Discrete Mathematics and Physics on the Planck-Scale

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Abstract

Starting from the hypothesis that both physics, in particular space-time and the physical vacuum, and the corresponding mathematics are discrete on the Planck scale we develop a certain framework in form of a '*cellular network*' consisting of cells interacting with each other via bonds. Both the internal states of the cells and the "strength" of the bonds are assumed to be dynamical variables. In section 3 the basis is laid for a version of '*discrete analysis*' which, starting from different, perhaps more physically oriented principles, manages to make contact with the much more abstract machinery of Connes et al. and may complement the latter approach. In section 4 a, as far as we can see, new concept of '*topological dimension*' in form of a '*degree of connectivity*' for graphs, networks and the like is developed. It is then indicated how this '*dimension*', which for continuous structures or lattices being embedded in a continuous background agrees with the usual notion of dimension, may change dynamically as a result of a '*phase transition like*' change in '*connectivity*' in the network. A certain speculative argument, along the lines of statistical mechanics, is supplied in favor of the naturalness of dimension 4 of ordinary (classical) space-time.

1 Introduction

There exists a certain suspicion in parts of the scientific community that nature may be "discrete" on the Planck scale. The point of view held by the majority is however, at least as far as we can see, that quantum theory as we know it holds sway down to arbitrarily small scales as an allembracing general principle, being applied to a sequence of increasingly fine grained effective field theories all the way down up to, say, string field theory. But even on that fundamental level one starts from strings moving in a continuous background. It is then argued that "discreteness" enters somehow through the backdoor via "quantisation".

The possibly most radical and heretical attempt, on the other side, is it to try to generate both gravity and quantum theory as secondary and derived concepts (in fact merely two aspects) of one and the same underlying more primordial theory instead of simply trying to quantise gravity, which is the canonical point of view (see e.g. [1]).

This strategy implies more or less directly that – as gravity is closely linked with the dynamics of (continuous) space-time – the hypothetical underlying more fundamental theory is supposed to live on a substratum which does not support from the outset something like continuous topological or geometrical structures. In our view these continuous structures should emerge as derived concepts via some sort of coarse graining over a relatively large number of "discrete" more elementary building blocks.

This program still leaves us with a lot of possibilities. For various reasons, which may become more plausible in the course of the investigation, we personally favor what we would like to call a "cellular network" as a realisation of this substratum, the precise definitions being given below. Without going into any details at the moment some of our personal motivations are briefly the following:

i) These systems are in a natural way discrete, the local state space at each site being usually finite or at least countable.

ii) Systems like these or their (probably better known) close relatives, the "cellular automata", are known to be capable of socalled "complex behavior", "pattern generation" and "selforganisation" in general while the underlying dynamical laws are frequently strikingly simple (a wellknown example being e.g. Conway's "game of life").

Remark: A beautiful introduction into this fascinating field is e.g. [2]. As a shorter review one may take the contribution of Wolfram (l.c.). More recent material can be found in the proceedings of the Santa Fee Institute, e.g. the article of Kauffman in [3], who investigates slightly different systems ("switching nets").

iii) Some people suspect (as also we do) that physics may be reducible at its very bottom to some sort of "information processing system" (cf. e.g. [4, 5]). Evidently cellular automata and the like are optimally adapted to this purpose.

iv) In "ordinary" field theory phenomena evolving in space-time are typically described by forming a fibre bundle over space-time (being locally homeomorphic to a product). In our view a picture like this can only be an approximate one. It conveys the impression that space-time is kind of an arena or stage being fundamentally different from the various fields and phenomena which evolve and interact in it. In our view these localised attributes, being encoded in the various field values, should rather be attributes of the – in the conventional picture hidden – infinitesimal neighborhoods of space-time points, more properly speaking, neighborhoods in a medium in which space-time is immersed as a lower dimensional "submanifold". To put it in a nutshell: We would prefer a medium in which what we typically regard as irreducible space-time points have an internal structure. To give a simple picture from an entirely different field: take e.g. a classical gas, consider local pressure, temperature etc. as collective coarse grained coordinates with respect to the infinitesimal volume elements, regard then the microscopic degrees of freedom of the particles in this small volume elements as the hidden internal structure of the "points" given by the values of the above collective coordinates (warning: this picture is of course not completely correct as the correspondence between the values of local pressure etc. and volume elements is usually not one-one). It will turn out that a discrete structure as alluded to above is a nice playground for modelling such features.

Remark: Evidently there are close ties between what we have said in iv) and certain foundational investigations in pure mathematics concerning the problem of the 'continuum', a catchword being e.g. "non-standard analysis".

A lot more could be said as to the general physical motivations and a lot more literature could be mentioned as e.g. the work of Finkelstein and many others (see e.g. [6, 7]. For further references cf. also the papers of Dimakis and Müller-Hoissen ([8]). Most similar in spirit is in our view however the approach of 't Hooft ([9]).

2 The Concept of the "Cellular Network"

While our primary interest is in the analysis of various partly long standing problems of current physics, which seem to beset physics many orders away from the Planck regime, we nevertheless claim that the understanding of the processes going on in the cellular network at Planck level will provide us with strong clues concerning the phenomena occurring in the "daylight" of "middle-energy-quantum-physics". In fact, as Planck scale physics is – possibly for all times – beyond the reach of experimental confirmation, this sort of serious speculation has to be taken as a substitution for experiments.

To mention some of these urgent problems of present day physics:
i) The unification of quantum theory and gravitation in general,
ii) the emergence of the universe, of space-time from "nothing" and its very early period of existence,
iii) the mystery of the seeming vanishing of the 'cosmological constant', which, in our view, is intimately related to the correct understanding of the nature of vacuum

fluctuations,
iv) the primordial nature of the "Higgs mechanism",
v) causality in quantum physics,
vi) 'potential' versus 'actual' existence in the quantum world and the quantum mechanical measurement problem.

