High Temperature Expansions and Dynamical Systems

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Abstract

We develop a resummed high-temperature expansion for lattice spin systems with long range interactions, in models where the free energy is not, in general, analytic. We establish uniqueness of the Gibbs state and exponential decay of the correlation functions. Then, we apply this expansion to the Perron-Frobenius operator of weakly coupled map lattices.

1 Introduction.

The theory of Gibbs states was originally developed for the mathematical analysis of equilibrium statistical mechanics. An interesting application of the theory was found by Sinai, Ruelle and Bowen in the 70's [41, 42, 38, 1] who applied it to the ergodic theory of uniformly hyperbolic dynamical systems. While this so called thermodynamic formalism has been very successful in ergodic theory, the Gibbs states that describe the statistics of such dynamical systems are quite simple from the point of view of statistical mechanics: they describe one dimensional spin systems with spins taking values in a finite set and interacting with exponentially decaying potentials. In particular, phase transitions, i.e. the coexistence of several Gibbs states for the same interaction, which are of major interest in statistical mechanics, are absent in such systems.

More recently, it has been realized that certain infinite dimensional dynamical systems possess attracting sets that are *extensive* in a suitable volume. This is believed to be the case for many classes of nonlinear parabolic partial differential equations on

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some spatial domain: the dimension of the attracting set (or a bound for it) increases to infinity as the domain becomes unbounded [43]. Discrete time dynamical systems, such as coupled maps, were introduced to model these phenomena [29]. Bunimovich and Sinai [5] showed that these systems give rise to a thermodynamic formalism for spin systems on a lattice of more than one dimension. Because of this last feature, the possibility of phase transitions is at least open. However, most rigorous analyses have so far been limited to weakly coupled maps which corresponds to "high temperature" in statistical mechanics. But, even in that regime, coupled maps give rise to interactions of a rather peculiar kind, and proving the absence of phase transitions for the latter is not completely trivial.

In statistical mechanics, the common wisdom is that "high temperature" leads to uniqueness and decay of correlations (mixing for lattice translations). Mathematically, one introduces a suitable Banach space of "interactions" parametrizing the Gibbs states and high temperature means small norm. For small norm the interaction uniquely determines the Gibbs state [10, 11]. However, the question of a suitable norm turns out to be subtle. One can distinguish between three properties: uniqueness of the Gibbs state, exponential mixing and analyticity (in the potentials) of the correlation functions. In the last case one has a complete control of the Gibbs state in terms of a convergent expansion, the high temperature expansion. In the other two cases, less detailed information is available.

As we shall recall in Section 2, there exist Banach spaces of interactions where exponential mixing holds, but analyticity does not hold, in general. It turns out that the interactions corresponding, in the thermodynamic formalism, to coupled maps belong to such "pathological" Banach spaces of interactions.

The purpose of this paper is twofold. First we wish to give a unified treatment of the high temperature states discussed above. We show how all the previous results (e.g. the uniqueness results of Dobrushin [10, 11]) and also the generalizations needed for dynamical systems can be obtained by a simple resummation of a high temperature expansion based on an idea of von Dreifus, Klein and Perez [17] used for disordered systems (such a resummation was used already by Fisher [20]). The resummation uses in an essential way the fact that the interactions are real and thus is not in conflict with the lack of analyticity.

Then we apply these results to the study of the ergodic theory of coupled $C^{1+\delta}$ circle maps. We rederive, generalize and complete the previous studies of such maps. We believe this is necessary due to some confusion and incorrect results in the litterature (see Section 4). Our approach is slightly novel since we derive the thermodynamic formalism without introducing Markov partitions and symbolic dynamics (which however exist in the models we consider). One motivation for this is that systems where one expects phase transitions (based on numerical and theoretical evidence [37, 33]) will, most probably, not have useful Markov partitions. In our formalism such systems (e.g. coupled bounded variation interval maps) nevertheless give rise to a thermodynamic formalism, but with potentials whose thermodynamic limit we are presently unable to control.

The paper is organized as follows. Section 2 contains a review of the theory of Gibbs states and of the various spaces of potentials. We also formulate and prove in Section

2 our main statistical mechanics theorem. This Section is completely independent of the dynamical system part of the paper. Section 3 defines coupled map lattices and states the main theorem, whose proof can be found in Section 4. We have tried to be rather self-contained in the statistical mechanics hoping the paper to be accessible to the dynamical systems community.

2 Resummed high-temperature expansions.

2.1. Lattice systems

We consider a lattice spin system: to each $i \in \mathbf{Z}^d$ we assign Ω_i, a copy of a finite set Ω_0 or more generally a compact metric space; we stick to the former, but the generalization to the latter case is straightforward. Given $X \subset \mathbf{Z}^d$, a spin configuration in X, denoted by s_X, is an element $s_X \in \Omega_X \equiv \times_{i\in X}\Omega_i$. An interaction is defined by a family $\Phi = (\Phi_X)$ of (continuous) functions indexed by finite subsets X of $\mathbf{Z}^d$:

$$\Phi_X : \Omega_X \to \mathbf{R} \tag{1}$$

We let $\|\Phi_X\|$ denote the sup norm of Φ_X. We consider translation invariant $\Phi's$: $\Phi_{X+i} = \tau_i\Phi_X$ where τ_i, $i \in \mathbf{Z}^d$ is the natural $\mathbf{Z}^d$ action on functions defined on Ω_X to functions defined on Ω_{X+i}. Given $\Lambda \subset \mathbf{Z}^d$, $|\Lambda| < \infty$, and a configuration $s'_{\Lambda^c} = (s'_i)_{i\in\Lambda^c}$ in Ω_{Λ^c}, the Hamiltonian in Λ (with boundary conditions s'_{Λ^c}) is defined as

$$\mathcal{H}(s_\Lambda|s'_{\Lambda^c}) = -\sum_{X\cap\Lambda\neq\emptyset} \Phi_X(s_{X\cap\Lambda} \vee s'_{X\cap\Lambda^c}) \tag{2}$$

where, if $s_X \in \Omega_X$, $s'_Y \in \Omega_Y$, for $X \cap Y = \emptyset$, $s_X \vee s'_Y$ is the obvious configuration in $\Omega_{X\cup Y}$. The associated (finite volume) Gibbs measure is a probability distribution on Ω_Λ:

$$\nu(s_\Lambda|s'_{\Lambda^c}) = Z^{-1}(\Lambda|s'_{\Lambda^c}) \exp(-\mathcal{H}(s_\Lambda|s'_{\Lambda^c})) \tag{3}$$

(we put a minus sign in (2) for convenience), with

$$Z(\Lambda|s'_{\Lambda^c}) = \sum_{s_\Lambda} \exp(-\mathcal{H}(s_\Lambda|s'_{\Lambda^c})). \tag{4}$$

See e.g. [38, 40, 19] for more details on the theory of Gibbs states.

2.2. Finite range interactions

Suppose first that Φ_X is of finite range, i.e., for some $R < \infty$, $\Phi_X = 0$ if the diameter of X, $d(X) > R$ (we put on $\mathbf{Z}^d$ the metric $|i| = \max_\alpha |i_\alpha|$). Then, if $\sup_X \|\Phi_X\|$ is small enough, a convergent high-temperature expansion (so-called polymer expansion) yields the following results (see [21, 4, 32, 40]):

1) The Gibbs state is unique i.e., the finite-volume Gibbs measures (3) converge, as $\Lambda \uparrow \mathbf{Z}^d$, independently of the boundary conditions to a unique measure on $\Omega_{\mathbf{Z}^d}$ satisfying certain consistency conditions (the DLR-equations).

2) The correlation functions in that Gibbs measure decay exponentially (see (12) below).

3) The free energy $F(\Phi)$ defined as $\lim_{\Lambda\to\infty} |\Lambda|^{-1} \log Z(\Lambda|s'_{\Lambda^c})$ and the correlation functions are analytic: Given finite subsets X_1,...,X_n and $\lambda \in \mathbf{C}^n$, define Φ^λ by $\Phi^\lambda_Y = \lambda_\alpha \Phi_Y$ for each $Y = X_\alpha + j$, for some $j \in \mathbf{Z}^d$ and some $\alpha = 1, \cdots, n$ and $\Phi^\lambda_Y = \Phi_Y$ for the other Y's. Then, $F(\Phi^\lambda)$ is analytic in the polydisc $|\lambda_\alpha| \leq \epsilon$, for ϵ small enough.

The smallness of $\|\Phi_X\|$ is however not a necessary condition for uniqueness and analyticity. For example, one dimensional finite range systems always satisfy all three conditions above. Dobrushin-Shlosman [15, 16] gave several equivalent conditions to be satisfied by the finite volume distributions (3) that guarantee that the Gibbs state is unique and that analyticity and exponential decay of correlations hold (they call these properties "complete analyticity"). These results were later rederived by Olivieri and Picco [34, 35] using expansion methods.

To formulate one of these conditions, used in [34, 35], it is convenient to cover $\mathbf{Z}^d$ by disjoint cubes of side L, called L-cubes. We fix such a covering. We choose L sufficiently large, and, in particular $L > R$, where R is the range of Φ^0. Then the condition becomes: There exists a function $f : \mathbf{N} \to \mathbf{R}$ such that

$$|\frac{Z(\Lambda|s^{xy})Z(\Lambda|s)}{Z(\Lambda|s^x)Z(\Lambda|s^y)} - 1| \leq f(|x-y|) \tag{5}$$

and

$$\lim_{n\to\infty} n^{2(d-1)} f(n) = 0. \tag{6}$$

Here, Λ is an arbitrary union of L-cubes and s an arbitrary configuration in Ω_{Λ^c}; $s^A_z = s_z$ for $z \notin A$, where $A = x, y$ or xy. (Actually, it is enough to check (5) for Λ's being subsets of a sufficiently large volume. Then, the fact that (5) holds for larger volumes follows from (13) below.)

2.3. Infinite range interactions

Many of these results extend to infinite range interactions but one has to carefully distinguish between different Banach spaces of interactions in which different results are valid. Without trying to be exhaustive, one has basically the following results: (see [40] for a review)

1) If

$$\|\Phi\|_1 = \sum_{0\in X} |X|\|\Phi_X\| \tag{7}$$

is small enough, then the Gibbs state is unique [10, 11, 42]

2) If, for some $\gamma > 0$,

$$\|\Phi\|_2 = \sum_{0 \in X} e^{\gamma d(X)} \|\Phi_X\| \tag{8}$$

is small enough, then the Gibbs state is unique and its correlation functions decay exponentially [23].

3) If, for some $\gamma > 0$,

$$\|\Phi\|_3 = \sum_{0 \in X} e^{\gamma |X|} \|\Phi_X\| \tag{9}$$

is small enough, then the Gibbs state is unique and its correlation functions are analytic [26].

The reasons for these different norms are as follows: usually, correlation functions do not decay faster than the interactions, so that something like (8) is needed to get exponential decay.

As for analyticity, Dobrushin and Martirosyan [13] have shown, by explicit counterexamples, that analyticity will not hold, (at least not uniformly in the volume, see [19], p.958), in a neighbourhood of zero, in any space larger than the one defined by (9): Let

$$\|\Phi\|_h = \sum_{0 \in X} h(|X|) \|\Phi_X\|. \tag{10}$$

Then, if $\lim_{n\to\infty} h(n)e^{-\gamma n} = 0, \forall \gamma > 0$, there exist (complex) interactions with arbitrarily small $\|\Phi\|_h$ such that the corresponding partition function vanishes for a sequence of cubes $\Lambda_n \to \infty$. (see [18], p.971 for a simple such counterexample).

¿From the point of view of polymer expansions, the norm (9) is quite natural, as was remarked by Brydges ([4] p.141): the exponential weight in (9), depending on the *size* of X, is needed to control the sum over small polymers lying inside a big one.

However, this norm is stronger than (7) and, sometimes, than (8): if the lattice dimension d is larger than one, and if X is, e.g., a large cube, $|X| >> d(X)$. So, for some Φ's, uniqueness and exponential decay hold but analyticity does not, and it seems that the first two properties cannot be proven using convergent polymer expansions (the existing proofs use Dobrushin's methods). The problem is that these expansions tend, when they converge, to yield analyticity almost automatically.

The reader should not be misled by the fact that standard high-temperature expansions *for spin* $\frac{1}{2}$ *systems* [21, 26, 9] seem to show analyticity for interactions that are small in a weaker norm than (9). Indeed, in these papers, the interactions are written in a special representation ("spin" or "gas" language) and the norm used for the interactions is quite different from the sup norm. This slightly confusing point is very nicely clarified in [18].

