EXISTENCE AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS IN SEMILINEAR CRITICAL PROBLEMS OF FOURTH ORDER

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Abstract. In this paper we consider the equation $\Delta^2 u = \lambda |u|^{q-2}u + |u|^{2^*-2}u \equiv f(u)$ in a smooth bounded domain $\Omega \subset \mathbf{R}^N$ with boundary conditions either $u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$ or $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$, where $N > 4$, $1 < q < 2$, $\lambda > 0$ and $2^* = 2N/(N-4)$. We prove the existence of λ_0 such that for $0 < \lambda < \lambda_0$ the above problems have infinitely many solutions. For the problem with the second boundary conditions, we prove the existence of a positive solution also in the supercritical case,i.e. when we have an exponent larger than 2^*. Moreover, in the critical case, we show the existence of at least two positive solutions.

1.- INTRODUCTION

In this paper we study the following fourth order problems

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2}u + |u|^{2^*-2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \end{cases} \tag{P1}$$

and

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2}u + |u|^{2^*-2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{P2}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, $N > 4$, $1 < q < 2$, $\lambda > 0$ and $2^* = 2N/(N-4)$, the critical Sobolev exponent for fourth order problems. Hereafter we denote $f(u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u$.

The critical growth in semilinear and quasilinear problems of second order has been extensively studied in the last years, starting with the seminal paper [6]. See [10] and [11] for an extensive list of references.

For fourth order equations there are some results for the case $q = 2$ and $2^* > q > 2$. See the references [9] and [16] for existence results and [18] for nonexistence theorems. Other results about existence and nonexistence, also for related systems, can be seen in [17] and in [12].

In this work one of the main points is to prove the existence of infinitely many solutions for problems $(P1)$ and $(P2)$, independently of the dimension. The proof of the existence of at least two positive solutions for problem $(P2)$ is another main point of our work.

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We study the existence of solutions understood as critical points of the energy functional,

$$J(u)=\frac{1}{2}\int_\Omega|\Delta u|^2dx-\frac{\lambda}{q}\int_\Omega|u|^qdx-\frac{1}{2^*}\int_\Omega|u|^{2^*}dx. \tag{1.1}$$

For the first problem, $(P1)$, J is defined in $W_0^{2,2}(\Omega)$; for the problem $(P2)$, J is defined in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$.

Then a critical point must be understood in the following way:

(1) $u\in W_0^{2,2}(\Omega)$ is a critical point associated to problem $(P1)$ if

$$0=\int_\Omega\Delta u\Delta\phi dx-\int_\Omega f(u)\phi dx\quad\text{for all}\quad\phi\in W_0^{2,2}(\Omega),$$

(2) $u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ is a critical point associated to problem $(P2)$ if

$$0=\int_\Omega\Delta u\Delta\phi dx-\int_\Omega f(u)\phi dx\quad\text{for all}\quad\phi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega).$$

In (1) integration by parts shows that the critical points of J are weak solutions of problem (P1). In (2) it is not directly clear why the second boundary condition must be satisfied by a critical point. We need some information about the regularity of such critical points.

Then the organization of the paper is the following. In section 2 we prove the regularity results that we need. Section 3 will be devoted to the proof of a *Local Palais-Smale condition*. The main tool here is the P.L. Lions concentration-compactness result. See [13], [14]. We will use at this point the result about the best constant of the Sobolev inclusion in [22]. The application of Ljusternik-Schnirelmann methods, allows us to establish the existence of infinitely many solutions for λ small enough. This is the contents of section 4. In section 5 we obtain a positive solution for (P2), also for *supercritical* problems, by classical methods and for $0<\lambda<\Lambda$. The section 6 contains the extension of a well known result by Brezis and Nirenberg [7] that we apply in section 7 to show the existence of a second positive solution for problem (P2). In the last section we obtain further results, for instance, some extensions to the quasilinear contexts.

2.- ABOUT REGULARITY

The regularity for problem $(P1)$ can be seen in the paper by S. Luckhaus [15].

The problem $(P2)$ must be considered in a different way because the second boundary condition is not included in the natural space $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$.

Consider the linear problem

$$\begin{cases}\Delta^2u=g(x)\text{ in }\Omega,\\ u|_{\partial\Omega}=0,\\ \Delta u|_{\partial\Omega}=0,\end{cases}\tag{PL2}$$

where we assume $g\in\mathcal{C}^\infty(\overline{\Omega})$. It is well known by the standard L^2 theory that (PL2) has a unique $\mathcal{C}^\infty$ solution. Then we can use, for instance, the classical Agmon-Douglis-Nirenberg *a priori* estimates,

$$||u||_{W^{4,p}}\le c||g||_p,\quad 1<p<\infty. \tag{2.1}$$

See [2].

Moreover, given $g \in L^r$, $\frac{1}{r} + \frac{1}{2^*} = 1$, the problem (PL2) has a unique weak solution, i.e. verifying

$$0 = \int_\Omega \Delta u \Delta \phi dx - \int_\Omega g\phi dx \quad \text{for all} \quad \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Approximate g in L^r by $\mathcal{C}_0^\infty(\Omega)$ functions and get

$$||u||_{W^{4,r}} \leq c||g||_r,$$

then in the sense of traces $u \in W^{4-(1/r),r}$ on $\partial\Omega$. Hence, integration by parts shows that u satisfies the second boundary condition. Therefore a critical point u of J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a weak solution of $(PL2)$.

Let u be a critical point of J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, i.e.,

$$0 = \int_\Omega \Delta u \Delta \phi dx - \int_\Omega f(u)\phi dx \quad \text{for all} \quad \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Then $g(x) = f(u(x)) \in L^r(\Omega)$, $\frac{1}{r} = 1 - \frac{1}{2^*}$, because $u \in L^{2^*}(\Omega)$, and the argument above shows that

$$||u||_{W^{4,r}} \leq c||g||_r.$$

Taking in particular as test functions $\phi \in \mathcal{C}_0^\infty(\Omega)$ we get that the equation is verified in Ω and the boundary conditions are satisfied in the sense of traces as above. This remark is crucial because now we can prove the following regularity lemma essentially contained in [22].

Theorem 2.1. *Assume that $\partial\Omega$ is a $\mathcal{C}^{4,\alpha}$ manifold. Let u be a critical point for J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Then $u \in \mathcal{C}^{4,\alpha}(\Omega) \cap \mathcal{C}^3(\overline{\Omega})$.

Proof. Let u be a critical point of J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Then by the previous remark u is a solution of the problem

$$\begin{cases} \Delta^2 u = a(x)u + g(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{PL}$$

where

$$a(x) = \begin{cases} 0 & \text{if} \quad u < 1 \\ \dfrac{f(u)}{u} & \text{if} \quad u \geq 1 \end{cases}$$

and

$$g(x) = \begin{cases} f(u) & \text{if} \quad u < 1 \\ 0 & \text{if} \quad u \geq 1. \end{cases}$$

It is obvious that $a \in L^{N/4}(\Omega)$ and $g \in L^\infty(\Omega)$. Moreover in [22] is proved the following

Claim: *For all $\epsilon > 0$ there exist $q_\epsilon \in L^{N/4}(\Omega)$ and $g_\epsilon \in L^\infty(\Omega)$ such that*

i) $||q_\epsilon||_{N/4} \leq \epsilon$

ii) $q_\epsilon(x)u(x) + g_\epsilon(x) = a(x)u(x) + g(x)$, $x \in \Omega$.

Let q_ϵ and g_ϵ be as in the claim, so that

$$\Delta^2 u = q_\epsilon u + g_\epsilon.$$

Consider the operator $(\Delta^2)^{-1}$ defined in the space of functions verifying our boundary conditions.

