On the connection between continuous Hahn polynomials and the Heisenberg algebra

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Abstract. We give an elementary proof for the connection between a system of continuous Hahn polynomials and symmetrizations of elements in the Heisenberg algebra (Bender, Mead and Pinsky 1986, 1987; Koornwinder 1989).

I. INTRODUCTION

Some years ago, while discussing the operator–ordering problem by the method of finite elements, Bender, Mead and Pinsky[1,2] pointed out an interesting connection between certain continuous Hahn polynomials and symmetrizations of elements in the Heisenberg algebra. Precisely, if $[q,p]=i$ and $T_{m,n}$ is the sum of all possible terms containing m factors of p and n factors of q, then

$$T_{n,n} = \frac{(2n)!}{2^n\, n!} S_n(T_{1,1}) \tag{1}$$

where

$$S_n(x) = (-i)^n\, {}_2F_1(-n, \frac{1-ix}{2}; 1; 2) \tag{2}$$

A very elegant proof of (1.1) has been given by Koornwinder[3]. It makes use of a transformation connecting the polynomials S_n to certain Meixner–Pollaczek polynomials, and of an operational formula involving these latter polynomials.

The aim of the present paper is to provide an elementary derivation of (1.1). It is based on a simple trick, which also leads almost immediately to the general result

$$T_{m,n} = (\frac{i}{2})^n \frac{(m+n)!}{n!\,(m-n)!} p^{m-n}\, {}_2F_1(-n, \frac{1+iT_{1,1}}{2}; m-n+1; 2), \quad m \geq n \tag{3a}$$

$$= (-\frac{i}{2})^m \frac{(n+m)!}{m!\,(n-m)!} q^{n-m}\, {}_2F_1(-m, \frac{1-iT_{1,1}}{2}; n-m+1; 2), \quad n \geq m \tag{3b}$$

II. DESCRIPTION OF THE METHOD.

We first recall[4] that $T_{m,n}$ is the coefficient of $\alpha^m\beta^n$ in the expansion of

$(\alpha p + \beta q)^{m+n}$, α and β being arbitrary parameters. Thus

$$\begin{aligned} T_{m,n} &= \frac{1}{m!\,n!}(\frac{\partial}{\partial\alpha})^m(\frac{\partial}{\partial\beta})^n(\alpha p+\beta q)^{m+n}|_{\alpha=\beta=0} \\ &= \frac{1}{m!\,n!}(\frac{\partial}{\partial\alpha})^m(\frac{\partial}{\partial\beta})^n(\frac{\partial}{\partial\lambda})^{m+n}e^{\lambda(\alpha p+\beta q)}|_{\alpha=\beta=\lambda=0} \\ &= \frac{(m+n)!}{m!\,n!}(\frac{\partial}{\partial\alpha})^m(\frac{\partial}{\partial\beta})^n e^{\alpha p+\beta q}|_{\alpha=\beta=0} \end{aligned} \tag{5}$$

whence

$$\sum_{m,n=0}^{\infty} \frac{\alpha^m \beta^n}{(m+n)!} T_{m,n} = e^{\alpha p + \beta q} \tag{6}$$

According to the BCH formula[5], we have

$$e^{\alpha p + \beta q} = e^{\alpha p} e^{\beta q} e^{\frac{i}{2}\alpha\beta} \tag{7a}$$

$$= e^{\beta q} e^{\alpha p} e^{-\frac{i}{2}\alpha\beta} \tag{7b}$$

Furthermore

$$\begin{aligned} e^{\alpha p} e^{\beta q} e^{\frac{i}{2}\alpha\beta} &= \sum_{k,l,r=0}^{\infty} \frac{\alpha^k}{k!} \frac{\beta^l}{l!} \frac{(\frac{i}{2}\alpha\beta)^r}{r!} p^k q^l \\ &= \sum_{m,n=0}^{\infty} \alpha^m \beta^n \left(\sum_{k=0}^{m\wedge n} \frac{\left(\frac{i}{2}\right)^k p^{m-k} q^{n-k}}{(m-k)!(n-k)!k!} \right) \end{aligned} \tag{8}$$