2.4. The main result

The analyticity results of Dobrushin and Shlosman were extended, for the analyticity part, by Dobrushin and Martirosyan [12] to infinite range interactions of the form

$$\Phi = \Phi^0 + \Phi^1 \tag{11}$$

where Φ^0 has finite range and is completely analytic, while $\|\Phi^1\|_3$ is small (actually, Dobrushin and Martirosyan discuss the Φ^0 part in terms of specifications instead of interactions; this could be done here too). In the dynamical systems problem of Section 3 we will encounter an interaction of even more general type, namely of the form (11), but with $\|\Phi^1\|_2$ (instead of $\|\Phi^1\|_3$) small. Our main result is

Theorem 1 *Let $\Phi = \Phi^0 + \Phi^1$, where Φ^0 is completely analytic. Then, there exist $\epsilon > 0$, $m > 0$, $C < \infty$ such that, if $\|\Phi^1\|_2 \leq \epsilon$, there is a unique Gibbs state μ for Φ and the correlation functions satisfy, for all F, G, with $F : \Omega_A \to \mathbf{R}$, $G : \Omega_B \to \mathbf{R}$, where $A, B \subset \mathbf{Z}^d$ are finite:*

$$|\langle FG\rangle - \langle F\rangle\langle G\rangle| \leq C\|F\|\|G\|e^{-md(A,B)}, \tag{12}$$

where $d(A,B)$ is the distance between the sets A and B and $\langle F\rangle = \int F d\mu$.

Remark 1. We shall discuss various extensions and variations of this result after giving the proof in Section 2.5.

Remark 2. Our proof is based on a high-temperature expansion instead of Dobrushin's method. We shall resum the expansion and use then, in an essential way, the fact that Φ_X is real, so that our proof applies even when analyticity does not hold. It is inspired by a recent work of von Dreifus, Klein and Perez [17] who developed a high-temperature expansion for disordered systems. Their problem was similar to ours: due to Griffiths' singularities [22], analyticity does not hold, in general, for disordered systems, while we have to circumvent the possibility of Dobrushin-Martirosyan singularities [13]. Actually, a simple version of our method appears already in Fisher's upper bound on the two-point function for the Ising model [20]. On the other hand, in a recent paper, Jiang and Mazel [28] consider real interactions similar to ours, and develop a convergent high-temperature expansion which yields uniqueness and exponential decay of the correlation functions, but for systems defined on a two-dimensional lattice (their method crucially depends on the fact that in two dimensions boundaries of volumes are proportional to their diameters).

2.5. The proof

Before starting with the proof, we will use another representation for the partition function of Φ^0 derived from (5, 6) by Olivieri and Pico. They show that if (5, 6) holds then the interaction Φ^0 "has a cluster (or polymer) expansion". Precisely, this means that the partition function admits the following representation. If Λ is a union of disjoint L-cubes, and $s \in \Omega_{\Lambda^c}$,

$$Z^0(\Lambda|s) = e^{f|\Lambda|} \prod_\alpha W(s_{\Lambda^c_\alpha}) \exp \sum_Y \phi_Y(s_{Y\cap\Lambda^c}) \tag{13}$$

where Z^0 refers to the partition function with interaction Φ^0, the product runs over connected components of $\Lambda^c = \cup_\alpha \Lambda^c_\alpha$ and the sum runs over connected sets of L-cubes Y, so that $Y \cap \Lambda \neq \emptyset$ (and ϕ_Y is a constant if $Y \cap \Lambda^c = \emptyset$). We define a subset $X \subset \mathbf{Z}^d$ to be connected if $\forall i, j \in X$ there exists a path $i(1), \cdots, i(\ell)$ with $i(1) = i, i(\ell) = j, i(k) \in X, |i(k) - i(k+1)| = 1, \forall k$, where $|i| = \max_\alpha |i_\alpha|$. $W \geq 0$ depends on s_i, for $i \in \Lambda^c_\alpha$, such that $d(i, \Lambda) \leq R$. We have the following bound: for any $\epsilon > 0$, there exist $L < \infty$, $\gamma > 0$ such that, for the expansion defined with L-cubes,

$$\sum_{0\in Y} \exp(\frac{\gamma d(Y)}{L}) \|\phi_Y\| \leq \epsilon \tag{14}$$

(there is no loss of generality in assuming that γ here is the same as in (8)).

Remark. Our condition (5) is Condition A in [34] (together with (3.7) in $d = 2$), or Condition C in [35] (see (2.65)). The representation (13) follows from (2.53), (2.56), (2.60) in [35], or, more precisely, from the extension of (2.53) to arbitrary boundary conditions. In [34, 35] volumes of various shapes are used, but one can always regroup terms and index them by L-cubes, as was done here for simplicity of notations. The reader may notice that (13) is what we get at high temperatures, the only difference being that a site is replaced here by an L-cube: f is the bulk free energy, the product over the W's gathers all the boundary terms, and the sum over ϕ_Y is the usual cluster expansion. The constant L is, in effect, of the order of the correlation length of the system. Note also that, if Φ^0 is a finite range one-dimensional interaction (i.e. it couples spins only along lines parallel to a lattice axis), then the transfer matrix formalism implies that (5) holds. In our application to coupled maps, Φ^0 will be exactly of that form (see Proposition 3 in Section 4).

Remark In the proofs, we shall denote by C or c a constant that may vary from place to place.

Proof of Theorem 1. We shall prove (12) where the expectation will be taken with respect to a finite volume Gibbs state with open boundary conditions (to simplify notations), i.e. the sum in (2) is restricted to $X \subset \Lambda$. The corresponding expectation is written $\langle\cdot\rangle_\Lambda$. Our bounds are uniform in Λ. We can then study the limit $\Lambda \uparrow \mathbf{Z}^d$, and prove (12) in that limit. The uniqueness of the Gibbs state can be shown in a similar way, and will be discussed at the end of the proof.

First, we write the LHS of (12) using duplicate variables:

$$\begin{aligned} & 2(\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda\langle G\rangle_\Lambda) \\ = & \sum_{s^i_\Lambda, i=1,2} (F(s^1_\Lambda) - F(s^2_\Lambda))(G(s^1_\Lambda) - G(s^2_\Lambda))\nu(s^1_\Lambda)\nu(s^2_\Lambda) \end{aligned}$$

$$= \quad Z(\Lambda)^{-2} \sum_{s^i_\Lambda, i=1,2} \tilde{F}\tilde{G} \exp(-\mathcal{H}(s^1_\Lambda) - \mathcal{H}(s^2_\Lambda)) \tag{15}$$

where $\tilde{F} = F(s^1_\Lambda) - F(s^2_\Lambda), \tilde{G} = G(s^1_\Lambda) - G(s^2_\Lambda)$.

We may replace Φ^1_X by $\Phi^1_X - \inf_{s_X} \Phi^1_X$, by adding a constant to the Hamiltonian. Thus, we may, without loss of generality, assume that $\Phi^1_X \geq 0$ for all X, and that Φ^1 still satisfies $\|\Phi^1\|_2 \leq 2\epsilon$. Now, we perform a usual high-temperature expansion on the Φ^1 part of $\mathcal{H}$:

$$\exp(\sum_{X, i=1,2} \Phi^1_X(s^i_X)) = \sum_{\mathcal{X}} \prod_{X \in \mathcal{X}} f_X \tag{16}$$

where the sum runs over sets $\mathcal{X}$ of subsets of Λ, and

$$f_X = \exp(\Phi^1_X(s^1_X) + \Phi^1_X(s^2_X)) - 1 \tag{17}$$

satisfies:

$$0 \leq f_X, \tag{18}$$

$$\sum_{0 \in X} e^{\gamma d(X)} \|f_X\| \leq C\epsilon \tag{19}$$

(Use (8) and $\|\Phi^1\|_2 \leq \epsilon$). Insert (16) in (15) and, for each term in (16), define $V = V(\mathcal{X}) = \underline{A} \cup \underline{B} \cup \underline{\mathcal{X}}$, where $\underline{\mathcal{X}} = \cup_{X \in \mathcal{X}} \underline{X}$ and for any $X \subset \mathbf{Z}^d$, $\underline{X}$ is the set of L-cubes intersected by X. We have

$$(15) = Z(\Lambda)^{-2} \sum_{\mathcal{X}} \sum_{s^i_V, i=1,2} \tilde{F}\tilde{G} \prod_{X \in \mathcal{X}} f_X \exp(-\mathcal{H}^0_V) \prod_{i=1,2} Z^0(\Lambda \backslash V | s^i_V) \tag{20}$$

where $\mathcal{H}^0_V = -\sum_{X \subset V} (\Phi^0_X(s^1_X) + \Phi^0_X(s^2_X))$, and $Z^0(\Lambda \backslash V | s^i_V)$ is the partition function with interaction Φ^0, s^i_V boundary condition in V, and open boundary conditions in Λ^c. Now, we use (13) for $Z^0(\Lambda \backslash V | s^i_V)$, and we define $\Psi_Y(s_{Y \cap V})$ by:

$$\phi_Y(s_{Y \cap V}) = \Psi_Y(s_{Y \cap V}) + \bar{\phi}_Y \tag{21}$$

where

$$\bar{\phi}_Y = \min_{s_{Y \cap V}} \phi_Y(s_{Y \cap V}) \tag{22}$$

so that $\Psi_Y \geq 0$. Then, we perform a high-temperature expansion of Ψ_Y:

$$\exp(\sum_{Y, i=1,2} \Psi_Y(s^i_{Y \cap V})) = \sum_{\mathcal{Y}} \prod_{Y \in \mathcal{Y}} g_Y \tag{23}$$

where $Y \cap (\Lambda \backslash V) \neq \emptyset$ and $\mathcal{Y}$ is a set of such Y's; $g_Y = \exp(\Psi_Y(s^1_{Y \cap V}) + \Psi_Y(s^2_{Y \cap V})) - 1$ satisfies:

$$0 \leq g_Y \tag{24}$$

$$\sum_{0\in Y} \exp(\frac{\gamma d(Y)}{L})\|g_Y\| \leq C\epsilon \tag{25}$$

(using (14)). Finally, we insert the result in (20). We get

$$(20) = Z(\Lambda)^{-2} \sum_{\mathcal{X},\mathcal{Y}} \sum_{s^i_V, i=1,2} \tilde{F}\tilde{G} \prod_{X\in\mathcal{X}} f_X \prod_{Y\in\mathcal{Y}} g_Y \exp(-\mathcal{H}^0_V)\mathcal{W} \exp(2\sum_Y \bar{\phi}(Y)) \tag{26}$$

where $\mathcal{W} = e^{2f|\Lambda\backslash V|}\prod_{\alpha,i=1,2} W(s^i_{V_\alpha})$ (V_α are the connected components of V), and the sum over Y runs over $Y\cap(\Lambda\backslash V)\neq\emptyset$.

Now comes the main observation: for each term of (26), decompose $V\cup\{Y|Y\in\mathcal{Y}\}$ into connected components. If $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ are in different components, then the corresponding term in (26) vanishes. Indeed, because of the factor $\tilde{F}\tilde{G}$, the summand is then odd under the interchange of s^1_i and s^2_i for all i in the component containing $\underline{\mathrm{A}}$ (or $\underline{\mathrm{B}}$). Here, we use the fact that, since $L > R$, R being the range of Φ_0, every $X\subset V$ contributing to the sum defining $\mathcal{H}^0_V$, belongs to a single connected component V_α.

