 \\ 0 & P_{2,2g+2} \end{pmatrix} = P_{1,2g+2} \oplus P_{2,2g+2} . \tag{1.33}$$

Here $P_{k,2g+2}$ is constructed as in (1.10) and (1.13) with (a, b) replaced by (a_k, b_k), $k = 1, 2$, respectively. Relations (1.28) and (1.33) then can be exploited to prove the implication

$$\mathrm{KM}_g(\rho) = 0 \Rightarrow \mathrm{TL}_g(a_k, b_k) = 0, \quad k = 1, 2, \tag{1.34}$$

that is, a solution ρ of the KM_g equations (1.16) yields two solutions (a_k, b_k), $k = 1, 2$ of the TL_g equations (1.15) related to one another by (1.31), (1.32), the discrete analog of Miura's transformation [67], familiar from the (m)KdV hierarchy. (According to a footnote in [71], the connection between the KM and TL lattices was mentioned by Hénon in a letter to Flaschka as early as 1973.) Incidentally, (1.28)–(1.32) illustrate the factorization of the Lax difference expression L alluded to earlier. The implication (1.34) was first systematically studied by Adler [1] using factorization techniques. Transformations between the KM and TL systems were studied earlier by Wadati [87] (see also [84]). In a recent paper [39] the converse of (1.34) was established. More precisely, assuming the existence of a solution (a_1, b_1) of the TL_g equations (1.15), that is,

$$\mathrm{TL}_g(a_1, b_1) = 0, \tag{1.35}$$

a solution ρ of the KM_g equations (1.16) and another solution (a_2, b_2) of the TL_g equations (1.15) are constructed,

$$\mathrm{KM}_g(\rho) = 0, \quad \mathrm{TL}_g(a_2, b_2) = 0 \tag{1.36}$$

related to each other by the Miura-type transformation (1.31), (1.32) (we refer to Section 7 for a detailed discussion of these facts). Equations (1.34) and especially (1.35), (1.36) yield the possibility of transferring classes of solutions (such as finite-gap solutions) from the Toda hierarchy to the KM hierarchy and vice versa.

Having illustrated items (i)–(iii) to some extent, we finally turn to a description of the content of each section. In Section 2 we develop an alternative recursive approach to the Toda hierarchy modeled after Al'ber [6]. In particular, we recursively compute the difference expressions P_{2g+2} in (1.13). We chose to develop this approach in detail since it most naturally leads to the fundamental Burchnall-Chaundy polynomials and hence to the underlying hyperelliptic curves in connection with the stationary Toda hierarchy. In addition it provides direct insight into the spectral properties of the underlying Lax operator as detailed later in Section 4.

Section 3 is devoted to the algebro-geometric approach to integrate nonlinear evolution equations and in particular to the Baker-Akhiezer (BA) function, the fundamental object of this approach. Historically, these techniques go back to the work of Baker [8], Burchnall and Chaundy [16], [17], and Akhiezer [5]. The modern approach was initiated by Its and Matveev [46] in connection with the KdV equation and further developed into a powerful machinery by Krichever (see, e.g., the review [58]) and others. We refer, in particular, to the extensive treatments in [10], [25], [26], [27], [62], [72], and [76]. In the special context of the Toda equations we refer to [2], [19], [26], [27], [56], [58], [62], [66], and [70]. Our own presentation starts with commuting difference expressions and their associated hyperelliptic curves and then develops the stationary algebro-geometric approach from first principles. In particular, we chose to follow Jacobi's classic representation of positive divisors of degree g of the hyperelliptic curve (1.21) [47] which was first applied to the KdV case by Mumford [73], Sect. III a).1 with subsequent extensions due to McKean [63]. The reader will find a meticulous account which provides more details on the BA-function than usually found in the literature (see, e.g., Theorem 3.5).

Spectral theoretic properties and Green's functions of self-adjoint $\ell^2(\mathbb{Z})$ realizations H of L are the main topic of Section 4. Assuming that L is defined in terms of stationary solutions (a, b) of the TL_g equations, we determine the spectrum of the Jacobi operator H in

Theorem 4.2 and provide a link between the 2×2 spectral matrix of H and our recursive approach to the stationary Toda hierarchy (cf. (4.32)–(4.36)). The latter result appears to be new and underscores the fundamental importance of the recursion formalism chosen in Section 2.

In Section 5 we continue our stationary algebro-geometric approach to the Toda hierarchy and provide a detailed derivation of the standard θ-function representation of all stationary TL_g solutions (cf. Theorem 5.2).

In Section 6 we finally complete the algebro-geometric approach to the Toda hierarchy. In addition to a detailed discussion of the time-dependent BA-function in Theorem 6.2 we derive the well-known (time-dependent) θ-function representation of the TL_r equations with g-gap initial conditions in Theorem 6.3. Our detailed account in Section 5 and 6 also leads to an alternative θ-function representation of b in Corollaries 5.6 and 6.5 which, much to our surprise, seems to have escaped notice in the literature thus far.

In Section 7 we turn to the KM hierarchy and its connection with the Toda hierarchy. In addition to developing a recursive approach to the KM hierarchy (which appears to be new) and in particular to a computation of Q_{2g+2} in (1.14), we describe at length the Miura-type transformation (1.31), (1.32) and especially the transfer of solutions from the Toda to the KM hierarchy in Theorem 7.2.

In analogy to Section 4, Section 8 establishes spectral properties of the self-adjoint realization D of M in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ associated with finite-gap solutions ρ of the KM_g equations. Theorem 8.2, in particular, reduces the spectral analysis of the Dirac-type difference operator D to that of the Jacobi operators H_k, the self-adjoint realizations of L_k, $k = 1, 2$ in $\ell^2(\mathbb{Z})$ (cf. (1.27), (1.29)–(1.32)) by using factorization (commutation) methods indicated in (1.28).

Finally, in Section 9 we complete the principal objective of this paper and derive all real-valued algebro-geometric quasi-periodic finite-gap solutions of the KM hierarchy in Theorems 9.3 and 9.5. Isospectral manifolds of finite-gap KM solutions are briefly considered in Remark 9.4 and a brief outlook on possible applications of these completely integrable lattice models ends this paper.

For convenience of the reader, and for the sake of being self-contained, we added Appendix A which summarizes basic facts on hyperelliptic curves and their θ-functions and defines the notation used in the main body of this paper. Appendix B records the principal results of Sections 3–5 in the important special case of periodic rather than quasi-periodic Jacobi operators by explicitly invoking Floquet theory.

Appendix C finally records the simplest explicit examples associated with genus $g = 0$ and 1.

2. The Toda Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we review the construction of the Toda hierarchy by using a recursive approach first advocated by Al'ber [6] and derive the Burchnall-Chaundy polynomials in connection with the stationary Toda hierarchy. Our recursive approach to the Toda hierarchy, though equivalent to the conventional one (see, e.g., [60], [70],[72], [77],[81],[82],[85]), markedly differs from the standard treatment. We have chosen to present the formalism below since it most naturally yields the Burchnall-Chaundy polynomials associated with the stationary Toda hierarchy and hence the underlying hyperelliptic curves for algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchies to be considered in Sections 6 and 9. Moreover, as shown in Section 4 (cf. (4.32)–(4.36)), this recursive approach provides a fundamental link to the spectral matrix of the underlying Lax operator.

We start by introducing some notations. In the following we denote by $\ell^p(M)$, where $1 \leq p \leq \infty$, $M = \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$, etc., the usual space of p-summable respectively bounded (if $p = \infty$) complex-valued sequences $f = \{f(m)\}_{m \in M}$ and by $\ell^p_{\mathbb{R}}(M)$ the corresponding restriction to real-valued sequences. The scalar product in the Hilbert space $\ell^2(M)$ will be denoted by

$$(f, g) = \sum_{n \in M} \overline{f(n)} g(n), \qquad f, g \in \ell^2(M). \tag{2.1}$$

Since $\ell^\infty(\mathbb{Z}) \subseteq \ell^p(\mathbb{Z})$ in a natural way it suffices to make all further definitions for $p = \infty$. In $\ell^\infty(\mathbb{Z})$ we introduce the shift operators

$$(S^\pm f)(n) = f(n \pm 1), \quad n \in \mathbb{Z},\ f \in \ell^\infty(\mathbb{Z}) \tag{2.2}$$

and in order to simplify notations we agree to use the short cuts

$$\begin{aligned} &f^\pm = S^\pm f, \quad \text{ that is, } f^\pm(n) = f(n \pm 1), \\ &(f+g)(n) = f(n) + g(n), \quad (fg)(n) = f(n)g(n), \quad n \in \mathbb{Z},\ f, g \in \ell^\infty(\mathbb{Z}) \end{aligned} \tag{2.3}$$

whenever convenient. Moreover, if $R : \ \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ denotes a difference expression, let

$$R = \{R(m,n)\}_{m,n \in \mathbb{Z}}, \quad R(m,n) = (\delta_m, R\delta_n) \tag{2.4}$$

denote its corresponding matrix representation with respect to the standard basis

$$e_m = \{\delta_m(n)\}_{n\in\mathbb{Z}}, \quad m \in \mathbb{Z}, \quad \delta_m(n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}. \tag{2.5}$$

In connection with (2.4) we define the diagonal and upper and lower triangular parts of R as follows

$$R_0 = \{R_0(m,n)\}_{m,n\in\mathbb{Z}}, \quad R_0(m,n) = \begin{cases} R(m,m), & m = n \\ 0, & m \neq n \end{cases},$$

$$R_\pm = \{R_\pm(m,n)\}_{m,n\in\mathbb{Z}}, \quad R_\pm(m,n) = \begin{cases} R(m,n), & \pm(n-m) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

Clearly,

$$R = R_+ + R_0 + R_-. \tag{2.7}$$

Given these notations one can now introduce the Toda hierarchy. Let

$$\begin{aligned} & a(t) = \{a(n,t)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \quad b(t) = \{b(n,t)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \ t \in \mathbb{R}, \\ & 0 \neq a(n,.),\ b(n,.) \in C^1(\mathbb{R}),\ n \in \mathbb{Z} \end{aligned} \tag{2.8}$$

and introduce the difference expressions $(L(t), P_{2g+2}(t))$ (the Lax pair) in $\ell^\infty(\mathbb{Z})$

$$L(t) = a(t)S^+ + a^-(t)S^- - b(t), \quad t \in \mathbb{R}, \tag{2.9}$$

$$\begin{aligned} P_{2g+2}(t) = -L(t)^{g+1} + \sum_{j=0}^{g} [g_j(t) + 2a(t)f_j(t)S^+]L(t)^{g-j} + f_{g+1}(t), \\ g \in \mathbb{N}_0,\ t \in \mathbb{R}, \end{aligned} \tag{2.10}$$

where $\{f_j(n,t)\}_{0\le j\le g+1}$ and $\{g_j(n,t)\}_{0\le j\le g}$ satisfy the recursion relations

$$\begin{aligned} & f_0 = 1,\ g_0 = -c_1, \\ & 2f_{j+1} + g_j + g_j^- + 2bf_j = 0, \quad 1 \le j \le g+1, \\ & g_{j+1} - g_{j+1}^- + 2[a^2 f_j + 2a^- f_j] + b[g_j - g_j^-] = 0, \quad 1 \le j \le g. \end{aligned} \tag{2.11}$$

Note that a enters in f_j and g_j only quadratically. Then the Lax equation

$$\frac{d}{dt}L(t) - [P_{2g+2}(t), L(t)] = 0, \quad t \in \mathbb{R} \tag{2.12}$$

(here $[.\,,\,.]$ denotes the commutator) is equivalent to

$$\begin{aligned}\mathrm{TL}_g(a(t),b(t))_1 &= \dot a(t) + a(t)[g_g^+(t) + g_g(t) + f_{g+1}^+(t) + f_{g+1}(t) \\ &\quad + 2b^+(t) f_g^+(t)] = 0, \\ \mathrm{TL}_g(a(t),b(t))_2 &= \dot b(t) + 2[b(t)(g_g(t) + f_{g+1}(t)) + a(t)^2 f_g^+(t) \\ &\quad - a^-(t)^2 f_g^-(t) + b(t)^2 f_g(t)] = 0, \quad t \in \mathbb{R}.\end{aligned} \tag{2.13}$$

Varying $g \in \mathbb{N}_0$ yields the Toda hierarchy

$$\mathrm{TL}_g(a,b) = (\mathrm{TL}_g(a,b)_1, \mathrm{TL}_g(a,b)_2)^T = 0, \quad g \in \mathbb{N}_0. \tag{2.14}$$

Explicitly, one obtains from (2.11),

$$\begin{aligned} f_1 &= -b + c_1, \\ g_1 &= -2a^2 - c_2, \\ f_2 &= a^2 + (a^-)^2 + b^2 - c_1 b + c_2, \\ g_2 &= 2a^2(b + b^+) - 2c_1 a^2 - c_3, \\ f_3 &= -(a^-)^2(b^- + 2b) - a^2(b^+ + 2b) - b^3 \\ &\quad + c_1(a^2 + (a^-)^2 + b^2) - c_2 b + c_3, \\ &\text{etc.} \end{aligned} \tag{2.15}$$

and hence from (2.13),

$$\mathrm{TL}_0(a,b) = \begin{pmatrix} \dot a - a(b - b^+) \\ \dot b - 2[(a^-)^2 - a^2] \end{pmatrix} = 0, \tag{2.16}$$

$$\mathrm{TL}_1(a,b) = \begin{pmatrix} \dot a - a[(a^+)^2 - (a^-)^2 + (b^+)^2 - b^2] \\ \dot b - 2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} + c_1 \begin{pmatrix} -a(b - b^+) \\ -2[(a^-)^2 - a^2] \end{pmatrix} = 0, \tag{2.17}$$

$$\begin{aligned}\mathrm{TL}_2(a,b) &= \begin{pmatrix} \dot a - a[b^3 - (b^+)^3 + 2(a^-)^2 b - 2(a^+)^2 b^+ + a^2(b - b^+) - (a^+)^2 b^{++} - (a^-)^2 b^- \\ \dot b - 2(a^-)^2[b^2 + bb^- + (b^-)^2 + (a^-)^2 + (a^{--})^2] + 2a^2[b^2 + bb^+ + (b^+)^2 + a^2 + (a^+)^2] \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} -a[(a^+)^2 - (a^-)^2 + (b^+)^2 - b^2] \\ -2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} + c_2 \begin{pmatrix} -a(b - b^+) \\ -2[(a^-)^2 - a^2] \end{pmatrix} = 0, \\ &\text{etc.}\end{aligned} \tag{2.18}$$

represent the first few equations of the Toda hierarchy. Here c_ℓ denote summation constants which naturally arise by solving the resulting difference equations for $g_{g+1,\ell}$ in (2.11). The corresponding homogeneous Toda equations obtained by taking all summation constants equal to zero, $c_\ell \equiv 0$, $\ell \in \mathbb{N}$, are then denoted by

$$\widetilde{\mathrm{TL}}_g(a,b) := \mathrm{TL}_g(a,b)\big|_{c_\ell \equiv 0,\ 1 \le \ell \le g} \tag{2.19}$$

and similarly we denote by $\tilde{P}_{2g+2} := P_{2g+2}(c_\ell \equiv 0)$, $\tilde{f}_j := f_j(c_\ell \equiv 0)$, $\tilde{g}_j := g_j(c_\ell \equiv 0)$ the corresponding homogeneous quantities. One verifies

$$P_{2g+2} = \sum_{m=0}^{g} c_{g-m} \tilde{P}_{2m+2}, \quad c_0 = 1. \tag{2.20}$$

Next we relate the homogeneous quantities $\tilde{f}_j$, $\tilde{g}_j$ to certain matrix elements of $L(t)^j$.

Lemma 2.1. *The homogeneous coefficients* $\{\tilde{f}_j(t)\}_{0\le j\le g+1}$ *and* $\{\tilde{g}_j(t)\}_{0\le j\le g}$ *satisfy*

$$\tilde{f}_j(n,t) = (\delta_n, L(t)^j \delta_n), \quad 0 \le j \le g+1, \ n \in \mathbb{Z}, \tag{2.21}$$

$$\tilde{g}_j(n,t) = -2a(n)(\delta_{n+1}, L(t)^j \delta_n), \quad 0 \le j \le g, \ n \in \mathbb{Z}, \tag{2.22}$$

where $\delta_n = \{\delta_{n,m}\}_{m\in\mathbb{Z}}$.

Proof. We abbreviate

$$\hat{f}_j(n) = (\delta_n, L^j \delta_n), \quad \hat{g}_j(n) = -2a(n)(\delta_{n+1}, L^j \delta_n). \tag{2.23}$$

Then

$$\hat{f}_{j+1} = (L\delta_n, L^j\delta_n) = -\frac{1}{2}(\hat{g}_j + \hat{g}_j^-) - b\hat{f}_j \tag{2.24}$$

and similarly,

$$\hat{g}_{j+1} = -b\hat{g}_j - 2a^2 \hat{f}_j^+ + \hat{h}_j = -b^+\hat{g}_j - 2a^2\hat{f}_j + \hat{h}_j^+, \tag{2.25}$$

where

$$\hat{h}_j(n) = -2a(n)a(n-1)(\delta_{n+1}, L^j \delta_{n-1}). \tag{2.26}$$

Eliminating $\hat{h}_j$ in (2.25) results in

$$\hat{g}_{j+1} - \hat{g}_{j+1}^- = -2[a^2\hat{f}_j^+ - (a^-)^2\hat{f}_j^-] - b[\hat{g}_j - \hat{g}_j^-]. \tag{2.27}$$

By inspection, (2.24) and (2.27) are equivalent to (2.11). In order to determine which solution of (2.11) has been found (i.e., determine the summation constants $c_1, \dots, c_g$) we temporarily assign the weight one to $a(n)$ and $b(n)$, $n \in \mathbb{Z}$. Then $\tilde{f}_j$ and $\tilde{g}_{j+1}$ have weight j and hence

$$c_0 = 1, \ c_j = 0, \quad 1 \le j \le g \tag{2.28}$$

completing the proof. ∎

Now we are in the position to reveal the connections with the usual approach to the Toda equations. It suffices to consider the homogeneous case.

Lemma 2.2. *The homogeneous Lax operator* $\tilde{P}_{2g+2}$ *satisfies*

$$\tilde{P}_{2g+2}(t) = [L(t)^{g+1}]_+ - [L(t)^{g+1}]_- \tag{2.29}$$

(cf. the notation in (2.7)*).*

Proof. We use induction on g. $g = 0$ is trivial. By (2.10) we need to show

$$\tilde{P}_{2g+2}(t) = \tilde{P}_{2g}(t)L(t) + [\tilde{g}_g(t) + 2a(t)\tilde{f}_g(t)S^+] - \tilde{f}_g(t)L(t) + \tilde{f}_{g+1}(t). \tag{2.30}$$

This can be done upon considering $(\delta_n, \tilde{P}_{2g+2}(t)\delta_n)$ and making case distinctions $m < n-1, m = n-1, m = n, m = n+1, m > n+1$ (for $m = n$ one has to use (2.11)). $\square$

Before we turn to a discussion of the stationary Toda hierarchy we briefly sketch the main steps leading to (2.10)–(2.13). If $\mathrm{Ker}(L(t) - z)$, $z \in \mathbb{C}$ denotes the two-dimensional nullspace of $L(t) - z$ (in the algebraic sense as opposed to the functional analytic one), we seek a representation of $P_{2g+2}(t)$ in $\mathrm{Ker}(L(t) - z)$ of the form

$$P_{2g+2}(t)\Big|_{\mathrm{Ker}(L(t)-z)} = 2a(t)F_g(z,t)S^+ + G_{g+1}(z,t), \tag{2.31}$$

where F_g and G_{g+1} are polynomials in z of the type

$$F_g(z,t) = \sum_{j=0}^{g} z^j f_{g-j}(t),\ G_{g+1}(z,t) = -z^{g+1} + \sum_{j=0}^{g} z^j g_{g-j}(t) + f_{g+1}(t), \tag{2.32}$$

with $f_\ell(t) = \{f_\ell(n,t)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, $g_\ell(t) = \{g_\ell(n,t)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. The Lax equation (2.12) restricted to $\mathrm{Ker}(L(t) - z)$ then yields

$$\begin{aligned} 0 &= \{\dot{L} - [P_{2g+2}, L]\}\Big|_{\mathrm{Ker}(L-z)} = \{\dot{L} + (L-z)P_{2g+2}\}\Big|_{\mathrm{Ker}(L-z)} \\ &= \Big\{a\Big[\frac{\dot{a}}{a} - \frac{\dot{a}^-}{a^-} + 2(b^+ + z)F_g^+ - 2(b+z)F_g + G_{g+1}^+ - G_{g+1}^-\Big]S^+ \\ &\quad + \Big[-\dot{b} + (b+z)\frac{\dot{a}^-}{a^-} + 2(a^-)^2F_g^- - 2a^2F_g^+ + (b+z)(G_{g+1}^- - G_{g+1})\Big]\Big\}\Big|_{\mathrm{Ker}(L-z)}. \end{aligned} \tag{2.33}$$

Hence one obtains

$$\frac{\dot a}{a} - \frac{\dot a^-}{a^-} = 2(b^+ + z)F_g^+ - 2(b+z)F_g + G_{g+1}^- - G_{g+1}^+, \tag{2.34}$$

$$\dot b = (b+z)\frac{\dot a^-}{a^-} + 2(a^-)^2 F_g^- - 2a^2 F_g^+ + (b+z)(G_{g+1}^- - G_{g+1}). \tag{2.35}$$

Upon summing (2.34) (adding $G_{g+1} - G_{g+1}$ and neglecting a trivial summation constant) one infers

$$\dot a = -a[2(b^+ + z)F_g^+ + G_{g+1}^+ + G_{g+1}], \quad g \in \mathbb{N}_0. \tag{2.36}$$

Insertion of (2.36) into (2.35) then implies

$$\dot b = -2[(b+z)^2 F_g + (b+z)G_{g+1} + a^2 F_g^+ - (a^-)^2 F_g^-], \quad g \in \mathbb{N}_0. \tag{2.37}$$

Insertion of (2.32) into (2.36) and (2.37) then produces the recursion relation (2.11) and the result (2.13). Relation (2.31) then yields (2.10). We omit further details and just record as an illustration a few of the polynomials F_g and G_{g+1},

$$\begin{aligned}
&F_0 = 1 = \tilde F_0,\\
&G_1 = -b - z = \tilde G_1,\\
&F_1 = c_1 - b + z = c_1 \tilde F_0 + \tilde F_1,\\
&G_2 = c_1(-b-z) + (a^-)^2 - a^2 + b^2 - z^2 = c_1 \tilde G_1 + \tilde G_2,\\
&F_2 = c_2 + c_1(-b+z) + a^2 + (a^-)^2 + b^2 - bz + z^2 = c_2 \tilde F_0 + c_1 \tilde F_1 + \tilde F_2,\\
&G_3 = c_2(-b-z) + c_1((a^-)^2 - a^2 + b^2 - z^2) + a^2 b^+ - (a^-)^2 b^-\\
&\qquad - 2(a^-)^2 b - b^3 - 2a^2 z - z^3 = c_2 \tilde G_1 + c_1 \tilde G_2 + \tilde G_3,
\end{aligned} \tag{2.38}$$

etc.

Remark 2.3. Since by (2.11), (2.32), a enters quadratically in F_g and G_{g+1}, the Toda hierarchy (2.13) (respectively (2.36), (2.37)) is invariant under the substitution

$$a(t) \to a_\epsilon(t) = \{\epsilon(n)a(n,t)\}_{n\in\mathbb{Z}}, \quad \epsilon(n) \in \{+1,-1\},\ n \in \mathbb{Z}. \tag{2.39}$$

This result should be compared with (the last part of) Lemma 3.1 and Lemma 4.1.

Finally, we specialize to the stationary Toda hierarchy characterized by $\dot a = \dot b = 0$ in (2.14) (respectively (2.13)), or more precisely, by commuting difference expressions

$$[P_{2g+2}, L] = 0 \tag{2.40}$$

of order $2g+2$ and 2, respectively. Equations (2.35) and (2.36) then yield

$$\begin{aligned}
&(b+z)(G_{g+1} - G^-_{g+1}) = 2(a^-)^2 F_g - 2a^2 F^+_g, && (2.41)\\
&G^+_{g+1} + G_{g+1} = -2(b^+ + z)F^+_g. && (2.42)
\end{aligned}$$

Because of (2.40) one computes

$$\begin{aligned}
\Big[P_{2g+2}\Big|_{\mathrm{Ker}(L-z)}\Big]^2 =& \Big[(2aF_gS^+ + G_{g+1})\Big|_{\mathrm{Ker}(L-z)}\Big]^2\\
=& \Big\{2aF_g[G^+_{g+1} + G_{g+1} + 2(b^+ + z)F^+_g]S^+\\
&+ G^2_{g+1} - 4a^2 F_g F^+_g\Big\}\Big|_{\mathrm{Ker}(L-z)} \qquad (2.43)\\
=& \{G^2_{g+1} - 4a^2 F_g F^+_g\}\Big|_{\mathrm{Ker}(L-z)} =: R_{2g+2}.
\end{aligned}$$

A simple calculation, using (2.41) and (2.42) then proves that R_{2g+2} is a lattice constant and hence a polynomial of degree $2g+2$ with respect to z:

$$\begin{aligned}
&(b+z)(R_{2g+2} - R^-_{2g+2})\\
&= (b+z)\{(G_{g+1} + G^-_{g+1})(G_{g+1} - G^-_{g+1}) - 4F_g[a^2F^+_g - (a^-)^2F^-_g]\}\\
&\overset{(2.41)}{=} -[G_{g+1} + G^-_{g+1} + 2(b+z)F_g]2[a^2F^+_g - (a^-)^2F^-_g] \overset{(2.42)}{=} 0. \qquad (2.44)
\end{aligned}$$

Thus one infers

$$R_{2g+2}(z) = \prod_{m=0}^{2g+1}(z - E_m), \quad \{E_m\}_{0\le m\le 2g+2} \subset \mathbb{C} \tag{2.45}$$

and, since $z \in \mathbb{C}$ is arbitrary, obtains the Burchnall-Chaundy polynomial (see [16], [17] in the case of differential expressions) relating P_{2g+2} and L,

$$P^2_{2g+2} = R_{2g+2}(L) = \prod_{m=0}^{2g+1}(L - E_m). \tag{2.46}$$

The resulting hyperelliptic curve K_g of (arithmetic) genus g obtained upon compactification of the curve

$$y^2 = R_{2g+2}(z) = \prod_{m=0}^{2g+1}(z - E_m) \tag{2.47}$$

will be the basic ingredient in our algebro-geometric treatment of the Toda and Kac-van Moerbeke hierarchies in the remainder of this paper.

The spectral theoretic content of the polynomials F_g and G_{g+1} is clearly displayed in (4.8), (4.19),(4.20) and especially in (4.32)–(4.36).

3. The Stationary Baker-Akhiezer Function

In this section we provide a major part of our thorough review of the algebro-geometric methods to construct quasi-periodic finite-gap solutions of the Toda hierarchy, a subject we shall complete in Section 6. As explained in the Introduction, the origins of our approach go back to a classic representation of positive divisors of degree g of K_g due to Jacobi [47] and its application to the KdV case by Mumford [73], Sect. III a).1 and subsequently McKean [63].

Although these finite-gap integration techniques (especially in the special case of spatially periodic solutions) have been discussed in several references on the subject, see, for instance, [2], [19], [26], [27], [56], [58], [62], [66], [70], [72], [84], we have chosen to give a detailed account. This decision is based both on the necessity of this material for our main Section 9 on algebro-geometric solutions of the Kac-van Moerbeke hierarchy and on the fact that we believe to be able to offer a simpler and more streamlined approach than the existing ones.

As indicated at the end of Section 2 (cf. (2.40), (2.45)–(2.47)), the stationary Toda hierarchy is intimately connected with pairs of commuting difference expressions (P_{2g+2}, L) of orders $2g+2$ and 2, respectively and hyperelliptic curves K_g obtained upon compactification of the curve

$$y^2 = R_{2g+2}(z) = \prod_{m=0}^{2g+1} (z - E_m) \tag{3.1}$$

described in detail in Appendix A (whose results and notations we shall freely use in the remainder of this paper). Since we are interested in real-valued Toda solutions and especially in their expressions in terms of the Riemann theta function associated with K_g, we shall make the assumption (cf. (A.1))

$$\{E_m\}_{0\le m\le 2g+1} \subset \mathbb{R}, \quad E_0 < E_1 < \cdots < E_{2g+1},\ g \in \mathbb{N}_0. \tag{3.2}$$

For a fixed but arbitrary point n_0 in $\mathbb{Z}$ consider

$$\{\hat\mu_j(n_0)\}_{1\le j\le g} \subset K_g,\ \tilde\pi(\hat\mu_j(n_0)) = \mu_j(n_0) \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le g \tag{3.3}$$

and

$$F_g(z,n_0)=\prod_{j=1}^{g}[z-\mu_j(n_0)], \tag{3.4}$$

$$G_{g+1}(z,n_0)=-\sum_{j=1}^{g}R_{2g+2}^{1/2}(\hat\mu_j(n_0))\prod_{\substack{k=1\\k\neq j}}^{g}\frac{[z-\mu_k(n_0)]}{[\mu_j(n_0)-\mu_k(n_0)]}-[z+b(n_0)]F_g(z,n_0), \tag{3.5}$$

where

$$b(n_0)=\sum_{j=1}^{g}\mu_j(n_0)-\frac{1}{2}\sum_{m=0}^{2g+1}E_m \tag{3.6}$$

and

$$\begin{aligned}&R_{2g+2}^{1/2}(\hat\mu_j(n_0))=\sigma_j(n_0)R_{2g+2}(\mu_j(n_0))^{1/2}=-G_{g+1}(\mu_j(n_0),n_0),\\&\hat\mu_j(n_0)=(\mu_j(n_0),-G_{g+1}(\mu_j(n_0),n_0)),\quad 1\leq j\leq g.\end{aligned} \tag{3.7}$$

Given (3.3)–(3.7) we define $F_g(z,n_0+1)$ by (cf. (2.43))

$$G_{g+1}(z,n_0)^2-4a(n_0)^2F_g(z,n_0)F_g(z,n_0+1)=R_{2g+2}(z), \tag{3.8}$$

where the constant $a(n_0)^2\neq 0$ has been introduced in (3.8) such as to guarantee that $F_g(z,n_0+1)$ is a monic polynomial in z (i.e., its highest coefficient is normalized to one). Since

$$a(n_0)^2F_g(z,n_0)F_g(z,n_0+1)\geq 0 \text{ for } z=E_{2j-1},E_{2j},E_{2g+1} \tag{3.9}$$

the left-hand-side of (3.9) has at least two zeros in $[E_{2j-1},E_{2j}]$. Thus $F_g(z,n_0+1)$ is of the form

$$F_g(z,n_0+1)=\prod_{j=1}^{g}[z-\mu_j(n_0+1)],\ \mu_j(n_0+1)\in[E_{2j-1},E_{2j}],\quad 1\leq j\leq g. \tag{3.10}$$

Equation (3.9) with $z=E_{2g+1}$ shows $a(n_0)^2\geq 0$ and hence $a(n_0)^2>0$ and one computes from (3.5) and (3.8) that

$$a(n_0)^2=\frac{1}{2}\sum_{j=1}^{g}R_{2g+2}^{1/2}(\hat\mu_j(n_0))\prod_{\substack{k=1\\k\neq j}}^{g}[\mu_j(n_0)-\mu_k(n_0)]^{-1}-\frac{1}{4}[b(n_0)^2+b^{(2)}(n_0)]>0, \tag{3.11}$$

where we used the notation

$$b^{(k)}(n_0)=\sum_{j=-1}^{g}\mu_j(n_0)^k-\frac{1}{2}\sum_{m=0}^{2g+1}E_m^k,\quad k\in\mathbb{N}, \tag{3.12}$$

(thus $b(n_0) = b^{(1)}(n_0)$). Introducing $\hat{\mu}_j(n_0+1)$ by

$$\hat{\mu}_j(n_0+1) = (\mu_j(n_0+1), G_{g+1}(\mu_j(n_0+1), n_0)), \quad 1 \le j \le g, \tag{3.13}$$

we have constructed the set $\{\hat{\mu}_j(n_0+1)\}_{1\le j\le g}$ from the set $\{\hat{\mu}_j(n_0)\}_{1\le j\le g}$ and get in addition,

$$\begin{aligned} G_{g+1}(z,n_0) = \sum_{j=1}^{g} R_{2g+2}^{1/2}(\hat{\mu}_j(n_0+1)) \prod_{\substack{k=1\\k\ne j}}^{g} \frac{[z-\mu_k(n_0+1)]}{[\mu_j(n_0+1)-\mu_k(n_0+1)]} \\ - [z+b(n_0+1)]F_g(z,n_0+1), \end{aligned} \tag{3.14}$$

with

$$b(n_0+1) = \sum_{j=1}^{g} \mu_j(n_0+1) - \frac{1}{2}\sum_{m=0}^{2g+1} E_m \tag{3.15}$$

and

$$G_{g+1}(z,n_0+1) = -G_{g+1}(z,n_0) - 2[z+b(n_0+1)]F_g(z,n_0+1). \tag{3.16}$$

Since by (3.5) and (3.14) $F_g(z,n_0)$ and $F_g(z,n_0+1)$ enter symmetrically in $G_{g+1}(z,n_0)$, we can reverse this process, that is, start with $\{\hat{\mu}_j(n_0+1)\}_{1\le j\le g}$ and (3.14) and determine $\{\hat{\mu}_j(n_0)\}_{1\le j\le g}$. Hence we obtain

Lemma 3.1. *Given $\{\hat{\mu}_j(n_0)\}_{1\le j\le g}$ satisfying (3.3) we can determine $\{\hat{\mu}_j(n)\}_{1\le j\le g}$ for all $n\in\mathbb{Z}$ satisfying again (3.3), that is,*

$$\tilde{\pi}(\hat{\mu}_j(n)) = \mu_j(n) \in [E_{2j-1}, E_{2j}], \quad 1\le j\le g,\ n\in\mathbb{Z}. \tag{3.17}$$

Moreover, we obtain two sequences $\{a(n)\}_{n\in\mathbb{Z}}$, $\{b(n)\}_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$ defined by

$$a(n)^2 = \frac{1}{2}\sum_{j=1}^{g} R_{2g+2}^{1/2}(\hat{\mu}_j(n)) \prod_{\substack{k=1\\k\ne j}}^{g} [\mu_j(n)-\mu_k(n)]^{-1} - \frac{1}{4}[b(n)^2 + b^{(2)}(n)] > 0, \tag{3.18}$$

$$b(n) = \sum_{j=1}^{g} \mu_j(n) - \frac{1}{2}\sum_{m=0}^{2g+1} E_m, \tag{3.19}$$

$$b^{(k)}(n) = \sum_{j=1}^{g} \mu_j(n)^k - \frac{1}{2}\sum_{m=0}^{2g+1} E_m^k, \quad k\in\mathbb{N}. \tag{3.20}$$

The sign of $a(n)$ is not determined by this procedure and can be chosen freely (cf. also Remark 2.3 and Lemma 4.1).