Put $A_\epsilon u = (\Delta^2)^{-1}(q_\epsilon u)$ and $h_\epsilon = (\Delta^2)^{-1} g_\epsilon$; then

$$u - A_\epsilon u = h_\epsilon, \quad \text{and} \quad u \in L^{2^*}.$$

or in a equivalent way

$$u = (I - A_\epsilon)^{-1} h_\epsilon.$$

Now, for ϵ small enough,

$$A_\epsilon : L^p \longrightarrow L^p \quad \text{for all} \quad p \geq 2^*,$$

with

$$||A_\epsilon||_{p,p} < \frac{1}{2}.$$

Indeed, by Hardy-Littlewood-Sobolev inequality we get

$$||A_\epsilon v||_p \leq c(p)||q_\epsilon v||_r, \quad \text{if} \quad \frac{1}{r} = \frac{1}{p} + \frac{4}{N},$$

and Hölder inequality provides the final estimate

$$||A_\epsilon v||_p \leq c(p)||q_\epsilon||_{N/4}||v||_p \leq \frac{1}{2}||v||_p,$$

if $c(p)\epsilon < 1/2$. In conclusion, for such an ϵ, we obtain

$$||(I - A_\epsilon)^{-1}||_{p,p} \leq 2,$$

and in turn,

$$||u||_p \leq ||(I - A_\epsilon)^{-1} h_\epsilon||_p \leq 2||h_\epsilon||_p \leq 2C(p)||g_\epsilon||_\infty,$$

in others words, $u \in L^p$ for all $p \in [1, \infty)$. This is sufficient to deduce the desired regularity, by the Sobolev inclusion and the Agmon-Douglis-Nirenberg estimates.

3.- THE LOCAL PALAIS-SMALE CONDITION BY THE LIONS CONCENTRATION-COMPACTNESS RESULTS

We denote $\mathbf{E}$ either $W_0^{2,2}(\Omega)$ or $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and define $||u||_{\mathbf{E}} = ||\Delta u||_2$.

A sequence $\{u_j\} \subset \mathbf{E}$ is said to be a *Palais-Smale sequence* for J, defined by (1.1), if

$$\begin{cases} J(u_j) \to c \\ J'(u_j) \to 0 \quad \text{in} \quad \mathbf{E}'. \end{cases} \tag{3.1}$$

If (3.1) implies the existence of a subsequence $\{u_{j_k}\} \subset \{u_j\}$ which converges strongly in $\mathbf{E}$, we say that J *verifies the Palais-Smale condition.*

If this strongly convergent subsequence exists only for some values of c, we say that J verifies a *local Palais-Smale condition.*

In our case, the main difficulty is the lack of compactness in the inclusion of $\mathbf{E}$ in L^{2^*}. Here, we shall prove a local Palais-Smale condition, which is sufficient for the problem.

The technical results used are based on a measure representation lemma, given by P.L. Lions in the proof of the concentration-compactness principle (see [13] and [14]).

Let $\{u_j\}$ be a bounded sequence in $\mathbf{E}$. Then, there is a subsequence, such that

$$u_j \rightharpoonup u \quad \text{weakly in} \quad \mathbf{E},$$

and

$$\left.\begin{aligned} |\Delta u_j|^2 &\rightharpoonup d\mu \\ |u_j|^{2^*} &\rightharpoonup d\nu \end{aligned}\right\} \text{ weakly-* in the sense of measures.}$$

If we take $\phi \in \mathcal{C}^\infty$, the Sobolev inequality applied to $u_j\phi$ gives:

$$(\int_\Omega |\phi|^{2^*} d\nu)^{\frac{1}{2^*}} S^{\frac{1}{2}} \le (\int_\Omega |\phi|^2 d\mu)^{\frac{1}{2}} + 2(\int_\Omega \langle \nabla\phi, \nabla u\rangle^2 dx)^{\frac{1}{2}} + (\int_\Omega |\Delta\phi|^2 |u|^2 dx)^{\frac{1}{2}}, \tag{3.2}$$

where $S = \inf\{||u||^2_{\mathbf{E}} : u \in \mathbf{E}, ||u||_{2^*} = 1\}$ is the best constant in the Sobolev inclusion.

The main idea is that if $u \equiv 0$ in (3.2), then we have a reverse Hölder inequality for two different measures, for which we have the following representation result (See P.L. Lions [13] and [14]):

Lemma 3.1. *Let μ, ν be two non-negative and bounded measures on $\overline{\Omega}$, such that for $1 \le p < r < \infty$ there exists some constant $C > 0$ for which*

$$(\int_\Omega |\phi|^r d\nu)^{\frac{1}{r}} \le C(\int_\Omega |\phi|^p d\mu)^{\frac{1}{p}} \quad \forall\phi \in \mathcal{C}^\infty(\overline{\Omega}), \quad \textit{with supp}(\phi) \quad \textit{bounded}$$

Then, there exist $\{x_j\}_{j\in I} \subset \overline{\Omega}$ and $\{\nu_j\}_{j\in I} \subset (0,\infty)$, where I is at most countable, such that:

$$\nu = \sum_{j\in I} \nu_j \delta_{x_j} \quad , \quad \mu \ge C^{-p} \sum_{j\in I} \nu_j^{\frac{p}{r}} \delta_{x_j},$$

where δ_{x_j} is the Dirac mass supported at x_j.

Remark If Ω is bounded then I is finite.

By application of this result to $v_j = u_j - u$, P.L.Lions obtains (see [13] and [14]):

Lemma 3.2. *Let $\{u_j\}$ be a weakly convergent sequence in $\mathbf{E}$ with weak limit u. Assume also*

i) $|\Delta u_j|^2$ *converges in the weak-* sense of measures to a measure μ*

ii) $|u_j|^{2^*}$ *converges in the weak-* sense of measures to a measure ν.*

Then, there exist $\{x_j\} \subset \overline{\Omega}$, $\quad j = 1, 2, ..., l$, such that:

$$\begin{cases} 1) \quad \nu = |u|^{2^*} + \sum_{j=1}^{l} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ 2) \quad \mu \ge |\Delta u|^2 + \sum_{j=1}^{N} \mu_j \delta_{x_j}, \quad \mu_j > 0, \\ 3) \quad \nu_j^{\frac{2}{2^*}} \le \dfrac{\mu_j}{S}, \end{cases} \tag{3.3}$$

The Lemma 3.2 allows us to prove the following important result.

Lemma 3.3. *Let $\{v_j\} \subset \mathbf{E}$ be a Palais-Smale sequence for J, defined by (1.1), that is:*

$$J(v_j) \to c \tag{3.4}$$

$$J'(v_j) \to 0 \quad in \quad \mathbf{E}' \tag{3.5}$$

If $c < \dfrac{2}{N} S^{\frac{N}{4}} - K\lambda^{\beta}$, where $\beta = \dfrac{2^*}{2^* - q}$ and K depends on q, N and Ω, then there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, which converges strongly in $\mathbf{E}$.

Proof. From (3.4), (3.5) we get that the sequence $\{v_j\}$ is bounded in $\mathbf{E}$. More precisely, by (3.5) we have

$$0 = \langle J'(v_j), v_j\rangle + 2^*\langle \epsilon_j, v_j\rangle, \quad ||\epsilon_j||_{\mathbf{E}'} \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore for $\delta > 0$ we have

$$\begin{aligned} c + \delta &\geq J(v_j) - \frac{1}{2^*}\langle J'(v_j), v_j\rangle - \langle \epsilon_j, v_j\rangle \\ &= (\frac{1}{2} - \frac{1}{2^*}) \int_\Omega |\Delta v_j|^2 dx - \lambda(\frac{1}{q} - \frac{1}{2^*}) \int_\Omega |v_j|^q dx - \langle \epsilon_j, v_j\rangle \end{aligned}$$

and the boundedness of $\{v_j\}$ in $\mathbf{E}$ follows easily from the Sobolev inequality.