So, for each non-vanishing term in (26), we can choose a "connected path" $P = (Z_i)^n_{i=1}$ where $Z_1 = \underline{A}, Z_n = \underline{B}$, each Z_i for $i\neq 1,n$ is either an $\underline{X}$ or a Y and Z_{i+1} is adjacent to Z_i (i.e. $d(Z_i,Z_{i+1})\leq 1$), $\forall i = 1,\cdots,n-1$. So we get a bound on (26):

$$|(26)| \leq Z(\Lambda)^{-2}4\|F\|\|G\| \sum_P \prod_{\underline{X}\in P}\|f_X\| \prod_{Y\in P}\|g_Y\| {\sum_{\mathcal{X},\mathcal{Y}}}^P \sum_{s^i_V} \prod_{X\in\mathcal{X},\underline{X}\notin P} f_X \prod_{X\in\mathcal{X},\underline{X}\in P}(1+f_X)$$
$$\prod_{Y\in\mathcal{Y}\backslash P} g_Y \prod_{Y\in P}(1+g_Y)\exp(-\mathcal{H}^0_V)\mathcal{W}\exp(2\sum_Y\bar{\phi}(Y)) \tag{27}$$

where we used the positivity of f_X, g_Y, $\mathcal{W}$ (in particular, in order to insert the products over $1+f_X$, $1+g_Y$), we bounded $\tilde{F},\tilde{G}$ by $4\|F\|\|G\|$, and we wrote $\sum^P$ to denote the sum over the pairs $(\mathcal{X},\mathcal{Y})$ for which the chosen path is P. Now, observe that, by resumming the expansions, the sum over $\mathcal{X},\mathcal{Y},s^i_V$ is less than $Z(\Lambda)^2$: we get an upper bound on $\sum^P_{\mathcal{X},\mathcal{Y}}$ by resumming first over all $\mathcal{Y}$ that are sets of Y's intersecting $\Lambda\backslash V$, with $Y\notin P$ and then over all $\mathcal{X}$ that are sets of X's in Λ, with $\underline{X}\notin P$ (since each term in (27) is positive). This upper bound is equal to $Z(\Lambda)^2$. So, we finally get

$$(27) \leq 4\|F\|\|G\|\sum_P\prod_{\underline{X}\in P}\|f_X\|\prod_{Y\in P}\|g_Y\| \tag{28}$$

Now, to get (12), we have to bound the sum over P. First use

$$d(A,B) \leq cL(1+\sum_{\underline{X}\in P} d(X)) + \sum_{Y\in P} d(Y) \tag{29}$$

for some $c<\infty$, where the factor cL enters because we consider $Z_1 = \underline{A}$, $Z_n = \underline{B}$, or $Z_i = \underline{X}$, which are unions of L-cubes. We choose $m = \frac{\gamma}{2cL}$, and use (29) to bound $e^{md(A,B)}$. Finally, we have to control the sum over P in (28). Using (29) and (15,20,26,27,28), we bound

$$e^{md(A,B)}|\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda\langle G\rangle_\Lambda| \leq 2\|F\|\,\|G\|\sum^\infty_{n=2}\sum_{(Z_i)^{n-1}_2}\prod^{n-1}_{i=2} h(Z_i)$$

where each Z_i is either a $\underline{X}$ or a Y, $h(Z) = \exp(\frac{\gamma d(X)}{2})\|f_X\|$ or $\exp(\frac{\gamma d(Y)}{2cL})\|g_Y\|$ and each Z_i is adjcacent to Z_{i-1}. We can bound the sum over (Z_i) by the (ordered) product

$$\prod_{i=2}^{n-1}(\sum_{Z_i}{}^{i-1}h(Z_i)),$$

where $\sum^{i-1}$ means that Z_i is adjacent to Z_{i-1}. Now,

$$\sum_{Z_i}{}^{i-1}h(Z_i) \leq C|X|(\sum_{0\in Z} h(Z))$$

if Z_{i-1} is an $\underline{X}$, or

$$\sum_{Z_i}{}^{i-1}h(Z_i) \leq C|Y|(\sum_{0\in Z} h(Z)),$$

if Z_{i-1} is a Y, since the number of translates of Z_i, $0 \in Z_i$ that are adjacent to Z_{i-1} (with $Z_{i-1} = \underline{X}$ or Y) is bounded by a constant times $|X|$ or $|Y|$, respectively. We then use the bounds

$$\sum_{0\in\underline{X}} |X| \exp(\frac{\gamma d(X)}{2})\|f_X\| \leq CL^d\epsilon, \tag{30}$$

and

$$\sum_{0\in Y} |Y| \exp(\frac{\gamma d(Y)}{2cL})\|g_Y\| \leq CL^d\epsilon, \tag{31}$$

which follow from (19,25) (in (30), we use the fact that the sum over $0 \in \underline{X}$ is bounded by CL^d times the sum over $0 \in X$, and $|X| \leq Cd(X)^d$ while in (31), we use $|Y| \leq Cd(Y)^d \leq C(2cL)^d(\frac{d(Y)}{2cL})^d$). Finally, for ϵ small enough, we bound (28) by a convergent geometric series:

$$e^{md(A,B)}|\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda\langle G\rangle_\Lambda| \leq 2\|F\|\,\|G\|\sum_{n=0}^{\infty}(CL^d\epsilon)^n$$

To prove the existence of the $\Lambda \uparrow \mathbf{Z}^d$ limit (in the sense of finite subsets ordered by inclusion) and the uniqueness of the Gibbs state, consider F as in (12) and two boundary conditions $s \in \Omega_{\Lambda^c}, s' \in \Omega_{\Lambda'^c}$. Compare

$$\langle F\rangle_\Lambda(s) - \langle F\rangle_{\Lambda'}(s') = \sum_{s^1_\Lambda, s^2_{\Lambda'}} \tilde{F}\nu(s^1_\Lambda|s)\nu(s^2_{\Lambda'}|s') \tag{32}$$

with Λ, Λ' such that $A \subset \Lambda \cap \Lambda'$, and do the same expansion as above. The only terms that do not vanish are such that the connected component containing $\underline{A}$ intersects $\partial(\Lambda \cap \Lambda')$, because $\tilde{F}$ is antisymmetric under the $s^1 \leftrightarrow s^2$ interchange. Hence, for each non-zero term, we may choose a path P connecting $\underline{A}$ to $\partial(\Lambda \cap \Lambda')$. By the arguments given above, the sum over those paths is bounded by $\|F\|\exp(-md(A, \partial(\Lambda \cap \Lambda')))$, and (32) goes to zero as $\Lambda, \Lambda' \uparrow \mathbf{Z}^d$. □

In the next sections, we shall need a simple extension of Theorem 1 to a situation where the interactions depend (weakly) on Λ. Assume that, for each finite $\Lambda \subset \mathbf{Z}^d$,

$\Phi_\Lambda = \Phi^0_\Lambda + \Phi^1_\Lambda$ where Φ^0_Λ, Φ^1_Λ satisfy the hypotheses of Theorem 1 uniformly in Λ. Assume also that

$$\|\Phi_{\Lambda X} - \Phi_{\Lambda' X}\| \leq \epsilon e^{-\gamma d(X,\partial\Lambda)} \tag{33}$$

for $X \subset \Lambda \subset \Lambda'$, so that $\Phi_X = \lim_{\Lambda\uparrow \mathbf{Z}^d} \Phi_{\Lambda X}$ exists. Then we have

Theorem 2 *Under the above assumptions, the conclusions of Theorem 1 hold; moreover,* $\lim_{\Lambda\uparrow\mathbf{Z}^d} \nu_\Lambda = \mu$*, where* ν_Λ *is defined by (2-4) with* Φ_X *replaced by* $\Phi_{\Lambda X}$ *and the sum in (2) is restricted to* $X \subset \Lambda$*.*

Proof We study the limit $\Lambda \uparrow \mathbf{Z}^d$ of ν_Λ. Let $\tilde\Lambda \subset \Lambda$ denote the largest cube containing the origin in $\mathbf{Z}^d$ such that $|\tilde\Lambda| \leq \frac{\gamma}{4} d(\Lambda)$ (so that $d(\tilde\Lambda) \to \infty$ if $d(\Lambda) \to \infty$). Write the Hamiltonian

$$\mathcal{H}(s_\Lambda) = \sum_{X\subset\tilde\Lambda} \Phi_X + \Phi_{\tilde\Lambda} + \sum_{X\not\subset\tilde\Lambda} \Phi_{\Lambda X}$$

where $\Phi_{\tilde\Lambda} = \sum_{X\subset\tilde\Lambda} \Phi_{\Lambda X} - \Phi_X$ can be regarded as an interaction coupling all the variables in $\tilde\Lambda$ and $\Phi_X = \lim_{\Lambda\uparrow\mathbf{Z}^d} \Phi_{\Lambda X}$. By (33), we have the trivial bound:

$$\|\Phi_{\tilde\Lambda}\| \leq \sum_{X\subset\tilde\Lambda} \|\Phi_{\Lambda X} - \Phi_X\| \leq \epsilon 2^{|\tilde\Lambda|} e^{-\gamma d(\tilde\Lambda,\partial\Lambda)} \leq C\epsilon e^{-\frac{\gamma}{3} d(\tilde\Lambda)}.$$

So, $\Phi_{\tilde\Lambda}$ satisfies an estimate similar to the one of Φ^1 (with a smaller γ). If we introduce this representation of $\mathcal{H}$ and expand $\Phi_{\tilde\Lambda}$, considering it as a part of Φ^1, we see that the only non-zero term must contain a path connecting A to $\partial\tilde\Lambda$. Since $d(\tilde\Lambda) \to \infty$ when $\Lambda \uparrow \mathbf{Z}^d$, the existence of the limit for ν_Λ follows. The rest is then as in the proof of Theorem 1. □

Remark 1. We used in an essential way the fact that Φ is real when we used the positivity of $f_X, g_X, \mathcal{W}$. This can always be arranged for Φ real by adding a constant but not, of course, for complex Φ. This "trick" of taking advantage of the positivity of the interactions was used e.g. in [2, 17, 20, 28]. The main interest of our method is that it yields uniqueness of the Gibbs state and exponential decay of correlations without using much combinatorics.

Remark 2. Here we first had L determined by the Φ_0 part of the interaction and then we chose ϵ small enough. However, in applications such as those of the next sections, the splitting of the interaction into Φ_0 and Φ_1 is somewhat arbitrary, and L and ϵ are not independent. All we need is that $L^d\epsilon$ be small enough, see (30,31).

Remark 3. The exponential decay (12) implies the decay of all the truncated correlation functions and, therefore, the fact the free energy and the correlation functions are C^∞ (in the same sense as analyticity was defined above); see e.g. [40], Sect. II 12. Of course, the bounds obtained in this way on the truncated correlation functions contain factorials that prevent us from drawing any conclusions about their analyticity.

Remark 4. Since we used $\|\Phi^1\|_2 \leq \epsilon$ only in the bounds (19,30), we can easily extend our results in several directions: First, one recovers uniqueness of the Gibbs state if we assume only $\|\Phi^1\|_1 \leq \epsilon$, and Φ^0 as before. Instead of (19), we have

$$\sum_{0\in X} |X| \|f_X\| \leq \mathcal{O}(\epsilon) \tag{34}$$

and (30) holds without the factor $\exp(\frac{\gamma}{2}d(X))$. Taking $\Phi^0 = 0$, this yields a simple proof of Dobrushin's uniqueness theorem. Using (30,31) (without $\exp(\frac{\gamma}{2}d(X))$) we can also bound

$$\sum_{i\in \mathbf{Z}^d} |\langle F\tau_i G\rangle - \langle F\rangle\langle G\rangle| < \infty \tag{35}$$

(just sum over $Z_n = \tau_i \underline{B}$). Therefore, one recovers the result of Gross [24] on the C^2 property of the free energy.

We may also consider potentials decreasing with a power law: introduce the norm

$$\|\Phi\|_4 = \sum_{0\in X} |X| e^{\gamma \ln d(X)} \|\Phi_X\| \tag{36}$$

Then, if $\|\Phi^1\|_4$ is small, we get an upper bound $d(A,B)^{-\gamma}$ in (12). Of course, here we need the factor $|X|$ in (36) to control the sum (30).

Finally, let us remark that we do not use the translation invariance of Φ except when we speak of the free energy, of analyticity or of differentiability. Hence, uniqueness of the Gibbs state and the decay of correlation (12) hold for arbitrary interactions on arbitrary lattices with

$$\sup_i \sum_{i\in X} |X| \|\Phi^1_X\| \quad \text{or} \quad \sup_i \sum_{i\in X} e^{\gamma d(X)} \|\Phi^1_X\|$$

small enough.

3 Coupled map lattices.

We consider the following class of dynamical systems. The phase space $\mathcal{M} = (S^1)^{\mathbf{Z}^d}$ i.e. $\mathcal{M}$ is the set of maps $\mathbf{z} = (z_j)_{j\in\mathbf{Z}^d}$ from $\mathbf{Z}^d$ to the circle. $\mathcal{M}$ inherits its topology and Borel σ-algebra $\mathcal{B}$ from S^1 : it is a compact metric space and we let m denote the product of Lebesgue measures on the S^1-factors. $\mathcal{M}$ carries a natural $\mathbf{Z}^d$-action, denoted by τ_i, $i \in \mathbf{Z}^d$.