Proof. It remains to prove the boundedness of a and b. But this follows immediately from (3.17). □

Remark 3.2. While the trace formula (3.19) for $b(n)$ is a standard result, the explicit representation (3.18) of $a(n)$ in terms of the Dirichlet data $\{\hat{\mu}_j(n)\}_{1\le j\le g}$ appears to be new to the best of our knowledge.

We remark that all formulas obtained thus far in this section still hold if n_0 is replaced by n. In addition, we note for later use

$$\begin{aligned} a(n)^2F_g(z,n+1) - a(n-1)F_g(z,n-1) + (b(n)+z)^2F_g(z,n) \\ = 2^{-1}(b(n)+z)G_{g+1}(z,n). \end{aligned} \tag{3.21}$$

For reasons to become obvious in the next section (cf. the spectral properties described in Theorem 4.2 in connection with the self-adjoint $\ell^2(\mathbb{Z})$ realization H associated with $L = aS^+ + a^-S^- - b$) we shall call $a = \{a(n)\}_{n\in\mathbb{Z}}$, $b = \{b(n)\}_{n\in\mathbb{Z}}$ of the type (3.18), (3.19) finite-gap sequences (respectively g-gap sequences whenever we want to emphasize the genus g of the underlying curve K_g).

Next, we define a meromorphic function $\phi(P,n)$ on K_g,

$$\begin{gathered} \phi(P,n) = \frac{-G_{g+1}(\tilde{\pi}(P),n) + R_{2g+2}^{1/2}(P)}{2a(n)F_g(\tilde{\pi}(P),n)} = \frac{-2a(n)F_g(\tilde{\pi}(P),n+1)}{G_{g+1}(\tilde{\pi}(P),n) + R_{2g+2}^{1/2}(P)}, \\ P = (z, \sigma R_{2g+2}(z)^{1/2}) = (\tilde{\pi}(P), R_{2g+2}^{1/2}(P)) \end{gathered} \tag{3.22}$$

and with the help of $\phi(P,n)$ another meromorphic function $\psi(P,n,n_0)$ on K_g, the stationary Baker-Akhiezer (BA) function

$$\psi(P,n,n_0) = \begin{cases} \prod_{m=n_0}^{n-1} \phi(P,m), & n \ge n_0+1 \\ 1, & n = n_0 \\ \prod_{m=n}^{n_0-1} \phi(P,m)^{-1}, & n \le n_0 - 1 \end{cases}. \tag{3.23}$$

As will turn out in the course of this section and in Sections 5, 6, and 9, these meromorphic functions are the fundamental ingredients of the finite-gap integration technique and $\phi(.,n)$, specifically, is the central object in our presentation of this material.

Lemma 3.3. *The function $\phi(P,n)$ satisfies the "Riccati-type" equation*

$$a(n)\phi(P,n) + a(n-1)\phi(P,n-1)^{-1} = b(n) + \tilde{\pi}(P), \quad n \in \mathbb{Z} \tag{3.24}$$

and the BA-function $\psi(P,n,n_0)$ satisfies the Jacobi equation

$$a(n)\psi(P,n+1,n_0) + a(n-1)\psi(P,n-1,n_0) = [b(n)+\tilde{\pi}(P)]\psi(P,n,n_0), \quad n,n_0 \in \mathbb{Z}. \tag{3.25}$$

Proof. (3.16) implies

$$a(n)\phi(P,n)+a(n-1)\phi(P,n-1)^{-1} = [-G_{g+1}(\tilde\pi(P),n)+R_{2g+2}^{1/2}(P)]F_g(\tilde\pi(P),n)^{-1} \\ -[G_{g+1}(\tilde\pi(P),n-1)+R_{2g+2}^{1/2}(P)]F_g(\tilde\pi(P),n)^{-1} = b(n)+\tilde\pi(P) \tag{3.26}$$

and (3.25) follows from (3.26) and

$$\phi(P,n)=\psi(P,n+1,n_0)/\psi(P,n,n_0). \qquad \square \tag{3.27}$$

We collect a few more useful relations which follow from (3.8) and (3.22), (3.23),

$$\phi(P,n)\phi(P^*,n) = F_g(\tilde\pi(P),n+1)/F_g(\tilde\pi(P),n), \tag{3.28}$$

$$\psi(P,n,n_0)\psi(P^*,n,n_0) = F_g(\tilde\pi(P),n)/F_g(\tilde\pi(P),n_0), \tag{3.29}$$

$$\phi(P,n)-\phi(P^*,n) = R_{2g+2}^{1/2}(P)/[a(n)F_g(\tilde\pi(P),n)], \tag{3.30}$$

$$\phi(P,n)+\phi(P^*,n) = -G_{g+1}(\tilde\pi(P),n)/[a(n)F_g(\tilde\pi(P),n)]. \tag{3.31}$$

It will be convenient later on to denote by $\phi_\pm(z,n)$, $\psi_\pm(z,n,n_0)$ the chart expressions (branches) of $\phi(P,n)$, $\psi(P,n,n_0)$ in the charts $(\Pi_\pm,\tilde\pi)$ (see (A.13)).

In order to analyze ϕ and the BA-function ψ further, it is convenient to express them in terms of the Riemann theta function associated with K_g. First we note that by (3.22) and (3.23), the divisors (ϕ) of ϕ and (ψ) of ψ are given by

$$(\phi(.,n)) = \mathcal{D}_{\underline{\hat\mu}(n+1)} - \mathcal{D}_{\underline{\hat\mu}(n)} + \mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-} \tag{3.32}$$

and

$$(\psi(.,n,n_0)) = \mathcal{D}_{\underline{\hat\mu}(n)} - \mathcal{D}_{\underline{\hat\mu}(n_0)} + (n-n_0)(\mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}) \tag{3.33}$$

(cf. our notation established in Appendix A). By Abel's theorem (cf. (A.43)), (3.33) yields

$$\begin{aligned}\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}(n)}) &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}(n_0)}) - (n-n_0)\underline{A}_{\infty_-}(\infty_+) \\ &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}(n_0)}) - 2(n-n_0)\underline{A}_{P_0}(\infty_+),\end{aligned} \tag{3.34}$$

where, for convenience only, from this point on we agree to fix the base point P_0 as the branch point $(E_0,0)$,

$$P_0 = (E_0,0). \tag{3.35}$$

Next we introduce the abbreviations,

$$\underline{z}(P,n) = \underline{\hat{A}}_{P_0}(P) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) - \underline{\hat{\Xi}}_{P_0} \in \mathbb{C}^g, \tag{3.36}$$

$$\underline{z}(n) = \underline{z}(\infty_+, n), \tag{3.37}$$

where

$$\underline{\hat{\Xi}}_{P_0} = (\hat{\Xi}_{P_0,1}, \dots, \hat{\Xi}_{P_0,g}), \quad \hat{\Xi}_{P_0,j} = \left[\frac{1+\tau_{j,j}}{2} - \sum_{\substack{k=1 \\ k\neq j}}^{g} \int_{a_k} \hat{A}_{P_0,j}\omega_k \right] \tag{3.38}$$

denotes a representative of the vector of Riemann constants $\underline{\Xi}_{P_0} = \underline{\hat{\Xi}}_{P_0} \mod (L_g)$. Since for any $Q_0 \in K_g$,

$$\underline{A}_{Q_0}(.) = \underline{A}_{P_0}(.) - \underline{A}_{P_0}(Q_0), \quad \underline{\Xi}_{Q_0} = \underline{\Xi}_{P_0} + (g-1)\underline{A}_{P_0}(Q_0), \tag{3.39}$$

$\underline{z}(P,n)$ is independent of the chosen base point P_0. For later purposes we recall that

$$2\underline{\Xi}_{P_0} = \underline{0} \mod (L_g) \tag{3.40}$$

and, due to (3.34), that

$$\underline{z}(\infty_-, n) = \underline{z}(\infty_+, n-1) \mod (\mathbb{Z}^g). \tag{3.41}$$

Next, consider the normal differential of the third kind $\omega^{(3)}_{\infty_+,\infty_-}$ which has simple poles at ∞_+ and ∞_-, corresponding residues $+1$ and -1, vanishing a-periods, and is holomorphic otherwise on K_g. Hence we have (cf. (A.37))

$$\omega^{(3)}_{\infty_+,\infty_-} = \frac{\prod_{j=1}^{g}(\tilde{\pi}-\lambda_j)d\tilde{\pi}}{R^{1/2}_{2g+2}}, \quad \omega^{(3)}_{\infty_-,\infty_+} = -\omega^{(3)}_{\infty_+,\infty_-}, \tag{3.42}$$

$$\int_{a_j} \omega^{(3)}_{\infty_+,\infty_-} = 0, \quad 1 \le j \le g, \tag{3.43}$$

$$U^{(3)}_j = \frac{1}{2\pi i}\int_{b_j} \omega^{(3)}_{\infty_+,\infty_-} = \hat{A}_{\infty_-,j}(\infty_+) = 2\hat{A}_{P_0,j}(\infty_+), \quad 1 \le j \le g, \tag{3.44}$$

where the numbers $\{\lambda_j\}_{1\le j\le g}$ are determined by the normalization (3.43).

Recalling that $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ are nonspecial by (3.17) and Lemma A.2, that is,

$$i(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = 0, \quad n \in \mathbb{Z}, \tag{3.45}$$

and that by (a special case of) Riemann's vanishing theorem

$$\theta(\underline{z}(P,n)) = 0 \text{ if and only if } P \in \{\hat{\mu}_j(n)\}_{1\le j\le g}, \tag{3.46}$$

the zeros and poles of ϕ and ψ as recorded in (3.32) and (3.33) suggest consideration of the expressions

$$\frac{\theta(\underline{z}(P,n+1))}{\theta(\underline{z},(P,n))}\exp\left[\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right] \tag{3.47}$$

and

$$\frac{\theta(\underline{z}(P,n))}{\theta(\underline{z}(P,n_0))}\exp\left[(n-n_0)\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right]. \tag{3.48}$$

Here we agree to use the same path of integration from P_0 to P on K_g in the Abel map $\underline{\hat{A}}_{P_0}(P)$ in $\underline{z}(P,n)$ and in the integral over $\omega^{(3)}_{\infty_+,\infty_-}$ in the exponents of (3.47) and (3.48). With this convention, both expressions (3.47), (3.48) are well-defined on K_g (due to (3.43), (3.44), and (A.27)) and we infer

$$\phi(P,n)=C(n)\frac{\theta(\underline{z}(P,n+1))}{\theta(\underline{z}(P,n))}\exp\left[\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right], \tag{3.49}$$

$$\psi(P,n,n_0)=C(n,n_0)\frac{\theta(\underline{z}(P,n))}{\theta(\underline{z}(P,n_0))}\exp\left[(n-n_0)\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right] \tag{3.50}$$

since ψ (and hence ϕ) is determined up to a constant by its zeros and poles. It remains to determine the constants $C(n)$, $C(n,n_0)$. Since the original literature appears to be vague at this point we shall dwell on this a bit. As a consequence of (3.29) one infers

$$\psi(\infty_+,n,n_0)\psi(\infty_-,n,n_0)=1 \tag{3.51}$$

and hence (3.37) and (3.49) yield

$$C(n,n_0)^2=\frac{\theta(\underline{z}(n_0))\theta(\underline{z}(n_0-1))}{\theta(\underline{z}(n))\theta(\underline{z}(n-1))}. \tag{3.52}$$

Because of

$$\phi(P,n)=\psi(P,n+1,n), \tag{3.53}$$

we get

$$C(n)=\left[\frac{\theta(\underline{z}(n-1))}{\theta(\underline{z}(n+1))}\right]^{1/2}, \tag{3.54}$$

where the determination of the square root will be given later (see (3.71)).

Next we collect a few useful results.

Lemma 3.4. *(i).* $\omega^{(3)}_{\infty_+,\infty_-}$ *satisfies*

$$\mathrm{Re}\left(\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right)\begin{cases}=0, & \tilde\pi(P)\in\bigcup_{j=0}^{g}[E_{2j},E_{2j+1}]\\ <0, & \tilde\pi(P)\in\mathbb{R}\backslash\bigcup_{j=0}^{g}[E_{2j}E_{2j+1}]\end{cases},\quad P\in\Pi_+ \tag{3.55}$$

(the sign in (3.55) *being reversed for* $P\in\Pi_-$*) and*

$$\lambda_j\in[E_{2j-1},E_{2j}],\quad 1\le j\le g. \tag{3.56}$$

(ii). Let $\mathcal{D}\in\sigma^g K_g$*. Then*

$$\begin{aligned}&i\,\mathrm{Im}[\underline{\hat\alpha}_{P_0}(\mathcal{D})+\underline{\hat\Xi}_{P_0}]=\underline{0}\mod (L_g)\\ &\textit{if and only if } \mathcal{D}=\mathcal{D}_{\underline{\hat\mu}}\left(=\sum_{j=1}^{g}\mathcal{D}_{\hat\mu_j}\right),\ \underline{\hat\mu}=(\hat\mu_1,\dots,\hat\mu_g),\\ &\qquad\tilde\pi(\hat\mu_j)\in[E_{2j-1},E_{2j}],\quad 1\le j\le g.\end{aligned} \tag{3.57}$$

(iii). Suppose $\hat\mu_j\in K_g$, $\tilde\pi(\hat\mu_j)=\mu_j\in[E_{2j-1},E_{2j}]$, $1\le j\le g$*. Then*

$$i\,\mathrm{Im}\left(\int_{P_0}^{\hat\mu}\omega^{(3)}_{\infty_+,\infty_-}\right)=-2\pi i\underline{\hat A}_{P_0}(\infty_+),\quad 1\le j\le g. \tag{3.58}$$

(iv). One has

$$\theta(\underline{z}(P,n))\theta(\underline{z}(P,n_0))>0 \textit{ for } \tilde\pi(P)<E_0 \textit{ or } \tilde\pi(P)>E_{2g+1},\quad n,n_0\in\mathbb{Z}, \tag{3.59}$$

in particular, $C(n)$ *in* (3.54) *is real-valued.*

Proof. (i). The normalization (3.43), that is,

$$\int_{a_j}\omega^{(3)}_{\infty_+,\infty_-}=-2\int_{E_{2j-1}}^{E_{2j}}\frac{\prod_{k=1}^{g}(z-\lambda_k)\,dz}{R_{2g+2}(z)^{1/2}}=0,\quad 1\le j\le g, \tag{3.60}$$

immediately yields (3.56). In order to prove (3.55) we assume $P\in\Pi_+$ and choose as integration path the lift of the straight line from $E_0+i\epsilon$ to $\tilde\pi(P)+i\epsilon$ and then take $\epsilon\downarrow 0$. (Since we are required to stay in the interior of $\hat K_g$, whenever we are to intersect some b_j-cycle we first go around a_j and then back on the other side of b_j. Since the parts on b_j cancel and due to our normalization (3.60) the a_j-periods of $\omega^{(3)}_{\infty_+,\infty_-}$ are zero, this does not alter the value of the integral in question.) The rest follows from (3.56) and (A.5).

(ii). First assume $\mathcal{D}=\mathcal{D}_{\underline{\hat\mu}}$ with $\tilde\pi(\hat\mu_j)\in[E_{2j-1},E_{2j}]$, $1\le j\le g$. Then, using (A.5), one can show that

$$i\,\mathrm{Im}[\hat A_{P_0,j}(\hat\mu_k)]=\frac{1}{2}\tau_{j,k}=i\,\mathrm{Im}\left[\int_{a_k}\hat A_{P_0,j}\omega_k\right] \tag{3.61}$$

by taking all integrals as lifts obtained from limits of straight line sequent of the type $\tilde{\pi}(P_1)+i\epsilon$ to $\tilde{\pi}(P_2)+i\epsilon$ as $\epsilon \downarrow 0$ for various points P_1, P_2. Again, in order to stay on $\hat{K}_g$, crossings of b_j-cycles can be avoided by adding contributions along a_j-cycles (which are real-valued) as in the proof of part (i). Equation (3.61) yields

$$i\,\mathrm{Im}[\underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\hat{\Xi}}_{P_0}] = \underline{0} \mod (L_g). \tag{3.62}$$

Next consider $\underline{\alpha}_{P_0}$ as a holomorphic map from $\sigma^g K_g$ to $J(K_g)$ and restrict $\underline{\alpha}_{P_0}$ to divisors $\mathcal{D}$ satisfying

$$\mathcal{D} = \mathcal{D}_{\underline{\hat{\mu}}},\ \tilde{\pi}(\underline{\hat{\mu}}_j) = \mu_j \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le g. \tag{3.63}$$

Denote this restriction by $\underline{\tilde{\alpha}}_{P_0}$. The set of divisors $\mathcal{D}$ in (3.63) is a connected submanifold of $\sigma^g K_g$ since it is isomorphic to $\times_{j=1}^g S^1$. Moreover, by the arguments leading to (3.62), the image of $\underline{\tilde{\alpha}}_{P_0}$ is a subset of $J = \{\underline{x} \in J(K_g) \big| i\,\mathrm{Im}(\underline{x} + \underline{\hat{\Xi}}_{P_0}) = \underline{0} \mod (L_g)\}$. Since

$$\mathrm{rank}(d\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}})) = g - i(\mathcal{D}_{\underline{\hat{\mu}}}) = g \tag{3.64}$$

as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial by (3.63) and Lemma A.2, $\underline{\tilde{\alpha}}_{P_0}$ is invertible and hence its image is all of J. Thus $\underline{\alpha}_{P_0}$ provides an isomorphism from $\{\mathcal{D}_{\underline{\hat{\mu}}} \in \sigma^g K_g \big| \tilde{\pi}(\hat{\mu}_j) = \mu_j \in [E_{2j-1}, E_{2j}],\ 1 \le j \le g\}$ onto J.
(iii). This is clear from (3.44).
(iv). Since

$$i\,\mathrm{Im}[\underline{z}(P,n)] = i\,\mathrm{Im}[\underline{\hat{A}}_{P_0}(P)] = \underline{0} \mod (L_g), \quad \tilde{\pi}(P) < E_0,\ \tilde{\pi}(P) > E_{2g+1}, \tag{3.65}$$

a combination of (A.27), (3.37), (3.57), and (3.65) yields

$$\mathrm{Im}[\theta(\underline{z}(P,n))\theta(\underline{z}(P,n_0))] = 0 \text{ for } \tilde{\pi}(P) < E_0,\ \tilde{\pi}(P) > E_{2g+1},\ n, n_0 \in \mathbb{Z}. \tag{3.66}$$

But then (3.59) immediately follows by considering $n = n_0$ and the fact that all zeros of $\theta(\underline{z}(P,n))$ occur precisely at $P = \hat{\mu}_j(n)$, $1 \le j \le g$ for all $n \in \mathbb{Z}$. □

Lemma 3.4 is the main ingredient for the following characterization of $\phi(P,n)$ and $\psi(P,n,n_0)$.

Theorem 3.5. *(i).*

$$\phi(P,n) = C(n)\frac{\theta(\underline{z}(P,n+1))}{\theta(\underline{z}(P,n))}\exp\left[\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-}\right], \tag{3.67}$$

$$\psi(P,n,n_0) = C(n,n_0)\frac{\theta(\underline{z}(P,n))}{\theta(\underline{z}(P,n_0))}\exp\left[(n-n_0)\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-}\right], \tag{3.68}$$

where $C(n)$, $C(n,n_0)$ are real-valued and

$$C(n) = C(n+1,n) = [\theta(\underline{z}(n-1))/\theta(\underline{z}(n+1))]^{1/2}, \tag{3.69}$$

$$C(n,n_0) = \begin{cases} \prod_{m=n_0}^{n-1} C(m), & n \geq n_0+1 \\ 1, & n = n_0 \\ \prod_{m=n}^{n_0-1} C(m)^{-1}, & n \leq n_0 - 1 \end{cases} = \left[\frac{\theta(\underline{z}(n_0))\theta(\underline{z}(n_0-1))}{\theta(\underline{z}(n))\theta(\underline{z}(n-1))}\right]^{1/2}. \tag{3.70}$$

In addition, the sign of $C(n)$ is opposite that of $a(n)$, that is,

$$\operatorname{sgn}[C(n)] = -\operatorname{sgn}[a(n)], \quad n \in \mathbb{Z}. \tag{3.71}$$

(ii). The function $\phi(P,n)$ and hence $\psi(P,n,n_0)$ is real-valued for all P such that $\pi(P) \in \mathbb{R}\backslash \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}]$ and all n, $n_0 \in \mathbb{Z}$.
(iii). Let $\lambda \in \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}]$. Then

$$\psi_{\pm}(\lambda,.,n_0) \in \ell^{\infty}(\mathbb{Z}). \tag{3.72}$$

(iv). Let $\lambda \in \mathbb{R}\backslash \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}]$. Then there exist constants M, $K(\lambda) > 0$ such that

$$|\psi_{\pm}(\lambda,n,n_0)| \leq Me^{\mp(n-n_0)K(\lambda)}. \tag{3.73}$$

Moreover,

$$\psi_{\pm}(\lambda,.,n_0) \in \ell^2((N_0,\pm\infty)), \quad N_0 \in \mathbb{Z}. \tag{3.74}$$

Proof. (i). Equations (3.67)–(3.70) follow from equations (3.23), (3.49)–(3.54), and Lemma 3.4 (iv) except for the sign correlation (3.71) of $C(n)$ and $a(n)$. The latter can be inferred as follows. Expanding $\phi(P,n)$ in (3.67) near $P = \infty_+$ on Π_+ yields in the chart $(\Pi_+\backslash\{(0,R_{2g+2}(0)^{1/2})\}, z = \zeta^{-1})$, integrating from P_0 to P along the lift of the corresponding straight line segment along the negative real axis,

$$\begin{aligned}\phi(P,n) &\underset{\zeta\to 0}{=} C(n)\left[\frac{\theta(\underline{z}(n+1))}{\theta(\underline{z}(n))} + O(\zeta)\right]\exp\{\ln[\tilde{a}\zeta + O(\zeta^2)]\} \\ &\underset{\zeta\to 0}{=} C(n)\frac{\theta(\underline{z}(n+1))}{\theta(\underline{z}(n))}\tilde{a}\zeta + O(\zeta^2),\end{aligned} \tag{3.75}$$

where $\tilde{a} < 0$ is an appropriate integration constant. Inserting this expansion into (3.24) (with n replaced by $n+1$) yields

$$a(n)\phi(P,n)^{-1} - \tilde{\pi}(P) \underset{\zeta\to 0}{=} O(1) \underset{\zeta\to 0}{=} \frac{a(n)}{C(n)\tilde{a}}\frac{\theta(\underline{z}(n))}{\theta(\underline{z}(n+1))}\zeta^{-1} - \zeta^{-1} + O(1), \tag{3.76}$$

that is,

$$a(n) = \tilde{a}C(n)\theta(\underline{z}(n+1))\theta(\underline{z}(n))^{-1}, \quad n \in \mathbb{Z}. \tag{3.77}$$

Since $[\theta(\underline{z}(n+1))/\theta(\underline{z}(n))] > 0$ by (3.59) we obtain (3.71).
(ii). This follows directly from (3.22) and (3.23), or alternatively, by

combining (i) and Lemma 3.4 (i), (iii), (iv).
(iii), (iv). Relations (3.72)–(3.74) are an immediate consequence of the quasi-periodicity and hence boundedness of $\theta(\underline{z}(P,n))$ with respect to $n \in \mathbb{Z}$ and Lemma 3.4 (i). $\square$

Relation (3.72) extends to all $z \in \mathbb{C}\backslash \bigcup_{j=0}^{g}[E_{2j}, E_{2j+1}]$. Since this is conveniently proved by invoking Weyl m-functions we postpone this fact to the next section.

4. Spectral Theory for Finite-Gap Jacobi Operators

In this section we shortly digress into spectral properties and Green's functions of self-adjoint $\ell^2(\mathbb{Z})$ realizations associated with finite-gap difference expressions.

We start with a general difference expression L of the type

$$L = aS^+ + a^- S^- - b \tag{4.1}$$

assuming $a,\, b \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $a(n) \neq 0$, $n \in \mathbb{Z}$ implying that L is in the limit point case near $\pm\infty$. Then the following Jacobi operator H in $\ell^2(\mathbb{Z})$ defined by

$$Hf = Lf, \quad f \in \mathcal{D}(H) = \ell^2(\mathbb{Z}) \tag{4.2}$$

is the unique self-adjoint realization associated with L in $\ell^2(\mathbb{Z})$. The corresponding Green's function $G(z,m,n)$ of H, defined by

$$((H-z)^{-1}f)(m) = \sum_{n\in\mathbb{Z}} G(z,m,n)f(n), \quad f \in \ell^2(\mathbb{Z}),\ z \in \mathbb{C}\backslash\sigma(H), \tag{4.3}$$

with $\sigma(H)$ denoting the spectrum of H, can be expressed as

$$G(z,m,n) = W(f_-(z), f_+(z))^{-1} \begin{cases} f_-(z,m)f_+(z,n), & m \leq n \\ f_+(z,m)f_-(z,n), & m \geq n \end{cases}, \quad z \in \mathbb{C}\backslash\sigma(H). \tag{4.4}$$

Here

$$f_\pm(z,.) \in \ell^2((N_0, \pm\infty)), \quad z \in \mathbb{C}\backslash\sigma(H),\ N_0 \in \mathbb{Z} \tag{4.5}$$

are weak solutions of

$$L\psi(z,n) = z\psi(z,n) \tag{4.6}$$

and

$$W(f,g)(n) = a(n)[f(n)g(n+1) - f(n+1)g(n)] \tag{4.7}$$

denotes the Wronskian of f and g.

By general principles, $G(z,n_0,n_0)$, $G(z,n_0{+}1,n_0{+}1)$, and $G(z,n_0,n_0{+}1)$ uniquely determine both sequences $\{a(n)^2\}_{n\in\mathbb{Z}}$ and $\{b(n)\}_{n\in\mathbb{Z}}$. These results are standard and a consequence of the 2×2 Weyl M-matrix associated with H,

$$M_{n_0}(z) = \begin{pmatrix} G(z,n_0,n_0) & G(z,n_0,n_0+1) \\ G(z,n_0,n_0+1) & G(z,n_0+1,n_0+1) \end{pmatrix}. \tag{4.8}$$

We recall the asymptotic behavior,

$$\begin{aligned} G(z,n,n) &\underset{z\to\infty}{=} -z^{-1} + O(z^{-2}), \\ G(z,n,n+1) &\underset{z\to\infty}{=} a(n)z^{-2} + O(z^{-3}), \quad n\in\mathbb{Z}. \end{aligned} \tag{4.9}$$

Moreover, the identity

$$[2a(n)G(z,n,n+1)-1]^2 = 1 + 4a(n)^2 G(z,n,n)G(z,n+1,n+1), \quad n\in\mathbb{Z} \tag{4.10}$$

proves that $G(z,n_0,n_0{+}1)$ is determined up to a sign by $a(n_0)$, $G(z,n_0,n_0)$, and $G(z,n_0+1,n_0+1)$.

Next, we define restrictions $H_{\pm,n_0}$ of H to $\ell^2([n_0\pm 1,\pm\infty))$ with a Dirichlet boundary condition at n_0,

$$H_{\pm,n_0} f = Lf, \quad f\in\mathcal{D}(H_{\pm,n_0}) = \{f\in\ell^2([n_0\pm 1,\pm\infty)) | f(n_0)=0\}. \tag{4.11}$$

Denoting by $G_{\pm,n_0}$ the corresponding Green's functions of $H_{\pm,n_0}$, the associated Weyl m-functions on $[n_0,\pm\infty)$ then read

$$\begin{aligned} m_{\pm,n_0}(z) &= G_{\pm,n_0}(z,n_0\pm 1,n_0\pm 1) = (\delta_{n_0\pm 1},(H_{\pm,n_0}-z)^{-1}\delta_{n_0\pm 1}) \\ &= \begin{cases} -a(n_0)^{-1}[f_+(z,n_0+1)/f_+(z,n_0)] \\ -a(n_0-1)^{-1}[f_-(z,n_0-1)/f_-(z,n_0)] \end{cases}, \end{aligned} \tag{4.12}$$

with $f_\pm(n,z)$ as in (4.5) and $\delta_m(n) = \delta_{m,n} = \begin{cases} 1, & m=n \\ 0, & m\neq n \end{cases}$.

The following is a simple but useful result concerning the spectral invariants of H.