By Lemma 3.2 there exists a subsequence which we continue to denote by $\{v_j\}$ such that

$$\begin{cases} v_j \rightharpoonup v \text{ weakly in } \mathbf{E}, \\ v_j \to v \text{ in } L^r, 1 < r < 2^*, \text{and a.e. in } \quad \overline{\Omega}, \\ |\Delta v_j|^2 \rightharpoonup d\mu \geq |\Delta v|^2 + \sum_{k \in J} \mu_k \delta_{x_k}, \\ |v_j|^{2^*} \rightharpoonup d\nu = |v|^{2^*} + \sum_{k \in J} \nu_k \delta_{x_k}. \end{cases} \tag{3.6}$$

Take $x_k \in \overline{\Omega}$ in the support of the singular part of $d\mu, d\nu$, and $\psi \in \mathcal{C}^\infty(\mathbf{R}^N)$, such that:

$$\begin{cases} \psi \equiv 1 \text{ on } B(x_k, \epsilon), \quad \psi \equiv 0 \text{ on } B(x_k, 2\epsilon)^c \\ |\nabla \psi| \leq \dfrac{2}{\epsilon}, \quad |\Delta \psi| \leq \dfrac{2}{\epsilon^2}. \end{cases} \tag{3.7}$$

Consider the sequence $\{\phi v_j\}$, where $\phi(x) = \psi(x)\chi_\Omega(x)$; it is obvious that this sequence is bounded in $\mathbf{E}$. Then, by the hypothesis (2.5),

$$\lim < F'(v_j), \phi v_j >= 0,$$

where $<, >$ is the duality pairing. Moreover

$$\int_\Omega \phi d\nu + \lambda \int_\Omega |v|^q \phi dx = \lim_{j \to \infty} \int_\Omega \Delta v_j \Delta(v_j \phi) dx$$

By (3.6), (3.7), the weak convergence and the Hölder inequality, we can estimate:

$$\lim_{j \to \infty} |\int_\Omega \Delta v_j \Delta(v_j \phi) dx| = \int_\Omega \phi d\mu + \lim_{j \to \infty} |\int_\Omega \Delta v_j [2\langle \nabla v_j, \nabla \phi\rangle + v_j \Delta \phi] dx|.$$

Now

$$\begin{aligned}
&0 \leq \lim_{j\to\infty} \Big| \int_\Omega \Delta v_j \langle \nabla v_j, \nabla \phi \rangle dx \Big| \\
&\leq \lim_{j\to\infty} \Big(\int_\Omega |\Delta v_j|^2 dx \Big)^{1/2} \Big(\int_\Omega |\nabla \phi|^2 |\nabla v_j|^2 dx \Big)^{1/2} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\nabla \phi|^2 |\nabla v|^2 dx \Big)^{1/2} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\nabla \phi|^N dx \Big)^{1/N} \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\nabla v|^{2N/(N-2)} dx \Big)^{(N-2)/2N} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\nabla v|^{2N/(N-2)} dx \Big)^{(N-2)/2N} \to 0 \quad \text{as} \quad \epsilon \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&0 \leq \lim_{j\to\infty} \Big| \int_\Omega \Delta v_j v_j \Delta \phi dx \Big| \\
&\leq \lim_{j\to\infty} \Big(\int_\Omega |\Delta v_j|^2 dx \Big)^{1/2} \Big(\int_\Omega |\Delta \phi|^2 |v_j|^2 dx \Big)^{1/2} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\Delta \phi|^2 |v|^2 dx \Big)^{1/2} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |\Delta \phi|^{N/2} dx \Big)^{2/N} \Big(\int_{B(x_k,\epsilon)\cap\Omega} |v|^{2^*} dx \Big)^{1/2^*} \\
&\leq C \Big(\int_{B(x_k,\epsilon)\cap\Omega} |v|^{2^*} dx \Big)^{1/2^*} \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Then,

$$0 = \lim_{\epsilon\to 0} \Big\{ \int_\Omega \phi d\nu + \lambda \int_\Omega |v|^q \phi dx - \int_\Omega \phi d\mu \Big\} = \nu_k - \mu_k.$$

By Lemma 3.2. $\mu_k \geq S \nu_k^{\frac{2}{2^*}}$, i.e. $\nu_k \geq S \nu_k^{\frac{2}{2^*}}$. Hence, either

$$\nu_k = 0,$$

or

$$\nu_k \geq S^{\frac{N}{4}} \tag{3.8}$$

We shall prove that (3.8) does not occur.

Assume for contradiction that there exists a k_0 with $\nu_{k_0} \neq 0$, *i.e.* $\nu_{k_0} \geq S^{\frac{N}{4}}$. From the hypotheses (3.4) and (3.5),

$$\begin{aligned}
c = \lim_{j\to\infty} J(v_j) &= \lim_{j\to\infty} \{ J(v_j) - \frac{1}{2} < J'(v_j), v_j > \} \\
&\geq \frac{2}{N} \int_\Omega |v|^{2^*} + \frac{2}{N} S^{\frac{N}{4}} + \lambda \Big(\frac{1}{2} - \frac{1}{q}\Big) \int_\Omega |v|^q. \qquad (3.9)
\end{aligned}$$

Because $1 < q < 2$, applying the Hölder inequality in (3.9), we have

$$c \geq \frac{2}{N} S^{\frac{N}{4}} + \frac{2}{N} \int_{\Omega} |v|^{2^*} - \lambda(\frac{1}{q} - \frac{1}{2})|\Omega|^{\frac{2^*-q}{2^*}} (\int_{\Omega} |v|^{2^*})^{\frac{q}{2^*}}.$$

Let $f(x) = c_1 x^{2^*} - \lambda c_2 x^q$. This function attains its absolute minimum (for $x > 0$) at the point $x_0 = \left(\frac{\lambda c_2 q}{2^* c_1}\right)^{\frac{1}{2^*-q}}$. Hence,

$$f(x) \geq f(x_0) = -K \lambda^{\frac{2^*}{2^*-q}},$$

which contradicts the hypothesis that $c < \frac{2}{N} S^{\frac{4}{N}} - K\lambda^{\beta}$. Hence, $\nu_k = 0 \quad \forall k$, and the proof is completed.

4.- EXISTENCE OF INFINITELY MANY SOLUTIONS

Let $\mathbf{E}$ be the Hilbert space defined in section 3. Let $\sum$ be the class of subsets of $\mathbf{E} - \{0\}$ which are closed and symmetric with respect to the origin. For $A \in \sum$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\{k \in \mathbf{N} : \exists \phi \in \mathbf{C}(A; \mathbf{R}^k - \{0\}), \phi(x) = -\phi(-x)\}.$$

If such a minimum is not defined then we consider $\gamma(A) = +\infty$. The main properties of the genus are the following (see [20] for the details):

Proposition 4.1. *Let $A, B \in \sum$. Then:*

1) *If there exists an odd function $f \in \mathbf{C}(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
2) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
3) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$.*
4) *If S^{N-1} is the sphere in $\mathbf{R}^N$, then $\gamma(S^{N-1}) = N$.*
5) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
6) *If $\gamma(B) < +\infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*
7) *If A is compact, then $\gamma(A) < +\infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$ where $N_\delta(A) = \{x \in \mathbf{E} : d(x, A) \leq \delta\}$.*
8) *If X_0 is a subspace of $\mathbf{E}$ with codimension K , and $\gamma(A) > K$, then $A \cap X_0 \neq \emptyset$.*

Assume that $1 < q < 2$ in (1.1). Then, by Sobolev's inequality we obtain:

$$J(u) \geq \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx - \frac{1}{2^* S^{\frac{2^*}{2}}} (\int_{\Omega} |\Delta u|^2 dx)^{\frac{2^*}{2}} - \frac{\lambda}{q} C_q (\int_{\Omega} |\Delta u|^2 dx)^{\frac{q}{2}}.$$

Consequently

$$J(u) \geq h(\|\Delta u\|_2), \tag{4.1}$$

where

$$h(x) = \frac{1}{2} x^2 - \frac{1}{2^* S^{\frac{2^*}{2}}} x^{2^*} - \frac{\lambda}{q} C_q x^q.$$

There exists $\lambda_o > 0$ such that, if $0 < \lambda < \lambda_o$, h attains a local minimum and a local maximum. Let R_0, R_1 be such that $r < R_0 < R < R_1$, where R is the value for which h attains its maximum and r is the value for which h attains its minimum, and $h(R_1) > h(r)$.