To describe the dynamics, we first fix a map $F : S^1 \to S^1$. We take F to be an expanding, orientation preserving $C^{1+\delta}$ map with $\delta > 0$. We describe F in terms of its lift to $\mathbf{R}$, denoted by f and chosen, say, with $f(0) \in [0, 1[$. We assume that

$$f'(x) > \lambda^{-1} \tag{1}$$

where $\lambda < 1$. Note that there exists an integer $k > 1$ such that

$$f(x+1) = f(x) + k \quad \forall x \in \mathbf{R} \tag{2}$$

We let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ denote the Cartesian product $\mathcal{F} = \mathrm{X}_{i \in \mathbf{Z}^d} F_i$ where F_i is a copy of F. $\mathcal{F}$ is called the uncoupled map.

The second ingredient in the dynamics is given by the coupling map $A : \mathcal{M} \to \mathcal{M}$. This is taken to be a small perturbation of the identity in the following sense. Let A_i be the projection of A on the i^{th} factor. We assume that A_i is C^1 in each z_j and satisfies (we use the parametization $z_j = e^{2\pi i x_j}$ on S^1, $x_j \in [0, 1[$ and we let a_j denote the lift of $A_j : A_j = e^{2\pi i a_j}$):

$$\|\frac{\partial a_i}{\partial x_j} - \delta_{ij}\|_\infty \leq \epsilon e^{-\beta|i-j|} \tag{3}$$

together with a Hölder condition

$$\left| \frac{\partial a_i}{\partial x_j}(\mathbf{x}) - \frac{\partial a_i}{\partial x_j}(\mathbf{y}) \right| \leq \epsilon \sum_k e^{-\beta(|i-j|+|i-k|)} |x_k - y_k|^\delta \tag{4}$$

for some $\epsilon, \beta, \delta > 0$, with $\mathbf{x} = (x_i)_{i \in \mathbf{Z}^d}$.

Furthermore, we suppose that A is $\mathbf{Z}^d$-invariant:

$$A = \tau_i \circ A \circ \tau_i^{-1} \ \ \forall i \in \mathbf{Z}^d, \tag{5}$$

although this is not essential, see Remark 2 in [3] and Remark 4 at the end of Section 2.

Example. An example often considered (see e.g. [3]) is the coupling map

$$a_j(\mathbf{x}) = x_j + 2\pi i \epsilon \sum_k g_{|j-k|}(x_j, x_k)$$

where g is a periodic $C^{1+\delta}$ function in both variables, with exponential falloff in $|j-k|$ as in (4). More general examples of such $A's$ can be found in [3] (where, however, we restricted ourselves to analytic maps).

The coupled map $T : \mathcal{M} \to \mathcal{M}$ is now defined by

$$T = A \circ \mathcal{F}.$$

We are looking for "natural" T-invariant measures on $\mathcal{M}$. For this, write, for $\Lambda \subset \mathbf{Z}^d$, $\mathcal{M}_\Lambda = (S^1)^\Lambda$, and let m_Λ be the product of Lebesgue measures. Let $\mathcal{B}_\Lambda \subset \mathcal{B}$ be the σ-algebra of subsets of $\mathcal{M}$ generated by the Borel sets of $\mathcal{M}_\Lambda$ (identified in the obvious way with subsets of $\mathcal{M}$).

Definition 1 *A Borel probability measure μ on $\mathcal{B}$ is a SRB measure if*

(a) μ is T-invariant

(b) The restriction μ_Λ of μ to $\mathcal{B}_\Lambda$ is absolutely continuous with respect to m_Λ for all $\Lambda \subset \mathbf{Z}^d$ finite.

Remark This is a natural extension of the notion of SRB measure to infinite dimensions, since each S^1 factor can be regarded as an unstable direction.

It is convenient to introduce the space of Hölder continuous functions $\mathcal{C}^\delta(\mathcal{M}_\Lambda)$ for $\Lambda \subset \mathbf{Z}^d, |\Lambda| < \infty$ equipped with the norm

$$\|G\|_\delta = \|G\|_\infty + \sup_{\mathbf{x},\mathbf{y}} \frac{|G(\mathbf{x}) - G(\mathbf{y})|}{\sum_{i\in\Lambda} |x_i - y_i|^\delta}$$

We also write $\mathcal{C}(\mathcal{M}_\Lambda)$ to denote the continuous functions on $\mathcal{M}_\Lambda$ with the sup norm.

Definition 2 *A Borel probability measure on $\mathcal{B}$ is regular if its restriction μ_Λ to $\mathcal{B}_\Lambda$ is absolutely continuous with respect to m_Λ for all $\Lambda \subset \mathbf{Z}^d$ finite and if $\log h_\Lambda \in \mathcal{C}^\delta(\mathcal{M}_\Lambda)$, where $h_\Lambda = \frac{d\mu_\Lambda}{dm_\Lambda}$:*

$$|\log \frac{h_\Lambda(\mathbf{x})}{h_\Lambda(\mathbf{y})}| \le C \sum_{i\in\Lambda} |x_i - y_i|^\delta \tag{6}$$

Remark A similar condition was introduced by Volevich [44]. If μ is a Gibbs state on $\mathcal{M}$ for some translation invariant interaction such that $\sum_{0\in X} \|\Phi_X\| < \infty$ and

$$|\Phi_X(\mathbf{x}) - \Phi_X(\mathbf{y})| \le C(X) \sum_{i\in X} |x_i - y_i|^\delta \tag{7}$$

with

$$\sum_{0\in X} C(X) < \infty, \tag{8}$$

then μ is regular. This follows easily from the DLR equations:

$$h_\Lambda(\mathbf{x}) = \int \nu(\mathbf{x}|\mathbf{z}) d\mu(\mathbf{z}) \tag{9}$$

where $\nu(\mathbf{x}|\mathbf{z})$ is defined as in (2.3). To get (6), using (9), by Jensen's inequality it is enough to bound

$$\begin{aligned} &\sup_{\mathbf{z}} |\log \frac{\nu(\mathbf{x}|\mathbf{z})}{\nu(\mathbf{y}|\mathbf{z})}| = \sup_{\mathbf{z}} |\mathcal{H}(\mathbf{x}|\mathbf{z}) - \mathcal{H}(\mathbf{y}|\mathbf{z})| \\ &\le \sup_{\mathbf{z}} \sum_{X\cap\Lambda\neq\phi} |\Phi_X(\mathbf{x}\vee\mathbf{z}) - \Phi_X(\mathbf{y}\vee\mathbf{z})| \end{aligned}$$

which, using (7, 8), is bounded by the RHS of (6).

We will prove

Theorem 3 *Let F and A satisfy the assumptions given above. Then there exists $\epsilon_0 > 0$ such that, for $\epsilon < \epsilon_0$, T has an SRB measure μ. Furthermore, μ is invariant and exponentially mixing under the space-time translations generated by τ_i and T: there exists $m > 0$, $C < \infty$, such that, $\forall B, D \subset \mathbf{Z}^d, |B|, |D| < \infty$ and $\forall G \in L^\infty(\mathcal{M}_B), \forall H \in \mathcal{C}^\delta(\mathcal{M}_D)$,*

$$|\int G \circ T^n H d\mu - \int G d\mu \int H d\mu| \leq C \|G\|_\infty \|H\|_\delta \min(|B|, |D|) e^{-m(n+d(B,D))}, \qquad (10)$$

where $d(B, D)$ is the distance between B and D.

Finally, for any regular measure ν, $T^{*N}\nu \to \mu$ weakly, as $N \to \infty$, T^* being the transpose of T.

Remark. We shall construct an expansion for the Perron-Frobenius operator of T, which will be quite similar to the expansion of the previous section, for Φ of the form (2.9), with Φ^0 one-dimensional and $\|\Phi^1\|_2$ small. Bunimovich and Sinai [5] have reduced the problem (for a one dimensional lattice of maps, and a slightly different model) to a spin system with exponentially decaying interactions of the same form. We shall rederive this estimate below (see Proposition 2). However, they did not discuss in detail the resulting statistical mechanical system (the reference to [12] is slightly misleading, since the latter authors assume $\|\Phi^1\|_3$ small, which probably does not hold in general here). The work of Bunimovich and Sinai was extended by Volevich [44, 45, 46], Pesin and Sinai [36], and by Jiang [27]. Our results are quite close to those of [46], where lattices of dimension $d > 1$ are considered. In that paper, the statistical mechanics is based on an extension to infinite range interactions of results of Dobrushin and Pecherski [14].

An extension of [5] is also claimed in [25, 8]. However, there the treatment of the statistical mechanics part of the problem is wrong. The authors essentially claim that uniqueness of the Gibbs state holds for any Φ, with $\|\Phi\|_2$ finite. Although this is true in one dimension, it certainly does not hold in higher dimensions. The Ising model, or any finite range model of this type that undergoes a phase transition, provide counterexamples. Moreover, the "proof" would also yield analyticity, which does not hold even under the correct assumption that Φ is of the form (2.9), with Φ^0 one dimensional and $\|\Phi^1\|_2$ small, because of the counterexamples in [13].

Finally, in [3], we constructed a convergent expansion for the Perron-Frobenius operator, but we needed stronger assumptions on the coupled map lattice (analyticity instead of smoothness). These assumptions, in the present context, would imply that $\|\Phi^1\|_3$ is small and that we could use the usual cluster expansion.

4 SRB measure from a Gibbs measure

The SRB measure μ is constructed using the Perron-Frobenius operator for T. We first introduce a finite volume cutoff version of T, $T_\Lambda : \mathcal{M}_\Lambda \to \mathcal{M}_\Lambda$, $\Lambda \subset \mathbf{Z}^d$, $|\Lambda| < \infty$. For definiteness, we let Λ be a cube centered at 0. Set $T_\Lambda = A_\Lambda \circ \mathcal{F}_\Lambda$ with $\mathcal{F}_\Lambda = \times_{i\in\Lambda} F_i$ and

$$A_\Lambda = R_\Lambda \circ A \circ \mathcal{C}_\Lambda \qquad (1)$$

where $\mathcal{C}_\Lambda : \mathcal{M}_\Lambda \to \mathcal{M}$ extends $\mathbf{z} \in \mathcal{M}_\Lambda$ to $\mathcal{M}$ by

$$\mathcal{C}_\Lambda(\mathbf{z})_i = \begin{cases} z_i & i \in \Lambda \\ 1 & i \notin \Lambda \end{cases}$$

(here $z_i, 1 \in S^1$, and the choice of 1 is arbitrary), and $R_\Lambda : \mathcal{M} \to \mathcal{M}_\Lambda$ is the restriction. It is more convenient to work in the covering space $\mathbf{R}^\Lambda$ of $\mathcal{M}_\Lambda$ and to introduce the corresponding lifts $a_\Lambda, f_\Lambda, t_\Lambda$ as maps from $\mathbf{R}^\Lambda$ to itself. Our assumptions for A then imply the following properties for a_Λ.

Lemma 1 *The maps a_Λ are $C^{1+\delta}$ diffeomeorphisms with the following properties: there exist $\beta > 0$, $C < \infty$, independent of Λ, such that $\forall \mathbf{x} \in \mathbf{R}^\Lambda$*

a) $a_\Lambda(\mathbf{x} + \mathbf{n}) = a_\Lambda(\mathbf{x}) + \mathbf{n} \ \ \forall \mathbf{n} \in \mathbf{Z}^\Lambda$

b) The derivative of a_Λ can be written as $Da_\Lambda(\mathbf{x}) = I + \epsilon_\Lambda(\mathbf{x})$, where I is the unit matrix, and ϵ_Λ is periodic function satisfying

$$|\epsilon_\Lambda(\mathbf{x})_{ij}| \le C\epsilon e^{-\beta|i-j|} \tag{2}$$

$$|\epsilon_\Lambda(\mathbf{x})_{ij} - \epsilon_\Lambda(\mathbf{y})_{ij}| \le C\epsilon \sum_{k\in\Lambda} e^{-\beta(|i-j|+|i-k|)}|x_k - y_k|^\delta \tag{3}$$

c) $\epsilon_\Lambda(\mathbf{x})_{ij} = \epsilon_{\Lambda'}(\mathbf{x} \vee \mathbf{1}_{\Lambda'\setminus\Lambda})_{ij}$ *for $i, j \in \Lambda \subset \Lambda'$.*

d) Let $\Lambda \subset \Lambda'$, $R_\Lambda \mathbf{y} = \mathbf{x}$, $\mathbf{y} \in \mathbf{R}^{\Lambda'}$ and $i, j \in \Lambda$. Then

$$|\epsilon_\Lambda(\mathbf{x})_{ij} - \epsilon_{\Lambda'}(\mathbf{y})_{ij}| \le C\epsilon e^{-\beta|i-j|}e^{-\beta d(i,\partial\Lambda)} \tag{4}$$

where d denotes the distance.

Furthermore a_Λ^{-1} satisfies a)-d) also, with possibly different β,C.