Some of these topics have been adressed by us recently in a somewhat tentative way, based partly on the assumption that nature behaves or can be imitated as a cellular network at its very bottom ([10]). The analysis was however hampered by the fact that the mathematical and technical details of the underlying discrete model were at that time not appropriately developed to a sufficiently high degree. Therefore we will concentrate in the following mainly on establishing the necessary (mostly mathematical) prerequisites on which the subsequent physical investigations can be safely based.

This is the more so necessary because one of our central hypotheses is that most of the hierarchical structure and fundamental building blocks of modern physics come into being via a sequence of *unfolding phase transitions* in this cellular medium. As far as we can see, the study of phase transitions in cellular networks is not yet very far developed, which is understandable given the extreme complexity of the whole field. Therefore a good deal of work should be, to begin with, devoted to a qualitative understanding of this intricate subject.

2.1 Definition(Cellular Automaton): A cellular automaton consists typically of a fixed regular array of cells $\{C_i\}$ sitting on the nodes $\{n_i\}$ of a regular lattice like, say, $\mathbb{Z}^d$ for some d. Each of the cells is characterized by its internal state s_i which can vary over a certain (typically finite) set $\mathcal{S}$ which is usually chosen to be the same for all lattice sites.

Evolution or dynamics take place in discrete steps τ and is given by a certain specific 'local law' LL:

$$s_i(t+\tau) = ll(\{s'_j(t)\} \quad \underline{S}(t+\tau) = LL(\underline{S}(t)) \tag{1}$$

where t denotes a certain "*clock time*" (not necessarily physical time), τ the elementary clock time interval, $\{s'_j\}$ the internal states of the nodes of a certain local neighborhood of the cell C_i, ll a map:

$$ll : \mathcal{S}^n \to \mathcal{S} \tag{2}$$

with n the number of neighbors occurring in (1), $\underline{\mathrm{S}}$(t) the global state at "time" t, LL the corresponding global map acting on the total state space $X := \{\underline{S}\}$. LL is called *reversible* if it is a bijective map of X onto itself.

Cellular automata of this type behave generically already very complicated (see [2]). But nevertheless we suspect they are still not complicated enough in order to perform the specific type of complex behavior we want them to do. For one, they are in our view too regular and rigid. For another, the occurring regular lattices inherit quasi automatically such a physically important notion like '*dimension*' from the underlying embedding space.

Our intuition is however exactly the other way round. We want to generate something like dimension (among other topological notions) via a dynamical process (of phase transition type) from a more primordial underlying model which, at least initially, is lacking such characteristic properties and features.

There exist a couple of further, perhaps subjective, motivations which will perhaps become more apparent in the following and which result in the choice of the following primordial model system:

2.2 Definition(Cellular Network): In the following we will mainly deal with the kind of system defined below:
i) "Geometrically" it is a *graph*, i.e. it consists of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$ where pictorially the bond b_{ik} connects the nodes n_i and n_k with $n_i \neq n_k$ implied (there are graphs where this is not so), furthermore, to each pair of nodes there exists at most one bond connecting them.

The graph is assumed to be *connected*, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds, and *regular*, that is it looks locally the same everywhere. Mathematically this means that the number of bonds being incident with a given node is the same over the graph (*order of a node*). We call the nodes which can be reached from a given node by making one step the *1-order-neighborhood* $\mathcal{U}_1$ and by not more than n steps $\mathcal{U}_n$.
ii) On the graph we implant a dynamics in the following way:

2.3 Definition(Dynamics): As for a cellular automaton each node n_i can be in a number of internal states $s_i \in \mathcal{S}$. Each bond b_{ik} carries a corresponding bond state $J_{ik} \in \mathcal{J}$. Then we assume:

$$s_i(t+\tau) = ll_s(\{s'_k(t)\}, \{J'_{kl}(t)\}) \tag{3}$$

$$J_{ik}(t+\tau) = ll_J(\{s'_l(t)\}, \{J'lm(t)\}) \tag{4}$$

$$(\underline{S}, \underline{J})(t+\tau) = LL((\underline{S}, \underline{J}(t)) \tag{5}$$

where ll_s, ll_J are two mappings (being the same all over the graph) from the state space of a local neighborhood of a given fixed node or bond to $\mathcal{S}, \mathcal{J}$, yielding the updated values of s_i and b_{ik}.

Remarks: i) The theory of graphs is developed in e.g. [11, 12].
ii) Synonyma for 'node' and 'bond' are e.g. 'site' and 'link'.
iii) It may be possible under certain circumstances to replace or rather emulate a cellular network of the above kind by some sort of extended cellular automaton (e.g.

by replacing the bonds by additional sites). The description will then however become quite cumbersome and involved.

What is the physical philosophy behind this picture? We assume the primordial substratum from which the physical universe is expected to emerge via a selforganisation process to be devoid of most of the characteristics we are usually accustomed to attribute to something like a manifold or a topological space. What we are prepared to admit is some kind of "*pregeometry*" consisting in this model under discussion of an irregular array of elementary grains and "direct interactions" between them, more specifically, between the members of the various local neighborhoods.

It is an essential ingredient of our approach that the strength of these direct interactions is of a dynamical nature and allowed to vary. In particular it can happen that two nodes or a whole cluster of nodes start to interact very strongly in the course of the evolution and that this type of *collective behavior* persists for a long time or forever (becomes *locked in*) or, on the other extreme, that the interaction between certain nodes becomes weak or vanishes.