We make the following truncation of the functional J. Take $\tau : \mathbf{R}^+ \to [0,1]$, nonincreasing and $\mathbf{C}^\infty$, such that

$$\begin{cases} \tau(x) = 1 & \text{if} \quad x \le R_0 \\ \tau(x) = 0 & \text{if} \quad x \ge R_1. \end{cases}$$

Let $\varphi(u) = \tau(\|\nabla u\|_p)$. We consider the truncated functional

$$\tilde{J}(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\varphi(u)dx - \frac{\lambda}{q}\int_\Omega |u|^q dx. \tag{4.2}$$

As in (4.1), $\tilde{J}(u) \ge \overline{h}(\|\Delta u\|_2)$, with

$$\overline{h}(x) = \frac{1}{2}x^2 - \frac{1}{2^* S^{\frac{2^*}{2}}}x^{2^*}\tau(x) - \frac{\lambda}{q}C_q x^q. \tag{4.3}$$

Observe that $\overline{h} = h$, for $x \le R_0$, and $\overline{h}(x) = \frac{1}{2}x^2 - \frac{\lambda}{q}C_q x^q$ for $x \ge R_1$.

The main properties of $\tilde{J}$, defined by (4.2), are the following:

Lemma 4.2.
1) $\tilde{J} \in \mathbf{C}^1(\mathbf{E}, \mathbf{R})$.
2) *If $\tilde{J}(u) \le 0$, then $\|\Delta u\|_2 < R_0$, and $J(v) = \tilde{J}(v)$ for all v in a small enough neighborhood of u.*
3) *There exists $A > 0$, such that, if $0 < \lambda < A$, then $\tilde{J}$ verifies a local Palais-Smale condition for $c \le 0$.*

Proof. 1) and 2) are immediate. To prove 3), observe that all the Palais-Smale sequences for $\tilde{J}$ with $c \le 0$ must be bounded; then, by Lemma 3.1, if λ verifies $\dfrac{1}{N}S^{\frac{N}{p}} - K\lambda^\beta \ge 0$ there exists a convergent subsequence.

Note that, if we find some negative critical value for $\tilde{J}$, then by 2) we have a negative critical value of J.

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional $\tilde{J}$.
The next lemma uses the same idea as in [11], and we include here the proof for the sake of completeness.

Lemma 4.3. *Given $n \in \mathbf{N}$, there is $\epsilon = \epsilon(n) > 0$, such that*

$$\gamma(\{u \in \mathbf{E} : \tilde{J}(u) \le -\epsilon\}) \ge n.$$

Proof. Fix n and let E_n be a n-dimensional subspace of $\mathbf{E}$. Take $u_n \in E_n$, with $||\Delta u_n||_2 = 1$. For $0 < \rho < R_0$, we have:

$$\tilde{J}(\rho u_n) = J(\rho u_n) = \frac{1}{2}\rho^2 - \frac{1}{2^*}\rho^{2^*}\int_\Omega |u|^{2^*} - \frac{\lambda}{q}\rho^q\int_\Omega |u|^q.$$

All the norms are equivalent in E_n. Define

$$\alpha_n = \inf\{\int_\Omega |u|^{2^*} : u \in E_n, \quad ||\Delta u_n||_2 = 1\} > 0,$$

$$\beta_n = \inf\{\int_\Omega |u|^q : u \in E_n, \quad ||\Delta u_n||_2 = 1\} > 0.$$

Hence $\tilde{J}(\rho u_n) \leq \frac{1}{2}\rho^2 - \frac{\alpha_n}{2^*}\rho^{2^*} - \frac{\lambda\beta_n}{q}\rho^q$, and we can choose ϵ (which depends on n), and $\eta < R_0$, such that $\tilde{J}(\eta u) \leq -\epsilon$ if $||\Delta u||_2 = 1$.

Let $S_\eta = \{u \in \mathbf{E} : ||\Delta u||_2 = \eta\}$ so that $S_\eta \cap E_n \subset \{u \in \mathbf{E} : \tilde{J}(u) \leq -\epsilon\}$; therefore, by Proposition 4.1,

$$\gamma(\{u \in \mathbf{E} : \tilde{J}(u) \leq -\epsilon\}) \geq \gamma(S_\eta \cap E_n) = n.$$

This lemma allows us to prove the existence of critical points.

Lemma 4.4. *Let $\Sigma_k = \{C \subset \mathbf{E} - \{0\} : C$ is closed, $C = -C$, $\gamma(C) \geq k\}$. Let $c_k = \inf_{C\in\Sigma_k} \sup_{u\in C} \tilde{J}(u)$, $K_c = \{u \in \mathbf{E} : \tilde{J}'(u) = 0, \tilde{J}(u) = c\}$, and suppose $0 < \lambda < A$, where A is the constant in Lemma 4.2. If $c = c_k = c_{k+1} = ... = c_{k+r}$, then $\gamma(K_c) \geq r + 1$. (In particular, the c_k's are critical values of J).*

Proof. In the proof, we will use Lemma 4.3, and a classical deformation lemma (see [20]).

For simplicity, put $\tilde{J}^{-\epsilon} = \{u \in \mathbf{E} : \tilde{J}(u) \leq -\epsilon\}$. By Lemma 4.3, $\forall k \in \mathbf{N}$, $\exists\epsilon(k) > 0$ such that $\gamma(\tilde{J}^{-\epsilon}) \geq k$.

Because $\tilde{J}$ is continuous and even, $\tilde{J}^{-\epsilon} \in \Sigma_k$; then, $c_k \leq -\epsilon(k) < 0, \forall k$. But $\tilde{J}$ is bounded from below; hence, $c_k > -\infty$ $\forall k$.

Let us assume that $c = c_k = ... = c_{k+r}$, and observe that $c < 0$; therefore, $\tilde{J}$ verifies the Palais-Smale condition in K_c. It is easy to see that K_c is compact.

Assume for contradiction that $\gamma(K_c) \leq r$. Thus there exists a closed and symmetric set U, with $K_c \subset U$, such that $\gamma(U) \leq r$. (We can choose $U \subset \tilde{J}^0$, because $c < 0$).

By the deformation lemma, we have an odd homeomorphism $\eta : \mathbf{E} \to \mathbf{E}$, such that $\eta(\tilde{J}^{c+\delta} - U) \subset \tilde{J}^{c-\delta}$, for some $\delta > 0$. (Again, we must choose $0 < \delta < -c$, because $\tilde{J}$ verifies the Palais-Smale condition on $\tilde{J}^0$, and we need $\tilde{J}^{c+\delta} \subset \tilde{J}^0$). By definition,

$$c = c_{k+r} = \inf_{C\in\Sigma_{k+r}} \sup_{u\in C} \tilde{J}(u).$$

Then, there exists $A \in \Sigma_{k+r}$, such that $\sup_{u\in A} \tilde{J}(u) < c + \delta$; i.e., $A \subset \tilde{J}^{c+\delta}$, and

$$\eta(A - U) \subset \eta(\tilde{J}^{c+\delta} - U) \subset \tilde{J}^{c-\delta}. \tag{4.4}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(\overline{A - U}) \geq \gamma(\overline{A - U}) \geq k$.