Lemma 4.1. *(See, e.g., [29], p. 141.) Suppose a, $b\in\ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $a(n)\neq 0$, $n\in\mathbb{Z}$ and introduce $a_\epsilon\in\ell^\infty_{\mathbb{R}}(\mathbb{Z})$ by*

$$a_\epsilon = \{\epsilon(n)a(n)\}_{n\in\mathbb{Z}}, \ \epsilon(n)\in\{+1,-1\}, \quad n\in\mathbb{Z}. \tag{4.13}$$

Define H_ϵ in $\ell^2(\mathbb{Z})$ as in (4.2) with L replaced by $L_\epsilon = a_\epsilon S^+ + a_\epsilon^- S^- - b$. Then H and H_ϵ are unitarily equivalent, that is, there exists a unitary operator U_ϵ in $\ell^2(\mathbb{Z})$ such that

$$H = U_\epsilon H_\epsilon U_\epsilon^{-1}. \tag{4.14}$$

Proof. U_ϵ is explicitly represented by the diagonal matrix

$$U_\epsilon = (\tilde{\epsilon}(n)\delta_{m,n})_{m,n\in\mathbb{Z}}, \quad \tilde{\epsilon}(n) \in \{+1,-1\},\ \tilde{\epsilon}(n)\tilde{\epsilon}(n+1) = \epsilon(n),\ n \in \mathbb{Z}. \quad \square \tag{4.15}$$

Next, let $c(z,n,n_0)$, $s(z,n,n_0)$, $z \in \mathbb{C}$ be a fundamental system of solutions of (4.6) satisfying

$$c(z,n_0,n_0) = s(z,n_0+1,n_0) = 1,\ c(z,n_0+1,n_0) = s(z,n_0,n_0) = 0. \tag{4.16}$$

Returning to our special g-gap sequences $a(n)$, $b(n)$ in (3.18), (3.19), the branches $\psi_\pm(z,n,n_0)$ of $\psi(P,n,n_0)$ in (3.23) satisfy

$$\psi_\pm(z,n,n_0) = c(z,n,n_0) + \phi_\pm(z,n_0)s(z,n,n_0) \tag{4.17}$$

and

$$W(\psi_-(z,.,n_0),\psi_+(z,.,n_0)) = a(n_0)[\phi_+(z,n_0) - \phi_-(z,n_0)], \tag{4.18}$$

with $\phi_\pm(z,n)$ the corresponding branches of $\phi(P,n)$ in (3.22). Taking into account (3.28)–(3.31) then enables one to further identify

$$G(z,n,n) = F_g(z,n)/R_{2g+2}(z)^{1/2}, \tag{4.19}$$

$$G(z,n,n+1) = \{1 - [G_{g+1}(z,n)/R_{2g+2}(z)^{1/2}]\}/[2a(n)], \tag{4.20}$$

$$\begin{aligned} m_{+,n}(z) &= G_{+,n}(z,n+1,n+1) = -\phi_+(z,n)/a(n) \\ &= [G_{g+1}(z,n) - R_{2g+2}(z)^{1/2}]/[2a(n)^2F_g(z,n)], \end{aligned} \tag{4.21}$$

$$\begin{aligned} m_{-,n}(z) &= G_{-,n}(z,n-1,n-1) = -1/[a(n-1)\phi_-(z,n-1)] \\ &= [G_{g+1}(z,n-1) + R_{2g+2}(z)^{1/2}]/[2a(n-1)^2F_g(z,n)]. \end{aligned} \tag{4.22}$$

We note that $a(n_0)$ and $G_{g+1}(z,n_0)$ determine the sign of $G(z,n_0,n_0+1)$ left open in (4.10).

We conclude this section with a summary of spectral properties of H, $H_{\pm,n}$ in connection with the g-gap sequences (3.18) and (3.19). Let $\sigma(.)$, $\sigma_{ac}(.)$, $\sigma_{sc}(.)$, and $\sigma_p(.)$ denote the spectrum absolutely continuous, singularly continuous, and point spectrum (set of eigenvalues), respectively. Then we have

Theorem 4.2. *Suppose a, $b \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$ are g-gap sequences satisfying* (3.18), (3.19) *and let H, $H_{\pm,n}$ be as in* (4.2), (4.11). *Then*

$$\sigma(H) = \sigma_{ac}(H) = \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}], \tag{4.23}$$

$$\sigma_{sc}(H) = \sigma_p(H) = \emptyset \tag{4.24}$$

and for all $n \in \mathbb{Z}$,

$$\sigma(H_{-,n} \oplus H_{+,n}) = \sigma(H) \cup \{\mu_j(n)\}_{1 \le j \le g}, \tag{4.25}$$

$$\sigma_{ac}(H_{\pm,n}) = \sigma(H), \ \sigma_{sc}(H_{\pm,n}) = \emptyset, \tag{4.26}$$

$$\sigma_p(H_{-,n} \oplus H_{+,n}) = \{\mu_j(n)\}_{1 \le j \le g} \cap \{\bigcup_{k=0}^{g} (E_{2k-1}, E_{2k})\}. \tag{4.27}$$

In addition, $\sigma(H)$ has uniform spectral multiplicity two whereas $\sigma(H_{\pm,n})$, $n \in \mathbb{Z}$ is simple.

Proof. Consider the trace of the Weyl M-matrix (4.8), that is,

$$T(z) = G(z, n_0, n_0) + G(z, n_0 + 1, n_0 + 1). \tag{4.28}$$

Then $\mathbb{C}\backslash\sigma(H)$ coincides with the holomorphy domain of T. This identifies $\sigma(H)$ in (4.23). For any $\lambda_0 \in \sigma_p(H)$ one must necessarily have

$$\lim_{\epsilon \downarrow 0} [i\epsilon T(\lambda_0 + i\epsilon)] < 0. \tag{4.29}$$

The explicit structure of $G(z, n, n)$ in (4.19) then proves the impossibility of (4.29) and hence $\sigma_p(H) = \emptyset$. In order to prove $\sigma_{sc}(H) = \emptyset$ we apply Theorem XIII.20 of [79] with $D = \ell_0(\mathbb{Z})$ (the subspace of sequences with at most finitely many elements being nonzero), $p = 2$ and $(a, b) = (E_{2j}+\epsilon, E_{2j+1}-\epsilon)$, $\epsilon > 0$. Upon letting $\epsilon \downarrow 0$ one infers the spectrum to be purely absolutely continuous on $[E_{2j}, E_{2j+1}]$. This proves (4.23) and (4.24). Next, define in $\ell^2((-\infty, n_0 - 1]) \oplus \ell^2([n_0 + 1, \infty))$

$$H_n^D = H_{-,n} \oplus H_{+,n}. \tag{4.30}$$

Then the Green's function $G_n^D(z, m, m')$ of H_n^D is given by

$$G_n^D(z, m, m') = G(z, m, m') - G(z, n, n)^{-1} G(z, m, n) G(z, n, m'),$$
$$z \in \mathbb{C}\backslash\{\mu_j(n)\}_{1 \le j \le g}, \ m, m' \in \mathbb{Z}\backslash\{n_0\}. \tag{4.31}$$

This proves (4.25) and (4.27) (cf. (4.19)). The relation (4.26) can again be proved by alluding to Theorem XIII.20 of [79]. Finally, self-adjoint half-line operators (like $H_{\pm,n}$) always have simple spectra, and uniform spectral multiplicity two of H follows from the existence of the two linearly independent branches $\psi_\pm(\lambda, n, n_0)$ for $\lambda \in \bigcup_{j=0}^{g} (E_{2j}, E_{2j+1})$. □

Alternatively, one can prove Theorem 4.2 directly, that is, by pure ODE techniques by explicitly computing the 2×2 spectral matrix of H given the Weyl M-matrix (4.8) (with entries (4.19), (4.20)) and by calculating the spectral function of $H_{\pm,n}$ given the corresponding Weyl m-functions $m_{\pm,n}$ in (4.21), (4.22). This also settles the multiplicity of the spectrum following [48] in the context of second-order differential operators in $L^2(\mathbb{R})$ (see also Appendices A–C of [38] for a short summary of the relevant spectral theoretic results). Here we only mention that if

$$d\rho_{n_0} = (d\rho_{n_0,j,k})_{1\le j,k\le 2} \tag{4.32}$$

denotes the (self-adjoint) matrix-valued spectral measure of H, related to the Weyl M-matrix (4.8) via

$$M_{n_0}(z) = \int_{\mathbb{R}} \frac{d\rho_{n_0}(\lambda)}{z-\lambda}, \tag{4.33}$$

one explicitly obtains from (4.19), (4.20) in the g-gap context of Theorem 4.2 that

$$\frac{d\rho_{n_0,1,1}(\lambda)}{d\lambda} = \begin{cases} \dfrac{F_g(n_0,\lambda)}{\pi i R_{2g+2}(\lambda)^{1/2}}, & \lambda\in\sigma(H) \\ 0, & \lambda\in\mathbb{R}\backslash\sigma(H) \end{cases}, \tag{4.34}$$

$$\frac{d\rho_{n_0,1,2}(\lambda)}{d\lambda} = \frac{d\rho_{n_0,2,1}(\lambda)}{d\lambda} = \begin{cases} \dfrac{-G_{g+1}(n_0,\lambda)}{2\pi i a(n_0) R_{2g+2}(\lambda)^{1/2}}, & \lambda\in\sigma(H) \\ 0, & \lambda\in\mathbb{R}\backslash\sigma(H) \end{cases}, \tag{4.35}$$

$$\frac{d\rho_{n_0,2,2}(\lambda)}{d\lambda} = \begin{cases} \dfrac{F_g(n_0+1,\lambda)}{\pi i R_{2g+2}(\lambda)^{1/2}}, & \lambda\in\sigma(H) \\ 0, & \lambda\in\mathbb{R}\backslash\sigma(H) \end{cases} \tag{4.36}$$

(cf. (A.5) for our conventions on $R_{2g+2}(\lambda)^{1/2}$).

Finally we relate the polynomials $F_g(z,n)$, $G_{g+1}(z,n)$ used in Section 3 to the homogeneous quantities $\tilde F_g(z,n)$, $\tilde G_{g+1}(z,n)$ (cf. (2.19)). We introduce the constants $c_j(E)$ via

$$R_{2g+2}(z)^{1/2} = -z^{g+1}\sum_{j=0}^{\infty} c_j(E) z^{-j}, \quad |z|>\|H\|, \tag{4.37}$$

implying

$$c_0(E)=1, \quad c_1(E) = -\frac{1}{2}\sum_{j=0}^{2g-1} E_j, \ \ etc. \tag{4.38}$$

Lemma 4.3. *Let $F_g(z,n)$, $G_{g+1}(z,n)$ be the polynomials defined in* (3.4), (3.5) *and let $a(n)^2$, $b(n)$ be defined as in* (3.18), (3.19). *Then we have*

$$F_g(z,n) = \sum_{\ell=0}^{g} c_\ell(E)\tilde F_\ell(z,n), \quad G_{g+1}(z,n) = \sum_{\ell=0}^{g} c_\ell(E)\tilde G_{\ell+1}(z,n). \tag{4.39}$$

Proof. ¿From (4.19), (4.20) we infer for $|z| > \|H\|$ using Neumann's expansion for the resolvent of H and Lemma 2.1 that

$$F_g(z,n) = -\frac{R_{2g+2}(z)^{1/2}}{z}\sum_{\ell=0}^{g}\tilde f_\ell(n)z^{-\ell}, \tag{4.40}$$

$$G_{g+1}(z,n) = R_{2g+2}(z)^{1/2}\Big(1-\frac{1}{z}\sum_{\ell=0}^{g}\tilde g_\ell(n)z^{-\ell}\Big), \tag{4.41}$$

which, together with (4.37), completes the proof. □

For interesting recent generalizations of almost periodic Jacobi operators to cases where formally $g\to\infty$, and Cantor spectra and solutions of infinite-dimensional Jacobi inversion problems are involved, we refer to [7], [80].

5. Quasi-Periodic Finite-Gap Solutions of the Stationary Toda Hierarchy

Given the detailed preparations in Section 3 we are now ready to derive the algebro-geometric finite-gap solutions of the stationary Toda hierarchy.

Starting with high-energy expansions for $\omega^{(3)}_{\infty_+,\infty_-}$ and $\theta(\underline{z}(P,n))$ we formulate

Lemma 5.1. *Given the canonical charts $(\Pi_\pm\backslash\{(0,R_{2g+2}(0)^{1/2})\}, z=\zeta^{-1})$ we obtain the following expansions for P near $\infty_\pm$.*
(i).

$$\exp\left[\int_{P_0}^{P}\omega^{(3)}_{\infty_+,\infty_-}\right] \underset{\zeta\to 0}{=} (\tilde a\zeta)^{\pm 1}\left[\sum_{\ell=0}^{\infty}\tilde b_\ell(-\zeta)^\ell\right]^{\pm 1}, \quad P \text{ near } \infty_\pm, \tag{5.1}$$

where $\tilde a$, $\{\tilde b_\ell\}_{\ell\in\mathbb{N}_0}$ only depend on K_g (i.e., on $\{E_m\}_{0\le m\le 2g+1}$) and

$$\tilde a < 0 \tag{5.2}$$

is an integration constant,

$$\begin{aligned}&\tilde b_0 = 1,\\ &\tilde b_1 = \sum_{j=1}^{g}\lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1}E_m,\\ &\textit{etc.}\end{aligned} \tag{5.3}$$

(ii).

$$\frac{\theta(\underline{z}(P,n+1))}{\theta(\underline{z}(P,n))} \underset{\zeta\to 0}{=} \frac{\theta\Big(\underline{z}\binom{n+1}{n}\Big)}{\theta\Big(\underline{z}\binom{n}{n-1}\Big)}\sum_{\ell=0}^{\infty}\tilde\theta_{\pm,\ell}(n)\zeta^\ell, \quad P \text{ near } \infty_\pm, \tag{5.4}$$

where

$$\tilde\theta_{\pm,0}(n) = 1,$$

$$\tilde\theta_{\pm,1}(n) = \pm\sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j}\ln\left[\frac{\theta\Big(\underline{w}+\underline{z}\binom{n+1}{n}\Big)}{\theta\Big(\underline{w}+\underline{z}\binom{n}{n-1}\Big)}\right]\Bigg|_{\underline{w}=\underline{0}}, \tag{5.5}$$

etc.,

and $c_j(k)$ *are defined in* (A.19).
(iii).

$$\begin{aligned}\phi(P,n) &\underset{\zeta\to 0}{=} \tilde a C(n)\frac{\theta(\underline{z}(n+1))}{\theta(\underline{z}(n))}\zeta \\ &+ \tilde a C(n)\frac{\theta(\underline{z}(n+1))}{\theta(\underline{z}(n))}\left\{-\tilde b_1 + \sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j}\ln\left[\frac{\theta(\underline{w}+\underline{z}(n+1))}{\theta(\underline{w}+\underline{z}(n))}\right]\Bigg|_{\underline{w}=\underline{0}}\right\}\zeta^2 \\ &+ O(\zeta^3), \quad P \text{ near } \infty_+.\end{aligned} \tag{5.6}$$

Proof. (i). Using the representation (3.42) for $\omega^{(3)}_{\infty_+,\infty_-}$ and Lemma 3.4 (i) one readily finds (5.1)–(5.3) as follows. First one expands

$$\omega^{(3)}_{\infty_+,\infty_-} \underset{\zeta\to 0}{=} \pm\zeta^{-1}\left\{1+\left[\frac{1}{2}\sum_{m=0}^{2g+1}E_m - \sum_{j=1}^{g}\lambda_j\right]\zeta + 0(\zeta^2)\right\}d\zeta, \quad P \text{ near } \infty_\pm \tag{5.7}$$

in a sufficiently small neighborhood of $\infty_\pm$ and integrates term by term. The remaining contribution to the integral in (5.1) is then absorbed into the integration constant $\tilde a$ by integrating along the lift of the straight line segment from E_0 to $\tilde\pi(\hat P)$ for some $\hat P$ near $\infty_\pm$ along the negative real axis.
(ii). This follows from (3.36), (3.37), (3.41), and from

$$(\hat{\underline{A}}_{P_0}\circ z^{-1})(\zeta) \underset{\zeta\to 0}{=} \pm\underline{c}(g)\zeta + O(\zeta^2) \mod (L_g) \quad \text{near } \infty_\pm \tag{5.8}$$

(cf. (A.24)).
Item (iii) is obvious from (i), (ii), and (3.67). ☐

Now we are in a position to derive the major result of this section expressing the g-gap sequences a, b in (3.18), (3.19) in terms of the θ-function associated with K_g.

Theorem 5.2. *The stationary* TL_g *solutions, or equivalently, the* g*-gap sequences* $a=\{a(n)\}_{n\in\mathbb{Z}}$*,* $b=\{b(n)\}_{n\in\mathbb{Z}}$ *in* (3.18), (3.19)*, are* *given by*

$$a(n) = \tilde{a}[\theta(\underline{z}(n-1))\theta(\underline{z}(n+1))/\theta(\underline{z}(n))^2]^{1/2}, \quad n \in \mathbb{Z}, \tag{5.9}$$

$$b(n) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\left[\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))}\right]\Bigg|_{\underline{w}=\underline{0}}, \quad n \in \mathbb{Z}. \tag{5.10}$$

Proof. Inserting expansion (5.6) near ∞_+ into (3.24) yields

$$\begin{aligned} 0 &= a(n-1)\phi(P, n-1)^{-1} - \pi(P) - b(n) + O(\zeta) \\ &= \frac{a(n-1)\theta(\underline{z}(n-1))}{\tilde{a}C(n-1)\theta(\underline{z}(n))\zeta}\Bigg\{1 + \Bigg[\tilde{b}_1 - \sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j}\ln\left(\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))}\right)\Bigg|_{\underline{w}=\underline{0}}\Bigg]\zeta \\ &\quad + O(\zeta^2)\Bigg\} - \zeta^{-1} - b(n) + O(\zeta), \end{aligned} \tag{5.11}$$

that is, one infers (3.77) (again) and (5.10). Equation 5.9 is clear from (3.69) and (3.77). □

Remark 5.3. Alternatively, one could have derived (5.10) by evaluating the integral

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{\partial \hat{K}_g} \tilde{\pi}(.)\, d\ln[\theta(\underline{z}(., n))] \\ &= \sum_{j=1}^{g} \mu_j(n) + \sum_{P \in \{\infty_\pm\}} \operatorname*{res}_{P}\{\tilde{\pi}(.)\, d\ln[\theta(\underline{z}(., n)]\} \end{aligned} \tag{5.12}$$

using the residue theorem. Since on the other hand a direct calculation shows that

$$I = \sum_{j=1}^{g}\int_{a_j} \tilde{\pi}\omega_j, \tag{5.13}$$

the trace relation (3.19) for $b(n)$ yields

$$b(n) = \sum_{j=1}^{g}\int_{a_j} \tilde{\pi}\omega_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j}\ln\left[\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))}\right]\Bigg|_{\underline{w}=\underline{0}}. \tag{5.14}$$

A comparison of (5.10) and (5.14) then reveals that

$$\sum_{j=1}^{g}\int_{a_j} \tilde{\pi}\omega_j = \sum_{j=1}^{g} \lambda_j. \tag{5.15}$$

Next we shall show that the BA-function is determined by the location of its poles on $K_g \backslash \{\infty_\pm\}$ and its behavior near $\infty_\pm$.

Lemma 5.4. *Let $\psi(.,n)$, $n \in \mathbb{Z}$ be meromorphic on K_g satisfying*

$$(\psi(.,n)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n-n_0)(\mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}). \tag{5.16}$$

Define a divisor $\mathcal{D}_0(n)$ by

$$(\psi(.,n)) = \mathcal{D}_0(n) - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n-n_0)(\mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}). \tag{5.17}$$

Then

$$\mathcal{D}_0(n) \in \sigma^g K_g,\ \mathcal{D}_0(n) > 0,\ \deg(\mathcal{D}_0(n)) = g. \tag{5.18}$$

Moreover, if $\mathcal{D}_0(n)$ is nonspecial for all $n \in \mathbb{Z}$, that is, if

$$i(\mathcal{D}_0(n)) = 0, \quad n \in \mathbb{Z} \tag{5.19}$$

then $\psi(.,n)$ is unique up to a constant multiple (which may depend on n).

Proof. By the Riemann-Roch theorem (see (A.42)) there exists at least one such function $\psi(.,n)$. If $\psi_j(.,n)$, $j=1,2$ are two such functions satisfying (5.17) with corresponding divisors $\mathcal{D}_{0,j}(n)$, $j=1,2$, one infers

$$(\psi_1(n)/\psi_2(n)) = \mathcal{D}_{0,1}(n) - \mathcal{D}_{0,2}(n). \tag{5.20}$$

Since $i(\mathcal{D}_{0,2}(n)) = 0$, $\deg(\mathcal{D}_{0,2}(n)) = g$ by hypothesis, (A.42) yields $r(-\mathcal{D}_{0,2}(n)) = 1$, $n \in \mathbb{Z}$ and hence $\psi_1(.,n)/\psi_2(.,n)$ is a constant on K_g. $\square$

One can use Lemma 5.4 to obtain an alternative proof of the fact that ϕ and ψ given by (3.67)–(3.70) coincide with the expressions (3.22), (3.23) and satisfy the Riccati and Jacobi equations (3.24) and (3.25), respectively. We shall use precisely this strategy in the t-dependent context to be discussed in the next section.

Remark 5.5. In the special case where $\mu_j(n_0) \in \{E_{2j-1}, E_{2j}\}$ for all $1 \leq j \leq g$ the following are equivalent.

(i).
$$\hat{\mu}_j(n_0+n) = \hat{\mu}_j(n_0-n)^*, \quad 1 \leq j \leq g,\ n \in \mathbb{Z}. \tag{5.21}$$

(ii).
$$\underline{z}(P, n_0+n) = -\underline{z}(P^*, n_0-n) \mod (L_g), \quad n \in \mathbb{Z}. \tag{5.22}$$

(iii).
$$a(n_0+n) = a(n_0-n+1),\ b(n_0+n) = b(n_0-n), \quad n \in \mathbb{Z}. \tag{5.23}$$

Next we derive an alternative and, to the best of our knowledge, novel θ-function representation of $b(n)$.

Corollary 5.6. *$b(n)$, $n \in \mathbb{Z}$ admits the representation*

$$b(n) = -E_0 + \tilde{a}\frac{\theta(\underline{z}(n-1))\theta(\underline{z}(P_0, n+1))}{\theta(\underline{z}(n))\theta(\underline{z}(P_0, n))} + \tilde{a}\frac{\theta(\underline{z}(n))\theta(\underline{z}(P_0(n-1))}{\theta(\underline{z}(n-1))\theta(\underline{z}(P_0, n))}. \tag{5.24}$$

Proof. It suffices to combine (3.24), (3.67), (3.69) (all at $P = P_0$), and (5.9). ☐

The results of this section, with the exception of Corollary 5.6, are well-known (they are contained, e.g., in Section 5 of [70] which is devoted to (stationary) θ-function representations of $\psi(.)$ and the coefficients (a, b) of L). Since they represent a special case of the time-dependent findings of Section 6 we postpone further references to the original literature to the next section.

Finally, for reasons of completeness, we also mention the following criterion for $\{a(n)\}_{n\in\mathbb{Z}}$, $\{b(n)\}_{n\in\mathbb{Z}}$ to be periodic of period $N \in \mathbb{N}$.

Theorem 5.7. *(See [58], Ch. 2.) A necessary and sufficient condition for the g-gap sequences $\{a(n)\}_{n\in\mathbb{Z}}$, $\{b(n)\}_{n\in\mathbb{Z}}$ in (3.18), (3.19) to be periodic of period $N \in \mathbb{N}$ is that $R_{2g+2}(z)$ is of the form*

$$R_{2g+2}(z)Q(z)^2 = \Delta(z)^2 - 1, \tag{5.25}$$

where $Q(.)$ and $\Delta(.)$ are polynomials. The period N is given by

$$N = \deg(Q) + g + 1. \tag{5.26}$$

Since we are not using Theorem 5.7 in our main text we defer its proof to Appendix B.

This completes our treatment of stationary quasi-periodic finite-gap sequences and we now turn to the t-dependent case.

6. Quasi-Periodic Finite-Gap Solutions of the Toda Hierarchy and the Time-Dependent Baker-Akhiezer Function

In this section we continue the construction of g-gap sequences for the Toda hierarchy and now treat the time-dependent case.

Our starting point will be a g-gap stationary solution $(a^{(0)}, b^{(0)})$ of the type (5.9), (5.10), that is,

$$a^{(0)}(n) = \tilde{a}[\theta(\underline{z}(n-1))\theta(\underline{z}(n+1))/\theta(\underline{z}(n))^2]^{1/2}, \tag{6.1}$$

$$b^{(0)}(n) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j}\ln\left[\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))}\right]\Bigg|_{\underline{w}=0} \tag{6.2}$$

satisfying (3.18), (3.19),

$$
\begin{aligned}
a^{(0)}(n)^2 = \frac{1}{2}\sum_{j=1}^{g} R_{2g+2}(\hat\mu_j^{(0)}(n))^{1/2} \prod_{\substack{k=1\\k\neq j}}^{g} [\mu_j^{(0)}(n) - \mu_k^{(0)}(n)]^{-1} \\
- \frac{1}{4}[b^{(0)}(n)^2 + b^{(0)(2)}(n)] > 0,
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
b^{(0)(k)}(n) &= \sum_{j=1}^{g} \mu_j^{(0)}(n)^k - \frac{1}{2}\sum_{m=0}^{2g+1} E_m^k, \quad k \in \mathbb{N}, \\
b^{(0)}(n) = b^{(0)(1)} &= \sum_{j=1}^{g} \mu_j^{(0)}(n) - \frac{1}{2}\sum_{m=0}^{2g+1} E_m,
\end{aligned}
\tag{6.4}
$$

where

$$
\tilde\pi(\hat\mu_j^{(0)}(n)) = \mu_j^{(0)}(n) \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le g,\ n \in \mathbb{Z}. \tag{6.5}
$$

This g-gap stationary solution $(a^{(0)}, b^{(0)})$ represents the initial condition for the following Toda flow (cf. (2.13)),

$$
\widehat{\mathrm{TL}}_r(a(t), b(t)) = 0, \quad (a(t_0), b(t_0)) = (a^{(0)}, b^{(0)}) \tag{6.6}
$$

for some $r \in \mathbb{N}_0$, whose explicit solution we seek below.

¿From our treatment in Section 3 we know that $(a^{(0)}, b^{(0)})$ is determined by the band edges $\{E_j\}_{0\le j\le 2g+1}$ and the Dirichlet eigenvalues $\{\hat\mu_j^{(0)}(n_0)\}_{1\le j\le g}$ at a fixed point n_0. Hence we will consider the following time evolution for the Dirichlet eigenvalues $\hat\mu_j(n_0, t)$

$$
\begin{aligned}
\frac{d}{dt}\mu_j(n_0, t) &= -2\hat F_r(\mu_j(n_0, t), n_0, t) \frac{R_{2g+2}^{1/2}(\hat\mu_j(n_0, t))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{g} [\mu_j(n_0, t) - \mu_\ell(n_0, t)]}, \\
\hat\mu_j(n_0, t_0) &= \hat\mu_j^{(0)}(n_0), \quad 1 \le j \le g,\ (n_0, t) \in \mathbb{Z}\times\mathbb{R}
\end{aligned}
\tag{6.7}
$$

(similar to those encountered in connection with the KdV hierarchy, see, e.g., [6], [12], [13]). Here $\hat F_r(z, n_0, t)$ has to be defined using (2.32) (cf. (2.11)) and $a(n_0, t)^2$, $b(n_0, t)$ have to be expressed in terms of $\hat\mu_j(n_0, t)$ (this can be done by comparing the coefficients in (4.40)). In order to stress the fact that the integration constants c_ℓ in F_r and F_g are different in general, we decided to indicate this by using the notation $\hat F_r$, $\hat G_{r+1}$, $\widehat{\mathrm{TL}}_r$, etc.

The appearance of the function $R_{2g+2}^{1/2}(.)$ in (6.7) indicates the natural way to interprete this system as a vector field on the (complex) manifold $K_g \times \cdots \times K_g = K_g^g$. Since we are interested in real-valued solutions $(a(t), b(t))$ of the Toda hierarchy, we restrict this vector field to the submanifold $\times_{j=1}^{g}\tilde\pi^{-1}([E_{2j-1},E_{2j}])$ which is isomorphic to the torus $S^1\times\cdots\times S^1 = T^g$. Standard theory for differential equations on C^∞ manifolds now implies the existence of a unique solution $\{\hat\mu_j(n_0,t)\}_{1\le j\le g}$ satisfying the initial condition $\hat\mu_j(n_0,t_0) = \hat\mu_j^{(0)}(n_0)$. An inspection of (6.7) using the charts (A.9), (A.10) confirms that the solution $\hat\mu_j(n,t)$ changes sheets whenever it hits E_{2j-1} or E_{2j} and its projection $\mu_j(n_0,t) = \tilde\pi(\hat\mu_j(n_0,t))$ remains trapped in $[E_{2j-1},E_{2j}]$ for all $t\in\mathbb{R}$. Thus up to this point we have

$$\hat\mu_j(n_0,.)\in C^\infty(\mathbb{R},K_g) \text{ and } \tilde\pi(\hat\mu_j(n_0,t))\in[E_{2j-1},E_{2j}],\quad 1\le j\le g,\ t\in\mathbb{R}, \tag{6.8}$$

and using (6.8) we can define polynomials $F_g(z,n,t)$, $G_{g+1}(z,n,t)$ as in Section 3 (cf. (3.4), (3.5)).

We start with calculating the time derivative of $F_g(z,n_0,t)$. By virtue of (6.7) we obtain

$$\frac{d}{dt}F_g(\mu_j(n_0,t),n_0,t) = -2\hat F_r(\mu_j(n,t),n_0,t)G_{g+1}(\mu_j(n,t),n_0,t),\quad 1\le j\le g. \tag{6.9}$$

Since two monic polynomials of (at most) degree $g-1$ coinciding at g points are equal we infer

$$\frac{d}{dt}F_g(z,n_0,t) = 2[F_g(z,n_0,t)\hat G_{r+1}(z,n_0,t) - \hat F_r(z,n_0,t)G_{g+1}(z,n_0,t)], \tag{6.10}$$

provided we can show that the right-hand-side of (6.10) is a polynomial of degree at most $g-1$. Dividing (6.10) by $R_{2g+2}(z)^{1/2}$, using (4.19), (4.20) shows that our assertion is equivalent to

$$G(z,n,n,t)\hat G_{r+1}(z,n,t) - \hat F_r(z,n,t)[1-2a(n)G(z,n,n+1,t)] = O(z^{-2}). \tag{6.11}$$

Next we invoke (2.31) and (4.4) (identifying $f_\pm = \psi_\pm$) to rewrite (6.11) as

$$(\delta_n,[(H(t)-z)^{-1},\hat P_{2r+2}(t)]\delta_n) = -2(\delta_n,\hat P_{2r+2}(t)(H(t)-z)^{-1}\delta_n) = O(z^{-2}). \tag{6.12}$$

Using the Neumann expansion for the resolvent of $H(t)$ establishes the claim (skew-symmetry of $\hat P_{2r+2}(t)$ implies that the $O(z^{-1})$-term cancels). As a consequence of (6.12) we note

$$\dot f_j(t) = -\hat f_{r+j+1}(t) - \sum_{\ell=0}^{j}[\hat f_{r-\ell}(t)g_{\ell+j}(t) - f_{\ell+j}(t)\hat g_{r-\ell}(t)] + \hat f_{r+1}(t)f_j(t). \tag{6.13}$$

To obtain the time derivative of $G_{g+1}(z, n_0, t)$ we use

$$G_{g+1}(z, n_0, t)^2 - 4a(n_0, t)^2 F_g(z, n_0, t) F_g(z, n_0 + 1, t) = R_{2g+2}(z) \tag{6.14}$$

as in (3.8). Again, evaluating the t-derivative first at $\mu_j(n_0, t)$, one obtains

$$\frac{d}{dt} G_{g+1}(z, n_0, t) = 4a(n_0, t)^2 [F_g(z, n_0, t)\hat{F}_r(z, n_0 + 1, t) - \hat{F}_r(z, n_0, t) F_g(z, n_0 + 1, t)] \tag{6.15}$$

for those $t \in \mathbb{R}$ such that $\mu_j(n_0, t) \notin \{E_{2j-1}, E_{2j}\}$, provided the right-hand-side of (6.15) is of degree at most $g - 1$. Since the exceptional set is discrete the identity will then follow for all $t \in \mathbb{R}$ by continuity. One proceeds as before and attempts to prove

$$\begin{aligned} &[\hat{G}_{r+1}(z, n+1, t) + \hat{G}_{r+1}(z, n, t) + 2(b(n+1, t) + z)\hat{F}_r(z, n, t)] G(z, n, n+1, t) \\ &+ 2a(n)[\hat{F}_r(z, n, t) G(z, n+1, n+1, t) - \hat{F}_r(z, n+1, t) G(z, n, n, t)] = O(z^{-2}). \end{aligned} \tag{6.16}$$

The factor in front of $G(z, n, n+1, t)$ is a constant in z as can be seen from (2.36) and the rest can be rewritten as

$$(\delta_n, [(H(t) - z)^{-1}, \hat{P}_{2r+2}(t)]\delta_{n+1}). \tag{6.17}$$

Again using the Neumann series for the resolvent of $H(t)$ proves the assertion (6.16) and hence (6.15).