Proof (2) and (3) are just a restatement of (3.3), (3.4) (we can take $C = 1$ here). By definition of a lift, $a_\Lambda(\mathbf{x} + \mathbf{n}) = a_\Lambda(\mathbf{x}) + \mathbf{m}(\mathbf{n})$ for some $\mathbf{m}(\mathbf{n}) \in \mathbf{Z}^\Lambda$. Write $a_\Lambda = id + b$ where id is the identity. Order points of Λ in an arbitrary way such that $\Lambda = \{i_\ell | \ell = 1, \cdots, |\Lambda|\}$. Then, given $\mathbf{x}, \mathbf{y} \in \mathbf{R}^\Lambda$, define $\mathbf{w}^{(n)}$ by letting let $w_{i_\ell}^{(n)}$ equal x_{i_ℓ} for $\ell \ge n$ and y_{i_ℓ} for $\ell < n$. Then,

$$b_i(\mathbf{x}) - b_i(\mathbf{y}) = \sum_{n=1}^{|\Lambda|} b_i(\mathbf{w}^{(n)}) - b_i(\mathbf{w}^{(n+1)})$$

and, from (2),

$$\begin{aligned} |b_i(\mathbf{w}^{(n)}) - b_i(\mathbf{w}^{(n+1)})| &\le \int_0^1 dt |\epsilon_{ii_n}((1-t)\mathbf{w}^{(n)} + t\mathbf{w}^{(n+1)})||x_{i_n} - y_{i_n}| \\ &\le C\epsilon e^{-\beta|i-i_n|}|x_{i_n} - y_{i_n}|. \end{aligned} \tag{5}$$

Hence,

$$|b_i(\mathbf{x}) - b_i(\mathbf{y})| \leq C\epsilon \sum_{j\in\Lambda} e^{-\beta|i-j|}|x_j - y_j|. \tag{6}$$

Let $|n_j| \leq 1$. Then, for ϵ small, $\mathbf{m}(\mathbf{n}) = \mathbf{n}$ and thus by iteration a) holds for all $\mathbf{n}$. Point c) follows from the definition (1) and d) follows from c) and (3).

It is now straightforward to show that a_Λ^{-1} exists and satisfies a)-d) too, (possibly with different β, C): Writing $Da_\Lambda^{-1} = 1 + \tilde{\epsilon}_\Lambda = (1+\epsilon_\Lambda)^{-1}$ and using (2) with $C = 1$, we get:

$$|\tilde{\epsilon}_\Lambda(\mathbf{x})_{ij}| \leq \sum_{n=1}^{\infty} |\epsilon_\Lambda^n(\mathbf{x})_{ij}| \leq \sum_{n=1}^{\infty} \epsilon^n \sum_{i_2,\cdots,i_{n-1}} \prod_{\ell=1}^{n-1} e^{-\beta|i_\ell - i_{\ell+1}|}$$

where $i_1 = i$, $i_n = j$. This is readily bounded by $\epsilon e^{-\tilde{\beta}|i-j|}$ for any $\tilde{\beta} < \beta$ and $C = C(\tilde{\beta}) < \infty$. This proves (2) for a_Λ^{-1}. To prove (3) for a_Λ^{-1}, we use (3) for ϵ_Λ and

$$|\epsilon_\Lambda^n(\mathbf{x})_{ij} - \epsilon_\Lambda^n(\mathbf{y})_{ij}| \leq \epsilon^n \sum_{i_2,\cdots,i_{n-1}} \sum_{\ell=1}^{n} \sum_k e^{-\beta|i_\ell - k|} \prod_{\ell=1}^{n-1} e^{-\beta|i_\ell - i_{\ell+1}|}|x_k - y_k|^\delta$$

with $i_1 = i, i_n = j$, which is bounded by

$$\leq (C\epsilon)^n \sum_k e^{-\tilde{\beta}(|i-j|+|i-k|)}|x_k - y_k|^\delta \tag{7}$$

with $\tilde{\beta} < \frac{\beta}{2}$ and C as above. The proof of (4) is similar. □

The Perron-Frobenius operator P_Λ for T_Λ is defined as usual by

$$\int G \circ T_\Lambda H dm_\Lambda = \int G P_\Lambda H dm_\Lambda \tag{8}$$

for $G \in L^\infty(\mathcal{M}_\Lambda)$, $H \in L^1(\mathcal{M}_\Lambda)$. Let us work in the covering space $\mathbf{R}^\Lambda$ and replace G, H by periodic functions denoted $g, h : g(\mathbf{x} + \mathbf{n}) = g(\mathbf{x})$, $\forall \mathbf{n} \in \mathbf{Z}^\Lambda$. Insert $t_\Lambda = a_\Lambda \circ f_\Lambda$ in (8) to get (with $C_\Lambda = [0,1]^\Lambda$)

$$\int_{C_\Lambda} g \circ t_\Lambda h d\mathbf{x} = \int_{C_\Lambda} g(a_\Lambda(\mathbf{x})) P_\Lambda^\circ h(\mathbf{x}) d\mathbf{x} \tag{9}$$

where P_Λ° is the Perron-Frobenius operator for f_Λ i.e.

$$P_\Lambda^\circ h(\mathbf{x}) = \sum_{\mathbf{s}} \frac{h(f_\Lambda^{-1}(\mathbf{x}+\mathbf{s}))}{\prod_{i\in\Lambda} f'(f^{-1}(x_i + s_i))}$$

where $\mathbf{s} \in \{0, \cdots, k-1\}^\Lambda$ (and k was introduced in (3.2)). Note that P_Λ° maps periodic functions into periodic functions because the sum is periodic even if the summands are not: indeed, (3.2) implies that $f^{-1}(x + 1 + k - 1) = f^{-1}(x + k) = f^{-1}(x) + 1$ (so that, if we add 1 to x_i, it amounts to a cyclic permutation of s_i) and that f' is periodic.

Since, for a periodic function k

$$\int_{a_\Lambda(C_\Lambda)} k(\mathbf{x})d\mathbf{x} = \int_{C_\Lambda} k(\mathbf{x})d\mathbf{x}$$

(both C_Λ and $a_\Lambda(C_\Lambda)$ are, up to their boundaries, fundamental domains for the action of $\mathbf{Z}^\Lambda$ on $\mathbf{R}^\Lambda$), we obtain the formula

$$P_\Lambda h(\mathbf{x}) = \det Da_\Lambda^{-1}(\mathbf{x}) \sum_{\mathbf{s}} \frac{h(\psi_{\Lambda\mathbf{s}}(\mathbf{x}))}{\prod_{i\in\Lambda} f'(\psi_{\Lambda\mathbf{s}}(\mathbf{x})_i)} \tag{10}$$

where we defined

$$\psi_{\Lambda\mathbf{s}}(\mathbf{x})_i = f^{-1}(a_\Lambda^{-1}(\mathbf{x})_i + s_i). \tag{11}$$

and (with a slight abuse of notations) we write P_Λ for the operator acting on periodic functions induced by the Perron-Frobenius operator defined by (8). Note that, by Lemma 4.1.a, $a_\Lambda^{-1}(\mathbf{x}+\mathbf{n}) = a_\Lambda^{-1}(\mathbf{x}) + \mathbf{n}$ and, so, P_Λ maps also periodic functions into periodic functions.

The invariant measure μ is constructed as a weak limit of the measures $T_\Lambda^{*N} m_\Lambda$, with T_Λ^* being the transpose of T_Λ, as $N \to \infty$ and $\Lambda \uparrow \mathbf{Z}^d$. These in turn are given by the Perron-Frobenius operator

$$T_\Lambda^{*N} m_\Lambda = (P_\Lambda^N 1) m_\Lambda$$

by putting $H = 1$ in (8). $P_\Lambda^N 1$ has a direct statistical mechanical interpretation which we now derive.

First, iterating (10), we get

$$(P_\Lambda^N 1)(\mathbf{x}) = \sum_{\mathbf{s}_1,\cdots\mathbf{s}_N} \prod_{t=1}^{N} \left[\det Da_\Lambda^{-1}(\psi_{\mathbf{s}_{t-1}} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x})) \prod_{i\in\Lambda} (f'(\psi_{\mathbf{s}_t} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x})_i))^{-1} \right] \tag{12}$$

where, for $t = 1$, the argument of Da_Λ^{-1} is $\mathbf{x}$, and we write $\psi_{\mathbf{s}}$ for $\psi_{\Lambda\mathbf{s}}$. ¿From now on, we shall consider $\mathbf{x} \in C_\Lambda$.

Next, we introduce a convenient notation: $\mathbf{x} \in C_\Lambda$ and $\mathbf{s}_1, \cdots, \mathbf{s}_N$ in (12) collectively define a configuration on a "space-time" lattice $\{0, \cdots, N\} \times \Lambda$. Thus, let $\mathbf{Z}_+^{d+1} = \mathbf{Z}_+ \times \mathbf{Z}^d, \mathbf{Z}_+$ being the non-negative integers. To any subset $X \subset \mathbf{Z}_+^{d+1}$ associate the configuration space $\Omega_X = \times_{\alpha\in X} \Omega_\alpha$ where, for $\alpha = (t, i)$, Ω_α equals [0,1] if $t = 0$, and equals $\{0, \cdots, k-1\}$ if $t > 0$. We could use the existence of a Markov partition for T_Λ to write $\mathbf{x}$ as a symbol sequence, as is usually done, e.g. in [5], but we shall not need this representation.

Let $\Lambda_N = \{0, \cdots, N\} \times \Lambda$ and $s = (\mathbf{x}, \mathbf{s}_1, \cdots, \mathbf{s}_N) \in \Omega_{\Lambda_N}$. We shall write $\mathbf{x}$,$\mathbf{s}$, as before, for elements of Ω_Λ, $\Lambda \subset \mathbf{Z}^d$, and use s to denote elements of Ω_Λ, $\Lambda \subset \mathbf{Z}_+^{d+1}$ .Then (12) reads

$$(P_\Lambda^N 1)(\mathbf{x}) = \sum_{\mathbf{s}_1\cdots\mathbf{s}_N} e^{-\mathcal{H}_{\Lambda_N}(s)} \tag{13}$$

with $e^{-\mathcal{H}_{\Lambda_N}}$ being the summand in (12).

The final step is to write $\mathcal{H}_{\Lambda_N}$ in terms of potentials. First, write $Da_\Lambda^{-1} = 1 + \tilde{\epsilon}_\Lambda$, where, by Lemma 1, $\tilde{\epsilon}_\Lambda$ satisfies (2)-(4). Then expand

$$\det Da_\Lambda^{-1} = \det(1 + \tilde{\epsilon}_\Lambda) = \exp[-\sum_{i\in\Lambda} v_{\Lambda i}]$$

where

$$v_{\Lambda i}(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\tilde{\epsilon}_\Lambda(\mathbf{x})^n)_{ii} \tag{14}$$

Lemma 2 *There exist* $\epsilon_1 > 0$, $C < \infty$ *such that, for* $\epsilon < \epsilon_1$,

$$\|v_{\Lambda i}\|_\infty \leq C\epsilon$$

and

$$|v_{\Lambda i}(\mathbf{x}) - v_{\Lambda i}(\mathbf{y})| \leq C\epsilon \sum_{k\in\Lambda} e^{-\frac{\beta}{2}|i-k|} |x_k - y_k|^\delta$$

and, for $\Lambda \subset \Lambda'$, $\mathbf{x} = R_\Lambda \mathbf{y}$, $i \in \Lambda$

$$|v_{\Lambda i}(\mathbf{x}) - v_{\Lambda' i}(\mathbf{y})| \leq C\epsilon e^{-\frac{\beta}{2} d(i,\partial\Lambda)}$$

uniformly in Λ, Λ'.

Proof Straightforward, using (2)-(4), and the bound (7), for $\tilde{\epsilon}_\Lambda$. $\Box$

Given $\alpha = (t, i) \in \Lambda_N$ we set

$$V_{\Lambda\alpha}(s) = v_{\Lambda i}(\psi_{\mathbf{s}_{t-1}} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x})) + \log f'((\psi_{\mathbf{s}_t} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x}))_i) \tag{15}$$

and hence

$$\mathcal{H}_{\Lambda_N}(s) = \sum_{\alpha\in\Lambda_N} V_{\Lambda\alpha}(s) \tag{16}$$

where $V_{\Lambda\alpha}$ depends on $s|_{\Lambda_N}$. The representation (16) still needs to be localized in order to express $\mathcal{H}$ in terms of potentials. We use the notation of Section 2: given $s \in \Omega_X$, $s' \in \Omega_Y$ for $X \cap Y = \emptyset$ we denote by $s \vee s'$ the corrresponding configuration in $X \cup Y$, and, if $Z \subset X$, we let $s_Z = s|_Z$. Also, let $0_X \in \Omega_X$ be $(0_X)_\alpha = 0$, $\forall \alpha \in X$ (note that 0 belongs to Ω_α for any α).