Consequently, $\eta(\overline{A - U}) \in \Sigma_k$. This contradicts (4.4) since

$$\eta(\overline{A - U}) \in \Sigma_k \quad \text{implies} \quad \sup_{u\in\eta(\overline{A-U})} \tilde{J}(u) \geq c_k = c.$$

The lemma above proves the following result:

Theorem 4.5. *Assume $1 < q < 2$. Then there exists a constant $A > 0$, such that for $0 < \lambda < A$, problems (P1) and (P2) admit infinitely many solutions.*

Proof. Integration by parts shows that any critical point is a solution of (P1). In fact, the boundary conditions are included in the choice $\mathbf{E} = W_0^{2,2}(\Omega)$. For problem (P2), the result is a consequence of the regularity result given in section 2.

5.- EXISTENCE OF A POSITIVE SOLUTION

Consider the problem

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2}u + |u|^{r-2}u \equiv f(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{$P3$}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, $N > 4$, $1 < q < 2$, $\lambda > 0$, $r > 2$. This means that we consider also *supercritical* problems. (Obviously, problem (P2) is a particular case of (P3)). The Laplacian case of (P3) has been already treated in [5].

Notice that for Δ^2 with these boundary conditions, the maximum principle holds as a consequence of the maximum principle for the Laplacian.

In this section, we will show the following result

Theorem 5.1. *There is a constant $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$, problem (P3) has a positive solution.*

The proof of the theorem is organized in several Lemmas.

Lemma 5.2. *Let v be the solution of the Dirichlet Problem*

$$\begin{cases} \Delta^2 v = \lambda + 1 \\ v|_{\partial\Omega} = 0 \\ \Delta v|_{\partial\Omega} = 0. \end{cases} \tag{S}$$

Then there exists a constant λ_0, such that for $\lambda < \lambda_0$ there is $T = T(\lambda) > 0$ for which the function $\overline{u} = Tv$ is a supersolution of (P3).

Proof. By simplicity we write

$$|u|^{r-2}u + \lambda |u|^{q-2}u = F(u).$$

Fixed λ consider the solution v of the Dirichlet problem (S). Then $0 < v < K$ in Ω. Define $\overline{u}(x) = Tv(x)$, where T is chosen in such a way that if:

a) $\Delta^2 \overline{u} = T(\lambda + 1)$

b) $F(\overline{u}) \leq \lambda T^{q-1} M^{q-1} + T^{r-1} M^{r-1}$, $M = \max\{1, \|v\|_\infty\}$. Then

$$\Delta^2 \overline{u} \geq F(TM) \geq F(\overline{u}) \tag{5.1}$$

Note that (5.1) is equivalent to

$$(\lambda + 1) \geq \lambda T^{q-2} M^{q-1} + T^{r-2} M^{r-1}, \tag{5.2}$$

M depends linearly on λ .

Define $\phi(T) = c_1\lambda T^{q-2} + c_2 T^{r-2}$ ($c_1 = M^{q-1}$, $c_2 = M^{r-1}$). Then

$$\lim_{T\to 0^+} \phi(T) = \lim_{T\to\infty} \phi(T) = \infty,$$

because $q-2 < 0 < r-2$. Thus ϕ attains a minimum in $[0,\infty)$. By elementary computations we have that $\phi'(T) \equiv -c_3\lambda T^{q-2-1} + c_4 T^{r-2-1} = 0$ in $T_0 = c_5\lambda^{\frac{1}{r-q}}$ where $c_5 = M^{-1}(r-q)^{-1}(2-q)$.

For the validity of (5,2), it suffices that

$$\phi(T_0) \le \lambda + 1,$$

that is

$$c_6\lambda^{\frac{r-2}{r-q}} < \lambda + 1.$$

Then, there is λ_0 such that for $0 < \lambda < \lambda_0$, $\overline{u}(x) = T_0 v$ is a supersolution of (P3).

Let ϕ_1 be the positive solution of the eigenvalue problem,

$$\begin{cases} \Delta^2 u = \lambda_1 u \\ u|_{\partial\Omega} = 0 \\ \Delta u|_{\partial\Omega} = 0, \end{cases}$$

with $\|\phi_1\|_\infty = 1$, corresponding to the first eigenvalue λ_1 of the operator Δ^2 with these boundary conditions. (We remark that in fact ϕ_1 coincides with the first eigenfunction of the Laplacian).

We have the following result.

Lemma 5.3. *(Construction of a subsolution) For t small enough $\underline{u} = t\phi_1$ is a subsolution of (P3), such that $\underline{u} \le \overline{u}$.*

Proof. Define $\underline{u}(x) = t\phi_1(x)$. Then

$$\Delta^2 \underline{u} = \lambda_1 t\phi_1,$$

and $\underline{u}$ is a subsolution of (P3) if

$$\lambda_1 t\phi_1 \le \lambda t^{q-1}\phi_1^{q-1} + t^{r-1}\phi_1^{r-1}.$$

In turn, for t small enough,

$$\lambda_1 t\phi_1 \le \lambda_1 (t\phi_1)^{q-1} \le \lambda (t\phi_1)^{q-1} + (t\phi)^{r-1} \tag{5.3}$$

Fix the supersolution $\overline{u}$, i.e. T. Then, for t small enough, we get

$$\Delta^2 \underline{u} = t\lambda_1\phi_1 \le t\lambda_1 \le F(TM) \le \Delta^2\overline{u}$$

by (5.1). Then $\underline{u}$ is a subsolution and by the weak comparison principle (which holds for problem (P3)), $\underline{u} \le \overline{u}$.

Proof of the Theorem 5.1. By iteration and comparison, we get a solution between the subsolution and the supersolution.

Remarks.-

1) The result is independent of the dimension for small λ.

2) Taking into account remark 1), it seems an interesting question to study the behavior of solutions of (P3) when $q \to 2$. Fix p and λ such that (P3) has a solution $u_q > 0$. Then

$$\lim_{q\to 2} ||u_q||_\infty = 0.$$

It suffices to observe that the constant c_5 in the proof of Lemma 5.2 goes to zero as $q \to 2$.

Proposition 5.4. *There is $\beta > 0$ such that for all $\lambda > \beta$ problem (P3) has no positive solution.*

Proof. Let $\phi_1 > 0$ be as in Lemma 5.3. Integration by parts shows

$$\int_\Omega (\lambda |u|^{q-1} + u^{r-1})\phi_1 dx = \int_\Omega \Delta^2 u \phi_1 dx = \lambda_1 \int_\Omega u\phi_1 dx.$$

But, for some $\alpha > 0$, $c\lambda^\alpha u \le u^{q-1} + u^{r-1} \quad \forall u > 0$, and then if u is a positive solution of (P3) necessarily $c\lambda^\alpha < \lambda_1$.

6.- ON A RESULT BY BREZIS-NIRENBERG

In the paper [7], Brezis and Nirenberg obtain a remarkable result showing that for very general functionals related with semilinear problems involving the Laplacian, the local minima in $\mathcal{C}^1$ are also local minima in $W_0^{1,2}$.

We need an extension of this result of Brezis and Nirenberg to our problems. The next theorem is stated solely for problem (P2) because later applications will be done only for (P2). The proof follows that of Brezis and Nirenberg for second order problems, but is included for the sake of completeness. We remark that this first result is also true for problem (P1).

Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx \tag{6.1}$$

and put $f(u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u$, $F(u) = \int_0^u f(s)ds$.

We recall that $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a critical point of J associated to problem $(P2)$ if

$$0 = \int_\Omega \Delta u \Delta\phi dx - \int_\Omega f(u)\phi dx, \quad \text{for all} \quad \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Define the class of functions

$$\mathcal{E}_0 = \{v \in \mathcal{C}^2(\overline{\Omega}) \quad | \, v(x) = 0, \, x \in \partial\Omega\}.$$

Then u_0 is a local minimizer of J in $\mathcal{E}_0$, (respectively $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$) if there is $\delta > 0$ such that $J(u_0) \le J(u_0 + v)$, for all $v \in \mathcal{E}_0$ such that $||v||_{\mathcal{C}^2} < \delta$ (respectively for all $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that $||v||_{W^{2,2}} < \delta$.)