Remark: Note that – in contrast to e.g. lattice field theory – for the time being the socalled '*clock time*' t is not standing on the same footing as, say, potential coordinates in the network (e.g. curves of bonds). We suppose anyhow that socalled '*physical time*' will emerge as sort of a secondary collective variable in the network, i.e. being different from the clock time (while being of course functionally related to it).

In our view this is consistent with the spirit of relativity. What Einstein was really teaching us is that there is a (dynamical) interdependence between what we experience as space and time, not that they are absolutely identical!

As can be seen from the definition of the cellular network it separates quite naturally into two parts of a different mathematical and physical nature. The first one comprises part i) of definition 2.2, the second one part ii) and definition 2.3. The first one is more static and "geometric" in character, the latter one conveys a more dynamical and topological flavor as we shall see in the following.

We begin in section 3 with a representation of what may be called discrete analysis on graphs and networks. This is followed in section 4 by an investigation of certain dynamical processes in networks of the defined type which resemble phase transitions and may induce *dimensional change*. Most importantly we develop a physically appropriate concept of'*dimension*' for such irregular discrete structures which may be of importance in a wider context.

3 Discrete Analysis on Networks

At first glance one would surmise that as an effect of discreteness something like a network will lack sufficient structure for such a discipline to exist, but this is not so. Quite the contrary, there are intimate and subtle relations to various recent branches of pure mathematics as e.g. '*cyclic (co)homology*', '*noncommutative de Rham complexes*', '*noncommutative geometry*' in general and the like (see e.g. [13]-[16]).

The general aim of these recent developements is it to generate something like a geometrical and differentiable structure within certain mathematical contexts which traditionally are not considered to support such structures. Particularly simple examples are discrete sets of, say, points, e.g. lattices. In a series of papers Dimakis and Müller-Hoissen have applied the general abstract machinery to models like these, having a possible bearing to, say, lattice field theory etc. (see e.g. [8] and further references there).

The fundamental object in these approaches is typically the socalled '*universal differential algebra*' or '*differential envelope*' which can be canonically constructed over any associative algebra and which is considered to be a generalisation or surrogate (depending on the point of view) of a differential structure in the ordinary cases.

As this notion may already indicate this scheme, paying tribute to its universality and generality, is sometimes relatively far away from the concrete physical models one is perhaps having in mind. In the case of networks, for example, the inevitable starting point is the '*maximally connected*' network or graph (also called a '*complete graph*' or in algebraic topology a '*simplex*'), i.e. any two nodes are directly connected by a bond.

As a consequence, the construction is lacking, at least initially, something which is of tantamount importance in physical models, i.e. a natural and physically motivated neighborhood structure. Typically the interesting physical models are relatively lowly connected, which implies that they usually exhibit a pronounced feeling of what is near by or far away on the network.

One can of course pull this general structure down to the level of the models one may have in mind by imposing '*relations*' between various classes of '*differential forms*' but anyway, given a concrete model this approach is relatively abstract and perhaps not the most transparent and direct one. Furthermore, as it stresses more the global algebraic relations, it does not naturally contain from the beginning something eminently geometrical like e.g. '*derivations at a point*' which, on the other side, are fundamental in ordinary differential topology.

There are other personal reasons to undertake to complement this elegant but more algebraic framework by an approach which carries, at least in our view, a more physical/geometric flavor and which is in some sense oriented "bottom up" instead of "top down".

We begin with the introduction of some useful concepts borrowed from algebraic topology and also known from graph theory (as to this we recommend the beautiful book of Lefschetz, [17].

At first we have to give the graph an '*orientation*':

3.1 Definition(Orientation): With the notions defined in definition 2.2 we say the bond b_{ik} points from node n_i to node n_k, the bond b_{ki} from n_k to n_i. We call n_i, n_k initial and terminal node of b_{ik} respectively. We assume the up to now formal relation:

$$b_{ik} = -b_{ki} \tag{6}$$

Remark: Note that orientation in the above (mathematical) sense is different from what is understood in many applications as '*directed bond*' in a network (as e.g. in typical "Kauffman nets", [3]). There a directed bond can typically "transport", say, a message only in one given fixed direction. That is, nets of this type behave, in physical terms, pronouncedly anisotropic locally. The defintion 3.1, on the other side, is rather implementing something like the orientation of curves.

3.2 Definition(Chain Complexes): We introduce, to begin with, the two vector spaces C_0, C_1 whose elements, *zero- and one-chains* are defined by up to now formal expressions

$$\underline{c}_0 := \sum f_i n_i \quad \underline{c}_1 := \sum g_{ik} b_{ik} \tag{7}$$

where the f_i's and g_{ik}'s range over a certain given field or ring, of in the simplest cases numbers (i.e.$\mathbb{Z}$,$\mathbb{R}$,$\mathbb{C}$), the n_i's and g_{ik}'s serve as generators of a free abelian group.

Remarks:i) Evidently one could in a next step choose much more general spaces like, say, groups or manifolds.
ii) Furthermore, for the time being, the f_i's and g_{ik}'s should not be confused with the s_i's and J_{ik}'s introduced in section 2. The f_i's and g_{ik}'s are e.g. allowed to vanish outside a certain given cluster of nodes in various calculations or, put differently, it may be convenient to deal only with certain subgraphs.
iii) The spaces C_0, C_1 are in fact only the first two members of a whole sequence of spaces.