Similarly, evaluating $(d/dt)F_g(z, n_0 + 1, t)$ at $z = \mu_j(n_0 + 1, t)$ (the zeros of $F_g(z, n_0 + 1, t)$), we see that (6.10) also holds with n_0 replaced by $n_0 + 1$ and finally that $\{\hat{\mu}_j(n_0 + 1, t)\}_{1 \le j \le g}$ satisfies (6.7) with initial condition $\hat{\mu}_j(n_0 + 1, t_0) = \hat{\mu}_j^{(0)}(n_0 + 1)$. Proceeding inductively we obtain this results for all $n \ge n_0$ and with a similar calculation (cf. Section 3) for all $n \le n_0$.

Summarizing, we have constructed the set $\{\hat{\mu}_j(n, t)\}_{1 \le j \le g}$ for all $(n, t) \in \mathbb{Z} \times \mathbb{R}$ such that

$$\begin{aligned} &\frac{d}{dt}\mu_j(n, t) = -2\hat{F}_r(\mu_j(n, t), n, t) \frac{R_{2g+2}(\hat{\mu}_j(n, t))^{1/2}}{\prod_{\substack{\ell=1 \\ \ell \ne j}}^{g} [\mu_j(n, t) - \mu_\ell(n, t)]}, \\ &\hat{\mu}_j(n, t_0) = \hat{\mu}_j^{(0)}(n), \quad 1 \le j \le g, \ (n, t) \in \mathbb{Z} \times \mathbb{R}, \end{aligned} \tag{6.18}$$

with

$$\hat{\mu}_j(n, .) \in C^\infty(\mathbb{R}, K_g) \text{ and } \tilde{\pi}(\hat{\mu}_j(n, t)) \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le g, \ (n, t) \in \mathbb{Z} \times \mathbb{R}. \tag{6.19}$$

As expected from the stationary g-gap outset in (6.1)–(6.5), the left-hand-side in (6.18) is t-independent for $r = g$, assuming the same summation constants $c_\ell = c_\ell(E)$, $1 \leq \ell \leq g$ in $\hat{F}_r$ and F_g (cf. Lemma 4.3).

Furthermore, we have corresponding polynomials $F_g(z,n,t)$ and $G_{g+1}(z,n,t)$ satisfying

$$\frac{d}{dt}F_g(z,n,t) = 2[F_g(z,n,t)\hat{G}_{r+1}(z,n,t) - \hat{F}_r(z,n,t)G_{g+1}(z,n,t)], \tag{6.20}$$

$$\frac{d}{dt}G_{g+1}(z,n,t) = 4a(n,t)^2[F_g(z,n,t)\hat{F}_r(z,n+1,t) - \hat{F}_r(z,n,t)F_g(z,n+1,t)]. \tag{6.21}$$

In order to see that (6.7) was indeed the correct choice one needs to calculate $\dot{a}$ and $\dot{b}$. Differentiating (6.14) involving (6.20), (6.21) and (3.16) yields (2.36). Differentiating (3.16) using (6.20), (6.21) and (3.21) yields (2.37). Thus $\widehat{\mathrm{TL}}_r(a,b) = 0$ and

$$\frac{d}{dt}L(t) - [\hat{P}_{2r+2}(t), L(t)] = 0, \quad t \in \mathbb{R}. \tag{6.22}$$

In addition to the function $\phi(P,n,t)$

$$\begin{aligned}\phi(P,n,t) &= \frac{-G_{g+1}(\tilde{\pi}(P),n,t) + R_{2g+2}^{1/2}(P)}{2a(n,t)F_g(\tilde{\pi}(P),n,t)} \\ &= \frac{-2a(n,t)F_g(\tilde{\pi}(P),n+1,t)}{G_{g+1}(\tilde{\pi}(P),n,t) + R_{2g+2}^{1/2}(P)},\end{aligned} \tag{6.23}$$

we now define the time-dependent BA-function $\psi(P,n,n_0,t,t_0)$, meromorphic on
$K_g \backslash \{\infty_+, \infty_-\}$, by

$$\begin{aligned}\psi(P,n,n_0,t,t_0) = \exp\left\{\int_{t_0}^{t} ds[2a(n_0,s)\hat{F}_r(z,n_0,s)\phi(P,n_0,s) + \hat{G}_{r+1}(z,n_0,s)]\right\}\times \\ \times \begin{cases}\prod_{m=n_0}^{n-1}\phi(P,m,t), & n \geq n_0+1 \\ 1, & n = n_0 \\ \prod_{m=n}^{n_0-1}\phi(P,m,t)^{-1}, & n \leq n_0 - 1\end{cases}.\end{aligned} \tag{6.24}$$

Straightforward calculations then imply

$$a(n,t)\phi(P,n,t) + a(n-1,t)\phi(P,n-1,t)^{-1} = b(n,t) + \tilde{\pi}(P), \ (n,t) \in \mathbb{Z}\times\mathbb{R}, \tag{6.25}$$

$$\begin{aligned}\frac{d}{dt}\phi(P,n,t) = &-2a(n,t)[\hat{F}_r(\tilde{\pi}(P),n,t)\phi(P,n,t)^2 + \hat{F}_r(\tilde{\pi}(P),n+1,t)] \\ &+ 2[b(n+1,t) + \tilde{\pi}(P)]\hat{F}_r(\tilde{\pi}(P),n,t)\phi(P,n,t) \\ &+ [\hat{G}_{r+1}(\tilde{\pi}(P),n+1,t) - \hat{G}_{r+1}(\tilde{\pi}(P),n,t)]\phi(P,n,t), \quad (n,t) \in \mathbb{Z}\times\mathbb{R}\end{aligned} \tag{6.26}$$

and similarly,

$$\begin{aligned}&a(n,t)\psi(P,n+1,n_0,t,t_0)+a(n-1,t)\psi(P,n-1,n_0,t,t_0)\\&=[b(n,t)+\tilde\pi(P)]\psi(P,n,n_0,t,t_0),\quad (n,n_0,t,t_0)\in\mathbb{Z}^2\times\mathbb{R}^2,\end{aligned}\tag{6.27}$$

$$\begin{aligned}&\frac{d}{dt}\psi(P,n,n_0,t,t_0)=2a(t,n)\hat F_r(\tilde\pi(P),n,t)\psi(P,n+1,n_0,t,t_0)\\&+\hat G_{r+1}(\tilde\pi(P),n,t)\psi(P,n,n_0,t,t_0),\quad (n,n_0,t,t_0)\in\mathbb{Z}^2\times\mathbb{R}^2.\end{aligned}\tag{6.28}$$

The analogs of (3.8) (for all $n_0\in\mathbb{Z}$) and (3.28)-(3.31) then extend to the present t-dependent situation.

Using (variants of) Lagrange's interpolation formula and the trace formula (3.19) for b, the flow (6.18) is easily seen to be linearized (straightened out) by the Abel map (i.e., $\frac{d}{dt}\,\underline{\hat\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}(n,t)})=\underline{U}_r$ for some $\underline{U}_r\in\mathbb{R}^g$) for $r=0,1$. Since this approach gets fairly cumbersome with increasing r, we omit further details at this point and postpone a proof of this fact until later (see Theorem 6.2) when alternative and more effective tools are available.

In order to express $\phi(P,n,t)$ and $\psi(P,n,n_0,t,t_0)$ in terms of the theta function of K_g we need a bit of notation. Let $\omega^{(2)}_{\infty_\pm,q}$ be the normalized Abelian dsk (i.e., with vanishing a-periods) with a single pole at $\infty_\pm$ of the form

$$\omega^{(2)}_{\infty_\pm,q}=[\zeta^{-2-q}+O(1)]\,d\zeta\text{ near }\infty_\pm,\quad q\in\mathbb{N}_0.\tag{6.29}$$

Given the summation constants $c_1,\dots,c_r$ in $\hat F_r$, see (2.15), (2.19), (2.20), and (2.38), we then define

$$\Omega^{(2)}_r=\sum_{q=0}^{r}(q+1)c_{r-q}(\omega^{(2)}_{\infty_+,q}-\omega^{(2)}_{\infty_-,q}),\quad c_0=1.\tag{6.30}$$

Since the $\omega^{(2)}_{\infty_\pm,q}$ were supposed to be normalized we have

$$\int_{a_j}\Omega^{(2)}_r=0,\quad 1\le j\le g.\tag{6.31}$$

Moreover, writing

$$\omega_j=\left(\sum_{m=0}^{\infty}d_{j,m}(\infty_\pm)\zeta^m\right)d\zeta=\pm\left(\sum_{m=0}^{\infty}d_{j,m}(\infty_+)\zeta^m\right)d\zeta\text{ near }\infty_\pm,\tag{6.32}$$

relation (A.35) yields

$$U^{(2)}_{r,j}=\frac{1}{2\pi i}\int_{b_j}\Omega^{(2)}_r=2\sum_{q=0}^{r}c_{r-q}d_{j,q}(\infty_+),\quad 1\le j\le g.\tag{6.33}$$

We also will have to employ the following slight generalization of Lemma 5.4.

Lemma 6.1. *Let $\psi(.,n,t)$, $(n,t) \in \mathbb{Z}\times\mathbb{R}$ be meromorphic on $K_g \backslash \{\infty_+,\infty_-\}$ with essential singularities at $\infty_\pm$ such that $\tilde{\psi}(.,n,t)$ defined by*

$$\tilde{\psi}(P,n,t) = \psi(P,n,t)\exp\Big[-(t-t_0)\int_{P_0}^{P}\Omega_r^{(2)}\Big] \tag{6.34}$$

is multivalued meromorphic on K_g and its divisor satisfies

$$(\tilde{\psi}(.,n,t)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)} + (n-n_0)(\mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}). \tag{6.35}$$

Define a divisor $\mathcal{D}_0(n,t)$ by

$$(\tilde{\psi}(.,n,t)) = \mathcal{D}_0(n,t) - \mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)} + (n-n_0)(\mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}). \tag{6.36}$$

Then

$$\mathcal{D}_0(n,t) \in \sigma^g K_g,\ \mathcal{D}_0(n,t) > 0,\ \deg(\mathcal{D}_0(n,t)) = g. \tag{6.37}$$

Moreover, if $\mathcal{D}_0(n,t)$ is nonspecial for all $(n,t)\in\mathbb{Z}\times\mathbb{R}$, that is, if

$$i(\mathcal{D}_0(n,t)) = 0, \quad (n,t)\in\mathbb{Z}\times\mathbb{R} \tag{6.38}$$

then $\psi(.,n,t)$ is unique up to a constant multiple (which may depend on n and t).

Since the proof is analogous to that of Lemma 5.4 we omit further details.

Given these preparations we obtain the following characterization of $\phi(P,n,t)$ and $\psi(P,n,n_0,t,t_0)$ in (6.23) and (6.24).

Theorem 6.2. *Introduce*

$$\underline{z}(P,n,t) = \underline{\hat{A}}_{P_0}(P) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)}) + (n-n_0)\underline{U}^{(3)} + (t-t_0)\underline{U}_r^{(2)} - \underline{\hat{\Xi}}_{P_0}, \tag{6.39}$$

$$\underline{z}(n,t) = \underline{z}(\infty_+,n,t). \tag{6.40}$$

Then we have

$$\phi(P,n,t) = C(n,t)\frac{\theta(\underline{z}(P,n+1,t))}{\theta(\underline{z}(P,n,t))}\exp\Big(\int_{P_0}^{P}\omega_{\infty_+,\infty_-}^{(3)}\Big), \tag{6.41}$$

$$\begin{aligned}\psi(P,n,n_0,t,t_0) = {}& C(n,n_0,t,t_0)\frac{\theta(\underline{z}(P,n,t))}{\theta(\underline{z}(P,n_0,t_0))}\times \\ &\times\exp\Big[(n-n_0)\int_{P_0}^{P}\omega_{\infty_+,\infty_-}^{(3)} + (t-t_0)\int_{P_0}^{P}\Omega_r^{(2)}\Big],\end{aligned} \tag{6.42}$$

where $C(n,t)$, $C(n,n_0,t,t_0)$ are real-valued,

$$C(n,t) = C(n+1,n,t,t) = \left[\frac{\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n+1,t))}\right]^{1/2}, \tag{6.43}$$

$$C(n,n_0,t,t_0) = \left[\frac{\theta(\underline{z}(n_0,t_0))\theta(\underline{z}(n_0-1,t_0))}{\theta(\underline{z}(n,t))\theta(\underline{z}(n-1,t))}\right]^{1/2}, \tag{6.44}$$

and the sign of $C(n,t)$ is opposite that of $a(n,t)$, that is,

$$\operatorname{sgn}[C(n,t)] = -\operatorname{sgn}[a(n,t)], \quad (n,t) \in \mathbb{Z} \times \mathbb{R}. \tag{6.45}$$

Moreover,

$$\underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t)}) = \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)}) - (n-n_0)\underline{U}^{(3)} - (t-t_0)\underline{U}_r^{(2)} \mod (L_g) \tag{6.46}$$

and hence the flows (6.7) *are linearized by the Abel map*

$$\frac{d}{dt}\underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t)}) = -\underline{U}_r^{(2)}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}. \tag{6.47}$$

Proof. First of all note that (6.41) and (6.42) are well defined due to (3.43), (3.44), (6.31), (6.33), and (A.27).

Denoting the right-hand-side of (6.42) by $\Psi(P,n,n_0,t,t_0)$, our goal is to prove $\psi = \Psi$. By inspection, one verifies

$$\Psi(P,n,n_0,t,t_0) = \Psi(P,n_0,n_0,t,t_0)\Psi(P,n,n_0,t,t). \tag{6.48}$$

Comparison of (3.23), (3.32), (3.67)–(3.69) and (6.23), (6.41)–(6.44) then yields

$$\psi(P,n+1,n,t,t) = \phi(P,n,t) = \Psi(P,n+1,n,t,t). \tag{6.49}$$

Moreover,

$$\psi(P,n,n_0,t,t) = \begin{Bmatrix} \prod_{m=n_0}^{n-1} \phi(P,m,t), & n \geq n_0+1 \\ 1, & n = n_0 \\ \prod_{m=n}^{n_0-1} \phi(P,m,t)^{-1}, & n \leq n_0 - 1 \end{Bmatrix} = \Psi(P,n,n_0,t,t). \tag{6.50}$$

By (6.48) it remains to identify

$$\psi(P,n_0,n_0,t,t_0) = \Psi(P,n_0,n_0,t,t_0). \tag{6.51}$$

This is a bit more involved. We start by noting that (6.23), (6.24), and (6.20) imply

$$\psi(P,n_0,n_0,t,t_0) = \exp\left\{\int_{t_0}^{t} ds[2a(n_0,s)\hat{F}_r(z,n_0,s)\phi(P,n_0,s) + \hat{G}_{r+1}(z,n_0,s)]\right\}$$
$$= \exp\left\{\int_{t_0}^{t} ds\left[\hat{F}_r(z,n_0,s)\frac{R_{2g+2}(P)^{1/2} - G_{g+1}(z,n_0,s)}{F_g(z,n_0,s)} + \hat{G}_{r+1}(z,n_0,s)\right]\right\} \tag{6.52}$$

In order to spot the zeros and poles of ψ on $K_g\backslash\{\infty_+,\infty_-\}$ we need to expand the integrand in (6.52) near its singularities (the zeros $\mu_j(n_0,s)$ of $F_g(z,n_0,s)$). Using (6.18) one obtains

$$\psi(P,n_0,n_0,t,t_0) = \exp\left\{\int_{t_0}^{t} ds\left[\frac{\frac{d}{ds}\mu_j(n_0,s)}{\mu_j(n_0,s)-\tilde{\pi}(P)} + O(1)\right]\right\}$$
$$= \begin{cases} [\mu_j(n_0,t)-\tilde{\pi}(P)]O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t) \neq \hat{\mu}_j(n_0,t_0) \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t) = \hat{\mu}_j(n_0,t_0) \\ [\mu_j(n_0,t_0)-\tilde{\pi}(P)]^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t_0) \neq \hat{\mu}_j(n_0,t_0) \end{cases} \tag{6.53}$$

with $O(1) \neq 0$. Hence all zeros and all poles of $\psi(P,n_0,n_0,t,t_0)$ on $K_g\backslash\{\infty_+,\infty_-\}$ are simple and the poles coincide with those of $\Psi(P,n_0,n_0,t,t_0)$. Next we need to identify the essential singularities of $\psi(P,n_0,n_0,t,t_0)$ at $\infty_\pm$. For this purpose we use (6.25) and rewrite (6.52) in the form

$$\psi(P,n_0,n_0,t,t_0) = \exp\left\{\int_{t_0}^{t} ds\left[\frac{1}{2}\frac{\frac{d}{ds}F_g(z,n_0,s)}{F_g(z,n_0,s)} + R_{2g+2}(P)^{1/2}\frac{\hat{F}_r(z,n_0,s)}{F_g(z,n_0,s)}\right]\right\}$$
$$= \left[\frac{F_g(z,n_0,t)}{F_g(z,n_0,t_0)}\right]^{1/2} \exp\left\{R_{2g+2}(P)^{1/2}\int_{t_0}^{t} ds\frac{\hat{F}_r(z,n_0,s)}{F_g(z,n_0,s)}\right\}. \tag{6.54}$$

We claim that

$$R_{2g+2}(P)^{1/2}\hat{F}_r(z,n,t)/F_g(z,n,t) = \mp\sum_{q=0}^{r} c_{r-q}\zeta^{-q-2} + O(1) \text{ for } P \text{ near } \infty_\pm. \tag{6.55}$$

By (2.20), in order to prove (6.55), it suffices to prove the homogeneous case $c_0 = 1$, $c_q = 0$, $1 \le q \le r$. Using (4.19), we may rewrite (6.55) in the form

$$\begin{aligned}\tilde{F}_r(z,n,t)/z^{r+1} &= z^{-1}\sum_{q=0}^{r}\tilde{f}_{r,r-q}(n,t)z^{q-r} \\ &\underset{z\to\infty}{=} -G(z,n,n,t) + O(z^{-r-1}).\end{aligned} \tag{6.56}$$

Since

$$G(z,n,n,t) = (\delta_n,(H(t)-z)^{-1}\delta_n), \tag{6.57}$$

the Neumann expansion for $(H(t)-z)^{-1}$ then shows that (6.56) is equivalent to

$$z^{-1}\sum_{q=0}^{r}\tilde f_{r,r-q}(n,t)z^{q-r} \underset{z\to\infty}{=} z^{-1}\sum_{q=0}^{r}(\delta_n, H(t)^q\delta_n)z^{-q}+O(z^{-r-2}). \tag{6.58}$$

But (6.58) is proven in (2.21) of Lemma 2.1. Given (6.55) we can apply Lemma 6.1 to conclude (6.51) since $\mathcal{D}_{\underline{\hat\mu}(n,t)}$ is nonspecial for all $(n,t)\in\mathbb{Z}\times\mathbb{R}$ by (6.8) and Lemma A.2. This yields (6.41), (6.42), and (6.46). Equations (6.43)–(6.45) are then proved as in Theorem 3.5 (i). □

For connections between complete integrability and linearizations on the Jacobian of a curve under very general assumptions (i.e., for generalized Toda systems in the sense of Lie Algebras) we refer, for instance, to [3], [4].

Finally, we conclude with the θ-function representation for the t-dependent g-gap solutions of the Toda hierarchy.

Theorem 6.3. *The solutions* $\{a(n,t)\}_{(n,t)\in\mathbb{Z}\times\mathbb{R}}$, $\{b(n,t)\}_{(n,t)\in\mathbb{Z}\times\mathbb{R}}$ *of the* $\widehat{\mathrm{TL}}_r$ *equations* (6.6) *with* g*-gap initial conditions* $\{a^{(0)}(n)\}_{n\in\mathbb{Z}}$, $\{b^{(0)}(n)\}_{n\in\mathbb{Z}}$ *in* (6.1)–(6.4) *are given by*

$$\begin{aligned} a(n,t)^2 &= \frac{1}{2}\sum_{j=1}^{g} R_{2g+2}(\hat\mu_j(n,t))^{1/2}\prod_{\substack{k=1\\k\neq j}}^{g}[\mu_j(n,t)-\mu_k(n,t)]^{-1} \\ &\quad -\frac{1}{4}[b(n,t)+b^{(2)}(n,t)]>0, \end{aligned} \tag{6.59}$$

$$\begin{aligned} b^{(k)}(n,t) &= \sum_{j=1}^{g}\mu_j(n,t)^k-\frac{1}{2}\sum_{m=0}^{2g+1}E_m^k, \quad k\in\mathbb{N}, \\ b(n,t)=b^{(1)}(n,t) &= \sum_{j=1}^{g}\mu_j(n,t)-\frac{1}{2}\sum_{m=0}^{2g+1}E_m, \end{aligned} \tag{6.60}$$

where $\{\hat\mu_j(n,t)\}_{1\le j\le g}$ *solves* (6.7). *Moreover, we have*

$$a(n,t)=\tilde a[\theta(\underline{z}(n-1,t))\theta(\underline{z}(n+1,t))/\theta(\underline{z}(n,t))^2]^{1/2}, \quad (n,t)\in\mathbb{Z}\times\mathbb{R}, \tag{6.61}$$

$$b(n,t)=\sum_{j=1}^{g}\lambda_j-\frac{1}{2}\sum_{m=0}^{2g+1}E_m-\sum_{j=1}^{g}c_j(g)\frac{\partial}{\partial w_j}\ln\left[\frac{\theta(\underline{w}+\underline{z}(n,t))}{\theta(\underline{w}+\underline{z}(n-1,t))}\right]\Bigg|_{\underline{w}=0}, \quad (n,t)\in\mathbb{Z}\times\mathbb{R}, \tag{6.62}$$

with $\tilde a<0$ *introduced in* (5.1), (5.2).

Proof. Equations (6.59) and (6.60) are obtained in precisely the same way as (3.18) and (3.19) taking into account (6.23), (6.25), (6.26), and

(6.14) (for $n_0 \in \mathbb{Z}$). Expressions (6.61) and (6.62) then follow as in Theorem 5.2. $\square$

Remark 6.4. (i). Since in the special case $r = 0$, that is, for the original Toda lattice equations, $\underline{U}_0^{(2)}$ simplifies to

$$\underline{U}_0^{(2)} = 2\underline{c}(g) \tag{6.63}$$

due to (6.32), (6.33), and (A.24), the expression for $b(n,t)$ in (6.62) can be rewritten in the familiar form

$$b(n,t) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \frac{1}{2}\frac{d}{dt}\ln\left[\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\right], \quad (n,t) \in \mathbb{Z}\times\mathbb{R},\ r = 0. \tag{6.64}$$

(ii). Furthermore, expanding equation (6.28) around $\infty_\pm$ (still for $r = 0$) shows that

$$\int_{P_0}^{P} \Omega_0^{(2)} = \mp\left[\frac{1}{\zeta} + \tilde{b}_1 + O(\zeta)\right] \quad \text{near } \infty_\pm, \quad r = 0, \tag{6.65}$$

where $\tilde{b}_1$ is defined in (5.3). Conversely, proving (6.28) as in the KdV case (by expanding both sides in (6.28) around $\infty_\pm$ and using Lemma 6.1) turns out to be equivalent to proving (6.65). However, since we are not aware of an independent proof of (6.65), we chose a different strategy in the proof of Theorem 6.2.

Moreover, in analogy to Corollary 5.6, $b(n,t)$ admits the alternative θ-function representation which, to the best of our knowledge, has thus far not been noted in the literature.

Corollary 6.5. *$b(n,t)$ admits the representation*

$$\begin{aligned} b(n,t) = &- E_0 + \tilde{a}\frac{\theta(\underline{z}(n-1,t))\theta(\underline{z}(P_0,n+1,t))}{\theta(\underline{z}(n,t))\theta(\underline{z}(P_0,n,t))} \\ &+ \tilde{a}\frac{\theta(\underline{z}(n,t))\theta(\underline{z}(P_0,n-1,t))}{\theta(\underline{z}(n-1,t))\theta(\underline{z}(P_0(n,t))}, \quad (n,t) \in \mathbb{Z}\times\mathbb{R}. \end{aligned} \tag{6.66}$$

Since the proof is identical to that of Corollary 5.6 we omit further details.

The θ-function representation of the BA-function $\psi(.,n,n_0,t,t_0)$ in (6.42) (except for a determination of $C(n,n_0,t,t_0)$, a minor point) can be found, for instance, in [2], [25], [26], [58] in the special case, $r = 0$, that is, for the original Toda lattice. Similarly, the θ-function representations for (a,b) in (6.61), (6.62) have first been derived by Krichever [56], again in the case $r = 0$. This result is reproduced (assuming periodicity of (a,b) with respect to n) with more details, for instance, in

[25], [26], [58], and, by somewhat different methods, derived in [2]. The periodic case was originally treated by Date and Tanaka [19] (and is reproduced with more details in [84], Ch. 4).

In accordance with the modern τ-function formulation of completely integrable systems (see, e.g., [11], [81], [82], [85] and the references therein), the results of this section clearly illustrate the possibility of simultaneously treating the entire Toda hierarchy by introducing infinitely many time variables $\underline{\underline{t}} = (t_0, t_1, t_2, \dots)$ and hence $a(n, \underline{\underline{t}})$, $b(n, \underline{\underline{t}})$, $\theta(n, \underline{\underline{t}}))$, etc., where the r-th coordinate t_r in $\underline{\underline{t}}$ is associated with the homogeneous TL_r system.

7. The Kac-van Moerbeke Hierarchy and its Relation to the Toda Hierarchy

This section is devoted to the Kac-van Moerbeke hierarchy and its connection with the Toda hierarchy. Using a commutation (sometimes also called a supersymmetric) approach one can show that the Kac-van Moerbeke (KM) hierarchy is a modified Toda hierarchy precisely in the way that the modified Korteweg-de Vries (mKdV) hierarchy is related to the Korteweg-de vries (KdV) hierarchy or more generally, the Drinfeld-Sokolov (DS) hierarchy is a modified version of the Gel'fand-Dickey (GD) hierarchy. The connection between these hierarchies and their modified counterparts is based on (suitable generalizations of) Miura-type transformations which in turn are based on factorizations of differential (respectively difference) expressions. The literature on this subject is too extensive to be quoted here in full. The interested reader can consult, for instance, [36], [37], [40], [41], [42], [84], Ch. 3, [87] and the references therein. In the present case of the Toda and modified Toda, respectively Kac-van Moerbeke hierarchies, the (discrete) analogy of Miura's transformation in connection with factorization methods was first systematically employed by Adler [1] and further developed in [39]. In particular, the approach presented in this section is essentially modeled after [39] where further details on the TL and KM system can be found. For an alternative approach to the modified Toda hierarchy we refer to [60], Ch. 4.

Let

$$\rho(t) = \{\rho(n,t)\}_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{R}}(\mathbb{Z}), \quad 0 \neq \rho(n,.) \in C^1(\mathbb{R}),\ n \in \mathbb{Z} \tag{7.1}$$

and define the "even" and "odd" parts of $\rho(t)$ by

$$\rho_e(n,t) = \rho(2n,t),\ \rho_o(n,t) = \rho(2n+1,t), \quad (n,t) \in \mathbb{Z}\times\mathbb{R}. \tag{7.2}$$

Next we consider the difference expressions in $\ell^\infty(\mathbb{Z})$ (respectively the bounded operators in $\ell^2(\mathbb{Z})$)

$$A(t) = \rho_o(t)S^+ + \rho_e(t), \ A(t)^* = \rho_o^-(t)S^- + \rho_e(t), \tag{7.3}$$

which enable us to define matrix-valued difference expressions $(M(t), Q_{2g+2}(t))$ (the Lax pair) in $\ell^\infty(\mathbb{Z}) \otimes \mathbb{C}^2$ as follows,

$$M(t) = \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \quad t \in \mathbb{R}, \tag{7.4}$$

$$Q_{2g+2}(t) = \begin{pmatrix} P_{1,2g+2}(t) & 0 \\ 0 & P_{2,2g+2}(t) \end{pmatrix} = P_{1,2g+2}(t) \oplus P_{2,2g+2}(t), \quad g \in \mathbb{N}_0, \ t \in \mathbb{R}. \tag{7.5}$$

Here $P_{k,2g+2}(t)$, $k = 1, 2$ are defined as in (2.10) respectively (2.31), that is,

$$P_{k,2g+2}(t) = -L_k(t)^{g+1} + \sum_{j=0}^{g} [g_{k,j}(t) + a_k(t) f_{k,j}(t) S^+] L_k(t)^j + f_{k,g+1}, \tag{7.6}$$

$$P_{k,2g+2}(t)\Big|_{\mathrm{Ker}(L_k(t)-z)} = 2a_k(t) F_{k,g}(z,t) S^+ + G_{k,g+1}(z,t), \quad k = 1, 2 \tag{7.7}$$

and $\{f_{k,g,j}(n,t)\}_{0\le j\le g}$, $\{g_{k,g+1,j}(n,t)\}_{0\le j\le g+1}$ and $F_{k,g}(z,t)$, $G_{k,g+1}(z,t)$ are defined as in (2.11) and (2.32) with

$$a_1(t) = \rho_e(t)\rho_o(t), \qquad b_1(t) = -\rho_e(t)^2 - \rho_o^-(t)^2, \tag{7.8}$$

$$a_2(t) = \rho_e^+(t)\rho_o(t), \qquad b_2(t) = -\rho_e(t)^2 - \rho_o(t)^2, \tag{7.9}$$

$$L_k(t) = a_k(t)S^+ + a_k^-(t)S^- - b_k(t), \qquad k = 1, 2. \tag{7.10}$$

One verifies the factorization,

$$L_1(t) = A(t)^* A(t), \quad L_2(t) = A(t) A(t)^*. \tag{7.11}$$

The corresponding Lax equation for the KM system then reads

$$\frac{d}{dt} M(t) - [Q_{2g+2}(t), M(t)] = 0, \quad t \in \mathbb{R} \tag{7.12}$$

and as in the Toda context (2.12), varying $g \in \mathbb{N}_0$ yields the KM hierarchy which we denote by

$$\mathrm{KM}_g(\rho) = 0, \quad g \in \mathbb{N}_0. \tag{7.13}$$

As in Section 2 (cf. (2.19)) we use the symbol "~" to distinguish between inhomogeneous and homogeneous KM equations, that is,

$$\widetilde{\mathrm{KM}}_g(\rho) := \mathrm{KM}_g(\rho)\Big|_{c_\ell \equiv 0, \ 1\le \ell \le g}, \tag{7.14}$$

with c_ℓ the summation constants of Section 2. In order to obtain explicit expressions for the KM equations (7.13) we proceed as follows. First we note that

$$\begin{aligned}\operatorname{Ker}(M(t)-w) &= \operatorname{Ker}(M(t)^2-w^2)\\ &= \operatorname{Ker}(L_1(t)-z)\oplus\operatorname{Ker}(L_2(t)-z), \quad w^2=z.\end{aligned} \tag{7.15}$$

(We shall only use (7.15) in an algebraic sense as in (2.31). However, using the methods in the proof of Theorems 2.1 and 2.3 of [41], (7.15) is easily seen to be valid in a functional analytic sense as well.) Since

$$\dot M - [Q_{2g+2}, M] = \begin{pmatrix} 0 & \dot A^* - P_{1,2g+2}A^* + A^*P_{2,2g+2} \\ \dot A - P_{2,2g+2}A + AP_{1,2g+2} & 0 \end{pmatrix}, \tag{7.16}$$

relations (7.15) and (7.3) yield after some computations,

$$\begin{aligned}\dot\rho_e &= 2\rho_e\rho_o^2(F_{1,g}^+ - F_{2,g}) - \rho_e(G_{1,g+1}-G_{2,g+1}),\\ \dot\rho_o &= -2\rho_o(\rho_e^+)^2(F_{1,g}^+ - F_{2,g}^+) + \rho_o(G_{1,g+1}-G_{2,g+1}), \quad g\in\mathbb{N}_0.\end{aligned} \tag{7.17}$$

(Here "$\cdot$" $= d/dt$.) Given $F_{k,g}$, $G_{k,g+1}$ from Section 2 with (a_k, b_k), $k=1,2$ defined in (7.8), (7.9), equations (7.17) yield the hierarchy of KM equations. In particular, introducing

$$\begin{aligned}\underline{\mathrm{KM}}_g(\rho) &= (\mathrm{KM}_g(\rho)_e,\ \mathrm{KM}_g(\rho)_o)^T\\ &:= \begin{pmatrix} \dot\rho_e - 2\rho_e\rho_o^2(F_{1,g}-F_{2,g}) + \rho_e(G_{1,g+1}-G_{2,g+1}) \\ \dot\rho_o - 2\rho_o(\rho_e^+)^2(F_{1,g}^+ - F_{2,g}^+) - \rho_o(G_{1,g+1}^+ - G_{2,g+1}) \end{pmatrix}\end{aligned} \tag{7.18}$$

one obtains the KM equations (7.13) by taking into account (7.2) in (7.18). Explicitly, one infers from (2.38), (7.2), (7.13), and (7.18),

$$\mathrm{KM}_0(\rho) = \dot\rho - \rho[(\rho^+)^2 - (\rho^-)^2] = 0, \tag{7.19}$$

$$\begin{aligned}\mathrm{KM}_1(\rho) = \dot\rho - \rho[&(\rho^+)^4 - (\rho^-)^4 + (\rho^{++})^2(\rho^+)^2 + (\rho^+)^2\rho^2 - \rho^2(\rho^-)^2\\ &- (\rho^-)^2(\rho^{--})^2] + c_1(-\rho)[(\rho^+)^2 - (\rho^-)^2 = 0,\\ &\text{etc.}\end{aligned} \tag{7.20}$$

Remark 7.1. In analogy to Remark 2.3 one infers that ρ_e and ρ_o enter F_g, G_{g+1} quadratically so that the KM hierarchy (7.13) is invariant under the substitution

$$\rho(t) \to \rho_\epsilon(t) = \{\epsilon(n)\rho(n,t)\}_{n\in\mathbb{Z}}, \quad \epsilon(n)\in\{+1,-1\},\ n\in\mathbb{Z}. \tag{7.21}$$

This result should be compared with Lemma 8.1.