We denote by B_ϵ the ball of radius ϵ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, centered at the origin.

Theorem 6.1. *Let $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a local minimizer of J in the $\mathcal{C}^2$ topology, then u_0 is a local minimizer in the $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ topology.*

Proof. The regularity Theorem 2.1 shows that the minimizer $u_0 \in \mathcal{C}^3(\overline{\Omega})$. By linearity on the differential operator, without loss of generality, we can assume that $u_0 = 0$. Then if the conclusion does not hold,

$$\text{for all} \quad \epsilon > 0, \quad \text{there exists} \quad v_\epsilon \in B_\epsilon, \quad \text{such that} \quad J(v_\epsilon) < J(0). \tag{6.2}$$

We shall proof that 0 is not a minimum in the $\mathcal{C}^2$ topology.

Consider the truncation

$$T_k(s) = \begin{cases} -k & \text{if} \quad s \le -k \\ s & \text{if} \quad -k < s < k \\ k & \text{if} \quad s \ge k. \end{cases}$$

Define $f_k(s) = f(T_k(s))$ and the corresponding primitive $F_k(u) = \int_0^u f_k(s)ds$. Take the functional

$$J_k(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F_k(u)dx, \tag{6.3}$$

so that for each $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$

$$\lim_{k\to\infty} J_k(u) = J(u). \tag{6.4}$$

Now, by (6.2) and (6.4), for all $\epsilon > 0$ there is a constant $k = k(\epsilon)$, such that

$$J_{k(\epsilon)}(v_\epsilon) < J(0). \tag{6.5}$$

A fortiori, if $w_\epsilon \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ verifies

$$J_k(w_\epsilon) = \min_{v \in B_\epsilon} J_k(v), \tag{6.6}$$

then

$$J_k(w_\epsilon) < J(0). \tag{6.7}$$

If

$$w_\epsilon \in \mathcal{C}^{3,\beta} \quad \text{and} \quad w_\epsilon \to 0 \quad \text{in} \quad \mathcal{C}^2\,, \tag{6.8}$$

then for some ϵ small enough

$$J(w_\epsilon) = J_k(w_\epsilon) < J(0),$$

which yields a contradiction since 0 is a local minimum in $\mathcal{C}^2$.

We will prove below that (6.8) holds.

According with the proof of Theorem 2.1, there exists a Lagrange multiplier $\mu_e \ge 0$, such that the Euler equation verified by w_ϵ is

$$(1 + \mu_\epsilon)\Delta^2 w_\epsilon = f_k(w_\epsilon), \tag{6.9}$$

and $w_\epsilon = 0$, $\Delta w_\epsilon = 0$ on $\partial\Omega$. Moreover we have the uniform estimate

$$|f_k(u)| \le c(1 + |u|^{2^*-1}). \tag{6.10}$$

Because $\{w_\epsilon\}$ converges to zero in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and therefore in $L^{2^*}(\Omega)$, for some subsequence there exists a function $h \in L^{2^*}(\Omega)$ such that $|w_\epsilon| \le h$ for all ϵ.

So we can write $|f_k(w_\epsilon)| \le C(1 + \alpha|w_\epsilon|)$ where $\alpha = h^{8/(N-4)}$.

Following the proof of Theorem 2.1 we get uniform estimates in $\mathcal{C}^{3,\beta}$, and the Ascoli lemma gives us (6.8).

The monotonicity of f in our problem plays a very important role.

If $\lambda_1 < \lambda < \lambda_2$ then the corresponding solutions of problem (P2) constructed in Theorem 5.1 verify,

i) $u_{\lambda_1} \leq u_{\lambda_2}$, by comparison.

ii) u_{λ_1}, (respectively u_{λ_2}), is a subsolution (respectively a supersolution) for problem

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2}u + |u|^{2^*-2}u \equiv f_\lambda(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{$P2_\lambda$}$$

and both are not solutions of $(P2_\lambda)$.

For simplicity of printing, denote $u_{\lambda_i} \equiv u_i$.

We can formulate the following result.

Theorem 6.2. *Let $0 < \lambda_1 < \lambda < \lambda_2$, and u_i be defined as above. Then there exists a solution u of $(P2_\lambda)$ such that $u_1 \leq u \leq u_2$ and moreover u is a local minimum of J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Proof. We consider the following truncation of f

$$\overline{f_\lambda}(x,u) = \begin{cases} f(u_1(x)), & \text{if } \ u < u_1(x) \\ f_\lambda(u), & \text{if } \ u_1(x) \leq u \leq u_2(x) \\ f(u_2(x)), & \text{if } \ u_2(x) < u, \end{cases}$$

and $\overline{F_\lambda}(x,u) = \int_0^u \overline{f_\lambda}(x,s)ds$. It is easy to show that the functional

$$\overline{J}(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega \overline{F}_\lambda(x,u)dx. \tag{6.11}$$

attains its minimum in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and that u verifies the problem

$$\begin{cases} \Delta^2 u = \overline{f}_\lambda(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0. \end{cases}$$

In particular u is regular. By monotonicity we see that

(1) $\Delta^2(u_1 - u) \leq f_\lambda(u_1) - \overline{f}_\lambda(x,u) \leq 0$

(2) $\Delta^2(u - u_2) \leq \overline{f}_\lambda(u) - f_\lambda(x,u_2) \leq 0$,

so that, $\Delta^2(u_1 - u) \leq 0$ and $\Delta^2(u - u_2) \leq 0$. By using twice the Hopf lemma for the Laplacian we have for some $\epsilon > 0$,

$$u_1(x) + \epsilon d(x,\partial\Omega) \leq u(x) \leq u_2(x) - \epsilon d(x,\partial\Omega). \tag{6.12}$$

Thus $J(u) = \overline{J}(u)$. Moreover, if we choose a ball in $\mathcal{E}_0$ of radius less than ϵ, i.e., $v \in \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{C}_0^1(\Omega)$ such that $||v - u||_{\mathcal{C}^2} < \epsilon$ we have that

$$J(v) = \overline{J}(v) \geq \overline{J}(u) = J(u),$$

that is, u is a local minimum of J in $\mathcal{C}^2$. But then, by Theorem 6.1, u is a local minimum of J in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

We will use Theorems 6.1 and 6.2 in the next section.

Remark.- Similar methods have been used by De Figueiredo, [8].

7.- EXISTENCE OF AT LEAST TWO POSITIVE SOLUTIONS

The existence of a second positive solution for (P2) depends on when we can apply some version of the *Mountain Pass Lemma.*

In fact for $\lambda > 0$ small enough, we can proceed as in [11] and obtain in this case a second positive solution. This result takes as starting point the minimum of the truncated functional discussed in section 4. The Palais-Smale condition is obtained depending on this minimum value.

We concentrate the attention in a global result, in the spirit of [1]. More precisely, if we define $\Lambda = \sup\{\lambda > 0 \,|\, (P2) \text{ has a positive solution}\}$, we obtain the following result.