3.3 Definition (Boundary): we now define a *boundary operator* by

$$\delta b_{ik} := n_k - n_i \tag{8}$$

which by linearity induces a linear map from C_1 to C_0:

$$\delta : C_1 \ni \sum g_{ik} b_{ik} \to \sum g_{ik}(n_k - n_i) \in C_0 \tag{9}$$

The kernel, Z_1 of this map, the 1-chains without '*boundary*', consist of the '*1-cycles*'. A typical example is a '*loop*', i.e. a sequence of bonds, $\sum_\nu b_{i_\nu k_\nu}$ s.t. $k_\nu = i_{\nu+1}$ and $k_n = i_1$. (However not every cycle is a loop!).

3.4 Definition(Coboundary): Analogously we can define the coboundary operator as a map from C_0 to C_1:

$$dn_i := \sum_k b_{ik} \tag{10}$$

where the sum extends over all bonds having n_i as initial node, and by linearity:

$$d : \sum_i f_i n_i \to \sum_i f_i (\sum_k b_{ik}) \tag{11}$$

Remark: Evidently one could in (8) and various other formulas choose slightly different conventions, i.e. define $dn_i := \sum_k b_{ki}$ etc.

We will now show that these two operations, well known in algebraic topology, can be fruitfully employed to create something like a discrete calculus. Evidently, the 0-chains can as well be considered as functions over the set of nodes; in this case we abbreviate them by f,g etc. chosen from a certain subclass of 0-chains $\mathcal{A} \subset C_0$ (e.g. of '*finite support*', L^1, $L^2 \ldots$). $\mathcal{A}$ is trivially a module over itself (pointwise multiplication) freely generated by the nodes $\{n_i\}$ which can be identified with the '*elementary functions*' $e_i := 1 \cdot n_i$.

3.5 Definition ("Quasi"Derivation): On a suitable class of node functions $\mathcal{A}$ we call d a quasiderivation and df a differential. That this is meaningful will become apparent from a rearrangement of (11):

With $f_i b_{ik}$ there occurs always $f_k b_{ki} = -f_k b_{ik}$ on the rhs of (11), hence we have:

3.6 Observation:

$$df = (\sum_i f_i n_i) = 1/2 \cdot \sum_{ik} (f_k - f_i) b_{ik} \tag{12}$$

We have still to show to what extent the operation d defined above has the properties we are expecting from an (exterior) derivation. The really crucial property in the continuum case is the (graded) Leibniz rule. This is in fact a subtle and interesting point. To see this we make a short aside about how discrete differentiation is usually expected to work.

Take the following definition:

3.7 Definition (Partial Forward Derivative and Partial Differential at Node (i)):

$$\nabla_{ik} f(i) := f(k) - f(i) \tag{13}$$

where n_i, n_k are '*nearest-neighbor-nodes*', i.e. being connected by a bond b_{ik}.

3.8 Corollary:

$$\nabla_{ik}(f \cdot g)(i) \;=\; (f \cdot g)(k) - (f \cdot g)(i)$$

$$= \nabla_{ik} f(i) \cdot g(i) + f(k) \cdot \nabla_{ik} g(i) \quad (14)$$

$$= \nabla_{ik} f(i) g(i) + f(i) \nabla_{ik} g(i) + \nabla_{ik} f(i) \nabla_{ik} g(i) \quad (15)$$

In other words the "derivation" ∇ does **not** obey the Leibniz rule. In fact, application of ∇ to, say, higher powers of f becomes increasingly cumbersome (nevertheless there is a certain systematic in it). One gets for example:

$$\nabla_{ik} f^{(n)}(i) = \nabla_{ik} f(i) \cdot \{f^{(n-1)}(k) + f^{(n-2)}(k) f(i) + \ldots + f(k) f^{(n-2)}(i) + f^{(n-1)}(i)\} \quad (16)$$

Due to the discreteness of the formalism and, as a consequence, the inevitable bilocality of the derivative there is no chance to get something as a true Leibniz rule on this level. (That this is impossible has also been stressed from a different point of view in e.g. example 2.1.1 of [14]).

In some sense it is considered to be one of the merits of the abstract algebraic framework (mentioned at the beginning of this section) that a graded Leibniz rule holds in that generalized case. The concrete network model under investigation offers a good opportunity to test the practical usefulness of concepts like these.

To write down something like a Leibniz rule an important structural element is still missing, i.e. the multiplication of node functions from, say, some $\mathcal{A}$ with the members of C_1, in other words a '*module structure over A*. One could try to make the following definition:

$$f \cdot b_{ik} := f(i) \cdot b_{ik} \quad b_{ik} \cdot f := f(k) \cdot b_{ik} \quad (17)$$

and extend this by linearity.

Unfortunately this "definition" does not respect the relation $b_{ik} = -b_{ki}$. We have in fact:

$$f(i) b_{ik} = f \cdot b_{ik} = -f \cdot b_{ki} = -f(k) b_{ki} = f(k) b_{ik} \quad (18)$$

which is wrong in general for non-constant f!

Evidently the problem arises from our geometrical intuition which results in the natural condition $b_{ik} = -b_{ki}$, a relation we however want to stick to. On the other side we can extend or embed our formalism in a way which looses the immediate contact with geometrical evidence but grants us with some additional mathematical structure. (This is in fact common mathematical practice).

We can define another relation between nodes, calling two nodes related if they are connected by a bond with a built-in direction from the one to the other. We write this in form of a *tensor product* structure. In the general tensor product $C_0 \otimes C_0$ we consider only the subspace $C_0 \hat{\otimes} C_0$ spanned by the elements $n_i \otimes n_k$ with n_i, n_k connected by a bond and consider $n_i \otimes n_k$ to be unrelated to $n_k \otimes n_i$, i.e. they are assumed to be linearly independent basis elements.