Now, given t, choose an arbitrary ordering of the points of $\{0, \ldots, t\} \times \mathbf{Z}^d = \{\alpha_\ell | \ell \in \mathbf{N}\}$ in such a way that $|\alpha_\ell|$ is a non-decreasing function of ℓ where, for $\alpha = (t', j)$

$$|\alpha| = \frac{1}{M}|t'| + \sum_{k=1}^{d} |j_k|. \tag{17}$$

M will be taken sufficiently large below (see (25)).

Consider $V_{\Lambda\alpha}$ in (16). Let $Y_\alpha^m = \{\alpha - \alpha_\ell | 0 \le \ell \le m\}$, and set, for $s \in \Omega_{Y_\alpha^m}$

$$\Phi^\Lambda_{Y_\alpha^m}(s) = V_{\Lambda\alpha}(s_{Y_\alpha^m} \vee 0_{(Y_\alpha^m)^c}) - V_{\Lambda\alpha}(s_{Y_\alpha^{m-1}} \vee 0_{(Y_\alpha^{m-1})^c}) \tag{18}$$

Note that the two configurations on the RHS of (18) differ only at site $\alpha - \alpha_m$. We have then the identity

$$V_{\Lambda\alpha}(s) = \sum_{m\ge 0} \Phi^\Lambda_{Y_\alpha^m}(s) + V_{\Lambda\alpha}(0) \tag{19}$$

and we define

$$\Phi_{\Lambda X}(s) = \Phi^\Lambda_{Y_\alpha^m}(s) \tag{20}$$

provided there exists m, α such that $X = Y_\alpha^m$, otherwise we set $\Phi_{\Lambda X}(s) = 0$ (note that there is at most one such pair (n, α)). $\Phi_{\Lambda X}$ depends only on s_X. Due to (16), (19), we have

$$\mathcal{H}_{\Lambda_N}(s) = \sum_{X \subset \Lambda_N} \Phi_{\Lambda X}(s) + \text{constant} \tag{21}$$

where constant $= \sum_{\alpha\in\Lambda_N} V_{\Lambda\alpha}(0)$. Note that (17) was chosen so that Y_α^m with $|\alpha_m| < 1$ is a one-dimensional interval of length at most M in the time direction.

Now write

$$\Phi_\Lambda = \Phi^0_\Lambda + \Phi^1_\Lambda \tag{22}$$

where Φ^0 collects the $X's$ with $|\alpha_m| < 1$ in (20) and Φ^1 all the others. We have the following basic bound:

Proposition 1 *There exist $\epsilon_2 > 0$, $C < \infty$, such that, if $\epsilon < \epsilon_2$, then $\Phi_{Y_\alpha^m}$ given by (18) is bounded by*

$$|\Phi^\Lambda_{Y_\alpha^m}(s)| \le C\epsilon'^{\delta_{0i_m}} \lambda'^{t_m} e^{-\beta'|i_m|} \tag{23}$$

where $\alpha_m = (t_m, i_m)$ $\lambda' = \lambda^{\delta/2}$, $\epsilon' = \epsilon^\delta$, $\beta' = \frac{\beta\delta}{3}$ where δ is as in (3.4). Moreover, for any $\Lambda' \supset \Lambda$,

$$|\Phi^\Lambda_{Y_\alpha^m}(s) - \Phi^{\Lambda'}_{Y_\alpha^m}(s)| \le C\epsilon' e^{-\beta' d(Y_\alpha^m, \partial\Lambda)} \tag{24}$$

Choose now M in (17) to be the smallest integer such that

$$\lambda'^M < \epsilon^\delta. \tag{25}$$

Then, we have

Proposition 2 *There exist $\gamma > 0$, $\epsilon_0 > 0$, $C < \infty$ such that, if $\epsilon < \epsilon_0$, Φ^1 satisfies the bound*

$$\sum_{\alpha \in X} e^{\gamma d(X)} \|\Phi^1_{\Lambda X}\| \leq C\epsilon^\delta \tag{26}$$

uniformly in α and Λ. Moreover,

$$|\Phi_{\Lambda X}(s) - \Phi_{\Lambda' X}(s)| \leq C\epsilon^\delta e^{-\gamma d(X, \partial\Lambda)}. \tag{27}$$

with $X \subset \Lambda \subset \Lambda'$.

Since Φ^0 has no coupling in spatial directions and has range M in the time direction we get the following decomposition of its partition function Z^0. We let for $i \in \mathbf{Z}^d$ $J_i = (\mathbf{Z}_+ \times i) \cap V$ so $V = \cup_i J_i$. Then

$$Z^0(V|s) = \prod_i Z^0(J_i|s) \tag{28}$$

Proposition 3 *There exist $\gamma > 0$, $C < \infty$ independent of M such that, if J is an interval of the form $[(i,t),(i,t+\ell M)]$ then*

$$Z^0(J|s) = \lambda^\ell W(s_-) W(s_+)(1 + g_J(s_-, s_+)) \tag{29}$$

where $s_+ = s_{((i,t+\ell M),(i,t+(\ell+1)M)]}$ and $s_- = s_{[(i,t-M),(i,t))}$ with

$$|g_J(s_-, s_+)| \leq C e^{-\frac{\gamma}{M}|J|} \tag{30}$$

Since this last Proposition is rather standard in statistical mechanics, we defer its proof to the Appendix. For the proof of the other Propositions, we need some Lemmas.

Lemma 3 *There exists $B < \infty$ such that $\psi_{\Lambda \mathbf{s}}$, given by (11), satisfies*

$$|\psi_{\Lambda \mathbf{s}}(\mathbf{x})_i - \psi_{\Lambda' \mathbf{s}'}(\mathbf{x}')_i| \leq \lambda |x_i - x_i'| + B\epsilon (\sum_{j \in \Lambda} |x_j - x_j'| e^{-\beta|i-j|} + e^{-\beta d(i, \partial\Lambda)}) \tag{31}$$

for $i \in \Lambda \subset \Lambda'$, and $s = s'|_\Lambda$. For $\Lambda = \Lambda'$, we get (31) without the term $e^{-\beta d(i,\partial\Lambda)}$.

Proof Proceeding as in (5, 6), and using Lemma 1 for a_Λ^{-1}, we get

$$|a_\Lambda^{-1}(\mathbf{x})_i - a_{\Lambda'}^{-1}(\mathbf{x}')_i| \leq |x_i - x_i'| + B\epsilon (\sum_{j \in \Lambda} |x_j - x_j'| e^{-\beta|i-j|} + e^{-\beta d(i,\partial\Lambda)})$$

Now, by (3.1), $(f^{-1})' < \lambda$ and this implies the claim. □

Iterating (31) then yields

Lemma 4 *Let $\psi_{\mathbf{s}} \equiv \psi_{\Lambda\mathbf{s}}$, $\psi'_{\mathbf{s}'} \equiv \psi_{\Lambda'\mathbf{s}'}$. Then, there exist $C < \infty$ such that*

$$
\begin{aligned}
&|\psi_{\mathbf{s}_{t_n}} \cdots \psi_{\mathbf{s}_{t_1}}(\mathbf{x})_i - \psi'_{\mathbf{s}'_{t_n}} \cdots \psi'_{\mathbf{s}'_{t_1}}(\mathbf{x}')_i| \\
&\leq \tilde{\lambda}^n (|x_i - x'_i| + C\epsilon \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|}|x_j - x'_j|) + C\epsilon e^{-\frac{\beta}{2}d(i,\partial\Lambda)}
\end{aligned}
\tag{32}
$$

where $i \in \Lambda \subset \Lambda'$, $s = s'|_\Lambda$ and $\tilde{\lambda} = \lambda^{1/2}$. For $\Lambda = \Lambda'$, we get (32) without the term $C\epsilon e^{-\frac{\beta}{2}d(i,\partial\Lambda)}$.

Proof of Lemma 4 The proof is done by induction. For $n = 1$, we can use Lemma 3. Using (31) and (32) for $n - 1$, we get

$$
\begin{aligned}
&|\psi_{\mathbf{s}_{t_n}} \cdots \psi_{\mathbf{s}_{t_1}}(\mathbf{x})_i - \psi'_{\mathbf{s}'_{t_n}} \cdots \psi'_{\mathbf{s}'_{t_1}}(\mathbf{x}')_i| \\
&\leq \lambda\, \tilde{\lambda}^{n-1} (|x_i - x'_i| + C\epsilon \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|}|x_j - x'_j|) + C\epsilon\lambda e^{-\frac{\beta}{2}d(i,\partial\Lambda)} \\
&+B\epsilon \left[\sum_{j\in\Lambda} \tilde{\lambda}^{n-1}|x_j - x'_j| e^{-\beta|i-j]} + C\epsilon\tilde{\lambda}^{n-1} \sum_{k,j\in\Lambda} e^{-\frac{\beta}{2}|j-k|}|x_k - x'_k| e^{-\beta|i-j|} \right. \\
&\left. +C\epsilon \sum_{j\in\Lambda} e^{-\frac{\beta}{2}d(j,\partial\Lambda)} e^{-\beta|i-j|} \right] + B\epsilon e^{-\beta d(i,\partial\Lambda)}
\end{aligned}
$$

Regrouping terms, and exchanging k, j in $\sum_{k,j}$, we get

$$
\begin{aligned}
\leq\ & \lambda\, \tilde{\lambda}^{n-1}|x_i - x'_i| \\
+\ & (\lambda\, \tilde{\lambda}^{n-1}C\epsilon + B\epsilon\, \tilde{\lambda}^{n-1} + BC\epsilon^2\, \tilde{\lambda}^{n-1} \sum_{k\in\Lambda} e^{-\frac{\beta}{2}|i-k|}) \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|}|x_j - x'_j| \\
+\ & (C\epsilon\lambda + BC\epsilon^2 \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|} + B\epsilon) e^{-\frac{\beta}{2}d(i,\partial\Lambda)}
\end{aligned}
$$

Now choose C large enough so that $\lambda C + \frac{B}{\lambda} + \mathcal{O}(\epsilon) < \tilde{\lambda} C$ (which is possible since $\tilde{\lambda} = \lambda^{1/2} > \lambda$) and we get (32) for n. $\square$

Remark It is easy to extend Lemmas 3 and 4 to the situation where $\Lambda = \Lambda'$ but $s \neq s'$ and to get, instead of (32),

$$
\begin{aligned}
|\ & \psi_{\mathbf{s}_{t_n}} \circ \cdots \circ \psi_{\mathbf{s}_{t_1}}(\mathbf{x})_i - \psi_{\mathbf{s}'_{t_n}} \circ \cdots \circ \psi_{\mathbf{s}'_{t_1}}(\mathbf{x}')_i| \\
\leq\ & \tilde{\lambda}^n (|x_i - x'_i| + C\epsilon \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|}|x_j - x'_j|) \\
+\ & \sum_{k=1}^{n} \tilde{\lambda}^{(n-k+1)} (|s_{(t_k,i)} - s'_{(t_k,i)}| + C\epsilon \sum_{j\in\Lambda} e^{-\frac{\beta}{2}|i-j|}|s_{(t_k,j)} - s'_{(t_k,j)}|)
\end{aligned}
\tag{33}
$$

Proof of Proposition 1 Using (15) and (18) we can write

$$
\Phi^{\Lambda}_{Y^m_\alpha}(s) = g(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\mathbf{y})) - g(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\bar{\mathbf{y}}))
$$

where $\alpha = (t, i)$, $l = t - t_m + 1$, $\bar{\mathbf{s}}_{t_k}$,$t_k = t, \dots, l$ is a configuration that coincides with $\mathbf{s}_{t_k}$ in Y^m_α and with 0 outside, $\mathbf{y} = \psi_{\bar{\mathbf{s}}_{t-t_m}} \circ \psi_{\mathbf{0}} \circ \cdots \circ \psi_{\mathbf{0}}(\mathbf{0})$ and $\bar{\mathbf{y}}$ is defined similarly, but with $\bar{\mathbf{s}}_{t-t_m}$ coinciding with $\mathbf{s}_{t-t_m}$ in Y^{m-1}_α and with 0 outside, so that $\bar{y}_j = y_j$ for $|i-j| < |i_m|$. Finally, g is given by (15).