Theorem 7.1. *If $\lambda \in I = (0, \Lambda)$, then problem (P2) has at least two positive solutions.*

Proof. The proof of the theorem will be done in several steps.
- Step 1.- Fix $\lambda \in I$ and consider the solution of (P2), u_0, obtained in Theorem 6.2, that is, u_0 is a local minimum of the functional

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F_\lambda(u) dx. \tag{7.1}$$

in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Define

$$g_\lambda(x,s) = \begin{cases} f_\lambda(u_0(x)+s) - f_\lambda(u_0(x)), & \text{if } s > 0 \\ 0, & \text{if } s \le 0, \end{cases}$$

consider the truncation of $g_\lambda(x,s)$,

$$\overline{g}_\lambda(x,s) = \begin{cases} f_\lambda(u_2(x)) - f_\lambda(u_0(x)), & \text{if } s \ge u_2(x) - u_0(x) \\ g_\lambda(x,s), & \text{if } u_2(x) - u_0(x) > s > 0 \\ 0, & \text{if } s \le 0, \end{cases}$$

and $G_\lambda(x,u) = \int_0^u g_\lambda(x,s)ds$, $\overline{G}_\lambda(x,u) = \int_0^u \overline{g}_\lambda(x,s)ds$, the respective primitives. Here $u_2 \equiv u_{\lambda_2}$ as in Theorem 6.2. It is easy to check that the functional

$$\Phi(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 dx - \int_\Omega \overline{G}_\lambda(x,v) dx \tag{7.2}$$

attains its absolute minimum in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ in some point $v_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. By the maximum principle we have that

$$u_2 - u_0 \ge v_0 \ge 0. \tag{7.3}$$

If $v_0 \ne 0$ in Ω we are done since

$$\Delta^2 v_0 = \overline{g}_\lambda(x, v_0) = g_\lambda(x, v_0),$$

and $u_1 = u_0 + v_0$ is the second positive solution of (P2), being

$$\Delta^2 u_1 = \Delta^2 v_0 + \Delta^2 u_0 = f_\lambda(u_0 + v_0) - f_\lambda(u_0) + f_\lambda(u_0) = f_\lambda(u_1).$$

Our problem is now reduced to the case when $v_0 \equiv 0$.

- STEP 2.- Assume that the minimum of Φ is attained only in $v_0 \equiv 0$. First, we want to show that $v_0 = 0$ is a local minimum in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of the functional

$$\Psi(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 dx - \int_\Omega G_\lambda(x, v)dx. \tag{7.4}$$

Now, we have $\Delta^2(u_2 - u_0) \geq f_\lambda(u_2) - f_\lambda(u_0) \geq 0$ and $u_2 - u_0 = 0$, $\Delta(u_2 - u_0) = 0$ on $\partial\Omega$. Thus, by the Hopf Lemma for the Laplacian we can conclude that for $\epsilon > 0$ small enough, if $h \in \mathcal{C}^2 \cap \mathcal{C}_0^1$ and $||h||_{\mathcal{C}^2} \leq \epsilon$, then

$$\Psi(h) = \Phi(h) \geq \Phi(v_0) \equiv \Phi(0) = \Psi(0).$$

This means that $v_0 = 0$ is a local minimum of Ψ in $\mathcal{C}^2 \cap \mathcal{C}_0^1$ and by Theorem 6.1 it is a local minimum in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

In the Step 2 above, we reduced the problem to the case where the minimum of Ψ is zero. All the critical points of Ψ are nonnegative.

To finish the proof, assume that $v_0 = 0$ is the unique critical point of Ψ. We shall prove that in this case the Mountain-Pass Theorem applies, giving a nontrivial solution, and hence a contradiction. This argument is developed in several steps.

- STEP 3.- We will prove the following lemma.

Lemma 7.2. *If $v_0 = 0$ is the only critical point of Ψ, then Ψ satisfies the Palais-Smale condition under the level $c_0 = \frac{2}{N}S^{N/4}$.*

Proof. Assume that $\{v_j\}_{j\in\mathbf{N}} \subset W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a Palais-Smale sequence under the level c_0 for the functional Ψ defined by (7.4), i.e.,

i) $\lim_{j\to\infty} \Psi(v_j) = c < c_0 = \frac{2}{N}S^{N/4}$,

ii) $\lim_{j\to\infty} \Psi'(v_j) = 0$ in the dual space of $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

In a similar way as in Lemma 3.3 we get the boundedness of the sequence in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, and as a consequence we get a convenient subsequence which converges weakly in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and verifies the inequalities (3.3). We also obtain that the mass for the singular part are as in (3.8), i.e., either $\nu_k = 0$, or $\nu_k \geq S^{N/4}$.

From i) and weak convergence we have,

$$0 = \lim_{j\to\infty} \int_\Omega \Delta v_j \Delta\phi dx - \int_\Omega g_\lambda(x, v_j)\phi dx = \int_\Omega \Delta v \Delta\phi dx - \int_\Omega g_\lambda(x, v)\phi dx$$

for all $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, hence v is a critical point of Ψ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, and so necessarily $v = 0$ by hypothesis. Then, if for some k, $\nu_k \neq 0$, we get

$$\frac{2}{N}S^{N/4} > c = \lim_{j\to\infty} \Psi(v_j) = \lim_{j\to\infty} \{\Psi(v_j) - \frac{1}{2} < \Psi'(v_j), v_j >\} \geq \frac{2}{N}S^{\frac{N}{4}},$$

and this is a contradiction. Thus the subsequence converges strongly in $L^{2^*}(\Omega)$, and in turn in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

The last result that we have to show is that there exists a Palais-Smale sequence below the critical level $\frac{2}{N}S^{N/4}$. More precisely

Lemma 7.3. *If $v_0 = 0$, local minimum of Ψ, is its unique critical point, then there exists a Palais-Smale sequence such that*

$$\lim_{j\to\infty} \Psi(v_j) = c < c_0 = \frac{2}{N} S^{N/4}.$$

Proof. We assume for simplicity of printing that $0 \in \Omega$. Consider the best constant of the Sobolev inclusion defined in (3.2). It is known [16] that for $\Omega = \mathbf{R}^N$ the best constant is attained by the following minimizers,

$$V_\epsilon(x) = K_1 \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^{\frac{N-4}{2}}, \quad \epsilon > 0, \quad K_1 = [N(N-4)(N^2-4)]^{(N-4)/8},$$

and V_ϵ verifies the problem $\Delta^2 u = u^{\frac{N+4}{N-4}}$ in $\mathbf{R}^N$ with $N > 4$ and

$$S^{\frac{N}{4}} = \int_{\mathbf{R}^N} |\Delta V_1|^2 dx = \int_{\mathbf{R}^N} |V_1|^{2^*} dx \tag{7.5}$$

The best constant is the same for equivalent norms. (See [13]).

The idea is to perform a truncation with a cutoff function $\rho(x) \geq 0$, smooth, such that, $\rho(x) = 1$ if $|x| < R$, $\rho(x) = 0$ if $|x| > 2R$; where we take $R > 0$ in such way that all x verifying $|x| \leq 2R$ belong to Ω. More precisely, define

$$v_\epsilon(x) = \rho(x) V_\epsilon(x). \tag{7.6}$$

For ϵ small enough, the concentration produced will give us that

$$\sup_{t\geq 0} \Psi(t v_\epsilon) = c_\epsilon < \frac{2}{N} S^{N/4}, \tag{7.7}$$

that is sufficient to have the result.