3.9 Observation: There exists an isomorphic embedding of C_1 onto the subspace generated by the antisymmetric elements in $C_0 \hat{\otimes} C_0$, i.e:

$$b_{ik} \to 1/2 \cdot (n_i \otimes n_k - n_k \otimes n_i) =: n_i \wedge n_k \quad (19)$$

generate an isomorphism by linearity between C_1 and the corresponding subspace $C_0 \wedge C_0 \subset C_0 \hat{\otimes} C_0$.

Proof: Both b_{ik} and $n_i \wedge n_k$ are linearly independent in there respective vector spaces.

In contrast to C_1 the larger $C_0 \hat{\otimes} C_0$ now supports a non-trivial bimodule structure:

3.10 Observation/Defintion (Bimodule): We can now define

$$\begin{aligned} f \cdot (n_i \otimes n_k) &:= f(i)(n_i \otimes n_k) && (20) \\ (n_i \otimes n_k) \cdot f &:= f(k)(n_i \otimes n_k) && (21) \end{aligned}$$

and extend this by linearity to the whole $C_0 \hat{\otimes} C_0$, making it into a bimodule over some $\mathcal{A} \in \mathcal{C}_l$.

Remark: Equivalently one could replace $n_i \otimes n_k$ by $e_i \otimes e_k$, the corresponding elementary functions.

3.11 Corollary: As a module over $\mathcal{A}$, $C_0 \hat{\otimes} C_0$ is generated by $C_0 \wedge C_0$.

Proof: It suffices to show that every $n_i \otimes n_k$ can be generated this way.

$$n_i \cdot (n_i \otimes n_k - n_k \otimes n_i) = n_i \otimes n_k \tag{22}$$

as $n_i \cdot n_k = 0$ for $i \neq k$.

With the b_{ik} so embedded in a larger space and identified with

$$1/2 \cdot (n_i \otimes n_k - n_k \otimes n_i) = n_i \wedge n_k \tag{23}$$

we are in the position to derive a graded Leibniz rule on the module (algebra) $\mathcal{A}$. Due to linearity and the structure of the respective spaces it suffices to show this for products of elementary functions $e_i = n_i$. We in fact have:

($i \neq k$ not nearest neighbors):

$$d(n_i \cdot n_k) = 0,\ dn_i \cdot n_k = n_i \cdot dn_k = 0 \tag{24}$$

($i \neq k$ nearest neighbors):

$$d(n_i \cdot n_k) = d(0) = 0 \ \text{ and} \tag{25}$$

$$\begin{aligned} dn_i \cdot n_k + n_i \cdot dn_k &= (\sum_{k'} b_{ik'}) \cdot n_k + n_i \cdot (\sum_{i'} b_{ki'}) && (26) \\ &= b_{ik} \cdot n_k + n_i \cdot b_{ki} && (27) \\ &= 1/2\{(n_i \otimes n_k - n_k \otimes n_i)n_k + n_i(n_k \otimes n_i - n_i \otimes n_k)\} && (28) \\ &= 1/2\{n_i \otimes n_k - n_i \otimes n_k\} = 0 && (29) \end{aligned}$$

$(i = k)$:

$$d(n_i^2) = d(n_i) = \sum_k b_{ik} \text{ and} \tag{30}$$

$$\begin{aligned} dn_i \cdot n_i + n_i \cdot dn_i &= (\sum_k b_{ik})\, n_i + n_i(\sum_k b_{ik}) && (31) \\ &= 1/2 \sum_k (n_i \otimes n_k - n_k \otimes n_i) = \sum_k b_{ik} = dn_i && (32) \end{aligned}$$

3.12 Conclusion: In the bimodule $C_0 \hat{\otimes} C_0$ generated by the elements b_{ik} over $\mathcal{A}$ the map d fulfills the Leibniz rule, i.e:

$$d(f \cdot g) = df \cdot g + f \cdot dg \tag{33}$$

¿From the above we see also that functions, i.e. elements from $\mathcal{A}$ and bonds or differentials of functions do no longer commute (more specifically, the two possible ways of imposing a module structure could be considered this way). We have for example:

3.13 Commutation Relations:
$(i \neq k$ not nearest neighbors$)$:

$$n_i \cdot dn_k = dn_k \cdot n_i = 0 \tag{34}$$

$(i \neq k$ nearest neighbors$)$.

$$\begin{aligned} n_i \cdot dn_k + dn_k \cdot n_i &= 1/2 \sum_{i'} \{ n_i (n_k \otimes n_{i'} - n_{i'} \otimes n_k) && (35) \\ &+ (n_k \otimes n_{i'} - n_{i'} \otimes n_k)\, n_i \} && (36) \\ &= -1/2 (n_i \otimes n_k - n_k \otimes n_i) = -b_{ik} && (37) \end{aligned}$$

$(i = k)$:

$$n_i \cdot dn_i + dn_i \cdot n_i = \sum_k b_{ik} = dn_i \tag{38}$$

Making contact with local differential topology on manifolds we can now formulate the following concepts:

3.14 Definition ((Co)Tangential Space):
i) We call the space spanned by the ∇_{ik} at node n_i the tangential space T_i.
ii) Correspondingly the space spanned by b_{ik} at node n_i is called the cotangential space T_i^*.
We can now consider the b_{ik}'s as linear forms over T_i via:

$$< b_{ik} | \nabla_{ij} >= \delta_{kj} \tag{39}$$

Another important relation we want to mention is the followng: $\delta\, d\, f$ is a map from $C_0 \to C_0$ and reads in detail:

3.15 Observation (Laplacian):

$$\delta\, d\, f = -\sum_i (\sum_k f(k) - n \cdot f(i))\, n_i =: -\Delta\, f \tag{40}$$

with n the number of nearest neighbors of n_i.