Using the Hölder continuity of f' and Lemma 2, we get

$$
\begin{aligned}
&|\Phi^\Lambda_{Y^m_\alpha}(s)| \le C|(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\mathbf{y}))_i - (\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\bar{\mathbf{y}}))_i|^\delta \\
&+ C\epsilon \sum_{k \in \Lambda} e^{-\frac{\beta}{2}|i-k|} |(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\mathbf{y}))_k - (\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\bar{\mathbf{y}})_k|^\delta
\end{aligned}
$$

Now, use (32) in Lemma 4 for $\Lambda = \Lambda'$ and the fact that $\bar{y}_j = y_j$ for $|i-j| < |i_m|$ to get (23). By definition, $\tilde{\lambda}^\delta = \lambda^{\delta/2} = \lambda'$.

For (24), we bound

$$
\begin{aligned}
&|\Phi^\Lambda_{Y^m_\alpha}(s) - \Phi^{\Lambda'}_{Y^m_\alpha}(s)| \\
&\le |g(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\mathbf{y})) - g(\psi'_{\bar{\mathbf{s}}'_t} \circ \cdots \circ \psi'_{\bar{\mathbf{s}}'_l}(\mathbf{y}))| \\
&+ |g(\psi_{\bar{\mathbf{s}}_t} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_l}(\bar{\mathbf{y}})) - g(\psi'_{\bar{\mathbf{s}}'_t} \circ \cdots \circ \psi'_{\bar{\mathbf{s}}'_l}(\bar{\mathbf{y}}))|
\end{aligned}
$$

where $\psi'_{\bar{\mathbf{s}}'} = \psi_{\Lambda' \bar{\mathbf{s}}'}$ and $\bar{\mathbf{s}}'_t$ is defined like $\bar{\mathbf{s}}_t$, but with Λ' instead of Λ. Now use the Hölder continuity of f' and Lemma 2 and then, use (32), where only the last term contributes. □

Proof of Proposition 2 The bounds (26), (27) follow from Proposition 1 because we excluded from Φ^1 the terms with $\delta_{0 i_m} = 0$ and $t_m < M$, and because of our choice of M in (25): To get (26), we use $d(X) \le C(t_m + |i_m|)$ which follows from (20) and from the definition of Y^m_α. □

Proof of Theorem 3 We shall construct μ as the weak limit of $T^{*N}_\Lambda m_\Lambda = (P^N_\Lambda 1) m_\Lambda$, as $N \to \infty, \Lambda \uparrow \mathbf{Z}^d$, and, for that, we shall use the results of Section 2, the representations (13, 21, 22) and the bounds of Proposition 2. Thus, let $G \in L^\infty(\mathcal{M}_B)$, $|B| < \infty$; we have

$$
\int G \circ T^N_\Lambda dm_\Lambda = \sum_{(\mathbf{s}_i)^N_1} \int_{C_\Lambda} G(\mathbf{x}) \exp(-\mathcal{H}_{\Lambda_N}(s)) d\mathbf{x} \tag{34}
$$

We shall first construct the unique Gibbs state $\bar{\mu}$ with Hamiltonian $\mathcal{H}$ on the following "mixed" phase space: the lattice is $\mathbf{Z}^{d+1}_+$, $\Omega_\alpha = S^1$ if $\alpha = (0, i)$, $i \in \mathbf{Z}^d$ and $\Omega_\alpha = \{0, \dots, k-1\}$ if $\alpha = (t, i)$, $t > 0$. Then, μ will be the restriction of $\bar{\mu}$ to the "time zero" phase space $\mathcal{M}$, as shown by (34). Since we want to use Theorem 2.2 (or, rather a trivial extension of it to the present setting), let us check now that our system satisfies the hypotheses of this theorem. We cover $\mathbf{Z}^{d+1}_+$ with L-cubes, $L = \ell_0 M$, where

$$
M e^{-\gamma \ell_0} \le \epsilon^\delta \tag{35}
$$

(here, it would be more natural to take, instead of cubes, segments parallel to the time axis, but we shall keep the notations of Section 2). We bound $\|\Phi^1_\Lambda\|_2 \le \epsilon$ uniformly in Λ

and we get (2.33) from Proposition 2 (with $\epsilon = C\epsilon^{\delta}$ of Proposition 2). We get (2.13,2.14) from (28) and Proposition 3 as follows. Let $f = \log\lambda$ and $\phi_Y = \log(1+g_J)$ for Y being and interval J as in Proposition 3 with $|J| = nL$, $n \geq 1$ and $\phi_Y = 0$ otherwise. Then (30,35) imply (2.14) (with ϵ replaced by $C\epsilon^{\delta}$), using

$$\sum_{n\geq 1} Lne^{\frac{\gamma Ln}{L}}e^{-\frac{\gamma Ln}{M}} = M\sum_{n\geq 1}\ell_0 ne^{-\gamma(\ell_0-1)n} \leq CMe^{-\gamma\ell_0}.$$

In order to be able to use estimates (2.30, 2.31) (see Remark 2.2), if we observe that $L = \ell_0 M$ with M given by (25) and ℓ_0 by (35) so that $M \leq C|\log\epsilon|, \ell_0 \leq C|\log\epsilon|$ and $L^{d+1}\epsilon^{\delta} << 1$ for ϵ small.

So, Theorem 2.2 proves the existence of μ. Its invariance under T and under the lattice translations follows by construction: Let $G \in \mathcal{C}(\mathcal{M}_B)$. Observe that since the limit $\Lambda \uparrow \mathbf{Z}^d, N \to \infty$ taken above can be taken in any order,

$$\mu^{\Lambda} = \lim_{N\to\infty}(P_{\Lambda}^{N}1)m_{\Lambda}$$

exists and is T_{Λ} invariant. On the other hand,

$$\lim_{\Lambda\uparrow\mathbf{Z}^d}\|G\circ T_{\Lambda} - G\circ T\|_{\infty} = 0 \tag{36}$$

since G is continuous. Therefore,

$$\begin{aligned}\int G\circ Td\mu = \lim_{\Lambda\uparrow\mathbf{Z}^d}\int G\circ T_{\Lambda}d\mu &= \lim_{\Lambda\uparrow\mathbf{Z}^d}\lim_{\Lambda'\uparrow\mathbf{Z}^d}\int G\circ T_{\Lambda}d\mu^{\Lambda'}\\ &= \lim_{\Lambda\uparrow\mathbf{Z}^d}\lim_{\Lambda'\uparrow\mathbf{Z}^d}\lim_{N\to\infty}\int G\circ T_{\Lambda}\circ T_{\Lambda'}^{N}dm_{\Lambda'}\end{aligned} \tag{37}$$

By (36), we may replace here T_{Λ} by $T_{\Lambda'}$ and we get

$$(37) = \lim_{\Lambda'\uparrow\mathbf{Z}^d}\lim_{N\to\infty}\int G\circ T_{\Lambda'}^{N+1}dm_{\Lambda'} = \int Gd\mu$$

The $\mathbf{Z}^d$-translation invariance of $\bar{\mu}$ implies the same invariance for μ.

On the other hand, since the limit $\Lambda \uparrow \mathbf{Z}^d, N \to \infty$ of (34) exists for any $G \in L^{\infty}(\mathcal{M}_B)$ with $|B| < \infty$, we see, by taking characteristic functions of sets of zero Lebesgue measure, that μ_{Λ} is absolutely continuous with respect to the Lebesgue measure, for any $|\Lambda| < \infty$.

The space-time exponential mixing of μ follows essentially from the exponential clustering of $\bar{\mu}$ which itself follows from Theorem 2.2: By an approximation argument, it is enough, to prove (3.10), to consider $G \in \mathcal{C}(\mathcal{M}_B)$, and H as in (3.10): Then, as in (37),

$$\begin{aligned}&\int G\circ T^n Hd\mu - \int Gd\mu\int Hd\mu\\ &= \lim_{\Lambda\uparrow\mathbf{Z}^d}\lim_{N\to\infty}(\int GP_{\Lambda}^{n}(HP_{\Lambda}^{N}1)dm_{\Lambda} - \int GP_{\Lambda}^{N}1dm_{\Lambda}\int HP_{\Lambda}^{N}1dm_{\Lambda}).\end{aligned} \tag{38}$$

Following equations (10,12,13), we get (replacing H by the periodic function h):

$$P_\Lambda^n(hP^N 1)(\mathbf{x}) = \sum_{\mathbf{s}_1,\cdots,\mathbf{s}_{N+n}} e^{-\mathcal{H}_{\Lambda_{N+n}}(s)} h(s) \tag{39}$$

where

$$h(s) \equiv h(\psi_{\mathbf{s}_n} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x})). \tag{40}$$

Proceeding as in (19), we write

$$h(s) = \sum_{m\geq 0} h_{Y_\alpha^m}(s) + \text{ constant} \tag{41}$$

where $\alpha = (i, n)$ for some $i \in D$, and

$$h_{Y_\alpha^m}(s) = h(s_{Y_\alpha^m} \vee 0_{(Y_\alpha^m)^c}) - h(s_{Y_\alpha^{m-1}} \vee 0_{(Y_\alpha^{m-1})^c})$$

Since h is Hölder continuous, proceeding as in the proof of Proposition 1, we get an estimate on $h_{Y_\alpha^m}$ similar to (23):

$$| \; h_{Y_\alpha^m}(s) \; | \leq C\|H\|_\delta \lambda'^{tm} e^{-\beta' d(i-i_m,D)} \tag{42}$$

Now, insert (39, 41) in (38), and use the exponential decay of correlation (2.12) for $\bar{\mu}$ with B being a subset of $\{0\} \times \mathbf{Z}^d$ and $A = Y_\alpha^m$, so that $d(A, B) \geq d(i - i_m, B) + (n - t_m)$, where the distance in the LHS is taken in $\mathbf{Z}_+^{d+1}$, while, on the RHS, it is taken in $\mathbf{Z}^d$. Then, from (42) and

$$\sum_{\ell\geq 0} \exp(-m(d(i-i_\ell, B) + (n - t_\ell)))\lambda'^{t_\ell} e^{-\beta' d(i-i_\ell,D)} \leq Ce^{-m'(n+d(B,D))} \min(|B|, |D|)$$

for some $m' > 0$, we get (3.10).

Finally, let ν be a regular state. We want to show that $\forall B \subset \mathbf{Z}^d, |B| < \infty, \forall G \in \mathcal{C}(\mathcal{M}_B)$,

$$\lim_{N\to\infty} \int G \circ T^N d\nu = \int G d\mu.$$

Since $\|G \circ T_\Lambda^N - G \circ T^N\|_\infty \to 0$, as $\Lambda \uparrow \mathbf{Z}^d$, it is enough to show that

$$\lim_{N\to\infty} \lim_{\Lambda\uparrow\mathbf{Z}^d} (\int G \circ T_\Lambda^N d\nu - \int G(P_\Lambda^N 1) dm_\Lambda) = 0. \tag{43}$$

Since $G \circ T_\Lambda^N$ is $\mathcal{B}_\Lambda$ measurable we may replace $d\nu$ by $h_\Lambda(\mathbf{x}_\Lambda)d\mathbf{x}_\Lambda$ where $h_\Lambda = \frac{d\nu_\Lambda}{dm_\Lambda}$.