The Miura-type relation between the TL and KM hierarchies, alluded to at the beginning of this section, is now obtain as follows.

The connection between $P_{k,2g+2}(t)$, $k=1,2$ and $Q_{2g+2}(t)$ is clear from (7.5), the corresponding connection between $L_k(t)$, $k=1,2$ and $M(t)$ is provided by the elementary observation

$$M(t)^2 = \begin{pmatrix} A(t)^*A(t) & 0 \\ 0 & A(t)A(t)^* \end{pmatrix} = L_1(t) \oplus L_2(t), \quad t \in \mathbb{R}. \tag{7.22}$$

Moreover, recalling the notation employed in (2.13), (2.14), that is,

$$\mathrm{TL}_g(a,b) = (\mathrm{TL}_g(a,b)_1, \mathrm{TL}_g(a,b)_2)^T, \tag{7.23}$$

one can verify that

$$\mathrm{TL}_g(a_k,b_k) = W_k \underline{\mathrm{KM}}_g(\rho), \quad k=1,2, \tag{7.24}$$

where $W_k(t)$ denote the matrix-valued difference expressions

$$W_1(t) = \begin{pmatrix} \rho_o(t) & \rho_e(t) \\ 2\rho_e(t) & 2\rho_o^-(t)S^- \end{pmatrix}, \; W_2(t) = \begin{pmatrix} \rho_o(t)S^+ & \rho_e^+(t) \\ 2\rho_e(t) & 2\rho_o(t) \end{pmatrix}. \tag{7.25}$$

Relation (7.24) is the analog of Miura's identity [67] between the KdV and mKdV hierarchy (which extends to GD and DS systems, see, e.g., [36], [37], [40], [41], [42] and the references therein). In particular, as systematically studied by Adler [1], the identity (7.24) yields the implication

$$\mathrm{KM}_g(\rho) = 0 \Rightarrow \mathrm{TL}_g(a_k,b_k) = 0, \quad k=1,2 \tag{7.26}$$

(since $\mathrm{KM}_g(\rho)=0 \Leftrightarrow \underline{\mathrm{KM}}_g(\rho)=0$), that is, given a solution ρ of the KM_g equation (7.13) (respectively (7.17)), one obtains two solutions, (a_1,b_1) and (a_2,b_2), of the TL_g equations (2.13) related to each other by the Miura-type transformations (7.8), (7.9). As mentioned briefly in the Introduction, a connection between the KM and Toda systems was known to Hénon in 1973. Moreover, transformations between KM and Toda lattices were investigated in some detail by Wadati [87] (see also [84]). Bäcklund transformations for the Toda hierarchy and connections with the KM system based on factorization techniques were also studied by Knill [53], [54].

The main result in [39] describes a method to reverse the implication in (7.26), that is, starting from a solution, say (a_1,b_1) of the TL_g equation (2.13), one constructs a solution ρ of the KM_g equation (7.17) and another TL_g solution (a_2,b_2) of (2.13) related to each other by the Miura-type transformation (7.8), (7.9). We now recall this construction.

Theorem 7.2. [39] *Assume (a_1, b_1) satisfies (2.8), $a_1(n,t) < 0$, $b_1(n,t) < 0$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$, and $\widehat{\mathrm{TL}}_r(a_1, b_1) = 0$ for some $f \in \mathbb{N}_0$. Suppose the associated self-adjoint realization $H_1(t)$ in $\ell^2(\mathbb{Z})$ of $L_1(t) = a_1(t)S^+ + a_1^-(t)S^- - b_1(t)$ is nonnegative for some (and hence for all) $t \in \mathbb{R}$, $H_1(t) \geq 0$, and $0 < \psi_{1,\pm}(n,.) \in C^1(\mathbb{R})$, $n \in \mathbb{Z}$ are positive weak solutions of*

$$H_1(t)\psi_{1,\pm}(t) = 0, \quad \dot{\psi}_{1,\pm}(t) = \hat{P}_{1,2r+2}(t)\psi_{1,\pm}(t), \quad t \in \mathbb{R}. \tag{7.27}$$

Define for $(n,t) \in \mathbb{Z} \times \mathbb{R}$,

$$\psi_{1,\sigma}(n,t) = \frac{1-\sigma(t)}{2}\psi_{1,-}(n,t) + \frac{1+\sigma(t)}{2}\psi_{1,+}(n,t), \tag{7.28}$$

$$\rho_{e,\sigma}(n,t) = -[-a_1(n,t)\psi_{1,\sigma}(n+1,t)/\psi_{1,\sigma}(n,t)]^{1/2}, \tag{7.29}$$

$$\rho_{o,\sigma}(n,t) = [-a_1(n,t)\psi_{1,\sigma}(n,t)/\psi_{1,\sigma}(n+1,t)]^{1/2}, \tag{7.30}$$

$$\rho_\sigma(n,t) = \begin{cases} \rho_{e,\sigma}(m,t), & n = 2m \\ \rho_{o,\sigma}(m,t), & n = 2m+1 \end{cases}, \tag{7.31}$$

$$a_{2,\sigma}(n,t) = \rho_{e,\sigma}(n+1,t)\rho_{o,\sigma}(n,t), \tag{7.32}$$

$$b_{2,\sigma}(n,t) = -\rho_{e,\sigma}(n,t)^2 - \rho_{o,\sigma}(n,t)^2, \tag{7.33}$$

where $\sigma : \mathbb{R} \to [-1,1]$, $\sigma \in C^1(\mathbb{R})$. Then $\rho_\sigma(t)$, $a_{2,\sigma}(t)$, $b_{2,\sigma}(t) \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $t \in \mathbb{R}$, $\rho_\sigma(n,t) \neq 0$, $a_{2,\sigma}(n,t) < 0$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$ and

$$\begin{aligned} &\widehat{\mathrm{KM}}_r(\rho_\sigma) = 0, \quad \widehat{\mathrm{TL}}_r(a_{2,\sigma}, b_{2,\sigma}) = 0 \\ &\textit{if and only if } \dot{\sigma} = 0 \textit{ or } W(\psi_{1,-}, \psi_{1,+}) = 0. \end{aligned} \tag{7.34}$$

The proof of Theorem 7.2, according to [39], can be reduced to identities of the form

$$\widehat{\mathrm{KM}}_r(\rho_\sigma)(n,t) = -\frac{1}{4}\dot{\sigma}(t)\rho_{e,\sigma}(m,t)^{-1}\psi_{1,\sigma}(m,t)^{-2}W(\psi_{1,-}, \psi_{1,+}), \quad t \in \mathbb{R}, \ n = 2m, \tag{7.35}$$

$$\widehat{\mathrm{KM}}_r(\rho_\sigma)(n,t) = \frac{1}{4}\dot{\sigma}(t)\rho_{o,\sigma}(m,t)^{-1}\psi_{1,\sigma}(m+1,t)^{-2}W(\psi_{1,-}, \psi_{1,+}), \quad t \in \mathbb{R}, \ n = 2m+1. \tag{7.36}$$

and similarly,

$$\widehat{\mathrm{TL}}_r(a_{2,\sigma}, b_{2,\sigma})(n,t) = \frac{1}{4}\dot\sigma(t) W(\psi_{1,-},\psi_{1,+}) \times$$
$$\times \begin{pmatrix} \psi_{1,\sigma}(n+1,t)^{-2}\left[-\dfrac{\rho_{o,\sigma}(n,t)}{\rho_{e,\sigma}(n+1,t)} + \dfrac{\rho_{e,\sigma}(n+1,t)}{\rho_{o,\sigma}(n,t)}\right] \\ 2[\psi_{1,\sigma}(n,t)^{-2} - \psi_{1,\sigma}(n+1,t)^{-2}] \end{pmatrix}, \quad (n,t) \in \mathbb{Z}\times\mathbb{R}. \tag{7.37}$$

(To be precise, Theorem 7.2 is proved in [39] in the case $r = 0$, i.e., for the original Toda and KM system. However, as shown in [36] and [40] in the case of the (m)KdV and GD, DS contexts, the proof directly extends to the entire hierarchy, $r \in \mathbb{N}_0$.)

Since the cases $\sigma = \pm 1$ with $\psi_{1,\pm}$ the branches of the BA-function associated with the finite-gap operator L_1 will be the most important ones for us in the following, we shall identify $\psi_{1,\pm 1} = \psi_{1,\pm}$, $\rho_{\pm 1} = \rho_\pm$, $a_{2,\pm 1} = a_{2,\pm}$, $b_{2,\pm 1} = b_{2,\pm}$, $L_{2,\pm} = L_{2,\pm}$, $H_{2,\pm 1} = H_{2,\pm}$, etc. for notational convenience throughout the rest of this paper. Moreover, in the special case where H_1 is critical (cf. Remark 7.3 (ii) below) and hence $\psi_{1,+} = \psi_{1,-} \equiv \psi_{1,0}$, we shall write, ρ_o, $a_{2,0}$, $b_{2,0}$, $L_{2,0}$, $H_{2,0}$, etc.

Theorem 7.2 has been used in [39] to derive the soliton solutions for the KM system given the corresponding solitons for the Toda system. In our final Section 9 we shall use Theorem 7.2 to derive the main objective of this paper, viz., the algebro-geometric quasi-periodic finite-gap solutions of the KM hierarchy, given the corresponding results of Sections 5 and 6 for the Toda hierarchy.

We conclude this section with a series of remarks further illustrating Theorem 7.2.

Remark 7.3. (i). Due to the Lax equation (2.12), $H_1(t)$ is well-known to be unitarily equivalent to $H_1(0)$ for all $t \in \mathbb{R}$. (Since $H_1(t)$ are bounded self-adjoint operators strongly continuous with respect to $t \in \mathbb{R}$, the Dyson expansion of the associated unitary propagator converges in the uniform operator topology, see, e.g., Theorem X.69 in [78].) Moreover, the Wronskian $W(\psi_{1,-},\psi_{1,+})$ is independent of $(n,t) \in \mathbb{Z}\times\mathbb{R}$. The existence of positive solutions $\psi_{1,\pm}$ satisfying (7.27) has been investigated in [39] and [43]. In our application of Theorem 7.2 in Section 9, $\psi_{1,\pm}$ will be the branches of the BA-function $\psi(P)$ and positivity of $\psi_{1,\pm}$ can be verified directly.
(ii). Depending on whether or not $H_1(0)$ (and hence $H_1(t)$ for all $t \in \mathbb{R}$, see [43]) is critical or subcritical, that is, whether or not $H_1(0)\psi = 0$ has a unique positive solution (up to constant multiples) or two linearly independent positive solutions, Theorem 7.2 yields a unique solution, ρ_0 (respectively $a_{2,0}$, $b_{2,0}$) or a one-parameter family ρ_σ (respectively

$a_{2,\sigma}$, $b_{2,\sigma}$) indexed by $\sigma \in [-1,1]$ (σ being t-independent) of $\widehat{\mathrm{KM}}_r$ (respectively $\widehat{\mathrm{TL}}_r$) solutions. Since $H_1(0) \geq 0$ implies the existence of at least one weak positive solution ψ of $H_1(0)\psi = 0$ (cf., e.g., [43] and the references therein) this case distinction is exhaustive. In addition, (a_1, b_1) and ρ_σ, $(a_{2,\sigma}, b_{2,\sigma})$ are all related by the Miura-type transformation (7.8), (7.9).
(iii). By Remark 2.3 and Lemma 4.1 we assumed $a_1 < 0$ without loss of generality in Theorem 7.2. The existence of weak solutions $\psi_{1,\pm} > 0$ satisfying $H_1\psi_{1,\pm} = 0$ then necessarily yields $b_1 < 0$.
(iv). If (a_1, b_1) are periodic (respectively quasi-periodic finite-gap in the sense of Section 6) then ρ_σ and $(a_{2,\sigma}, b_{2,\sigma})$ are periodic (respectively quasi-periodic finite-gap) if and only if $\sigma = \pm 1$ (or if $H_1(0)$ is critical). This will be the case in our final Section 9 where we construct the algebro-geometric quasi-periodic finite-gap solutions of the KM hierarchy.

In the special case of the stationary KM hierarchy characterized by $\dot\rho = 0$ in (7.13) (respectively (7.17)), or more precisely, by commuting matrix difference expressions of the type

$$[Q_{2g+2}, M] = 0, \tag{7.38}$$

the analogs of (2.46) and (2.47) then read

$$Q_{2g+2}^2 = \prod_{m=0}^{2g+1} (M - E_m^{1/2})(M + E_m^{1/2}) = \prod_{m=0}^{2g+1} (M^2 - E_m) \tag{7.39}$$

and

$$y^2 = \prod_{m=0}^{2g+1} (w - E_m^{1/2})(w + E_m^{1/2}) = \prod_{m=0}^{2g+1} (w^2 - E_m). \tag{7.40}$$

We note that the curve (7.40) becomes singular if and only if $E_m = 0$ for some $0 \leq m \leq 2g+1$. In the self-adjoint case where $0 \leq E_0 < E_1 < \cdots < E_{2g+1}$ this happens if and only if $E_0 = 0$ (i.e., if and only if H_1 and hence H_2 are critical).

8. Spectral Theory for Finite-Gap Dirac-Type Difference Operators

In this section we briefly study spectral properties of self-adjoint $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ realizations associated with finite-gap Dirac-type difference expressions.

Assuming $\rho \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $\rho(n) \neq 0$, $n \in \mathbb{Z}$ we start by introducing the general matrix-valued difference expression M by

$$M = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad A = \rho_o S^+ + \rho_e, \quad A^* = \rho_o^- S^- + \rho_e,$$
$$\rho_e(n) = \rho(2n), \quad \rho_o(n) = \rho(2n+1), \quad n \in \mathbb{Z}. \tag{8.1}$$

We denote by D the unique self-adjoint realization associated with M in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$,

$$Df = Mf, \quad f \in \mathcal{D}(D) = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2. \tag{8.2}$$

The analog of Lemma 4.1 then reads

Lemma 8.1. *Let $\rho \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $\rho(n) \neq 0$, $n \in \mathbb{Z}$ and introduce $\rho_\epsilon \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$ by*

$$\rho_\epsilon = \{\epsilon(n)\rho(n)\}_{n\in\mathbb{Z}}, \ \epsilon(n) \in \{+1, -1\}, \quad n \in \mathbb{Z}. \tag{8.3}$$

Define D_ϵ in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ as in (8.2) with M replaced by $M_\epsilon = \left(\begin{smallmatrix} 0 & A_\epsilon^* \\ A_\epsilon & 0 \end{smallmatrix}\right)$ and ρ by ρ_ϵ. Then D and D_ϵ are unitarily equivalent, that is, there exists a unitary operator U_ϵ in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ such that

$$D = U_\epsilon D_\epsilon U_\epsilon^{-1}. \tag{8.4}$$

Proof. U_ϵ is explicitly represented by

$$U_\epsilon = \begin{pmatrix} U_{1,\epsilon} & 0 \\ 0 & U_{2,\epsilon} \end{pmatrix}, \ U_{k,\epsilon} = (\tilde\epsilon_k(n)\delta_{m,n})_{m,n\in\mathbb{Z}}, \quad k = 1, 2, \tag{8.5}$$

$\tilde\epsilon_1(n+1)\tilde\epsilon_2(n) = \epsilon(2n+1)$, $\tilde\epsilon_1(n)\tilde\epsilon_2(n) = \epsilon(2n)$, $\tilde\epsilon_1(n)\tilde\epsilon_2(n-1) = \epsilon(2n-1)$, $n \in \mathbb{Z}$. ☐

Next, we summarize the spectral properties of H_1, $H_{2,\sigma}$ and D_σ. Since in this paper we are interested in quasi-periodic finite-gap operators we shall restrict ourselves to the case $\sigma = \pm 1$ (this includes the case where H_1 (and hence $H_{2,\sigma}$) is critical and hence $D_+ = D_- \equiv D_0$ as a limiting case). To fix our notation assume $a_1, b_1 \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $a_1(n) < 0$, $b_1(n) < 0$, $n \in \mathbb{Z}$ and define L_1, H_1 as in (4.1), (4.2). Assuming $H_1 \geq 0$ let $\psi_1 > 0$ be a weak solution of $L_1\psi_1 = 0$ and define

$$\rho_e(n) = -[-a_1(n)\psi_1(n+1)/\psi_1(n)]^{1/2}, \tag{8.6}$$
$$\rho_o(n) = [-a_1(n)\psi_1(n)/\psi_1(n+1)]^{1/2}, \tag{8.7}$$
$$\rho(n) = \begin{cases} \rho_e(m), & n = 2m \\ \rho_o(m), & n = 2m+1 \end{cases}, \tag{8.8}$$
$$a_2(n) = \rho_e(n+1)\rho_o(n), \tag{8.9}$$
$$b_2(n) = -\rho_e(n)^2 - \rho_o(n)^2. \tag{8.10}$$

Given (8.6)–(8.10) one defines L_2, H_2, A, M, and D as in (4.1), (4.2), (7.3), (8.1), and (8.2).

Theorem 8.2. *Suppose a_1, $b_1 \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$ are g-gap sequences satisfying (3.18), (3.19), and $a_1(n) < 0$, $b_1(n) < 0$, $n \in \mathbb{Z}$. Define L_1, H_1 as in (4.1), (4.2), suppose $H_1 \geq 0$, and let $\psi_{1,\pm}(n) = \psi_{1,\pm}(0, n, n_0)$ be the branches of the stationary BA-function $\psi_1(Q_0, n, n_0)$, $Q_0 = (0, R_{2g+2}(0)^{1/2})$ of L_1 in (3.23) (respectively (3.68)). Then*

$$\psi_{1,\pm}(n) > 0, \quad n \in \mathbb{Z} \tag{8.11}$$

and we may define $\rho_{e,\pm}$, $\rho_{o,\pm}$, $\rho_\pm$, $a_{2,\pm}$, $b_{2,\pm}$, $L_{2,\pm}$, $H_{2,\pm}$, $A_\pm$, $M_\pm$, and $D_\pm$ as in (8.6)–(8.10), (4.1), (4.2), (7.3), (8.1), and (8.2). Then $a_{2,\pm}$, $b_{2,\pm}$, $\rho_\pm \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $a_{2\pm}(n) < 0$, $\rho_\pm(n) \neq 0$, $n \in \mathbb{Z}$ and

$$\sigma(H_1) = \sigma(H_{2,\pm}) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \quad E_0 \geq 0, \tag{8.12}$$

$$\begin{aligned} \sigma(D_\pm) &= \bigcup_{\substack{j=-g-1 \\ j\neq 0}}^{g+1} \sum\nolimits_j , \\ \sum\nolimits_j &= [E^{1/2}_{2(j-1)}, E^{1/2}_{2j-1}],\ \sum\nolimits_{-j} = -\sum\nolimits_j , \quad 1 \leq j \leq g+1, \end{aligned} \tag{8.13}$$

$$\sigma_{sc}(H_1) = \sigma_{sc}(H_{2,\pm}) = \sigma_{sc}(D_\pm) = \sigma_p(H_1) = \sigma_p(H_{2,\pm}) = \sigma_p(D_\pm) = \emptyset. \tag{8.14}$$

In addition, H_1, $H_{2,\pm}$, and $D_\pm$ all have uniform spectral multiplicity two.

Proof. By Lemma 3.4 (i) and (iv), $\psi_1(P, n, n_0)$ given by (3.68) satisfies

$$\psi_1(P, n, n_0) > 0 \text{ for } \tilde\pi(P) \leq E_0. \tag{8.15}$$

Hence one infers $a_{2,\pm}$, $b_{2,\pm}$, $\rho_\pm \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$. Theorem 4.2, the spectral theorem, and the identities

$$D^2_\pm = H_1 \oplus H_{2,\pm}, \tag{8.16}$$

$$\sigma_3 D_\pm \sigma_3^{-1} = -D_\pm, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.17}$$

then prove (8.12)–(8.14). ☐

We note that the spectral gap $(-E_0^{1/2}, E_0^{1/2})$ of $D_\pm$ "closes" if and only if H_1 is critical, that is, if and only if $E_0 = 0$.

For $\sigma \in (-1, 1)$ one can show in contrast to (8.14) that

$$\sigma_p(H_{2,\sigma}) = \sigma_p(D_\sigma) = \{0\}, \quad \sigma \in (-1, 1). \tag{8.18}$$

In fact, using identities of the type

$$(D-w)^{-1} = \begin{pmatrix} w(H_1-w^2)^{-1} & A^*(H_2-w^2)^{-1} \\ A(H_1-w^2)^{-1} & w(H_2-w^2)^{-1} \end{pmatrix}, \; w^2 \in \mathbb{C}\backslash\{\sigma(H_1)\cup\sigma(H_2)\}, \tag{8.19}$$

one can reduce the spectral analysis of general Dirac-type operators D to that of H_1 and H_2. Moreover, noting that H_1 and H_2 are essentially isospectral, that is,

$$\sigma(H_1)\backslash\{0\} = \sigma(H_2)\backslash\{0\} \tag{8.20}$$

and that $H_1|_{\mathrm{Ker}(H_1)^\perp}$ and $H_2|_{\mathrm{Ker}(H_2)^\perp}$ are unitarily equivalent, a complete spectral analysis of D in terms of that of H_1 and $\mathrm{Ker}(H_2)$ can be given. Since here we are mainly concerned with finite-gap operators H_k, $k=1,2$ and D, these considerations are beyond the scope of this paper. The interested reader may find an exhaustive discussion of such topics, for instance, in [20], [36], [41], [42], [83].

9. Quasi-Periodic Finite-Gap Solutions of the Kac-van Moerbeke Hierarchy

In our final section we shall complete our main goal and construct the algebro-geometric quasi-periodic finite-gap solutions of the Kac-van Moerbeke hierarchy. Given the extensive preparations in Sections 3–8, our final task will be relatively straightforward.

We start with some notations. Let (a_1,b_1), $a_1(n)<0$, $b_1(n)<0$, $n\in\mathbb{Z}$ be the stationary g-gap solution (3.18), (3.19) and denote the corresponding Dirichlet eigenvalues and divisor by $\{\mu_{1,j}(n)\}_{1\le j\le g}$ and $\mathcal{D}_{\underline{\hat{\mu}}_1(n)}$ etc. Given (a_1,b_1), $\{\hat{\mu}_j(n)\}_{1\le j\le g}$ we define L_1, H_1, $\phi_1(P,n)$, $\psi_1(P,n,n_0)$, $\underline{z}_1(P,n)$, and $\underline{z}_1(n)$ as in (4.1), (4.2), (3.22), (3.23) (respectively (3.67)–(3.70)), (3.36), and (3.37)). Next, identifying the branches $\psi_{1,\pm}(0,n,n_0)$ of $\psi_1(Q_0,n,n_0)$, $Q_0=(0,R_{2g+2}^{1/2}(Q_0))$ with $\psi_{1,\pm}(n)$ in Theorem 7.2, and noticing

$$\psi_1(P,n,n_0)>0, \quad \tilde{\pi}(P)\le E_0 \tag{9.1}$$

as a consequence of Lemma 3.4 (i), (iv) and Theorem 3.5 (i), enables one to construct $(a_{2,\pm},b_{2,\pm})$, $\rho_\pm$ as in Theorem 8.2. For convenience we list these formulas below.

$$\begin{aligned}\rho_{e,\pm}(n) &= -[-a_1(n)\psi_{1,\pm}(n+1)/\psi_{1,\pm}(n)]^{1/2},\\ \rho_{o,\pm}(n) &= [-a_1(n)\psi_{1,\pm}(n)/\psi_{1,\pm}(n+1)]^{1/2},\end{aligned} \tag{9.2}$$

$$\rho_{\pm}(n) = \begin{cases}\rho_{e,\pm}(m), & n=2m\\ \rho_{o,\pm}(m), & n=2m+1\end{cases}, \tag{9.3}$$

$$a_{2,\pm}(n) = \rho_{e,\pm}(n+1)\rho_{o,\pm}(n), \tag{9.4}$$

$$b_{2,\pm}(n) = -\rho_{e,\pm}(n)^2 - \rho_{o,\pm}(n)^2. \tag{9.5}$$

Given $(a_{2,\pm}, b_{2,\pm})$, $\rho_{\pm}$ one then defines $L_{2,\pm}$, $H_{2,\pm}$, $\phi_{2,\pm}(P,n)$, $\psi_{2,\pm}(P,n,n_0)$, $A_{\pm}$, and finally $M_{\pm}$, $D_{\pm}$ as in the context of Theorem 8.2 using (4.1), (4.2), (3.22), (3.23) (respectively (3.67)–(3.70)), (7.3), and (8.1), (8.2). Moreover, defining

$$\phi_{1,\pm}(n) = -\rho_{e,\pm}(n)/\rho_{o,\pm}(n), \quad n\in\mathbb{Z}, \tag{9.6}$$

one verifies

$$a_1\phi_{1,\pm} + (a_1^-/\phi_{1,\pm}^-) = b_1, \quad \phi_{1,\pm} > 0 \tag{9.7}$$

and a comparison with the Riccati-type equation (3.24) then yields that

$$\phi_{1,\pm}(n) = \phi_{1,\pm}(0,n) \tag{9.8}$$

are the branches of $\phi_1(Q_0,n)$, $Q_0 = (0, R_{2g+2}^{1/2}(Q_0))$. In particular, introducing

$$\psi_{1,\pm}(n) = \begin{cases}\prod_{m=n_0}^{n-1}\phi_{1,\pm}(n), & n\geq n_0+1\\ 1, & n=n_0\\ \prod_{m=n}^{n_0-1}\phi_{1,\pm}(n)^{-1}, & n\leq n_0-1\end{cases}, \tag{9.9}$$

one verifies that

$$L_1\psi_{1,\pm} = 0, \quad \psi_{1,\pm} > 0 \tag{9.10}$$

and hence

$$\psi_{1,\pm}(n) = \psi_{1,\pm}(0,n,n_0) \tag{9.11}$$

are indeed the branches of $\psi_1(Q_0,n,n_0)$, $Q_0 = (0, R_{2g+2}^{1/2}(Q_0))$. Next, defining

$$\phi_{2,\pm,\mp}(n) = -\rho_{0,\pm}(n)/\rho_{e,\pm}(n+1), \quad n\in\mathbb{Z} \tag{9.12}$$

and

$$\psi_{2,\pm,\mp}(n) = \begin{cases} \prod_{m=n_0}^{n-1} \phi_{2,\pm,\mp}(n), & n \geq n_0 + 1 \\ 1, & n = n_0 \\ \prod_{m=n}^{n_0-1} \phi_{2,\pm,\mp}(n)^{-1}, & n \leq n_0 - 1 \end{cases}, \tag{9.13}$$

one verifies

$$a_{2,\pm}\phi_{2,\pm,\mp} + (a_{2,\pm}^{-}/\phi_{2,\pm,\mp}) = b_{2,\pm} \tag{9.14}$$

and

$$L_{2,\pm}\psi_{2,\pm,\mp} = 0. \tag{9.15}$$

In order to derive the θ-function representation for $(a_{2,\pm}, b_{2,\pm})$ and especially for $\rho_\pm$, we first recall that for (a_1, b_1), $\phi_1(Q_0)$, $\psi_1(Q_0)$ from Theorems 3.5 and 5.2,

$$a_1(n) = \tilde{a}[\theta(\underline{z}_1(n+1))\theta(\underline{z}_1(n-1))/\theta(\underline{z}_1(n))^2]^{1/2}, \tag{9.16}$$

$$b_1(n) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\left[\frac{\theta(\underline{w} + \underline{z}_1(n))}{\theta(\underline{w} + \underline{z}_1(n-1))}\right]\Bigg|_{\underline{w}=0}, \tag{9.17}$$

$$\phi_1(Q_0, n) = \left[\frac{\theta(\underline{z}_1(n-1))}{\theta(\underline{z}_1(n+1))}\right]^{1/2} \frac{\theta(\underline{z}_1(Q_0, n+1))}{\theta(\underline{z}_1(Q_0, n))} \exp\left(\int_{P_0}^{Q_0} \omega_{\infty_+,\infty_-}^{(3)}\right), \tag{9.18}$$

$$\psi_1(Q_0, n, n_0) = C(n, n_0) \frac{\theta(\underline{z}_1(Q_0, n))}{\theta(\underline{z}_1(Q_0, n_0))} \exp\left[(n - n_0)\int_{P_0}^{Q_0} \omega_{\infty_+,\infty_-}^{(3)}\right],$$

$$\underline{z}_1(P, n) = \underline{\hat{A}}_{P_0}(P) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n)}) - \underline{\hat{\Xi}}_{P_0}, \quad \underline{z}_1(n) = \underline{z}_1(\infty_+, n), \quad Q_0 = (0, R_{2g+2}^{1/2}(Q_0)). \tag{9.19}$$

Here $C(n, n_0)$ is defined in (3.69), (3.70) and the square roots in (9.16), (9.18) and hence in

$$C(n) = C(n+1, n) = \left[\frac{\theta(\underline{z}_1(n+1))}{\theta(\underline{z}_1(n))}\right]^{1/2} \tag{9.20}$$

are all positive (cf. (3.59), (3.71), (3.77), and $\tilde{a} < 0$).