Proof:

$$\begin{aligned}\delta\, d\, f &= 1/2 \sum_{ik} (f(k) - f(i))(n_k - n_i) &(41)\\ &= 1/2 \sum_{ik} (f(k)\, n_k + f(i)\, n_i - f(i)\, n_k - f(k)\, n_i) &(42)\\ &= -\sum_i (\sum_k f(k) - n \cdot f(i))\, n_i &(43)\end{aligned}$$

Having now established the first steps in setting up this particular version of discrete calculus one can proceed in various directions. First, one can develop a discrete Lagrangian variational calculus , derive Euler-Lagrange-equations and Noetherian theorems and the like and compare our approach with other existing schemes in discrete mathematics.

Second, one can continue the above line of reasoning and proceed to higher differential forms.

3.16 Definition/Observation: Higher tensor products of differential forms at a node n_i can be defined as *multilinear forms*:

$$< b_{ik_1} \otimes \cdots \otimes b_{ik_n} | (\nabla_{il_1}, \cdots, \nabla_{il_n}) > := \delta_{k_1 l_1} \times \cdots \times \delta_{k_n l_n} \tag{44}$$

and linear extension.

A comparison of our scheme with the ordinary approach, performed within the framework of the universal differential algebra, is, on the other side, a subtle and delicate point and would lead us a little bit astray at the moment. The deeper reason is the following:

In contrast to the universal differential algebra mentioned above, where every two nodes are connected by a bond, this is not so for our '*reduced*' calculus. As a consequence certain operations are straightforward to define in the former approach. However, descending afterwards to the lower-connected more realistic models is tedious in general and not always particularly transparent. That is, this method does not really save calculational efforts in typical concrete cases (for a discussion of certain simple examples see [8]).

The mathematical "triviality " of the differential envelope is reflected by the trivialty of the corresponding '*(co)homology groups*' of the maximally connected graph (simplex). This trivialty is then broken by deleting graphs in the reduction process.

To mention a typical situation: Take e.g. the subgraphs of a graph G consisting of, say, four nodes n_i, n_k, n_l, n_m and all the bonds between them which occur in G.

In the case G being a simplex (i.e. non-reduced case) all these subgraphs are geometrically/topologically equivalent. An important consequence of this is that the four nodes can be connected by a '*path*', i.e. a sequence of consecutive bonds with each node being passed only once, the effect being that one can naturally define a multiplication in this scheme via '*concatenation*', e.g:

$$(n_i \otimes n_k) \cdot (n_l \otimes n_m) = n_i \otimes n_k \cdot n_l \otimes n_m \tag{45}$$

with $n_k \cdot n_l = \delta_{kl} \cdot n_k$. Correspondingly, all the higher differentials can be generated by the 1-forms via concatenation. The reason why it is sufficient to concatenate only at the extreme left and right of a '*word*' stems exactly from the simplex-character of each subgraph.

In typical reduced cases all this is no longer the case; the combinatorical topology becomes non-trivial. To give an example: Take as G a regular graph with the order of the nodes (number of incident bonds) $n = 3$. In the analogous case of 4-node-subgraphs there exists now a kind of subgraph the nodes of which cannot be concatenated in the above way. Take e.g. the subgraph with bonds existing only between, say, $n_1\, n_4$, $n_2\, n_4$, $n_3\, n_4$. I.e., one has the 1-forms $b_{14},\ b_{24},\ b_{34}$ or:

$$n_1 \otimes n_4,\ n_2 \otimes n_4,\ n_3 \otimes n_4 \tag{46}$$

but there is no obvious way to generate the corresponding reduced subgraph by concatenating them sequentially(!). The loophole is that one has to define multiplication differently (this can in fact be done and all the higher differentials generated that way without employing the universal differential envelope).

Nevertheless, the situation is much more involved in the more realistic cases. As this highly interesting feature, which we have not yet found discussed in this particular context in the literature known to us, deserves a more careful analysis of its own we prefer to present it elsewhere and proceed in the next section with the developement of a concept of '*dimension*' in networks and graphs which reflects the '*degree of connectivity*' and has a bearing on physical concepts like '*interaction*' and '*phase transitions*'.

4 Intrinsic Dimension in Networks and other Discrete Systems

There exist a variety of concepts in modern mathematics which generalize the notion of '*dimension*' one is accustomed to in e.g. differential topology or linear algebra. In fact, '*topological dimension*' is a notion which seems to be even closer to the underlying intuition (cf. e.g. [18]).

Apart from the purely mathematical concept there is also a physical aspect of something like dimension which has e.g. pronounced effects on the behavior of, say, many-body-systems, especially their microscopic dynamics and, most notably, their possible '*phase transitions*'.

But even in the case of e.g. lattice systems they are usually considered as embedded in an underlying continuous background space (typically euclidean) which supplies the concept of ordinary dimension so that the intrinsic dimension of the discrete array itself does usually not openly enter the considerations.

Anyway, it is worthwhile even in this relatively transparent situations to have a closer look on where attributes of something like dimension really come into the physical play. Properties of models of, say, statitical mechanics are almost solely derived from the structure of the microscopic interactions of their constituents. This is more or less the only place where dimensional aspects enter the calculations.

Naive reasoning might suggest that it is the number of nearest neighbors (in e.g. lattice systems) which reflects in an obvious way the dimension of the underlying space and influences via that way the dynamics of the system. However, this surmise, as we will show in the following, does not reflect the crucial point which is considerably more subtle.

This holds the more so for systems which cannot be consided as being embedded in a smooth regular background and hence do not get their dimension from the embedding space. A case in point is our primordial network in which in which Planck-scale-physics is assumed to tkae place. In our approach it is in fact exactly the other way round: Smooth space-time is assumed to emerge via a phase transition and after some '*coarse graining*' from this more fundamental structure.