We have

$$\int G \circ T_\Lambda^N h_\Lambda dm_\Lambda = \int_{C_\Lambda} g P_\Lambda^N h_\Lambda d\mathbf{x}_\Lambda$$

$$= \sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} g(\mathbf{x}_B) \exp(-\mathcal{H}_{\Lambda_N}(s)) h_\Lambda(s) d\mathbf{x}_\Lambda \tag{44}$$

where $h_\Lambda(s)$ is defined by (40), with n replaced by N. With the same notation, introduce $h_\Lambda(\bar{s})$ where

$$\bar{s}_\alpha = 0 \ \ \forall\alpha = (t,i), \ \ t \le \frac{N}{2} \ \text{ and } \ d(i,B) \le N \tag{45}$$

and $\bar{s}_\alpha = s_\alpha$ otherwise. Let

$$\bar{h}_\Lambda(\bar{s}) = (\sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} h_\Lambda(\bar{s}) e^{-\mathcal{H}_{\Lambda_N}(s)} d\mathbf{x}_\Lambda)^{-1} h_\Lambda(\bar{s}) \tag{46}$$

Insert the identity

$$h_\Lambda(s) = (1 - \frac{\bar{h}_\Lambda(\bar{s})}{h_\Lambda(s)}) h_\Lambda(s) + \bar{h}_\Lambda(\bar{s}) \tag{47}$$

in (44). Since

$$\sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} h_\Lambda(s) e^{\mathcal{H}_{\Lambda_N}(s)} d\mathbf{x}_\Lambda = \int_{C_\Lambda} P_\Lambda^N h_\Lambda d\mathbf{x}_\Lambda = \int_{C_\Lambda} h_\Lambda d\mathbf{x}_\Lambda = 1, \tag{48}$$

(46) implies that $\frac{\bar{h}_\Lambda(\bar{s})}{h_\Lambda(s)} = \frac{h_\Lambda(\bar{s})}{h_\Lambda(s)} (\int \frac{h_\Lambda(\bar{s})}{h_\Lambda(s)} d\rho_\Lambda)^{-1}$, where $d\rho_\Lambda$ is a probability measure. So, we can bound

$$|1 - \frac{\bar{h}_\Lambda(\bar{s})}{h_\Lambda(s)} | \le \exp(2 \sup_s |\log h_\Lambda(s) - \log h_\Lambda(\bar{s})|) - 1. \tag{49}$$

Now, since ν is regular

$$|\log h_\Lambda(s) - \log h_\Lambda(\bar{s})| \le C \sum_i |\psi_{\mathbf{s}_N} \circ \cdots \circ \psi_{\mathbf{s}_1}(\mathbf{x})_i - \psi_{\bar{\mathbf{s}}_N} \circ \cdots \circ \psi_{\bar{\mathbf{s}}_i}(\bar{\mathbf{x}})_i|^\delta \tag{50}$$

Now we use (33) and the definition (45) of $\bar{s}$ to bound each term in (50) by $C\lambda^{\mathcal{O}(N)}$ if $d(i,B) \le N$ and by $C\lambda^{\mathcal{O}(N)} e^{-\frac{\delta\beta}{3}(d(i,B)-N)}$ if $d(i,B) > N$. Hence, (50) is bounded by $C(|B|)\lambda^{\mathcal{O}(N)}$. Combining this with (49), (47), (48) and since G is bounded, we have

$$\begin{aligned} & | \int G \circ T_\Lambda^N d\nu - \int G(P_\Lambda^N 1) dm_\Lambda| \\ \le & \ | \sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} g(\mathbf{x}_B) \exp(-\mathcal{H}_{\Lambda_N}(s)) \bar{h}_\Lambda(\bar{s}) d\mathbf{x}_\Lambda - \sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} g(\mathbf{x}_B) \exp(-\mathcal{H}_{\Lambda_N}(s)) d\mathbf{x}_\Lambda| \\ + & \ C(|B|) \|G\|_\infty \lambda^{\mathcal{O}(N)} \end{aligned} \tag{51}$$

Since g depends only on $\mathbf{x}_B$ and $\bar{h}_\Lambda(\bar{s})$ depends only on s_α, with $\alpha = (t,i), t > \frac{N}{2}$ or $d(i,B) > N$ the first term in (51) is also bounded by $CG\lambda^{\mathcal{O}(N)}$, using the exponential decay of the Gibbs state defined by $\mathcal{H}$ (uniformly in Λ and N) proven above and the normalization

$$\sum_{(\mathbf{s}_i)_{i=1}^N} \int_{C_\Lambda} \exp(-\mathcal{H}_{\Lambda_N}(s)) \bar{h}_\Lambda(\bar{s}) d\mathbf{x}_\Lambda = 1$$

which follows from (46). $\square$

Appendix

Proof of Proposition 3 We need to study a one-dimensional system with potentials Φ^0_Y, $Y \subset \mathbf{Z}_+$, $Y = [t_1, t_1+1, \cdots, t_2]$, $|t_2 - t_1| < M$, $t_1 \geq 1$ satisfying

$$|\Phi^0_Y| \leq C\lambda'^{|Y|} \tag{A.1}$$

for $\lambda' < 1$ (we suppress here the Λ dependence of Φ).

Let $\mathcal{T}$ be the transfer matrix for Φ^0, i.e. $\mathcal{T}$ is the linear operator on functions on $S \equiv \{0, 1, \cdots, k-1\}^M$ (i.e. on $\mathbf{R}^{|S|}$) given by

$$(\mathcal{T}f)(s) = \sum_{t\in S} \mathcal{T}(s,t)f(t) \tag{A.2}$$

with

$$\mathcal{T}(s,t) = a(s)a(t)e^{U(s,t)} \tag{A.3}$$

where

$$a(s) = \exp(\frac{1}{2}\sum_{Y\subset[1,M]} \Phi^0_Y(s)) \tag{A.4}$$

and

$$U(s,t) = \sum_Y \Phi^0_Y(s \vee t) \tag{A.5}$$

where $Y \subset [1, 2M]$ with $M, M+1 \in Y$ and $(s\vee t)_i$; equals s_i if $i \leq M$ and t_{i-M} if $i > M$.

Let J be as in the Proposition; we have

$$Z^0(J|s) = (f_{s_-}, \mathcal{T}^\ell h_{s_+}) \tag{A.6}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbf{R}^{|S|}$, and

$$f_{s_-}(s) = a(s)e^{U(s_-,s)}$$

$$h_{s_+}(t) = a(t)e^{U(t,s_+)} \tag{A.7}$$

We can apply the Perron-Frobenius theory since $\mathcal{T}$ has strictly positive entries. $\mathcal{T}$ has a non-degenerate largest eigenvalue λ with left and right eigenvectors u and v:

$$u\mathcal{T} = \lambda u$$

$$\mathcal{T}v = \lambda v \tag{A.8}$$

The vectors v and u can be chosen with positive entries: $u(s), v(s) > 0$, $\forall s \in S$ and we normalize them by

$$\sum_{s\in S} u(s)v(s) = 1 > \tag{A.9}$$

We may write $\mathcal{T} = \lambda Q + R$ where $Q(s,t) = v(s)u(t)$, $Q^2 = Q$ and $QR = RQ = 0$.

Hence $\mathcal{T}^\ell = \lambda^\ell Q + R^\ell$, and

$$(f_{s_-}, \mathcal{T}^\ell h_{s_+}) = \lambda^\ell (f_{s_-}, v)(u, h_{s_+}) \left(1 + \frac{(f_{s_-}, R^\ell h_{s_+})}{\lambda^\ell (f_{s_-}, v)(u, h_{s_+})} \right) \tag{A.10}$$

So define $W(s_-) = (f_{s_-}, v), W(s_+) = (u, h_{s_+})$, and

$$g_J(s_-, s_+) = \frac{(f_{s_-}, R^\ell h_{s_+})}{\lambda^\ell (f_{s_-}, v)(u, h_{s_+})} \tag{A.11}$$

so that, using (A.6), $Z(J|s_-, s_+)$ is of the form (4.29). To estimate g_J it is convenient to introduce the matrix

$$P(s,t) = \frac{\mathcal{T}(s,t)v(t)}{\lambda v(s)} \tag{A.12}$$

Since $P(s,t) > 0, \sum_t P(s,t) = 1$, we can view $P(s,t)$ as the transition probability (from state s to state t) of a Markov chain. P has a unique stationary distribution $pP = p$, where $p(s) = u(s)v(s)$. Write

$$P = \tilde{Q} + \tilde{R} \tag{A.13}$$

where

$$\tilde{Q}(s,t) = \frac{Q(s,t)v(t)}{v(s)} = u(t)v(t) = p(t) \tag{A.14}$$

and

$$\tilde{R}(s,t) = \frac{R(s,t)v(t)}{\lambda v(s)} \tag{A.15}$$

We have $\tilde{Q}\tilde{R} = \tilde{R}\tilde{Q} = 0$ and $P^\ell = \tilde{Q} + \tilde{R}^\ell$.

A standard result in the ergodic theory of Markov chains implies that

$$\sum_t | \sum_s q(s) P^\ell(s,t) - p(t)| \leq 2(1-\gamma)^\ell \tag{A.16}$$

where

$$\gamma = \min_{s,s',t} \frac{P(s',t)}{P(s,t)}, \tag{A.17}$$

for any q, with $q(s) \geq 0$, $\sum_s q(s) = 1$. Postposing the proof of (A.16), we rewrite it, using (A.13,A.14), as

$$\sum_t | \sum_s q(s) \tilde{R}^\ell(s,t)| \leq 2(1-\gamma)^\ell \tag{A.18}$$

We use (A.11,A.15) to write

$$\begin{aligned} |g_J(s_-, s_+)| &= \left| \frac{(f_{s_-} v, \tilde{R}^\ell h_{s_+} v^{-1})}{(f_{s_-}, v)(u, h_{s_+})} \right| \\ &\leq 2(1-\gamma)^\ell \frac{\max_t h_{s_+}(t) v(t)^{-1}}{\min_s h_{s_+}(s) v(s)^{-1}}, \end{aligned} \tag{A.19}$$

using (A.18) with

$$q(s) = \frac{f_{s_-}(s)v(s)}{(f_{s_-}, v)} = \frac{f_{s_-}(s)v(s)}{\sum_t f_{s_-}(t)v(t)},$$

and using $\sum_s u(s)v(s) = 1$ to bound $(u, h_{s_+}) \geq \min_s h_{s_+}(s)v(s)^{-1}$.

Now, we need to show that $\gamma > 0$ uniformly in M and we have to bound the last factor in (A.19) by a constant independent of M. Using (A.12,A.17)

$$\gamma = \min_{s,s',t} \frac{\mathcal{T}(s',t)v(s)}{\mathcal{T}(s,t)v(s')} = \min_{s,s',t} \frac{\mathcal{T}(s',t)\sum_r \mathcal{T}(s,r)v(r)}{\mathcal{T}(s,t)\sum_r \mathcal{T}(s',r)v(r)}$$

where we use (A.8) in the last equality. So, since $v(r) > 0$,

$$\gamma \geq \min_{s',t,s,r} \frac{\mathcal{T}(s',t)\mathcal{T}(s,r)}{\mathcal{T}(s,t)\mathcal{T}(s',r)} \geq e^{-4\|U\|}$$

by (A.3). Using (A.1,A.5), we see that $\|U\| \leq C_1(\lambda')$, uniformly in M. Now consider

$$\frac{h_{s_+}(t)v(s)}{h_{s_+}(s)v(t)} \leq \max_{s,t,r} \frac{h_{s_+}(t)\mathcal{T}(s,r)}{h_{s_+}(s)\mathcal{T}(t,r)} \leq e^{4\|U\|} \tag{A.20}$$

by (A.3,A.7). Inserting (A.20) in (A.19) yields (4.30).

We are left with the proof of (A.16). Since $pP = p$ and since $\Sigma p(s) = \Sigma q(s) = 1$, the LHS of (A.16) can be written as

$$\sum_t |\sum_{s,s'}(P^\ell(s,t) - P^\ell(s',t))q(s)p(s')|,$$

and it is enough to prove that

$$\sum_t |P^\ell(s,t) - P^\ell(s',t)| \leq 2(1-\gamma)^\ell \tag{A.21}$$

for all s, s'. Let $f_{s,s'}(t) = \text{ sign } (P^\ell(s,t) - P^\ell(s',t))$. Then, the LHS of (A.21) equals

$$\begin{aligned} & |(P^\ell f)(s) - (P^\ell f)(s')| \\ = \; & |\sum_z P(s,z)(P^{\ell-1}f)(z) - P(s',z)(P^{\ell-1}f)(z)| \\ \leq \; & \frac{1}{2}(\sum_z |P(s,z) - P(s',z)|) \sup_{z,z'} |(P^{\ell-1}f)(z) - (P^{\ell-1}f)(z')| \end{aligned} \tag{A.22}$$

where, to get the last inequality, we use

$$|\mu_1(F) - \mu_2(F)| \leq \frac{1}{2}(\sum_z |\mu_1(z) - \mu_2(z)|) \sup_{z,z'} |F(z) - F(z')|$$

(see e.g. [41] Lemma 5.1.8) where μ_1, μ_2 are the probability measures $\mu_1(z) = P(s,z)$, $\mu_2(z) = P(s',z)$ and $F = P^{\ell-1}f$.

Since $P(s,z), P(s,z')$ are probability measures,

$$\frac{1}{2}\sum_z |P(s,z) - P(s',z)| = \sum_z{}^+ (P(s,z) - P(s',z))$$

where the last sum runs over positive terms

$$= \sum_z{}^+ (1 - \frac{P(s',z)}{P(s,z)}) P(s,z) \leq 1 - \gamma \tag{A.23}$$

by definition (A.17) of γ.

Inserting (A.23) in (A.22) and iterating yields (A.21), using in the last step

$$\sup_{z,z'} |f_{s,s'}(z) - f_{s,s'}(z')| \leq 2.$$

□

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