so that, for $n \le m$

$$\begin{aligned} T_{m,n} &= (m+n)! i^{-n} p^{m-n} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)^k (ip)^{n-k} q^{n-k}}{(m-k)!(n-k)!k!} \\ &= (m+n)! i^{-n} p^{m-n} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)^k (i\overline{pq})^{n-k}}{(m-k)!(n-k)!k!} \end{aligned} \tag{9}$$

where $\overline{AB}$ means that, for commuting operators A and B, $(\overline{AB})^n$ has to be interpreted as $A^n B^n$.

Now, by mathematical induction, it is easy to show that, if $[A,B]=1$, then

$$(\overline{AB})^n = (AB)_n \tag{10}$$

where, as usual, $(a)_n$ stands for $a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Thus, eq. (2.5) takes the form

$$T_{m,n} = (m+n)!\, i^{-n} p^{m-n} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)^k (ipq)_{n-k}}{(m-k)!(n-k)!k!} \quad , \quad n \le m \tag{11}$$

which, since $ipq = (1 + iT_{1,1})/2$, coincides with (1.3a). Eq. (1.3b) follows from (2.2) and (2.3b); for $m = n$, we recover (1.1). This concludes our proof.

For the sake of completeness, we quote the further formula, which generalizes (2.6)

$$(\alpha - \beta AB)_n = n!\, g_n^{(\alpha,\beta)}(\overline{AB}) \tag{12}$$

where the polynomials $g_n^{(\alpha,\beta)}(x)$ are defined by the generating function

$$(1-t)^{-\alpha} e^{[1-(1-t)^{-\beta}]x} = \sum_{n=0}^{\infty} t^n g_n^{(\alpha,\beta)}(x) \tag{13}$$

or, equivalently, by the recurrence formula

$$g_{n+1}^{(\alpha,\beta)}(x) = \frac{1}{n+1} \sum_{r=0}^{n} [\alpha - \frac{(\beta+1)_r}{r!} \beta x] g_{n-r}^{(\alpha,\beta)}(x) \tag{14}$$

which is readily obtained from (2.9) by differentiating both sides with respect to t. To prove (2.8), it suffices to observe that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha - \beta AB)_n = (1-t)^{-\alpha} [(1-t)^{-\beta}]^{-AB} \\ = (1-t)^{-\alpha} &\sum_{n=0}^{\infty} \frac{(\overline{AB})^n}{n!} [1-(1-t)^{-\beta}]^n = (1-t)^{-\alpha} e^{[1-(1-t)^{-\beta}]\overline{AB}} \end{aligned} \tag{15}$$

In particular, when $\alpha = 0$ and $\beta = -1$, $g_n^{(\alpha,\beta)}(x)$ is the Laguerre polynomial $L_n^{-1}(x)$, and eq. (2.8) becomes

$$(-AB)_n = n! \, L_n^{-1}(\overline{AB}). \tag{16}$$

If, in (2.8), we replace AB by a c-number x, we get

$$(\alpha - \beta x)_n = n! \, g_n^{(\alpha,\beta)}(\tilde{x}), \qquad (\tilde{x})^r \equiv (x)_r \tag{17}$$

This is the expansion of the polynomial $(\alpha - \beta x)_n$ in terms of the polynomials $(x)_r$.

As a final remark, it may be noted that the orthogonality condition[2]

$$\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{sech}(\frac{\pi x}{2}) S_m(x) S_n(x) = \delta_{mn} \tag{18}$$

is a particular case of the more general integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \, B(\alpha - i\beta x, \lambda + 1 - \alpha + i\beta x) \, S_{m,\lambda'}^{(\alpha',\beta')}(x) S_{n,\lambda}^{(\alpha,\beta)}(x) = \begin{cases} 0 & , \quad \text{m¡n} \\ \frac{\pi n!}{2^{\lambda} (\lambda'+1)_n \beta'^{n} \beta^{-(n+1)}} & , \quad m=n \end{cases} \tag{19}$$

where

$$S_{n,\lambda}^{(\alpha,\beta)}(x) = (-i)^n \, {}_2F_1(-n, \alpha - i\beta x; \lambda + 1; 2) \tag{20}$$

Without loss of generality, we assume $\beta > 0$.