In the following we explicitly need the branches $\phi_{1,\pm}(z, n)$ of $\phi_1(P, n)$. To fix notations we abbreviate

$$A_{E_0}^{+}(z) = \int_{P_0}^{Q} \underline{\omega}, \ \int_{E_0}^{z} \omega_{\infty_+,\infty_-}^{(3)+} = \int_{P_0}^{Q} \omega_{\infty_+,\infty_-}^{(3)} < 0, \quad Q = (z, R_{2g+2}(z)^{1/2}) \in \Pi_+, \ \tilde{\pi}(Q) = z \leq E_0, \tag{9.21}$$

where the path of integration from P_0 to Q is along the lift of the straight line segment from E_0 to $z(\leq E_0)$. (As in the proof of Lemma 3.4, whenever the integration path meets the cycle b_k we first move along

b_k until we hit the intersection point with a_k. Then we follow a_k and return on the other side of b_k before we continue the straight line path. The contributions on b_k cancel and the contribution from a_k is irrelevant in (9.24) below due to the $\mathbb{Z}^g$-periodicity of the Riemann theta function in (A.27) and the normalization (3.43) of $\omega^{(3)}_{\infty_+,\infty_-}$.) Moreover, we use the notation

$$\underline{z}_{1,\pm}(z,n) = \pm\underline{A}^+_{E_0}(z) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n)}) - \underline{\hat{\Xi}}_{P_0}, \tag{9.22}$$
$$\underline{z}_{1,\pm}(n) = \underline{z}_{1,\pm}(-\infty,n),\ \underline{z}_{1,+}(n) = \underline{z}_1(n) \tag{9.23}$$

for the branches of $\underline{z}_1(P,n)$, $\tilde{\pi}(P) = z \le E_0$. The branches $\phi_{1,\pm}(z,n)$ of $\phi_1(P,n)$ then read explicitly,

$$\phi_{1,\pm}(z,n) = \left[\frac{\theta(\underline{z}_1(n-1))}{\theta(\underline{z}_1(n+1))}\right]^{1/2} \frac{\theta(\underline{z}_{1,\pm}(z,n+1))}{\theta(\underline{z}_{1,\pm}(z,n))} e^{\pm\int_{E_0}^{z}\omega^{(3)+}_{\infty_+,\infty_-}}. \tag{9.24}$$

Together with (9.2), (9.12), and

$$\phi_{1,\pm}(z,n) = \psi_{1,\pm}(z,n+1)/\psi_{1,\pm}(z,n) \tag{9.25}$$

this allows one to compute

$$\begin{aligned}
\rho_{e,\pm}(n) &= -[-a_1(n)\phi_{1,\pm}(0,n)]^{1/2} \\
&= -\left[-\tilde{a}\frac{\theta(\underline{z}_1(n-1))\theta(\underline{z}_{1,\pm}(0,n+1))}{\theta(\underline{z}_1(n))\theta(\underline{z}_{1,\pm}(0,n))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}} && (9.26)\\
\rho_{o,\pm}(n) &= [-a_1(n)\phi_{1,\pm}(0,n)^{-1}]^{1/2} \\
&= \left[-\tilde{a}\frac{\theta(\underline{z}_1(n+1))\theta(\underline{z}_{1,\pm}(0,n))}{\theta(\underline{z}_1(n))\theta(\underline{z}_{1,\pm}(0,n+1))}\right]^{1/2} e^{\mp\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}}, && (9.27)\\
\phi_{2,\pm,\mp}(n) &= -\rho_{o,\pm}(n)/\rho_{e,\pm}(n+1) \\
&= \left[\frac{\theta(\underline{z}_{1,\pm}(0,n))}{\theta(\underline{z}_{1,\pm}(0,n+2))}\right]^{1/2} \frac{\theta(\underline{z}_1(n+1))}{\theta(\underline{z}_1(n))} e^{\mp\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}}. && (9.28)
\end{aligned}$$

Next we define

$$\underline{z}_{2,\pm}(P,n) = \underline{\hat{A}}_{P_0}(P) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}) - \underline{\hat{\Xi}}_{P_0}, \tag{9.29}$$
$$\underline{z}_{2,\pm}(n) = \underline{z}_{2,\pm}(\infty_+,n), \tag{9.30}$$

where

$$\begin{aligned}
&\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n)}) - \epsilon_\pm\underline{A}_{P_0}(Q_0) - \underline{A}_{P_0}(\infty_+), \\
&\epsilon_+ = -\epsilon_- = \pm 1 \text{ for } Q_0 \in \Pi_\pm,\ Q_0 = (0, R^{1/2}_{2g+2}(Q_0)) && (9.31)
\end{aligned}$$

describes the connection between the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}$ of $H_{2,\pm} = A_\pm A_\pm^*$ and $\mathcal{D}_{\underline{\hat{\mu}}_1(n)}$ of $H_1 = A_\pm^* A_\pm$. (In the special case where H_1 is critical and hence $E_0 = 0$, that is, $P_0 = Q_0$, one obtains $\hat{\mu}_{2,+,j}(n) = \hat{\mu}_{2,-,j}(n) \equiv \hat{\mu}_{2,0,j}(n)$, $1 \le j \le g$ and the sign ambiguity in (9.31) vanishes.) The branches of $z_{2,\pm}(P,n)$ are then denoted by

$$\underline{z}_{2,\epsilon,\epsilon'}(z,n) = \epsilon' \underline{A}_{E_0}^{+}(z) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\epsilon}(n)}) - \underline{\hat{\Xi}}_{P_0}, \tag{9.32}$$

$$\underline{z}_{2,\epsilon,\epsilon'}(n) = \underline{z}_{2,\epsilon,\epsilon'}(-\infty,n),\ \underline{z}_{2,\epsilon,+}(n) = \underline{z}_{2,\epsilon}(n),\ \ \epsilon,\epsilon' \in \{+,-\} \tag{9.33}$$

and $\phi_{2,+,-}$, $\phi_{2,-,+}$ in (9.28) are seen to be branches of the following meromorphic function $\phi_{2,\pm}(P,n)$ on K_g,

$$\phi_{2,\pm}(P,n) = \left[\frac{\theta(\underline{z}_{2,\pm}(n-1))}{\theta(\underline{z}_{2,\pm}(n+1))}\right]^{1/2} \frac{\theta(\underline{z}_{2,\pm}(P,n+1))}{\theta(\underline{z}_{2,\pm}(P,n))} e^{\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-}}, \tag{9.34}$$

by noticing that

$$\underline{z}_{2,\pm,\mp}(0,n) = \underline{z}_{1,+}(n) = \underline{z}_1(n),\ z_{2,\pm,+}(n-1) = \underline{z}_{2,\pm}(n-1) = \underline{z}_{1,\pm}(0,n). \tag{9.35}$$

In particular, we may rewrite and extend (9.28) in the form

$$\phi_{2,\epsilon,\epsilon'}(n) = \left[\frac{\theta(\underline{z}_{2,\epsilon}(n-1))}{\theta(\underline{z}_{2,\epsilon}(n+1))}\right]^{1/2} \frac{\theta(\underline{z}_{2,\epsilon,-\epsilon'}(n+1))}{\theta(\underline{z}_{2,\epsilon,-\epsilon'}(n))} e^{\epsilon' \int_{E_0}^{0} \omega^{(3)+}_{\infty_+,\infty_-}},\ \ \epsilon,\epsilon' \in \{+,-\}. \tag{9.36}$$

The divisor of $\phi_{2,\pm}(P,n)$ thus reads

$$(\phi_{2,\pm}(.,n)) = \mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n+1)} - \mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)} + \mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-} \tag{9.37}$$

in analogy to that of $\phi_1(P,n)$ (cf. (3.32))

$$(\phi_1(.,n)) = \mathcal{D}_{\underline{\hat{\mu}}_1(n+1)} - \mathcal{D}_{\underline{\hat{\mu}}_1(n)} + \mathcal{D}_{\infty_+} - \mathcal{D}_{\infty_-}. \tag{9.38}$$

Given (9.36) respectively (9.26)–(9.33) we can now express $(a_{2,\pm}, b_{2,\pm})$ and $\rho_\pm$ in terms of θ-functions as follows.

Theorem 9.1. *Let (a_1,b_1) in (9.16), (9.17) be the g-gap sequences associated with $H_1 = A_\pm^* A_\pm$. Then the sequences $(a_{2,\pm}, b_{2,\pm})$ associated with $H_{2,\pm} = A_\pm A_\pm^*$ are explicitly given by*

$$a_{2,\pm}(n) = \tilde{a}[\theta(\underline{z}_{2,\pm}(n+1))\theta(\underline{z}_{2,\pm}(n-1))/\theta(\underline{z}_{2,\pm}(n))^2]^{1/2} \quad n \in \mathbb{Z}, \tag{9.39}$$

$$b_{2,\pm}(n) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln \left[\frac{\theta(\underline{w} + \underline{z}_{2,\pm}(n))}{\theta(\underline{w} + \underline{z}_{2,\pm}(n-1))}\right]\Bigg|_{\underline{w}=\underline{0}}, \quad n \in \mathbb{Z}. \tag{9.40}$$

In particular, $H_{2,\pm}$ are isospectral to H_1 and hence $(a_{2,\pm}, b_{2,\pm})$ are g-gap sequences associated with the same hyperelliptic curve K_g as (a_1, b_1) and with nonspecial Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}$ satisfying

$$\mu_{2,\pm,j}(n) = \tilde{\pi}(\hat{\mu}_{2,\pm,j}(n)) \in [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq g, \ n \in \mathbb{Z} \tag{9.41}$$

and

$$\begin{aligned} \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}) &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n)}) \mp \underline{A}^{+}_{E_0}(0) - \underline{A}_{P_0}(\infty_+) \\ &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n_0)}) - 2(n-n_0)\underline{A}_{P_0}(\infty_+) \mp \underline{A}^{+}_{E_0}(0) - \underline{A}_{P_0}(\infty_+) \end{aligned} \tag{9.42}$$

Moreover, we have $H_{2,+} = H_{2,-}$, $a_{2,+} = a_{2,-}$, $b_{2,+} = b_{2,-}$, $\hat{\mu}_{2,+,j} = \hat{\mu}_{2,-,j}$, $1 \leq j \leq g$, etc. if and only if H_1 is critical, that is, if and only if $E_0 = 0$ (i.e., $P_0 = Q_0$).

Proof. Equation (9.39) is clear from (9.4), (9.26), (9.27), and (9.35). That $H_{2,\pm}$ are isospectral to H_1 has been proven in Theorem 8.2 and the nonspeciality of $\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n)}$ (cf. Lemma A.2) together with (9.41) is a consequence of Lemma 3.4 (ii) and the fact that

$$i\,\mathrm{Im}[\underline{\hat{A}}_{P_0}(Q_0)] = i\,\mathrm{Im}[\underline{\hat{A}}_{P_0}(\infty_+)] = \underline{0} \mod (L_g). \tag{9.43}$$

Equation (9.42) directly follows from (9.31) and (9.21). Equation (9.40) for $b_{2,\pm}$ finally can be derived from an expansion of $\phi_{2,\pm}(P,n)$ near $P = \infty_+$ exactly as in the proof of Theorem 5.2. $\square$

Remark 9.2. Equations (9.31) respectively (9.42) illustrate the effect of commutation (i.e., $H_1 = A_\pm^* A_\pm \to H_{2,\pm} = A_\pm A_\pm^*$) as translations by $\mp\underline{A}^{+}_{E_0}(0) - \underline{A}_{P_0}(\infty_+) = -\epsilon_\pm \underline{A}_{P_0}(Q_0) - \underline{A}_{P_0}(\infty_+)$, $\epsilon_+ = -\epsilon_- = \pm 1$ for $Q \in \Pi_\pm$ on the Jacobi variety. This clearly resembles the differential operator case pioneered by Burchnall and Chaundy [16], [17] and put into the context of Bäcklund transformations for the KdV equation in [28], [30], [37], [42] and discussed in connection with the spectral theory of Hill's equation in [63]–[65].

Depending on whether or not H_1 (and hence $H_{2,\pm}$) is critical, that is, whether or not $E_0 = 0$, the corresponding Dirac-type operator $D_\pm = \begin{pmatrix} 0 & A_\pm^* \\ A_\pm & 0 \end{pmatrix}$ in Theorem 8.2 has a spectral gap containing 0 and hence altogether $2g+1$ spectral gaps (if H_1, $H_{2,\pm}$ are subcritical, that is, if $E_0 > 0$) or precisely $2g$ spectral gaps (if H_1, $H_{2,\pm}$ are critical, i.e., if $E_0 = 0$). Accordingly we call the corresponding sequence $\rho_\pm$ (respectively ρ_0) a $(2g+1)$-gap (respectively $2g$-gap) sequence associated with $D_\pm$ (respectively D_0). The explicit θ-function characterization of $\rho_\pm$ (respectively ρ_0) then can be summarized as follows.

Theorem 9.3. *The* $(2g+1)$*-gap and* $2g$*-gap sequences* $\rho_\pm$ *and* ρ_0 *associated with* $D_\pm = \begin{pmatrix} 0 & A_\pm^* \\ A_\pm & 0 \end{pmatrix}$ *and* $D_0 = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$ *are given by*

$$\rho_\pm(n) = \begin{cases} \rho_{e,\pm}(m), & n = 2m \\ \rho_{o,\pm}(m), & n = 2m+1 \end{cases}, \quad \rho_0(n) = \begin{cases} \rho_{e,0}(m), & n = 2m \\ \rho_{o,0}(m), & n = 2m+1 \end{cases}, \quad n \in \mathbb{Z}, \tag{9.44}$$

where

$$\begin{aligned} \rho_{e,\pm}(n) &= -\left[-\tilde{a}\frac{\theta(\underline{z}_1(n-1))\theta(\underline{z}_{1,\pm}(0,n+1))}{\theta(\underline{z}_1(n))\theta(\underline{z}_{1,\pm}(0,n))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^0 \omega^{(3)+}_{\infty_+,\infty_-}} \\ &= -\left[-\tilde{a}\frac{\theta(\underline{z}_{2,\pm}(n))\theta(\underline{z}_{2,\pm,\mp}(0,n-1))}{\theta(\underline{z}_{2,\pm}(n-1))\theta(\underline{z}_{2,\pm,\mp}(0,n))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^0 \omega^{(3)+}_{\infty_+,\infty_-}}, \end{aligned} \tag{9.45}$$

$$\begin{aligned} \rho_{o,\pm}(n) &= \left[-\tilde{a}\frac{\theta(\underline{z}_1(n+1))\theta(\underline{z}_{1,\pm}(0,n))}{\theta(\underline{z}_1(n))\theta(\underline{z}_{1,\pm}(0,n+1))}\right]^{1/2} e^{\mp\frac{1}{2}\int_{E_0}^0 \omega^{(3)+}_{\infty_+,\infty_-}} \\ &= \left[-\tilde{a}\frac{\theta(\underline{z}_{2,\pm}(n-1))\theta(\underline{z}_{2,\pm,\mp}(0,n+1))}{\theta(\underline{z}_{2,\pm}(n))\theta(\underline{z}_{2,\pm,\mp}(0,n))}\right]^{1/2} e^{\mp\frac{1}{2}\int_{E_0}^0 \omega^{(3)+}_{\infty_+,\infty_-}}, \\ & \qquad n \in \mathbb{Z} \end{aligned} \tag{9.46}$$

and $\rho_{e,0}(n)$*,* $\rho_{o,0}(n)$ *are obtained from* (9.45), (9.46) *by taking* $E_0 = 0$.

Proof. It suffices to combine (9.3), (9.26), (9.27), and (9.35). □

Isospectral manifolds in connection with Toda flows (including non-Abelian generalizations) have attracted a lot of interest (see, e.g., [9], [14], [33], [68], [69], [70] and the references therein). In the present finite-gap case the situation is analogous to the (m)KdV case and briefly summarized below.

Remark 9.4. For the fixed hyperelliptic curve K_g (cf. (3.1)), Lemma 3.1 shows that all g-gap sequences (a,b) associated with the Jacobi operator H are parameterized by the initial conditions

$$\hat{\mu}_j(n_0) = (\mu_j(n_0), R_{2g+2}(\hat{\mu}_j(n_0))^{1/2}), \ \mu_j(n_0) \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le g, \tag{9.47}$$

or equivalently, by the pairs

$$\{(\mu_j(n_0), \sigma_j(n_0))\}_{1\le j\le g}, \ \mu_j(n_0) \in [E_{2j-1}, E_{2j}], \ \sigma_j(n_0) = \pm \text{ for } \hat{\mu}_j(n_0) \in \Pi_\pm, \quad 1 \le j \le g. \tag{9.48}$$

(Here we omit $\sigma_j(n_0)$ in the special case where $\mu_j(n_0) \in \{E_{2j-1}, E_{2j}\}$.) With this restriction in mind, (9.48) represents the product of g circles S^1 when varying $\mu_j(n_0)$ (independently from $\mu_\ell(n_0)$, $\ell \neq j$) in $[E_{2j-1}, E_{2j}]$, $1 \le j \le g$. In other words, the isospectral set of all g-gap sequences (a,b) associated with H can be identified with the

g-dimensional torus $T^g = \times_{j=1}^g S^1$. Theorem 5.2 then provides a concrete realization of T^g. By Theorem 9.3 the same applies to the set of all $(2g+1)$-gap (respectively $2g$-gap) sequences ρ associated with the Dirac-type operator D. More precisely, assuming H_1 (and hence $H_{2,\pm}$) to be subcritical (and thus $0 \in \mathbb{R}\backslash\sigma(D_\pm)$), the isospectral set of all $(2g+1)$-gap sequences ρ (in connection with the nonsingular hyperelliptic curve K_{2g+1} of genus $2g+1$, cf. (7.40)) is again parameterized bijectively by the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}_1(n_0)}$ (respectively by the analog of (9.48)) as is demonstrated in (9.42). In particular, ρ_+ and ρ_- in (9.44)–(9.46) represent two independent (yet equivalent) concrete realizations of the isospectral manifold T^g of all $(2g+1)$-gap sequences ρ associated with D. In the case where H_1 (and hence $H_{2,\pm}$) is critical (i.e., $E_0 = 0$ and thus $0 \in \sigma(D_0)$) the (fixed) curve K_{2g+1} of (arithmetic) genus $2g+1$ (cf. (7.40)) is singular, yet $\mathcal{D}_{\underline{\hat{\mu}}_1(n_0)}$ still parameterizes the corresponding isospectral set of $2g$-gap sequences ρ in a one-to-one and onto fashion. In particular, ρ_0 in (9.44)–(9.46) then represents a concrete realization of the isospectral torus T^g of all $2g$-gap sequences ρ associated with D.

Finally we briefly treat the t-dependent case. Our starting point is a solution $(a_1(t), b_1(t))$, $a_1(n,t) < 0$, $b_1(n,t) < 0$, $(n,t) \in \mathbb{Z}\times\mathbb{R}$ of the $\widehat{\mathrm{TL}}_r$ equations (6.6) with g-gap initial conditions $(a_1^{(0)}, b_1^{(0)})$ at $t = t_0$ and θ-function representation as described in Theorem 6.3. Next we note that as in (9.1) the t-dependent BA-function $\psi_1(P,n,n_0,t,t_0)$ in (6.24), respectively (6.42), shares the positivity condition

$$\psi_1(P,n,n_0,t,t_0) > 0, \quad \tilde{\pi}(P) \le E_0 \tag{9.49}$$

and by (6.27), (6.28) satisfies

$$L_1(t)\psi_1(P,t) = 0, \quad t \in \mathbb{R}, \tag{9.50}$$

$$\frac{d}{dt}\psi_1(P,t) = \hat{P}_{1,2r+2}\psi_1(P,t), \quad r \in \mathbb{N}_0,\ t \in \mathbb{R}. \tag{9.51}$$

At this point one can follow the stationary considerations in (9.2)–(9.15), (9.21)–(9.38) step by step. Especially, $\underline{z}_{1,\epsilon}(z,n)$, $\underline{z}_{2,\epsilon,\epsilon'}(z,n)$ are now replaced by

$$\begin{aligned} \underline{z}_{1,\epsilon}(z,n,t) &= \epsilon \underline{A}_{E_0}^+(z) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n,t)}) - \underline{\hat{\Xi}}_{P_0}, \\ \underline{z}_1(n,t) &= \underline{z}_{1,+}(-\infty,n,t), \end{aligned} \tag{9.52}$$

$$\begin{aligned} \underline{z}_{2,\epsilon,\epsilon'}(z,n,t) &= \epsilon' \underline{A}_{E_0}^+(z) - \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\epsilon}(n,t)}) - \underline{\hat{\Xi}}_{P_0}, \\ \underline{z}_{2,\epsilon}(n,t) &= \underline{z}_{2,\epsilon,+}(-\infty,n,t), \quad \epsilon,\epsilon' \in \{+,-\}, \end{aligned} \tag{9.53}$$

with (cf. (3.44), (6.33), (6.46))

$$\begin{aligned} \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n,t)}) &= \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n,t)}) \mp \underline{A}^+_{E_0}(0) - \underline{\hat{A}}_{P_0}(\infty_+) \\ &= \underline{\hat{\alpha}}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_1(n_0,t_0)}) - (n-n_0)\underline{U}^{(3)} - (t-t_0)\underline{U}^{(2)}_r \mp \underline{A}^+_{E_0}(0) - \underline{\hat{A}}_{P_0}(\infty_+) \end{aligned} \tag{9.54}$$

The divisors $\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm(n,t)}}$ are all nonspecial, that is,

$$i(\mathcal{D}_{\underline{\hat{\mu}}_{2,\pm}(n,t)}) = 0, \quad (n,t) \in \mathbb{Z} \times \mathbb{R} \tag{9.55}$$

by exactly the same argument as in the proof of Theorem 9.1.

Introducing $L_1(t)$, $H_1(t)$, $\rho_\pm(t)$, $A_\pm(t)$, $H_{2,\pm}(t)$, $M_\pm(t)$, $D_\pm(t)$, $a_{2,\pm}(t)$, $b_{2,\pm}(t)$ according to (7.3), (7.4), (7.8)–(7.11), (7.29)–(7.33), (8.1), (8.2) we may briefly summarize our t-dependent results applying Theorem 7.2 as follows.

Theorem 9.5. *The θ-function representation of the g-gap solutions $(a_{2,\pm}(t), b_{2,\pm}(t))$ of the $\widehat{\mathrm{TL}}_r$ equations read*

$$a_{2,\pm}(n,t) = \tilde{a}[\theta(\underline{z}_{2,\pm}(n+1,t))\theta(\underline{z}_{2,\pm}(n-1,t))/\theta(\underline{z}_{2,\pm}(n,t))^2]^{1/2}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}, \tag{9.56}$$

$$b_{2,\pm}(n,t) = \sum_{j=1}^{g} \lambda_j - \frac{1}{2}\sum_{m=0}^{2g+1} E_m - \sum_{j=1}^{g} c_j(g)\frac{\partial}{\partial w_j} \ln\left[\frac{\theta(\underline{w} + \underline{z}_{2,\pm}(n,t))}{\theta(\underline{w} + \underline{z}_{2,\pm}(n-1,t))}\right]\Bigg|_{\underline{w}=0}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}. \tag{9.57}$$

Similarly, the $(2g+1)$-gap and $2g$-gap solutions $\rho_\pm$ and ρ_0 of the $\widehat{\mathrm{KM}}_r$ equations are given by

$$\rho_\pm(n,t) = \begin{cases} \rho_{e,\pm}(m,t), & n = 2m \\ \rho_{o,\pm}(m,t), & n = 2m+1 \end{cases}, \quad \rho_0(n,t) = \begin{cases} \rho_{e,0}(m,t), & n = 2m \\ \rho_{o,0}(m,t), & n = 2m+1 \end{cases}, \tag{9.58}$$

where

$$\begin{aligned} \rho_{e,\pm}(n,t) &= -\left[-\tilde{a}\frac{\theta(\underline{z}_1(n-1,t))\theta(\underline{z}_{1,\pm}(0,n+1,t))}{\theta(\underline{z}_1(n,t))\theta(\underline{z}_{1,\pm}(0,n,t))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}} \\ &= -\left[-\tilde{a}\frac{\theta(\underline{z}_{2,\pm}(n,t))\theta(\underline{z}_{2,\pm,\mp}(0,n-1,t))}{\theta(\underline{z}_{2,\pm}(n-1,t))\theta(\underline{z}_{2,\pm,\mp}(0,n,t))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}}, \end{aligned} \tag{9.59}$$

$$\begin{aligned} \rho_{o,\pm}(n,t) &= \left[-\tilde{a}\frac{\theta(\underline{z}_1(n+1,t))\theta(\underline{z}_{1,\pm}(0,n,t))}{\theta(\underline{z}_1(n,t))\theta(\underline{z}_{1,\pm}(0,n+1,t))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}} \\ &= \left[-\tilde{a}\frac{\theta(\underline{z}_{2,\pm}(n-1,t))\theta(\underline{z}_{2,\pm,\mp}(0,n+1,t))}{\theta(\underline{z}_{2,\pm}(n,t))\theta(\underline{z}_{2,\pm,\mp}(0,n,t))}\right]^{1/2} e^{\pm\frac{1}{2}\int_{E_0}^{0}\omega^{(3)+}_{\infty_+,\infty_-}} \end{aligned} \tag{9.60}$$

and $\rho_{e,0}(n,t)$*,* $\rho_{o,0}(n,t)$ *are obtained from* (9.59)*,* (9.60) *by taking* $E_0 = 0$.

We end up with a brief outlook at possible applications of the results of this section. In many respects the construction of all real-valued algebro-geometric quasi-periodic finite-gap solutions of the KM hierarchy is by no means the end of the story but rather the beginning of the next chapter in view of possible applications of this material. For instance, in addition to the applications described in [88], it appears tempting to transfer results on the Toda shock problem (see, e.g., [23], [45], [59], [86] and the references therein) to that of the KM system and to search for connections between algorithms for eigenvalue computation of real matrices with the Toda flows (see, e.g., [21], [24], [34] and the references therein). Similarly the solution of certain discrete Peierls models for quasi-one-dimensional conducting polymers in connection with finite-gap Toda solutions (see, e.g., [57], [58] and the references therein and in Ch. 8 of [10]) and especially the phenomenon of soliton excitations in conducting polymers (such as polyacetylen) and Fermion number fractionization (see, e.g., the reviews [44], [75]), where the underlying model Hamiltonians are related to the Dirac-type expression (7.4), offer a variety of applications for finite-gap solutions of the KM hierarchy.

Appendix A. Hyperelliptic Curves of the Toda-Type and Theta Functions

We briefly summarize our basic notation in connection with hyperelliptic Toda curves and their theta functions as employed in Sections 3, 5, 6, and 9. For background information on this standard material we refer, for instance, to [31], [32], [55], [74].

Consider the points

$$\{E_m\}_{0\leq m\leq 2g+1} \subset \mathbb{R}, \; E_0 < E_1 < \cdots < E_{2g+1}, \quad g \in \mathbb{N}_0 \tag{A.1}$$

and define the cut plane

$$\Pi = \mathbb{C}\backslash \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}] \tag{A.2}$$

with the holomorphic function

$$R_{2g+2}(.)^{1/2} : \begin{cases} \Pi \to \mathbb{C} \\ z \to [\prod_{m=0}^{2g+1}(z - E_m)]^{1/2} \end{cases} \tag{A.3}$$

on it. We extend $R_{2g+2}^{1/2}$ to all of $\mathbb{C}$ by

$$R_{2g+2}(\lambda)^{1/2} = \lim_{\epsilon\downarrow 0} R_{2g+2}(\lambda + i\epsilon)^{1/2}, \quad \lambda \in \mathbb{C}\backslash\Pi, \tag{A.4}$$

with the sign of the square root chosen according to

$$R_{2g+2}(\lambda)^{1/2} = \begin{cases} -|R_{2g+2}(\lambda)^{1/2}|, & \lambda \in (E_{2g+1}, \infty) \\ (-1)^{g+j+1}|R_{2g+2}(\lambda)^{1/2}|, & \lambda \in (E_{2j+1}, E_{2j+2}),\ 0 \le j \le g-1 \\ (-1)^{g}|R_{2g+2}(\lambda)^{1/2}|, & \lambda \in (-\infty, E_0) \\ (-1)^{g+j+1} i|R_{2g+2}(\lambda)^{1/2}|, & \lambda \in (E_{2j}, E_{2j+1}),\ 0 \le j \le g \end{cases}. \tag{A.5}$$

Next we define the set

$$M = \{(z, \sigma R_{2g+2}(z)^{1/2}) | z \in \mathbb{C},\ \sigma \in \{-,+\}\} \cup \{\infty_+, \infty_-\},\ \infty_\pm = (\infty, \pm\infty) \tag{A.6}$$

and

$$B = \{(E_m, 0)\}_{0 \le m \le 2g+1}\ , \tag{A.7}$$

the set of branch points. M becomes a Riemann surface upon introducing the charts (U_{P_0}, ζ_{P_0}) defined as follows:

$$P_0 = (z_0, \sigma_0 R_{2g+2}(z_0)^{1/2}) \text{ or } P_0 = \infty_\pm,\ P = (z, \sigma R_{2g+2}(z)^{1/2}) \in U_{P_0} \subset M,$$
$$V_{P_0} = \zeta_{P_0}(U_{P_0}) \subset \mathbb{C}. \tag{A.8}$$

$\underline{P_0 \notin \{B \cup \{\infty_+, \infty_-\}\}.}$

$$\begin{aligned} U_{P_0} =& \{P \in M \big| |z - z_0| < C,\ \sigma R_{2g+2}(z)^{1/2} \text{ the branch obtained by straight line} \\ & \quad \text{analytic continuation starting from } z_0\}, \quad C = \min_m |z_0 - E_m|, \\ V_{P_0} =& \{\zeta \in \mathbb{C} \big| |\zeta| < C\}, \end{aligned}$$

$$\zeta_{P_0} : \begin{cases} U_{P_0} \to V_{P_0} \\ P \to z - z_0 \end{cases}, \quad \zeta_{P_0}^{-1} : \begin{cases} V_{P_0} \to U_{P_0} \\ \zeta \to (z_0 + \zeta,\ \sigma R_{2g+2}(z_0 + \zeta)^{1/2}) \end{cases}. \tag{A.9}$$

$\underline{P_0 = (E_{m_0}, 0).}$

$$U_{P_0} = \Big\{P \in M \Big| |z - E_{m_0}| < C_{m_0}\Big\},\ C_{m_0} = \min_{m \ne m_0} |E_m - E_{m_0}|,$$
$$V_{P_0} = \{\zeta \in \mathbb{C} \big| |\zeta| < C_{m_0}^{1/2}\},$$

$$\zeta_{P_0}:\begin{cases} U_{P_0}\to V_{P_0} \\ P\to\sigma(z-E_{m_0})^{1/2}\end{cases}, \qquad \begin{aligned}&(z-E_{m_0})^{1/2}=|(z-E_{m_0})^{1/2}|e^{i\arg(z-E_{m_0})/2},\\ &\arg(z-E_{m_0})\in\begin{cases}[0,2\pi), & m_0 \text{ even}\\ (-\pi,\pi], & m_0 \text{ odd}\end{cases},\end{aligned}$$

$$\zeta_{P_0}^{-1}:\begin{cases} V_{P_0}\to U_{P_0} \\ \zeta\to(E_{m_0}+\zeta^2,\ \zeta[\prod_{m\neq m_0}(E_{m_0}-E_m+\zeta^2)]^{1/2})\end{cases},$$

$$\Big[\prod_{m\neq m_0}(E_{m_0}-E_m+\zeta^2)\Big]^{1/2}=(-1)^g i^{-m_0-1}\Big|\Big[\prod_{m\neq m_0}(E_{m_0}-E_m)\Big]^{1/2}\Big|\times \tag{A.10}$$

$$\times\Big[1+\frac{1}{2}\zeta^2\sum_{m\neq m_0}(E_{m_0}-E_m)^{-1}+O(\zeta^4)\Big].$$

$\underline{P_0=\infty_\pm.}$

$U_{P_0}=\{P\in M\Big||z|>C_\infty\}$, $C_\infty=\max_m|E_m|$, $V_{P_0}=\Big\{\zeta\in\mathbb{C}\Big||\zeta|<C_\infty^{-1}\Big\}$,

$$\zeta_{P_0}:\begin{cases} U_{P_0}\to V_{P_0} \\ P\to z^{-1} \\ \infty_\pm\to 0\end{cases}, \quad \zeta_{P_0}^{-1}:\begin{cases} V_{P_0}\to U_{P_0} \\ \zeta\to(\zeta^{-1}\pm\zeta^{-g-1}[\Pi_m(1-\zeta E_m)]^{1/2}) \\ 0\to\infty_\pm\end{cases},$$

$$\Big[\Pi_m(1-\zeta E_m)\Big]^{1/2}=-1+\frac{1}{2}\zeta\sum_m E_m+0(\zeta^2). \tag{A.11}$$

It will also be useful to introduce the subsets $\Pi_\pm\subset M$ (upper and lower sheets)

$$\Pi_\pm=\{(z,\pm R_{2g+2}(z)^{1/2})\in M|z\in\Pi\} \tag{A.12}$$

and the associated charts

$$\zeta_\pm:\begin{cases}\Pi_\pm\to\Pi \\ P\to z\end{cases}. \tag{A.13}$$

The topology introduced by the charts (A.9)–(A.11) is Hausdorff and second countable (finitely many of them cover M). In addition, $\Pi_\pm$ are connected (being homeomorphic to Π) and so are their closures $\overline{\Pi}_\pm$. Moreover, since $M=\overline{\Pi}_+\cup\overline{\Pi}_-$ and $\overline{\Pi}_+$ and $\overline{\Pi}_-$ have points in common, M is connected and (A.9)–(A.11) define a complex structure on M. We shall denote the resulting Riemann surface (curve) by K_g. Topologically, K_g is a sphere with g handles and hence has genus g.