We proceed to prove (7.7). We can get the following estimates,

$$\int_\Omega |\Delta v_\epsilon|^2 dx = \int_{\mathbf{R}^N} |\Delta V_1|^2 dx + O(\epsilon^{N-4}) \tag{7.8}$$

$$\int_\Omega |v_\epsilon|^{2^*} dx = \int_{\mathbf{R}^N} |V_1|^{2^*} dx + O(\epsilon^N) \tag{7.9}$$

and for some positive k,

$$\int_\Omega |v_\epsilon|^r dx = \begin{cases} k\epsilon^{\frac{(N-4)r}{2}} + o(\epsilon^{\frac{(N-4)r}{2}}) & \text{if} \quad r < \dfrac{N}{N-4} \\ k\epsilon^{N-\frac{(N-4)r}{2}} |\log\epsilon| + o(\epsilon^{N-\frac{(N-4)r}{2}} |\log\epsilon|) & \text{if} \quad r = \dfrac{N}{N-4} \\ k\epsilon^{N-\frac{(N-4)r}{2}} + o(\epsilon^{N-\frac{(N-4)r}{2}}) & \text{if} \quad r > \dfrac{N}{N-4} \end{cases} \tag{7.10}$$

The key for the estimate (7.7) is

$$G_\lambda(x,s) \geq \frac{1}{2^*} s^{2^*} + u_0(x) s^{2^*-1} + C u_0(x)^{2^*-\gamma} s^\gamma, \quad \gamma \in \left(\frac{N}{N-4}, \frac{N+4}{N-4}\right), \tag{7.11}$$

which is a consequence of the following calculus inequality:

If $r>2$ then given $\gamma\in(1,r-1)$ there exists a constant $C>-\infty$ such that

$$\inf_{t>0}\{\frac{(1+t)^r-(1+t^r+rt+rt^{r-1})}{t^\gamma}\}\ge C.$$

Now, from (7.11) we have

$$\Psi(tv_\epsilon)\le\frac{t^2}{2}\int\limits_\Omega|\Delta v_\epsilon|^2dx-\frac{t^{2^*}}{2^*}\int\limits_\Omega|v_\epsilon|^{2^*}dx-m_1t^{2^*-1}\int\limits_\Omega|v_\epsilon|^{2^*-1}dx+|C|m_1^{2^*-\gamma}t^\gamma\int\limits_\Omega|v_\epsilon(x)|^\gamma dx,$$

(here we use that $0<m_1=\inf\limits_{x\in B_{2R}}u_0(x)$.) Then

$$\Psi(tv_\epsilon)\le\frac{t^2}{2}\int\limits_{\mathbf{R}^N}|\Delta V_1|^2dx-\frac{t^{2^*}}{2^*}\int\limits_{\mathbf{R}^N}|V_1|^{2^*}dx-m_1t^{2^*-1}k\epsilon^{\frac{N-4}{2}}+o(\epsilon^{\frac{N-4}{2}}).$$

Consider the function

$$h_\epsilon(t)=\frac{t^2}{2}\int_{\mathbf{R}^N}|\Delta V_1|^2dx-\frac{t^{2^*}}{2^*}\int_{\mathbf{R}^N}|V_1|^{2^*}dx-Ct^{2^*-1}\epsilon^{\frac{N-4}{2}}+o(\epsilon^{\frac{N-4}{2}}).$$

When $\epsilon=0$, h_o attains its maximum in $[0,\infty)$ at t_o and $h_o(t_o)=\dfrac{2}{N}S^{N/4}$ (because of the relationship between V_1 and the best Sobolev constant S).

It is clear that $h_\epsilon(t)<h_o(t)$; hence we conclude that

$$\max_{t>0}h_\epsilon(t)<h_o(t_o)=\frac{2}{N}S^{N/4}.$$

To finish the proof, we need to analyze the influence of the error term.

If we denote by t_ϵ the point where h_ϵ attaint its maximum, it is easily seen that $0<t_\epsilon<t_o$ and $t_\epsilon\to t_o$ as $\epsilon\to0$. Therefore, we can write $t_\epsilon=t_ox_\epsilon$, where $x_\epsilon\to1$ as $\epsilon\to0$. Taking into account that $h'_\epsilon(t_\epsilon)=0$, we get

$$t_ox_\epsilon\int_{\mathbf{R}^N}|\Delta V_1|^2dx-t_o^{2^*-1}x_\epsilon^{2^*-1}\int_{\mathbf{R}^N}|V_1|^{2^*}dx=C(2^*-1)t_o^{2^*-2}x_\epsilon^{2^*-2}\epsilon^{\frac{N-4}{2}}.$$

Using the precise value of t_o, after some computations we arrive to

$$1-x_\epsilon^{2^*-2}=Ax_\epsilon^{2^*-3}\epsilon^{\frac{N-4}{4}},$$

where

$$A=C(2^*-1)\frac{(\int_{\mathbf{R}^N}|\Delta V_1|^2dx)^{\frac{-1}{2^*-2}}}{(\int_{\mathbf{R}^N}|V_1|^{2^*}dx)^{1-\frac{1}{2^*-2}}}.$$

By Taylor's expansion:

$$(1-x_\epsilon)(2^*-2)x_\epsilon^{2^*-3}+o(1-x_\epsilon)=Ax_\epsilon^{2^*-3}\epsilon^{\frac{N-4}{2}}.$$

Therefore, $1 - x_\epsilon = M\epsilon^{\frac{N-4}{2}} + o(\epsilon^{\frac{N-4}{2}})$, for $M = \dfrac{A}{2^* - 2}$.

Finally, this identity allows us to prove that

$$h_\epsilon(t_\epsilon) = \frac{2}{N} S^{N/4} - C t_o^{2^*-1} \epsilon^{\frac{N-4}{2}} + o(\epsilon^{\frac{N-4}{2}})$$

and the conclusion follows at once.

- STEP 4.- Assume that v_0 is the unique critical point of Ψ.

Consider the function $w_\epsilon = r_\epsilon v_\epsilon$, with r_ϵ large enough, such that $\Psi(w_\epsilon) < 0$ and the mini-max value

$$c_\epsilon = \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} \Psi(\gamma(t)),$$

where

$$\mathcal{P} = \{\gamma : [0,1] \longrightarrow W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \ : \text{continuous}, \gamma(0) = 0,\, \gamma(1) = w_\epsilon\}.$$

Because $v_0 = 0$ is the local minimum, then $0 \le c_\epsilon < \frac{2}{N} S^{\frac{N}{4}}$. If $c_\epsilon > 0$ the Mountain Pass Lemma by Ambrosetti and Rabinowitz, [4], give us a second positive critical point, in contradiction with the hypothesis. In the case $c_\epsilon = 0$, we get the same contradiction by using the result by Pucci-Serrin, [19].

This contradiction finishes the proof.

Remark We can say that the solutions constructed in sections 4 and 5 correspond to the *sublinear* term, because, for instance, when $\lambda \to 0$ they converge to the trivial solution. The same behavior is obtained for $q \to 2$ in the case of the minimal positive solution obtained in section 5.

The second positive solution obtained in Theorem 7.1, however, tends to a Dirac mass as $\lambda \to 0$. This behavior was obtained for solutions of the p-laplacian in [11].

8.- FURTHER RESULTS

The results given above can be generalized without difficulty to second members of the form $f(x, u)$, where f is increasing in u, verifies some regularity, growth and oddness properties. We prefer to avoid more technicalities.

We have, on the other hand, several remarks about possible applications of the methods above to similar cases.

A) Subcritical Problems. Obviously all the results obtained above also hold for the problems

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2} u + |u|^{r-2} u \equiv f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, & \end{cases} \tag{S1}$$

and

$$\begin{cases} \Delta^2 u = \lambda |u|^{q-2} u + |u|^{r-2} u \equiv f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & \\ \Delta u|_{\partial\Omega} = 0, & \end{cases} \tag{S2}$$

where $1 < q < 2 < r < 2^*$.

In these cases there is compactness and all arguments are much easier.

B) Quasilinear Problems. The results in Section 4 can be extended to the problems

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-2}u + |u|^{p^*-2}u \equiv f(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \end{cases} \tag{$Q1$}$$

and

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-2}u + |u|^{p^*-2}u \equiv f(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{$Q2$}$$

where $N > 2p$, $1 < q < p$ and $p^* = \frac{Np}{N-2p}$.

The mini-max theorem for quasilinear equations of second order can be seen in [10], and here only a minor change gives the result.

The existence result in Section 5 can be obtained also for the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-2}u + |u|^{r-2}u \equiv f(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{P}$$

where $1 < q < p < r$, since a maximum principle is verified by reiteration of the maximum principle for the Laplacian and the monotonicity of the function $h(s) = |s|^{p-2}s$.

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