4.1 Problem: Formulate an intrinsic notion of dimension for model theories without making recourse to the dimension of some embedding space.

In a first step we will show that graphs and networks as introduced in the preceding sections have a natural metric structure. We have already introduced a certain neighborhood structure in a graph with the help of the minimal number of consecutive bonds connecting two given nodes.

In a connected graph any two nodes can be connected by a sequence of bonds. Without loss of generality one can restrict oneself to '*paths*'. One can then define the length of a path (or sequence of bonds) by the number l of consecutive bonds making up the path.

4.2 Observation/Definition: Among the paths connecting two arbitrary nodes there exists at least one with minimal length which we denote by $d(n_i, n_k)$. This d has the properties of a '*metric*', i.e:

$$d(n_i, n_i) \quad = \quad 0 \tag{47}$$

$$d(n_i, n_k) = d(n_k, n_i) \quad (48)$$

$$d(n_i, n_l) \leq d(n_i, n_k) + d(n_k, n_l) \quad (49)$$

(The proof is more or less evident).

4.3 Corollary: With the help of the metric one gets a natural neighborhood structure around any given node, where $\mathcal{U}_m(n_i)$ comprises all the nodes with $d(n_i, n_k) \leq m$, $\partial\mathcal{U}_m(n_i)$ the nodes with $d(n_i, n_k) = m$.

With the help of the above neighborhood structure we can now develop the concept of an intrinsic dimension on graphs and networks. To this end one has at first to realize what property really matters physically (e.g. dynamically) independently of the model or embedding space.

4.4 Observation: The crucial and characteristic property of, say, a graph or network which may be associated with something like dimension is the number of '*new nodes*' in $\mathcal{U}_{m+1}$ compared to $\mathcal{U}_m$ for m sufficiently large or $m \to \infty$. The deeper meaning of this quantity is that it measures the kind of '*wiring*' or '*connectivity*' in the network and is therefore a '*topological invariant*'.

Remark: In the light of what we have learned in the preceding section it is tempting to relate the number of bonds branching off a node, i.e. the number of nearest neighbors or order of a node, to something like dimension.

On the other side there exist quite a few different lattices with a variety of number nearest neighbors in, say, two- or three- dimensional euclidean space. What however really matters in physics is the embedding dimension of the lattice (e.g. with respect to phase transitions) and only to a much lesser extent the number of nearest neighbors.

In contrast to the latter property dimension reflects the degree of connectivity and type of wiring in the network.

In many cases one expects the number of nodes in $\mathcal{U}_m$ to grow like some power D of m for increasing m. By the same token one expects the number of new nodes after an additional step to increase proportional to m^{D-1}. With $|\cdot|$ denoting number of nodes we hence have:

$$|\mathcal{U}_{m+1}| - |\mathcal{U}_m| = |\partial\mathcal{U}_{m+1}| = f(m) \quad (50)$$

with

$$f(m) \sim m^{D-1} \quad (51)$$

for m large.

4.5 Definition: The intrinsic dimension D of a regular (infinite) graph is given by

$$D - 1 := \lim_{m\to\infty} (\ln f(m)/\ln m) \text{ or} \quad (52)$$

$$D := \lim_{m\to\infty}(\ln|\mathcal{U}_m|/\ln m) \tag{53}$$

That this definition is reasonable can be seen by applying it to ordinary cases like regular translation invariant lattices.

4.6 Observation For regular lattices D coincides with the dimension of the euclidean embedding space D_E.

Proof: It is instructive to draw a picture of the consecutive series of neighborhoods of a fixed node for e.g. a 2-dimensional Bravais lattice. It is obvious and can also be proved that for m sufficiently large the number of nodes in $\mathcal{U}_m$ goes like a power of m with the exponent being the embedding dimension D_E as the euclidean volume of $\mathcal{U}_m$ grows with the same power.

Remarks:i) For $\mathcal{U}_m$ to small the number of nodes may deviate from an exact power law which in general becomes only correct for sufficiently large m.
ii) The number of nearest neighbors, on the other side, does not(!) influence the exponent but rather enters in the prefactor. In other words, it influences $|\mathcal{U}_m|$ for m small but drops out asymptotically by taking the logarithm. For an ordinary Bravais lattice with N_C the number of nodes in a unit cell one has asymptotically:

$$|\mathcal{U}_m| \sim N_C \cdot m^{D_E} \quad \text{and hence:} \tag{54}$$

$$D = \lim_{m\to\infty}(\ln(N_C \cdot m^{D_E})/\ln m) = D_E + \lim_{m\to\infty}(N_C/\ln m) = D_E \tag{55}$$

independently of N_C.

Matters become much more interesting and subtle if one studies more general graphs than simple lattices. Note that there exists a general theorem showing that practically every graph can be embedded in $\mathbb{R}^3$ and still quite a lot in $\mathbb{R}^2$ ('*planar graphs*').

The point is however that this embedding is in general not invariant with respect to the euclidean metric. But something like an apriori given euclidean length is unphysical for the models we are after anyhow. This result has the advantage that one can visualize many graphs already in, say, $\mathbb{R}^2$ whereas their intrinsic dimension may be much larger.

An extreme example is a '*tree graph*', i.e. a graph without '*loops*'. In the following we study an infinite, regular tree graph with node order 3, i.e. 3 bonds branching off each node. The absence of loops means that the '*connectivity*' is extremely low which results in an exceptionally high '*dimension*' as we will see.