Next, consider the holomorphic sheet exchange map (involution)

$$*:\begin{cases} K_g\to K_g \\ (z,\sigma R_{2g+2}(z)^{1/2})\to(z,\sigma R_{2g+2}(z)^{1/2})^*=(z,-\sigma R_{2g+2}(z)^{1/2}) \\ \infty_\pm\to\infty_\pm^*=\infty_\mp\end{cases} \tag{A.14}$$

and the two meromorphic projection maps

$$\tilde{\pi}: \begin{cases} K_g \to \mathbb{C}\cup\{\infty\} \\ (z, \sigma R_{2g+2}(z)^{1/2} \to z \\ \infty_\pm \to \infty \end{cases}, \quad R_{2g+2}^{1/2}: \begin{cases} K_g \to \mathbb{C}\cup\{\infty\} \\ (z, \sigma R_{2g+2}(z)^{1/2}) \to \sigma R_{2g+2}(z)^{1/2}. \\ \infty_\pm \to \infty \end{cases} \tag{A.15}$$

$\tilde{\pi}$ has poles of order 1 at $\infty_\pm$ and two simple zeros at $(0, \pm R_{2g+2}(0)^{1/2})$ if $R_{2g+2}(0) \neq 0$ or a double zero at $(0,0)$ if $R_{2g+2}(0) = 0$ (i.e., if $0 \in \{E_m\}_{0\le m\le 2g+1}$) and $R_{2g+2}^{1/2}$ has poles of order $g+1$ at $\infty_\pm$ and $2g+2$ simple zeros at $(E_m 0)$, $0 \le m \le 2g+1$. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P),\ R_{2g+2}^{1/2}(P^*) = -R_{2g+2}^{1/2}(P), \quad P \in K_g. \tag{A.16}$$

Thus K_g is a two-sheeted ramified covering of the Riemann sphere $\mathbb{C}_\infty (\cong \mathbb{C}\cup\{\infty\})$, K_g is compact (since $\tilde{\pi}$ is open and $\mathbb{C}_\infty$ is compact), and K_g is hyperelliptic (since it admits the meromorphic function $\tilde{\pi}$ of degree two).

Using our local charts one infers that for $g \in \mathbb{N}$, $d\tilde{\pi}/R_{2g+2}^{1/2}$ is a holomorphic differential on K_g with zeros of order $g-1$ at $\infty_\pm$ and hence

$$\eta_j = \frac{\tilde{\pi}^{j-1} d\tilde{\pi}}{R_{2g+2}^{1/2}}, \quad 1 \le j \le g \tag{A.17}$$

form a basis for the space of holomorphic differentials on K_g.

Next we introduce a canonical homology basis $\{a_j, b_j\}_{1\le j\le g}$ for K_g as follows. The cycle a_ℓ starts near $E_{2\ell-1}$ on Π_+ surrounds $E_{2\ell}$ counterclockwise thereby changing to Π_-, and returns to the starting point encircling $E_{2\ell-1}$ changing sheets again. The cycle b_ℓ surrounds E_0, $E_{2\ell-1}$ counterclockwise (once) on Π_+. The cycles are chosen so that their intersection matrix reads

$$a_j \circ b_k = \delta_{j,k}, \quad 1 \le j, k \le g. \tag{A.18}$$

Introducing the invertible matrix C in $\mathbb{C}^g$,

$$\begin{aligned} &C = (C_{j,k})_{1\le j,k\le g},\ C_{j,k} = \int_{a_k} \eta_j = 2\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}\,dz}{R_{2g+2}(z)^{1/2}} \in \mathbb{R}, \\ &\underline{c}(k) = (c_1(k), \dots, c_g(k)),\ c_j(k) = C_{j,k}^{-1}, \end{aligned} \tag{A.19}$$

the normalized differentials ω_j, $1 \le j \le g$,

$$\omega_j = \sum_{\ell=1}^{g} c_j(\ell)\eta_\ell,\ \int_{a_k} \omega_j = \delta_{j,k}, \quad 1 \le j, k \le g \tag{A.20}$$

form a canonical basis for the space of holomorphic differentials on K_g. The matrix τ in $\mathbb{C}^g$ of b-periods,

$$\tau = (\tau_{j,k})_{1\le j,k\le g}, \quad \tau_{j,k} = \int_{b_k} \omega_j \tag{A.21}$$

satisfies

$$\tau_{j,k} = \tau_{k,j}, \quad 1 \le j,k \le g, \tag{A.22}$$

$$\tau = iT, \quad T > 0. \tag{A.23}$$

In the charts $(U_{\infty_\pm}, \zeta_{\infty_\pm} \equiv \zeta)$ induced by $1/\tilde{\pi}$ near $\infty_\pm$ one infers

$$\begin{aligned} \underline{\omega} &= \mp \sum_{j=1}^{g} \underline{c}(j) \frac{\zeta^{g-j}\, d\zeta}{[\Pi_m (1-\zeta E_m)]^{1/2}} \\ &= \pm\Big\{\underline{c}(g) + \zeta\Big[\frac{1}{2}\underline{c}(g) \sum_{m=0}^{2g+1} E_m + \underline{c}(g-1)\Big] + O(\zeta^2)\Big\}\, d\zeta. \end{aligned} \tag{A.24}$$

Associated with the homology basis $\{a_j, b_j\}_{1\le j\le g}$ we also recall the canonical dissection of K_g along its cycles yielding the simply connected interior $\hat{K}_g$ of the fundamental polygon $\partial\hat{K}_g$ given by

$$\partial\hat{K}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g^{-1} b_g^{-1}. \tag{A.25}$$

The Riemann theta function associated with K_g is defined by

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^g} \exp[2\pi i(\underline{n},\underline{z}) + \pi i(\underline{n},\tau\underline{n})], \quad \underline{z} = (z_1,\ldots,z_g) \in \mathbb{C}^g, \tag{A.26}$$

where $(\underline{u},\underline{v}) = \sum_{j=1}^{g} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^g$. It has the fundamental properties

$$\begin{aligned} &\theta(z_1,\ldots,z_{j-1},-z_j,z_{j+1},\ldots,z_g) = \theta(\underline{z}), \\ &\theta(\underline{z}+\underline{m}+\tau\underline{n}) = \exp[-2\pi i(\underline{n},\underline{z}) - \pi i(\underline{n},\tau\underline{n})]\theta(\underline{z}), \quad \underline{m},\underline{n}\in\mathbb{Z}^g. \end{aligned} \tag{A.27}$$

A divisor $\mathcal{D}$ on K_g is a map $\mathcal{D}: K_g \to \mathbb{Z}$, where $\mathcal{D}(P) \ne 0$ for only finitely many $P \in K_g$. The set of all divisors on K_g will be denoted by $\mathrm{Div}(K_g)$. With L_g we denote the period lattice

$$L_g := \{\underline{z}\in\mathbb{C}^g \,|\, \underline{z} = \underline{m}+\tau\underline{n},\ \underline{m},\underline{n}\in\mathbb{Z}^g\} \tag{A.28}$$

and the Jacobi variety $J(K_g)$ is defined by

$$J(K_g) = \mathbb{C}^g/L_g. \tag{A.29}$$

The Abel maps $\underline{A}_{P_0}(.)$ respectively $\underline{\alpha}_{P_0}(.)$ are defined by

$$\underline{A}_{P_0} : \begin{cases} K_g \to J(K_g) \\ P \to \underline{A}_{P_0}(P) = \int_{P_0}^{P} \underline{\omega} \mod (L_g) \end{cases}, \tag{A.30}$$

$$\underline{\alpha}_{P_0} : \begin{cases} \mathrm{Div}(K_g) \to J(K_g) \\ \mathcal{D} \to \underline{\alpha}_{P_0}(\mathcal{D}) = \sum_{P \in K_g} \mathcal{D}(P) \underline{A}_{P_0}(P) \end{cases}, \tag{A.31}$$

with $P_0 \in K_g$ a fixed base point. (In the main text we agree to fix $P_0 = (E_0, 0)$ for convenience.)

In connection with (A.25) we shall also need the maps

$$\underline{\hat{A}}_{P_0} : \begin{cases} \hat{K}_g \to \mathbb{C}^g \\ P \to \int_{P_0}^{P} \underline{\omega} \end{cases}, \quad \underline{\hat{A}}_{P_0} : \begin{cases} \mathrm{Div}(K_g) \to \mathbb{C}^g \\ \mathcal{D} \to \sum_{P \in \hat{K}_g} \mathcal{D}(P) \underline{\hat{A}}_{P_0}(P) \end{cases}, \tag{A.32}$$

with path of integration lying in $\hat{K}_g$.

Let $\mathcal{M}(K_g)$ and $\mathcal{M}^1(K_g)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on K_g. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(K_g)$ at a point $Q_0 \in K_g$ is defined by

$$\operatorname*{res}_{Q_0}(\nu) = \frac{1}{2\pi i} \int_{\gamma_{Q_0}} \nu, \tag{A.33}$$

where γ_{Q_0} is a counterclockwise oriented smooth simple closed contour encircling Q_0 but no other pole of ν. Holomorphic differentials are also called (Abelian) differentials of the first kind (dfk), (Abelian) differentials of the second kind (dsk) $\omega^{(2)} \in \mathcal{M}^1(K_g)$ are characterized by the property that all their residues vanish. They are normalized, for instance, by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad 1 \leq j \leq g. \tag{A.34}$$

If $\omega^{(2)}_{P_1,n}$ is a dsk on K_g whose only pole is $P_1 \in \hat{K}_g$ with principal part $\zeta^{-n-2}\, d\zeta$, $n \in \mathbb{N}_0$ near P_1 and $\omega_j = (\sum_{m=0}^{\infty} d_{j,m}(P_1)\zeta^m)\, d\zeta$ near P_1, then

$$\int_{b_j} \omega^{(2)}_{P_1,n} = \frac{2\pi i}{n+1} d_{j,n}(P_1). \tag{A.35}$$

Any meromorphic differential $\omega^{(3)}$ on K_g not of the first or second kind is said to be of the third kind (dtk). A dtk $\omega^{(3)} \in \mathcal{M}^1(K_g)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad 1 \leq j \leq g. \tag{A.36}$$

A normal dtk $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \hat{K}_g$, $P_1 \neq P_2$ by definition has simple poles at P_1 and P_2 with residues $+1$ at P_1 and -1 at P_2 and vanishing a-periods. If $\omega^{(3)}_{P,Q}$ is a normal dtk associated with P, $Q \in \hat{K}_g$, holomorphic on $K_g \backslash \{P, Q\}$, then

$$\int_{b_j} \omega^{(3)}_{P,Q} = 2\pi i \int_Q^P \omega_j, \quad 1 \leq j \leq g, \tag{A.37}$$

where the path from Q to P lies in $\hat{K}_g$ (i.e., does not touch any of the cycles a_j, b_j).

We shall always assume (without loss of generality) that all poles of dsk's and dtk's on K_g lie on $\hat{K}_g$ (i.e., not on $\partial\hat{K}_g$).

For $f \in \mathcal{M}(K_g)\backslash\{0\}$, $\omega \in \mathcal{M}^1(K_g)\backslash\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \mathrm{Div}(K_g)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(K_g)\backslash\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \mathrm{Div}(K_g) | \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(g-1),\ f \in \mathcal{M}(K_g)\backslash\{0\},\ \omega \in \mathcal{M}^1(K_g)\backslash\{0\}, \tag{A.38}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in K_g} \mathcal{D}(P)$. It is custom to call (f) (respectively (ω)) a principal (respectively canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(K_g) | f = 0 \text{ or } (f) \geq \mathcal{D}\},\ r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.39}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(K_g) | \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\},\ i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}), \tag{A.40}$$

($i(\mathcal{D})$ the index of specialty of $\mathcal{D}$) one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \mathrm{Div}(K_g)$, $\omega \in \mathcal{M}^1(K_g)\backslash\{0\}$. Then*
(i).

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad g \in \mathbb{N}_0. \tag{A.41}$$

(ii) (Riemann-Roch theorem).

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - g + 1, \quad g \in \mathbb{N}_0. \tag{A.42}$$

(iii) (Abel's theorem). $\mathcal{D} \in \mathrm{Div}(K_g)$, $g \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{P_0}(\mathcal{D}) = \underline{0}. \tag{A.43}$$

(iv) (Jacobi's inversion theorem). Assume $g \in \mathbb{N}$, then $\underline{\alpha}_{P_0} : \mathrm{Div}(K_g) \to J(K_g)$ is surjective.

For notational convenience we agree to abbreviate

$$\mathcal{D}_Q : \begin{cases} K_g \to \{0,1\} \\ P \to \begin{cases} 1, & P = Q \\ 0, & P \neq Q \end{cases} \end{cases} \tag{A.44}$$

and, for $\underline{Q} = (Q_1, \dots, Q_g) \in \sigma^g K_g$ ($\sigma^n K_g$ the n-th symmetric power of K_g),

$$\mathcal{D}_{\underline{Q}} : \begin{cases} K_g \to \{0, 1, \dots, g\} \\ P \to \begin{cases} m & \text{if } P \text{ occurs } m \text{ times in } \{Q_1, \dots, Q_g\} \\ 0 & \text{if } P \notin \{Q_1, \dots, Q_g\} \end{cases} \end{cases}. \tag{A.45}$$

Moreover, $\sigma^n K_g$ can be identified with the set of positive divisors $0 < \mathcal{D} \in \mathrm{Div}(K_g)$ of degree n.

Lemma A.2. *Let $\mathcal{D}_{\underline{Q}} \in \sigma^g K_g$, $\underline{Q} = (Q_1, \dots, Q_g)$. Then*

$$1 \le i(\mathcal{D}_{\underline{Q}}) = s(\le g/2) \tag{A.46}$$

if and only if there are s pairs of the type $(P, P^*) \in \{Q_1, \dots, Q_g\}$ (this includes, of course, branch points for which $P = P^*$).

We emphasize that most results in this appendix immediately extend to the case where $\{E_m\}_{0 \le m \le 2g+1} \subset \mathbb{C}$. (In this case τ is no longer purely imaginary as stated in (A.23) but has a positive definite imaginary part.)

Appendix B. Periodic Jacobi Operators

Due to the extensive attention paid in the literature to the theory of periodic Jacobi matrices (see, e.g., [2], [9], [19], [22], [51], [52], [62], [68]–[70]) we shall now summarize the highlights of this special case. Throughout this appendix we shall use (and extend) the notation established in Section 4 in connection with (bounded) Jacobi operators.

In addition to the assumption a, $b \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$, $a(n) \neq 0$, $n \in \mathbb{Z}$ in (4.1), (4.2) we now add the periodicity condition

$$a(n+N) = a(n), \; b(n+N) = b(n), \quad n \in \mathbb{Z} \tag{B.1}$$

for some $N \in \mathbb{N}$. (In most formulas below we tacitly avoid the trivial case $N = 1$ (cf. Appendix C) but assume $N \ge 2$ instead.) We agree to abbreviate

$$A = \prod_{n=1}^{N} a(n) = \prod_{n=1}^{N} a(n_0 + n), \;\; B = \sum_{n=1}^{N} b(n) = \sum_{n=1}^{N} b(n_0 + n), \quad n_0 \in \mathbb{Z}. \tag{B.2}$$

Given the fundamental system of solutions $c(z, n, n_0)$, $s(z, n, n_0)$ (see (4.16)) of (4.6) one defines the fundamental matrix

$$\begin{aligned} \Phi(z, n, n_0) &= \begin{pmatrix} c(z, n, n_0) & s(z, n, n_0) \\ c(z, n+1, n_0) & s(z, n+1, n_0) \end{pmatrix} \\ &= \begin{cases} U_n(z) \cdots U_{n_0+1}(z), & n \geq n_0 + 1 \\ 1, & n = n_0 \\ U_{n+1}^{-1}(z) \cdots U_{n_0}^{-1}(z), & n \leq n_0 - 1 \end{cases}, \end{aligned} \tag{B.3}$$

where

$$\begin{aligned} U_m(z) &= \frac{1}{a(m)} \begin{pmatrix} 0 & a(m) \\ -a(m-1) & z + b(m) \end{pmatrix}, \\ U_m(z)^{-1} &= \frac{1}{a(m-1)} \begin{pmatrix} z + b(m) & -a(m) \\ a(m-1) & 0 \end{pmatrix}. \end{aligned} \tag{B.4}$$

Since

$$W(c(z, ., n_0), s(z, ., n_0)) = a(n_0), \tag{B.5}$$

an arbitrary solution $\psi(z)$ of (4.6) is of the type

$$\psi(z, n) = \psi(z, n_0) c(z, n, n_0) + \psi(z, n_0 + 1) s(z, n, n_0), \tag{B.6}$$

or equivalently,

$$\begin{pmatrix} \psi(z, n) \\ \psi(z, n+1) \end{pmatrix} = \Phi(z, n, n_0) \begin{pmatrix} \psi(z, n_0) \\ \psi(z, n_0 + 1) \end{pmatrix}. \tag{B.7}$$

Moreover, one infers

$$\det[\Phi(z, n, n_0)] = \frac{a(n_0)}{a(n)}, \tag{B.8}$$

$$\Phi(z, n, n_0) = \Phi(z, n, n_1) \Phi(z, n_1, n_0), \tag{B.9}$$

$$\Phi(z, n, n_0)^{-1} = \Phi(z, n_0, n). \tag{B.10}$$

The monodromy matrix $M(z, n)$ is then defined by

$$M(z, n) = \Phi(z, n + N, n) \tag{B.11}$$

and hence

$$M(z, n) = \Phi(z, n, n_0) M(z, n_0) \Phi(z, n, n_0)^{-1} \tag{B.12}$$

and

$$\det[M(z,n)] = 1. \tag{B.13}$$

The Floquet discriminant $\Delta(z)$ defined by

$$\Delta(z) = \mathrm{Tr}[M(z,n)]/2 \tag{B.14}$$

is independent of n (cf. (B.12)) and the Floquet multipliers $m_\pm(z)$ (the eigenvalues of $M(z,n)$) then read

$$m_\pm(z) = \Delta(z) \pm [\Delta(z)^2 - 1]^{1/2}. \tag{B.15}$$

Again by (B.12) they are independent of n and satisfy

$$m_+(z)m_-(z) = 1, \; m_+(z) + m_-(z) = 2\Delta(z). \tag{B.16}$$

Let $\{\tilde{E}_\ell\}_{0\le\ell\le 2N-1}$ be the zeros of $\Delta(z)^2 - 1$ and write

$$\Delta(z)^2 - 1 = \frac{1}{4A^2}\prod_{\ell=0}^{2N-1}(z - \tilde{E}_\ell) \tag{B.17}$$

and

$$\Delta(z) \mp 1 = \frac{1}{2A}\prod_{j=1}^{N}(z - E_j^\pm). \tag{B.18}$$

The zeros $\{E_j^\pm\}_{1\le j\le N}$ turn out to be the eigenvalues of the following periodic respectively antiperiodic Jacobi matrices $\tilde{H}_{n_0}^\pm$ in $\mathbb{C}^N$. More generally, define $\tilde{H}_{n_0}^\theta$ in $\mathbb{C}^N$ associated with the boundary conditions

$$a(n_0+N)\psi(n_0+N) = e^{i\theta}a(n_0)\psi(n_0), \; \psi(n_0+N+1) = e^{i\theta}\psi(n_0+1), \; 0 \le \theta < 2\pi \tag{B.19}$$

by

$$\tilde{H}_{n_0}^\theta = \begin{pmatrix} -b(n_0+1) & a(n_0+1) & 0 & \cdots & & 0 & e^{-i\theta}a(n_0+N) \\ a(n_0+1) & -b(n_0+2) & \ddots & & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & & \ddots & \ddots & & -b(n_0+N-1) & a(n_0+N-1) \\ e^{i\theta}a(n_0+N) & 0 & \cdots & 0 & & a(n_0+N-1) & -b(n_0+N) \end{pmatrix},$$

$$0 \le \theta < 2\pi. \tag{B.20}$$

One infers that $\tilde{H}_{n_0}^\theta$ and $\tilde{H}_{n_0}^{2\pi-\theta}$ are antiunitarily equivalent. The periodic respectively antiperiodic Jacobi matrices $\tilde{H}_{n_0}^\pm$ alluded to above are then defined by

$$\tilde{H}_{n_0}^+ = \tilde{H}_{n_0}^0, \; \tilde{H}_{n_0}^- = \tilde{H}_{n_0}^\pi. \tag{B.21}$$

The eigenvalues of $\tilde{H}^{\theta}_{n_0}$ are then given by

$$(m_+(z) - e^{i\theta})(m_-(z) - e^{i\theta}) = 0, \text{ that is, } \Delta(z) = \cos(\theta). \tag{B.22}$$

They are simple for $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and at most twice degenerate for $\theta = 0, \pi$. In the latter case one infers

$$E_1^{\pm} < E_1^{\mp} \leq E_2^{\mp} < E_2^{\pm} \leq E_3^{\pm} < \cdots < E_{N-1}^{(-1)^{N-1}} \leq E_N^{(-1)^{N-1}} < E_N^{(-1)^N}, \\ \operatorname{sgn}(A) = \pm(-1)^N, \tag{B.23}$$

and $\{\tilde{E}_\ell\}_{0 \leq \ell \leq 2N-1}$ coincides with the corresponding sequence in (B.23). Another way to express these facts is to invoke the theory of direct integral decompositions (see, e.g., [79], Sect. XIII. 16)

$$\ell^2(\mathbb{Z}) \cong \int_{[0,2\pi)}^{\oplus} \frac{d\theta}{2\pi} \ell^2((n_0+1, n_0+N)), \; H \cong \int_{[0,2\pi)}^{\oplus} \frac{d\theta}{2\pi} \tilde{H}^{\theta}_{n_0} \tag{B.24}$$

and $\cong$ denotes unitary equivalence. In particular, the spectrum $\sigma(H)$ of H is characterized by

$$\sigma(H) = \{\lambda \in \mathbb{R} \big| |\Delta(\lambda)| \leq 1\} = \bigcup_{j=0}^{N-1} [\tilde{E}_{2j}, \tilde{E}_{2j-1}] \tag{B.25}$$

(see Theorem 4.2 for additional information).

Returning to the square root $[\Delta(z)^2 - 1]^{1/2}$ in (B.15), we shall consider it as a fixed branch defined as follows,

$$\begin{aligned} &[\Delta(\lambda)^2 - 1]^{1/2} = -\operatorname{sgn}(A) |[\Delta(\lambda)^2 - 1]^{1/2}|, \quad \lambda > \tilde{E}_{2N-1}, \\ &[\Delta(\lambda)^2 - 1]^{1/2} = \lim_{\epsilon \downarrow 0} [\Delta(\lambda + i\epsilon)^2 - 1]^{1/2}, \quad \lambda \in \mathbb{R}, \end{aligned} \tag{B.26}$$

assuming $[\Delta(z)^2 - 1]^{1/2}$ to be analytic in $\mathbb{C} \backslash \bigcup_{j=0}^{N-1} [\tilde{E}_{2j}, \tilde{E}_{2j+1}]$. As a consequence one obtains

$$|m_+(z)| \leq 1, \; |m_-(z)| \geq 1 \tag{B.27}$$

and the (normalized) Floquet functions $\psi_{\pm}(z, n, n_0)$ in (4.17) then satisfy

$$\psi_{\pm}(z, n+N, n_0) = m_{\pm}(z) \psi_{\pm}(z, n, n_0) \tag{B.28}$$

and (4.18). In addition, one infers (cf. (4.4), (4.17)–(4.19))

$$\begin{aligned}\phi_\pm(z,n_0)=\phi_\pm(z,n_0+N)&=\frac{m_\pm(z)-c(z,n_0+N,n_0)}{s(z,n_0+N,n_0)}\\&=\frac{c(z,n_0+N+1,n_0)}{m_\pm(z)-s(z,n_0+N+1,n_0)},\end{aligned} \tag{B.29}$$

$$W(\psi_-(z,.,n_0),\,\psi_+(z,.,n_0))=\frac{2a(n_0)[\Delta(z)^2-1]^{1/2}}{s(z,n_0+N,n_0)}, \tag{B.30}$$

$$G(z,n,n)=\frac{s(z,n+N,n)}{2a(n)[\Delta(z)^2-1]^{1/2}}=\frac{\prod_{j=1}^{N-1}[z-\mu_j(n)]}{\left\{\prod_{\ell=0}^{2N-1}[z-\tilde E_\ell]\right\}^{1/2}}, \tag{B.31}$$

$$\psi_+(z,n,n_0)\psi_-(z,n,n_0)=\frac{a(n_0)s(z,n+N,n)}{a(n)s(z,n_0+N,n_0)}=\prod_{j=1}^{N-1}\left[\frac{z-\mu_j(n)}{z-\mu_j(n_0)}\right]. \tag{B.32}$$

If all spectral gaps of H are "open", that is, the spectra of $\tilde H^\pm_{n_0}$ are both simple, we have

$$g=N-1,\quad \left[\prod_{\ell=0}^{2N-1}(z-\tilde E_\ell)\right]^{1/2}=R_{2g+2}(z)^{1/2}=2A[\Delta(z)^2-1]^{1/2}, \tag{B.33}$$

see (A.4), (A.5). In the case where some spectral gaps "close" we introduce the index sets

$$J'=\{1\le j'\le N-1|\tilde E_{2j'-1}=\tilde E_{2j'}\},\ J=\{0,1,\dots,2N-1\}\backslash\{j',j'+1|j'\in J'\} \tag{B.34}$$

and define

$$Q(z)=\frac{1}{2A}\prod_{j'\in J'}(z-\tilde E_{2j'-1}),\quad R_{2g+2}(z)=\prod_{j\in J}(z-\tilde E_j). \tag{B.35}$$

In order to establish the connection with the notation employed in the main text and in Appendix A we agree to identify

$$\{\tilde E_j\}_{j\in J}\text{ and }\{E_m\}_{0\le m\le 2g+1}. \tag{B.36}$$

Then one infers

$$\begin{aligned}g=N-1-|J'|=N-1-\deg(Q)=(|J|-2)/2,\\ [\Delta(z)^2-1]^{1/2}=R_{2g+2}(z)^{1/2}Q(z),\end{aligned} \tag{B.37}$$

where $|J|,|J'|$ abbreviates the cardinality of J,J'.