Eq. (2.15) is easily obtained by writing $S_{n,\lambda}^{(\alpha,\beta)}(x)$ as an Eulerian integral[6]

$$\begin{aligned} S_{n,\lambda}^{(\alpha,\beta)}(x) = (-i)^n \frac{\Gamma(\lambda+1)}{\Gamma(\alpha - i\beta x)\Gamma(\lambda + 1 - \alpha + i\beta x)} \int_0^1 \mathrm{d}t \, t^{\alpha - 1 - i\beta x} (1-t)^{\lambda - \alpha + i\beta x} (1-2t)^n \\ \mathrm{Re}\alpha > 0 \quad , \quad \mathrm{Re}(\lambda - \alpha) > -1 \end{aligned} \tag{21}$$

or, by putting $t = (1 + e^{\frac{u}{\beta}})^{-1}$, as a Fourier integral

$$S_{n,\lambda}^{(\alpha,\beta)}(x) = (-i)^n \frac{\Gamma(\lambda+1)}{\Gamma(\alpha - i\beta x)\Gamma(\lambda+1-\alpha+i\beta x)} \frac{1}{2^{\lambda+1}\beta} \int_{-\infty}^{\infty} \mathrm{d}\, u e^{\frac{\lambda-2\alpha+1}{2\beta}u} (\mathrm{sech}\frac{u}{2\beta})^{\lambda+1} \cdot (\tanh\frac{u}{\beta})^n e^{ixu} \tag{22}$$

Since $S_{m,\lambda'}^{(\alpha',\beta')}(x) = (-i)^m \sum_{r=0}^m \frac{(-m)_r(\alpha'-i\beta' x)_r}{(\lambda'+1)_r r!} 2^r$, and

$$\int_{-\infty}^{\infty} \mathrm{d}x\, (\alpha' - i\beta' x)_r e^{iux} = (-\beta')^r 2\pi\, \delta^{(r)}(u) + \ldots \tag{23}$$

where the dots denote terms involving $\delta^{(s)}(u), s = 0, 1, \ldots, r-1$, it is obvious that the integral in (2.15) is zero for $m < n$, because the integrand in (2.18) vanishes as u^n at $u = 0$. If $m = n$, only the term with $r = n$ contributes to the integral, and a straightforward calculation leads to the required result.

As a matter of fact, formulas (2.15) and (2.16) are the orthogonality relations (shifted in the complex plane) and the hypergeometric series definition for special Meixner–Pollaczek polynomials (see, for instance, eqs. (2.3) and (2.4) in reference 3), since $S_{n,\lambda}^{(\alpha,\beta)}(x) = (-i)^n P_n^{(\frac{\lambda+1}{2})}(-\beta x + \frac{i}{2}(\lambda+1) - i\alpha; \frac{\pi}{2})$.

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[1] C.M. Bender, L. R. Mead and S.S. Pinsky, Phys. Rev. Lett. **56**, 2445 (1986).

[2] C. M. Bender, L. R. Mead, and S.S. Pinsky, J. Math. Phys. **28**, 509(1987).

[3] T. H. Koornwinder, J. Math. Phys. **30**, 767 (1989).

[4] B. Sakita, *Quantum Theory of Many-Variable Systems and Fields* (World Scientific Publishing Company, Singapore, 1985), p.62

[5] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967), eq.(4.15).

[6] W.N. Bayley, *Generalized Hypergeometric Series* (Cambridge U. P., Cambridge, 1935), p.4.