Starting from an arbitrary node we can construct the neighborhoods $\mathcal{U}_m$ and count the number of nodes in $\mathcal{U}_m$ or $\partial\mathcal{U}_m$. $\mathcal{U}_1$ contains 3 nodes which are linked with the reference node n_0. There are 2 other bonds branching off each of these nodes. Hence in $\partial\mathcal{U}_2 = \mathcal{U}_2 \backslash \mathcal{U}_1$ we have $3 \cdot 2$ nodes and by induction:

$$|\partial\mathcal{U}_{m+1}| = 3 \cdot 2^m \tag{56}$$

which implies

$$D - 1 := \lim_{m\to\infty}(\ln|\partial\mathcal{U}_{m+1}|/\ln m) = \lim_{m\to\infty}(m \cdot ln2/\ln m + 3/\ln m) = \infty \tag{57}$$

Hence we have:

4.7 Observation(Tree): The intrinsic dimension of an infinite tree is ∞ and the number of new nodes grows exponentially like some $n(n-1)^m$ with m (n being the node order).

Remark: $D = \infty$ is mainly a result of the absence of loops(!), in other words: there is exactly one path, connecting any two nodes. This is usually not so in other graphs, e.g. lattices, where the number of new nodes grows at a much slower pace (whereas the number of nearest neighbors can nevertheless be large). This is due to the existence of many loops s.t. many of the nodes which can be reached from, say, a node of $\partial\mathcal{U}_m$ by one step are already contained in $\mathcal{U}_m$ itself.

We have seen that for, say, lattices the number of new nodes grows like some fixed power of m while for, say, trees m occurs in the exponent. The borderline can be found as follows:

4.8 Observation: If for $m \to \infty$ the average number of new nodes per node contained in $\partial\mathcal{U}_m$, i.e:

$$|\mathcal{U}_{m+1}|/|\mathcal{U}_m| \geq 1 + \varepsilon \tag{58}$$

for some $\varepsilon \geq 0$ then we have exponential growth, in other words, the intrinsic dimension is ∞.

Proof: If the above estimate holds for all $m \geq m_0$ we have by induction:

$$|\mathcal{U}_m| \geq |\mathcal{U}_{m_0}| \cdot (1 + \varepsilon)^{m - m_0} \tag{59}$$

Potential applications of this concept of intrinsic dimension are manifold. Our main goal is it to develop a theory which explains how our classical space-time and what we call the '*physical vacuum*' has emerged from a more primordial and discrete background via some sort of phase transition.

In this context we can also ask in what sense space-time dimension 4 is exceptional, i.e. whether it is merely an accident or whether there is a cogent reason for it.

As the plan of this paper was mainly to introduce and develop the necessary conceptual tools and to pave the ground, the bulk of the investigation in this particular direction shall be presented elsewhere as it is part of a detailed analysis of the (statitical) dynamics on networks as introduced above, their possible phase transitions, selforganisation, emergence of patterns and the like.

In this paper, which is to have a more technical flavor, we will only supply a speculative and heuristic argument in favor of space- dimension 3. As we emphasized in section 2 also the bond states, modelling the strength of local interactions between neighboring nodes, are in our model theory dynamical variables. In extreme cases these couplings may completely vanish or become extremely strong between certain nodes.

It may now happen that in the course of evolution a local island (or several of them) emerges as a fluctuation in a, on large scales, unordered or lowly connected network (e.g. a tree-like structure) where couplings between nodes are switched on which have been uncoupled before or, on the other side, become very strong.

One important effect of this scenario (among others) is that there may occur now a lot of local loops in this island while the state outside is more or less loopless. This may have the consequence that the intrinsic dimension within this island may become substantially lower than outside, say, finite as compared to (nearly) infinity.

If this '*nucleation center*' is both sufficiently large and its local state '*dynamically favorable*' in a sense to be specified (note that a concept like '*entropy*' or something like that would be of use here) it may start to grow and trigger a global phase transition.

As a result of this phase transition a relatively smooth and stable submanifold (in the language of synergetics an '*order parameter manifold*') may come into being which displays certain properties we would attribute to space-time.

Under these premises we could now ask what is the probability for such a specific and sufficiently large spontaneous fluctuation? As we are at the moment talking about heuristics and qualitative behavior we make the following thumb-rule-like assumptions:

i) In the primordial network '*correlation lengths*' are supposed to be extremely short (more or less nearest neighbor), i.e. the strengths of the couplings are fluctuating more or less independently.
ii) A large fluctuation of the above type implies that a substantial fraction of the couplings in the island passes a certain threshold, i.e. becomes sufficiently strong and cooperative. The probability per individual bond for this to happen be p. Let L be the diameter of the nucleation center, $const \cdot L^d$ the number of nodes or bonds in this island for some d. The probability for such a fluctuation is then roughly:

$$W_d = const \cdot p^{(L^d)} \tag{60}$$

iii) We know from our experience with phase transitions that there are favorable dimensions, i.e. the nucleation centers may fade away if either they themselves are too small or the dimension of the system is too small. Apart from certain non-generic models $d = 3$ is typically the threshold dimension.

iv) On the other side we can compare the relative probabilities for the occurrence of sufficiently large spontaneous fluctuations for various d's. One has:

$$W_{d+1}/W_d \sim p^{(L^{d+1})}/p^{(L^d)} = p^{L^d(L-1)} \tag{61}$$

Take e.g. $d=3$, $L=10$, $p=1/2$ one gets:

$$W_4/W_3 \sim 2^{-(9\cdot 10^3)} \tag{62}$$

In other words, provided that this crude estimate has a grain of truth in it, one may at least get a certain clue that space-dimension 3 is both the threshold dimension and, among the class of in principle allowed dimensions (i.e. $d \geq 3$) the one with the dominant probability.

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