Next we shall give the

Proof of Theorem 5.7. First we claim that in the periodic case $\omega^{(3)}_{\infty_+,\infty_-}$ is explicitly given by

$$\omega^{(3)}_{\infty_+,\infty_-} = \frac{\operatorname{sgn}(A)\Delta'\,d\tilde{\pi}}{NR^{1/2}_{2g+2}Q} = \frac{\operatorname{sgn}(A)\Delta'\,d\tilde{\pi}}{N[\Delta^2-1]^{1/2}}. \tag{B.38}$$

For a proof of (B.38) we only need to check that it is appropriately normalized, that is, all its a-periods vanish. This is seen from

$$\begin{aligned}\left|\int_{a_j}\omega^{(3)}_{\infty_+,\infty_-}\right| &= \frac{2}{N}\left|\int_{E_{2j-1}}^{E_{2j}}\frac{dz\Delta'(z)}{|[\Delta(z)^2-1]^{1/2}|}\right| = \frac{2}{N}\Big|\ln\{\Delta(z)+[\Delta(z)^2-1]^{1/2}\}\Big|_{z=E_{2j-1}}^{E_{2j}}\Big| \\ &= 0, \quad 1\le j\le g.\end{aligned} \tag{B.39}$$

For the b-periods, one computes

$$\begin{aligned}\int_{b_j}\omega^{(3)}_{\infty_+,\infty_-} &= \frac{2i}{N}\sum_{k=1}^{j}\left|\int_{E_{2k}}^{E_{2k+1}}\frac{dz\Delta'(z)}{[\Delta(z)^2-1]^{1/2}}\right| = \frac{2i}{N}\sum_{k=1}^{j}\Big|\arcsin[\Delta(z)]\Big|_{z=E_{2k}}^{E_{2k+1}}\Big| \\ &= 2\pi i(j/N), \quad 1\le j\le g.\end{aligned} \tag{B.40}$$

By (3.44) this implies

$$2N\underline{\hat{A}}_{P_0}(\infty_+) = \underline{0} \mod (L_g) \tag{B.41}$$

which completes the proof. □

Next, we indicate a systematic approach to high-energy expansions of $c(z,n,n_0)$ and $s(z,n,n_0)$. First we note that (B.10) yields

$$\begin{aligned} s(z,n+1,n_0) &= a(n_0)a(n)^{-1}c(z,n_0,n), \\ s(z,n,n_0) &= -a(n_0)a(n)^{-1}s(z,n_0,n), \\ c(z,n,n_0) &= a(n_0)a(n)^{-1}s(z,n_0+1,n), \\ c(z,n+1,n_0) &= -a(n_0)a(n)^{-1}c(z,n_0+1,n)\end{aligned} \tag{B.42}$$

and (B.7) implies

$$\begin{aligned} &s(z,n,n_0+1) = -a(n_0+1)a(n_0)^{-1}c(z,n,n_0), \\ &c(z,n,n_0-1) = -a(n_0-1)a(n_0)^{-1}s(z,n,n_0), \\ &s(z,n,n_0-1) = c(z,n,n_0) + [b(n_0)+z]a(n_0)^{-1}s(z,n,n_0), \\ &c(z,n,n_0+1) = s(z,n,n_0) + [b(n_0+1)+z]a(n_0+1)^{-1}c(z,n,n_0).\end{aligned} \tag{B.43}$$

Next we define the Jacobi matrix $J_{n_0}(k)$ in $\mathbb{C}^k$

$$J_{n_0}(k) = \begin{pmatrix} -b(n_0+1) & a(n_0+1) & 0 & \cdots & 0 \\ a(n_0+1) & -b(n_0+2) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -b(n_0+k-1) & a(n_0+k-1) \\ 0 & \cdots & 0 & a(n_0+k-1) & -b(n_0+k) \end{pmatrix} \tag{B.44}$$

and introduce

$$P_{n_0}(n,k) = \frac{1}{n}\{\mathrm{Tr}[J_{n_0}(k)^n] - \sum_{j=1}^{n-1} P_{n_0}(j,k)\,\mathrm{Tr}[J_{n_0}(k)^{n-j}]\}. \tag{B.45}$$

One then obtains

$$s(z,n_0+k+1,n_0) = \frac{\det[z-J_{n_0}(k)]}{\prod_{n=1}^k a(n_0+n)} = \frac{z^k - \sum_{\ell=1}^k P_{n_0}(\ell,k) z^{k-\ell}}{\prod_{n=1}^k a(n_0+n)}, \ k\in\mathbb{N}. \tag{B.46}$$

Explicitly, one computes

$$\begin{aligned} \mathrm{Tr}[J_{n_0}(k)] &= -\sum_{n=n_0+1}^{n_0+k} b(n), \\ \mathrm{Tr}[J_{n_0}(k)^2] &= \sum_{n=n_0+1}^{n_0+k} b(n)^2 + 2\sum_{n=n_0+1}^{n_0+k-1} a(n)^2, \\ \mathrm{Tr}[J_{n_0}(k)^3] &= -\sum_{n=n_0+1}^{n_0+k} b(n)^3 - 3\sum_{n=n_0+1}^{n_0+k-1} a(n)^2[b(n)+b(n+1)], \\ &\text{etc.} \end{aligned} \tag{B.47}$$

Using (B.42) and (B.43) one can extend (B.46) to $k \leq -1$ and to corresponding results for $c(z,n,n_0)$. A direct calculation yields

$$\begin{aligned} c(z,n_0+k+1,n_0) &= -\frac{a(n_0)z^{k-1}}{\prod_{n=1}^k a(n_0+n)}\left[1+z^{-1}\sum_{n=2}^{k} b(n_0+n) + O(z^{-2})\right], \\ c(z,n_0-k,n_0) &= \frac{z^k}{\prod_{n=1}^k a(n_0-n)}\left[1+z^{-1}\sum_{n=0}^{k-1} b(n_0-n) + O(z^{-2})\right], \\ s(z,n_0+k+1,n_0) &= \frac{z^k}{\prod_{n=1}^k a(n_0+n)}\left[1+z^{-1}\sum_{n=1}^{k} b(n_0+n) + O(z^{-2})\right], \\ s(z,n_0-k,n_0) &= -\frac{a(n_0)z^{k-1}}{\prod_{n=1}^k a(n_0-n)}\left[1+z^{-1}\sum_{n=1}^{k-1} b(n_0-n) + O(z^{-2})\right], \quad k\in\mathbb{N}. \end{aligned} \tag{B.48}$$

We emphasize that (B.42)–(B.48) hold for general (not necessarily periodic or finite-gap) Jacobi operators. In the following we shall apply

(B.48) to the periodic case. Equations (B.15), (B.27) yield the expansion

$$\begin{aligned} m_\pm(z) &\underset{|z|\to\infty}{=} (1\mp 1)\Delta(z) \pm \frac{1}{2\Delta(z)} + O(\Delta(z)^{-3}) \\ &\underset{|z|\to\infty}{=} (z^N/A)^{\mp 1}[1+O(z^{-1})] \end{aligned} \tag{B.49}$$

and (B.29) and (B.48) then imply

$$\phi_\pm(z,n) \underset{|z|\to\infty}{=} [a(n)/z]^{\pm 1}\left[1 \mp z^{-1} b\binom{n+1}{n} + O(z^{-2})\right]. \tag{B.50}$$

The relation

$$\psi_\pm(z,n,n_0) = \begin{cases} \prod_{m=n_0}^{n-1}\phi_\pm(z,m), & n\geq n_0+1 \\ 1, & n=n_0 \\ \prod_{m=n}^{n_0-1}\phi_\pm(z,m)^{-1}, & n\leq n_0-1 \end{cases} \tag{B.51}$$

then yields

$$\begin{aligned} \psi_{\underset{(-)}{+}}(z,n_0+k,n_0) &= \left[z^{-k}\prod_{n=0}^{k-1}a(n_0+n)\right]^{\overset{+}{(-)}1}\left[1 \underset{(+)}{-} z^{-1}\sum_{n=1(0)}^{k(k-1)} b(n_0+n) + O(z^{-2})\right], \\ \psi_{\underset{(-)}{+}}(z,n_0-k,n_0) &= \left[z^{-k}\prod_{n=1}^{k}a(n_0-n)\right]^{\overset{-}{(+)}1}\left[1 \underset{(+)}{-} z^{-1}\sum_{n=1(0)}^{k(k-1)} b(n_0-n) + 0(z^{-2})\right], \\ &\qquad k\in\mathbb{N}. \end{aligned} \tag{B.52}$$

Expansions (B.50)–(B.52) also hold in the general case if $\psi_\pm$ are the solutions of (4.6) which are in $\ell^2((0,\pm\infty))$.

These expansions can now be employed to explicitly compute $\tilde{a}$, $\tilde{b}_1$, in Lemma 5.1 (i).

Lemma B.1. *In the periodic case one obtains*

$$\tilde{a} = -|A|^{1/N}, \tag{B.53}$$

$$\tilde{b}_1 = B/N, \tag{B.54}$$

and

$$B = \frac{1}{2}\sum_{\ell=0}^{2N-1}\tilde{E}_\ell. \tag{B.55}$$

Proof. Combining (B.28), (B.52), and (5.1) yields

$$\begin{aligned} m_\pm(z,n_0) &= \psi_\pm(z,n_0+N,n_0) = (A/z^N)^{\pm 1}[1\mp z^{-1}B + O(z^{-2})] \\ &= \operatorname{sgn}(A)(-\tilde{a}/z)^{\pm N}[1\mp z^{-1}N\tilde{b}_1 + O(z^{-2})] \end{aligned} \tag{B.56}$$

and hence (B.53) (noting $\tilde{a} < 0$) and (B.54). Combining (B.54) and (5.3) (accounting for the possibility of closing spectral gaps) then yields (B.55). □

Appendix C. Examples, $g = 0, 1$

In this Appendix we illustrate the two simplest examples in connection with genus $g = 0$ and 1.

We start with $\boldsymbol{g = 0}$:

Let $N \in \mathbb{N}$ be fixed and consider

$$a(n) = a, \ b(n) = b, \quad n \in \mathbb{Z}. \tag{C.1}$$

One then verifies the following series of formulas,

$$\phi_{\pm}(z, n) = (2a)^{-1}(z + b) \pm [(2a)^{-2}(z + b)^2 - 1]^{1/2}, \tag{C.2}$$

$$\psi_{\pm}(z, n, n_0) = \{(2a)^{-1}(z + b) \pm [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^{(n-n_0)}, \tag{C.3}$$

$$\begin{aligned} s(z, n, n_0) = \{&\{(2a)^{-1}(z + b) + [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^{(n-n_0)} \\ &- \{(2a)^{-1}(z + b) - [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^{(n-n_0)}\} \times \\ &\times \{2[(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^{-1}, \end{aligned} \tag{C.4}$$

$$c(z, n, n_0) = -s(z, n - 1, n_0), \tag{C.5}$$

$$\begin{aligned} \Delta(z) = &\frac{1}{2}\{(2a)^{-1}(z + b) + [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^N \\ &+ \frac{1}{2}\{(2a)^{-1}(z + b) - [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^N, \end{aligned} \tag{C.6}$$

$$m_{\pm}(z) = \{(2a)^{-1}(z + b) \pm [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^N, \tag{C.7}$$

$$A = a^N,\ B = Nb, \tag{C.8}$$

$$\tilde{E}_0 = -2|a| - b,\ \tilde{E}_{2j+1} = \tilde{E}_{2j+2} = \mu_j(n) = -2|a|\cos(j\pi/N) - b,$$
$$0 \le j \le N-2,\ n \in \mathbb{Z},\ \tilde{E}_{2N-1} = 2|a| - b, \tag{C.9}$$

$$J' = \{1, 2, \dots, N-1\},\ J = \{0, 2N-1\}, \tag{C.10}$$

$$\begin{aligned} &E_0 = -2|a| - b,\ E_1 = 2|a| - b, \\ &|a| = (E_1 - E_0)/4,\ b = -(E_0 + E_1)/2, \end{aligned} \tag{C.11}$$

$$H = a(S^+ + S^-) - b,\ \mathcal{D}(H) = \ell^2(\mathbb{Z}), \tag{C.12}$$

$$\sigma(H) = [E_0, E_1] = [-2|a| - b,\ 2|a| - b], \tag{C.13}$$

$$R_2(z) = (z - E_0)(z - E_1), \tag{C.14}$$

$$\tilde{a} = -|a|,\ \tilde{b}_1 = -b. \tag{C.15}$$

Concerning the t-dependence of the branches of the BA-function ψ in the simplest case where $r = 0$, that is, for the original Toda system, one obtains

$$\psi_\pm(z, n, n_0, t, t_0) = \{(2a)^{-1}(z + b) \pm [(2a)^{-2}(z + b)^2 - 1]^{1/2}\}^{(n - n_0)} \times$$
$$\times \exp[\pm(t - t_0)R_2(z)^{1/2}]. \tag{C.16}$$

Finally, assuming $a_1(n) = a < 0,\ b_1(n) = b < 0,\ n \in \mathbb{Z}$ and $H_1 \ge 0$, that is, $|b| \ge 2|a|$, one computes,

$$\begin{aligned} &\rho_{e,\pm}(n) = -\{\frac{1}{2}|b| \pm \frac{1}{2}[b^2 - 4a^2]^{1/2}\}^{1/2}, \\ &\rho_{o,\pm}(n) = -\rho_{e,\mp}(n), \end{aligned} \tag{C.17}$$

$$\rho_\pm(n) = \begin{cases} -\{\frac{1}{2}|b| \pm \frac{1}{2}[b^2 - 4a^2]^{1/2}\}^{1/2}, & n = 2m \\ \{\frac{1}{2}|b| \mp \frac{1}{2}[b^2 - 4a^2]^{1/2}\}^{1/2}, & n = 2m+1 \end{cases}, \tag{C.18}$$

$$a_{2,\pm}(n) = a,\ b_{2,\pm}(n) = b, \tag{C.19}$$

etc.

Next we turn to the case $\boldsymbol{g = 1}$:

We suppose

$$E_0 < E_1 < E_2 < E_3,\ R_4(z) = \prod_{m=0}^{3} (z - E_m) \tag{C.20}$$

and introduce the following notations.

$$k = \left[\frac{(E_2 - E_1)(E_3 - E_0)}{(E_3 - E_1)(E_2 - E_0)}\right]^{1/2} \in (0,1),\ k' = \left[\frac{(E_3 - E_2)(E_1 - E_0)}{(E_3 - E_1)(E_2 - E_0)}\right]^{1/2} \in (0,1), \tag{C.21}$$

such that $k^2 + k'^2 = 1$,

$$\bar{u}(z) = \left[\frac{(E_3 - E_1)(E_0 - z)}{(E_3 - E_0)(E_1 - z)}\right]^{1/2}, \ C = \frac{2}{[(E_3 - E_1)(E_2 - E_0)]^{1/2}}, \tag{C.22}$$

and Jacobi's integral of the first

$$F(z,k) = \int_0^z \frac{dx}{[(1-x^2)(1-k^2x^2)]^{1/2}}, \tag{C.23}$$

second

$$E(z,k) = \int_0^z dx \left[\frac{1-x^2}{1-k^2x^2}\right]^{1/2}, \tag{C.24}$$

and third kind

$$\Pi(z,\alpha^2,k) = \int_0^z \frac{dx}{(1-\alpha^2x^2)[(1-x^2)(1-k^2x^2)]^{1/2}}, \quad \alpha^2 \in \mathbb{R}, \tag{C.25}$$

respectively. (We refer, e.g., to [18] for details on Jacobi elliptic integrals.) We define

$$K(k) = F(1,k), \ E(k) = E(1,k), \ \Pi(\alpha^2,k) = \Pi(1,\alpha^2,k) \tag{C.26}$$

and note that all square roots are assumed to be positive for $x \in (0,1)$. We observe that $E(z,k)$ has a simple pole at ∞ while $\Pi(z,\alpha^2,k)$ has simple poles at $z = \pm\alpha^{-1}$.

Given these concepts we can now express the basic objects in connection with the elliptic curve K_1 in terms of the quantities (C.21)–(C.26). We list a series of results below.

The Abelian dfk ω_1 reads in the charts $(\Pi_\pm, z)$,

$$\omega_1 = \frac{dz}{\pm 2CK(k)R_4(z)^{1/2}} \tag{C.27}$$

and one computes

$$\tau_{1,1} = \int_{b_1} \omega_1 = iK(k')/K(k). \tag{C.28}$$

The Abel map A_{P_0} reads

$$A_{P_0}(P) = \pm\frac{F(\bar{u}(z),k)}{2K(k)} \quad \mathrm{mod}\ (L_1), \ P = (z, \pm R_4(z)^{1/2}) \tag{C.29}$$

and hence

$$A_{P_0}(\infty_+) = \frac{F\left(\left(\frac{E_3-E_1}{E_3-E_0}\right)^{1/2}, k\right)}{2K(k)} \mod (L_1). \tag{C.30}$$

The Riemann constant is base point independent and given by

$$\Xi = \frac{1-\tau_{1,1}}{2} \mod (L_1). \tag{C.31}$$

Moreover, one computes

$$\omega^{(3)}_{\infty_+,\infty_-} = \frac{(z-\lambda_1)\,dz}{\pm R_4(z)^{1/2}}, \ \lambda_1 = E_0 + \frac{E_1-E_0}{K(k)}\Pi\left(\frac{E_2-E_1}{E_2-E_0}, k\right), \tag{C.32}$$

$$\int_{b_1} \omega^{(3)}_{\infty_+,\infty_-} = 2\pi i\left[K(k)^{-1}F\left(\left(\frac{E_3-E_1}{E_3-E_0}\right)^{1/2}, k\right)+1\right], \tag{C.33}$$

$$\begin{aligned}\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-} = \pm C(E_1-E_0)\Bigg\{ &\left[1-K(k)^{-1}\Pi\left(\frac{E_2-E_1}{E_2-E_0}, k\right)\right]F(\bar u(z),k) \\ &-\Pi\left(\bar u(z), \frac{E_3-E_0}{E_3-E_1}, k\right)\Bigg\}, \quad P=(z,\pm R_4(z)^{1/2}).\end{aligned} \tag{C.34}$$

Restricting ourselves to the case $r=0$ (i.e., the original Toda system) one obtains for

$$\Omega^{(2)}_0 = \omega^{(2)}_{\infty_+,0} - \omega^{(2)}_{\infty_-,0} \tag{C.35}$$

(cf. (6.30)) the explicit relations

$$2\pi i U^{(2)}_{0,1} = \int_{b_1} \Omega^{(2)}_0 = 4\pi i c_1(1) = \frac{2\pi i}{CK(k)}, \quad r=0, \tag{C.36}$$

$$\begin{aligned}\int_{P_0}^{P} \Omega^{(2)}_0 = \frac{1}{2}C(E_2-E_0)\Bigg\{&(E_3-E_1)K(k)^{-1}E(k)F(\bar u(z),k) \\ &-(E_3-E_1)E(\bar u(z),k) - (E_3-E_0)\left[1-\frac{E_3-E_0}{E_3-E_1}\bar u(z)^2\right]^{-1}\times \\ &\times \bar u(z)[1-\bar u(z)^2]^{1/2}[1-k^2\bar u(z)^2]^{-1/2}\Bigg\}, \\ &r=0,\ P=(z,\pm R_4(z)^{1/2}).\end{aligned} \tag{C.37}$$

The relation

$$A_{P_0}(\hat\mu_1(n,t)) = A_{P_0}(\hat\mu_1(n_0,t_0)) - 2(n-n_0)A_{P_0}(\infty_+) - 2(t-t_0)c_1(1) \tag{C.38}$$

then yields

$$\mu_1(n,t) = E_1\Big\{1-\Big(\tfrac{E_2-E_1}{E_2-E_0}\Big)\tfrac{E_0}{E_1}\operatorname{sn}^2\Big[2K(k)\delta_1+2(n-n_0)F\Big(\Big(\tfrac{E_3-E_1}{E_3-E_0}\Big)^{1/2},k\Big)+2C^{-1}(t-t_0)\Big]\Big\}\times \\ \times\Big\{1-\Big(\tfrac{E_2-E_1}{E_2-E_0}\Big)\operatorname{sn}^2\Big[2K(k)\delta_1+2(n-n_0)F\Big(\Big(\tfrac{E_3-E_1}{E_3-E_0}\Big)^{1/2},k\Big)+2C^{-1}(t-t_0)\Big]\Big\}^{-1}, \tag{C.39}$$

where we abbreviated

$$A_{P_0}(\hat{\mu}_1(n_0,t_0)) = \Big(-\delta_1+\frac{\tau_{1,1}}{2}\Big) \quad \bmod\ (L_1) \tag{C.40}$$

and

$$\operatorname{sn}(w)=z,\ w=\int_0^z \frac{dx}{[(1-x^2)(1-k^2x^2)]^{1/2}} = F(z,k). \tag{C.41}$$

The corresponding sheet of $\hat{\mu}_1(n,t)$ can be read off the sign of $\operatorname{sn}\big[2K(k)\delta_1+ \dots\big]$.

Finally we recall that

$$\theta(z)=\vartheta_3(z)=\sum_{n\in\mathbb{Z}}\exp[2\pi inz+\pi i\tau_{1,1}n^2]. \tag{C.42}$$

The results (C.27)–(C.41) now enable one to express all objects like $a_1(n,t)$, $b_1(n,t)$, $a_{2,\pm}(n,t)$, $b_{2,\pm}(n,t)$, $\rho_\pm(n,t)$, (for $r=0$) in terms of the quantities (C.21)–(C.26), (C.41), and (C.42). We omit further details at this point.

Acknowledgments

W. B. is indebted to the Department of Mathematics of the University of Missouri, Columbia for the hospitality extended to him during stays in the springs of 1990 and 1993. Furthermore, he gratefully acknowledges financial supports for both stays by the Amt der Steiermärkischen Landesregierung and the Technical University of Graz, Austria in 1993.

F. G. is indebted to the Department of Mathematical Sciences of the University of Trondheim, NTH, Norway for the extraordinary hospitality extended to him during one month stays in the summers of 1993 and 1994.

Support by the Norwegian Research Council (F. G. and H. H.) is gratefully acknowledged.

G. T. gratefully acknowledges a kind invitation from the Department of Mathematics of the University of Missouri, Columbia (December 1, 1993 to March 31, 1994) where parts of this research were performed.

This stay was supported by a Fellowship of the Austrian Ministry of Science and a Fellowship of the Technical University of Graz, Austria.

References

1. M. Adler, *On the Bäcklund transformation for the Gel'fand-Dickey equations*, Commun. Math. Phys. **80**, 517–527 (1981).
2. M. Adler, L. Haine, and P. van Moerbeke, *Limit matrices for the Toda flow and periodic flags for loop groups*, Math. Ann. **296**, 1–33 (1993).
3. M. Adler and P. van Moerbeke, *Completely integrable systems, Euclidean Lie algebras, and curves*, Adv. Math. **38**, 267–317 (1980).
4. M. Adler and P. van Moerbeke, *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*, Adv. Math. **38**, 318–379 (1980).
5. N. I. Akhiezer, *A continuous analogue of orthogonal polynomials on a system of intervals*, Sov. Math. Dokl. **2**, 1409–1412 (1961).
6. S. J. Al'ber, *Associated integrable systems*, J. Math. Phys. **32**, 916–922 (1991).
7. A. J. Antony and M. Krishna, *Inverse spectral theory for Jacobi matrices and their almost periodicity*, Proc. Ind. Acad. Sci. (to appear).
8. H. F. Baker, *Note on the foregoing paper, "Commutative ordinary differential operators"*, by J. L. Burchnall and T. W. Chaundy, Proc. Roy. Soc. London **A118**, 584–593 (1928).
9. D. Bättig, B. Grébert, J. C. Guillot, and T. Kappeler, *Fibration of the phase space of the periodic Toda lattice*, preprint, 1992.
10. E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.
11. M. Bergvelt and A. P. E. ten Kroode, *τ functions and zero curvature equations of Toda-AKNS type*, J. Math. Phys. **29**, 1308–1320 (1988).
12. B. Birnir, *Complex Hill's equation and the complex periodic Korteweg-de Vries equations*, Commun. Pure Appl. Math. **39**, 1–49 (1986).
13. B. Birnir, *Singularities of the complex Korteweg-de Vries flows*, Commun. Pure Appl. Math. **39**, 283–305 (1986).
14. A. M. Bloch, H. Flaschka, and T. Ratiu, *A convexity theorem for isospectral manifolds of Jacobi matrices in a compact Lie algebra*, Duke Math. J. **61**, 41–65 (1990).
15. O. I. Bogoyavlenskii, *Algebraic constructions of integrable dynamical systems–extensions of the Volterra system*, Russian Math. Surv. **46:3**, 1–64 (1991).
16. J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. Ser. 2, **21**, 420–440 (1923).
17. J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. Roy. Soc. London **A118**, 557–583 (1928).
18. P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.
19. E. Date and S. Tanaka, *Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice*, Progr. Theoret. Phys. **56**, 457–465 (1976).
20. P. A. Deift, *Applications of a commutation formula*, Duke Math. J. **45**, 267–310 (1978).

21. P. A. Deift and L. C. Li, *Generalized affine Lie algebras and the solution of a class of flows associated with the QR eigenvalue algorithm*, Commun. Pure Appl. Math. **42**, 963–991 (1989).
22. P. Deift and L. C. Li, *Poisson geometry of the analog of the Miura maps and Bäcklund-Darboux transformations for equations of Toda type and periodic Toda flows*, Commun. Math. Phys. **143**, 201–214 (1991).
23. P. Deift, T. Kriecherbauer, and S. Venakides, *Forced lattice vibrations – a videotext*, MSRI preprint no. 003-95, 1994.
24. P. Deift, L. C. Li, and C. Tomei, *Matrix factorizations and integrable systems*, Commun. Pure Appl. Math. **42**, 443–521 (1989).
25. B. A. Dubrovin, *Theta functions and non-linear equations*, Russian Math. Surv. **36:2**, 11–92 (1981).
26. B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Integrable Systems I*, in "*Dynamical Systems IV*", V. I. Arnol'd and S. P. Novikov (eds.), Springer, Berlin, 1990.
27. B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russian Math. Surv. **31:1**, 59–146 (1976).
28. F. Ehlers and H. Knörrer, *An algebro-geometric interpretation of the Bäcklund-transformation for the Korteweg-de Vries equation*, Comment. math. Helvetici **57**, 1–10 (1982).
29. G. Eilenberger, *Solitons*, Springer, Berlin, 1983.
30. N. M. Ercolani and H. Flaschka, *The geometry of the Hill equation and of the Neumann system*, Phil. Trans. R. Soc. Lond. **A315**, 405–422 (1985).
31. H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer, New York, 1992.
32. J. D. Fay, *Theta Functions on Riemann Surfaces*, Lecture Notes in Mathematics, A. Dold and B. Eckmann (eds.), **352**, Springer, Berlin, 1973.
33. L. Faybusovich, *Toda flows and isospectral manifolds*, Proc. Amer. Math. Soc. **115**, 837–847 (1992).
34. L. Faybusovich, *Rational functions, Toda flows, and LR-like algorithms*, Linear Algebra Appl. **203–204**, 359–381 (1994).
35. H. Flaschka, *On the Toda lattice. II*, Progr. Theoret. Phys. **51**, 703–716 (1974).
36. F. Gesztesy, *Some applications of commutation methods*, in "*Schrödinger Operators*", H. Holden and A. Jensen (eds.), Lecture Notes in Physics **345**, Springer, Berlin, 1989, pp. 93–117.
37. F. Gesztesy, *Quasi-periodic, finite-gap solutions of the modified Korteweg-de Vries equation*, in "*Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, Volume 1*", S. Albeverio, J. E. Fenstad, H. Holden, and T. Lindstrøm (eds.), Cambridge University Press, Cambridge, 1992, pp. 428–471.
38. F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. **117**, 401–446 (1993).
39. F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, *On the Toda and Kac-van Moerbeke systems*, Trans. Amer. Math. Soc. **339**, 849–868 (1993).
40. F. Gesztesy, D. Race, K. Unterkofler, and R. Weikard, *On Gelfand-Dickey and Drinfeld-Sokolov systems*, Revs. Math. Phys. **6**, 227–276 (1994).
41. F. Gesztesy, W. Schweiger, and B. Simon, *Commutation methods applied to the mKdV-equation*, Trans. Amer. Math. Soc. **324**, 465–525 (1991).

42. F. Gesztesy and R. Svirsky, *(m)KdV solitons on the background of quasi-periodic finite-gap solutions*, Memoirs Amer. Math. Soc. (to appear).
43. F. Gesztesy and Z. Zhao, *Critical and subcritical Jacobi operators defined as Friedrichs extensions*, J. Diff. Eqs. **103**, 68–93 (1993).
44. A. J. Heeger, S. Kivelson, J. R. Schrieffer, and W. P. Su, *Solitons in conducting polymers*, Revs. Mod. Phys. **60**, 781–850 (1988).
45. B. L. Holian, *Shock waves in the Toda lattice: Analysis*, Phys. Rev. **A24**, 2595–2623 (1981).
46. A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Theoret. Math. Phys. **23**, 343–355 (1975).
47. C. G. T. Jacobi, *Über eine neue Methode zur Integration der hyperelliptischen Differentialgleichung und über die rationale Form ihrer vollständigen algebraischen Integralgleichungen*, J. Reine Angew. Math. **32**, 220–226 (1846).
48. I. S. Kac, *On the multiplicity of the spectrum of a second-order differential operator*, Sov. Math. Dokl. **3**, 1035–1039 (1962).
49. M. Kac and P. van Moerbeke, *On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices*, Adv. Math. **16**, 160–169 (1975).
50. M. Kac and P. van Moerbeke, *On some periodic Toda lattices*, Proc. Nat. Acad. Sci. USA **72**, 1627–1629 (1975).
51. M. Kac and P. van Moerbeke, *A complete solution of the periodic Toda problem*, Proc. Nat. Acad. Sci. USA **72**, 2879–2880 (1975).
52. Y. Kato, *On the spectral density of periodic Jacobi matrices*, in "*Non-linear Integrable Systems–Classical Theory and Quantum Theory*, M. Jimbo and T. Miwa (eds.), World Scientific, 1983, pp. 153–181.
53. O. Knill, *Factorization of random Jacobi operators and Bäcklund transformations*, Commun. Math. Phys. **151**, 589–605 (1993).
54. O. Knill, *Renormalization of random Jacobi operators*, Commun. Math. Phys. **164**, 195–215 (1993).
55. A. Krazer, *Lehrbuch der Thetafunktionen*, Chelsea Publ. Comp., New York, 1970.
56. I. M. Krichever, *Algebraic curves and non-linear difference equations*, Russian Math. Surv. **33:4**, 255–256 (1978).
57. I. M. Krichever, *The Peierls model*, Funct. Anal. Appl. **16**, 248–263 (1982).
58. I. M. Krichever, *Nonlinear equations and elliptic curves*, Revs. Sci. Tech. **23**, 51–90 (1983).
59. T. Kriecherbauer, *Forced lattice vibrations*, Ph.D. Thesis, Courant Institute, NYU, 1994.
60. B. A. Kupershmidt, *Discrete Lax equations and differential-difference calculus*, Asterisqué **123**, 1985.
61. S. V. Manakov, *Complete integrability and stochastization of discrete dynamical systems*, Sov. Phys. JETP **40**, 269–274.
62. H. P. McKean, *Integrable systems and algebraic curves*, in "*Global Analysis*", M. Grmela and J. E. Marsden (eds.), Lecture Notes in Mathematics **755**, Springer, Berlin, 1979, pp. 83–200.
63. H. P. McKean, *Variation on a theme of Jacobi*, Commun. Pure Appl. Math. **38**, 669–678 (1985).

64. H. P. McKean, *Geometry of KdV (1): Addition and the unimodular spectral classes*, Rev. Mat. Iberoamericana **2**, 235–261 (1986).
65. H. P. McKean, *Geometry of KdV (2): Three examples*, J. Stat. Phys. **46**, 1115–1143 (1987).
66. H. P. McKean, *Hill and Toda curves*, Commun. Pure Appl. Math. **33**, 23–42 (1980).
67. R. M. Miura, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9**, 1202–1204 (1968).
68. P. van Moerbeke, *The spectrum of Jacobi matrices*, Invent. Math. **37**, 45–81 (1976).
69. P. van Moerbeke, *About isospectral deformations of discrete Laplacians*, in "*Global Analysis*", M. Grmela and J. E. Marsden (eds.), Lecture Notes in Mathematics **755**, Springer, Berlin, 1979, pp. 313–370.
70. P. van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves*, Acta Math. **143**, 93–154 (1979).
71. J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16**, 197–220 (1975).
72. D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations*, Intl. Symp. Algebraic Geometry, 115–153, Kyoto, 1977.
73. D. Mumford, *Tata Lectures on Theta II*, Birkhäuser, Boston, 1984.
74. R. Narasimhan, *Compact Riemann Surfaces*, Birkhäuser, Basel, 1992.
75. A. J. Niemi and G. W. Semenoff, *Fermion number fractionization in quantum field theory*, Phys. Rep. **135**, 99–193 (1986).
76. S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.
77. A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras, Volume I*, Birkhäuser, Basel, 1990.
78. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-adjointness*, Academic Press, New York, 1975.
79. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.
80. M. Sodin and P. Yuditskii, *Infinite-dimensional Jacobi inversion problem, almost-periodic Jacobi matrices with homogeneous spectrum, and Hardy classes of character-automorphic functions*, preprint, 1994.
81. K. Takasaki, *Initial value problem for the Toda lattice hierarchy*, in "*Advanced Studies in Pure Mathematics 4*", K. Okamoto (ed.), North-Holland, Amsterdam, 1984, pp. 139–163.
82. T. Takebe, *Toda lattice hierarchy and conservation laws*, Commun. Math. Phys. **129**, 281–318 (1990).
83. B. Thaller, *Normal forms of an abstract Dirac operator and applications to scattering theory*, J. Math. Phys. **29**, 249–267 (1988).
84. M. Toda, *Theory of Nonlinear Lattices*, 2nd enlarged ed., Springer, Berlin, 1989.
85. K. Ueno and K. Takasaki, *Toda lattice hierarchy*, in "*Advanced Studies in Pure Mathematics 4*", K. Okamoto (ed.), North-Holland, Amsterdam, 1984, pp. 1–95.
86. S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Commun. Pure Appl. Math. **44**, 1171–1242 (1991).

87. M. Wadati, *Transformation theories for nonlinear discrete systems*, Suppl. Progr. Theoret. Phys. **59**, 36–63 (1976).
88. V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, *Nonlinear stage of parametric wave excitation in a plasma*, JETP Lett. **19**, 151–152 (1974).

INSTITUTE FOR THEORETICAL PHYSICS, TECHNICAL UNIVERSITY OF GRAZ, A-8010 GRAZ, AUSTRIA.
E-mail address: bulla@fampds01.tu-graz.ac.at

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA.
E-mail address: mathfg@mizzou1.missouri.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, THE NORWEGIAN INSTITUTE OF TECHNOLOGY, UNIVERSITY OF TRONDHEIM, N-7034 TRONDHEIM, NORWAY.
E-mail address: holden@imf.unit.no

INSTITUTE FOR THEORETICAL PHYSICS, TECHNICAL UNIVERSITY OF GRAZ, A-8010 GRAZ, AUSTRIA AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA.
E-mail address: mathgr42@mizzou1.missouri.edu