ON CLASSICAL ELECTRODYNAMICS OF POINT PARTICLES AND MASS RENORMALIZATION

Some preliminary results

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Abstract. *We consider the problem of finding rigorous results for the dynamics of a classical charged point particle interacting with the electromagnetic field, as described by the standard Maxwell–Lorentz equations. Some results are given for the corresponding linearized system, i.e. the so called dipole approximation, in the presence of a mechanical linear restoring force. We regularize the system by taking a form factor for the particle (Pauli–Fierz model) and study the limit of the particle's motion as the regularization is removed. We prove that (i) if the regularization is removed but mass is not renormalized the motion is trivial (i.e. the particle does not move at all); (ii) if the regularization is removed and mass is renormalized, the particle's motion corresponding to smooth initial data for the field has a well defined nontrivial limit; (iii) in the case of vanishing initial field the limit motion satisfies exactly the Abraham–Lorentz–Dirac equation; (iv) for generic initial data the limit motion is runaway.*

1. Introduction

Concerning the interaction of a charged point particle with the electromagnetic field in classical electrodynamics, the following quotation from E. Nelson might be appropriate: "With suitable ultraviolet and infrared cutoffs, this is a dynamical system of finitely many degrees of freedom and we have global existence and uniqueness.... Is it an exaggeration to say that nothing whatever is known about the behavior of this system as the cutoffs are removed, and there is not one single theorem that has been proven?"[1] (p. 65). In the present paper we give some preliminary results on the limit motion of the particle when the cutoff is removed. The main limitation is due to the fact that, instead of the complete Maxwell–Lorentz system (namely Maxwell equations for the field with a current due to the particle's motion, and the relativistic Newton equation for the particle with Lorentz force

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due to the field), we consider the corresponding linearized, and thus non–relativistic, version, i.e. the so called dipole approximation. This is indeed a model very much studied both in its classical and in its quantum version (see e.g. [2, 3]), which presumably allows to give preliminary information also on the complete non linear system. Moreover, we recall that a large part of physical effects ranging from Thomson scattering to Lamb shift (see e.g. [4, 5]) are well described within such an approximation.

As is well known, in the Coulomb gauge the only unknowns are the vector potential $\mathbf{A}$ for the field (with $\mathrm{div}\mathbf{A} = 0$) and the particle's position $\mathbf{q}$; the formal equations of motion are

$$\begin{aligned} \frac{1}{c^2}\ddot{\mathbf{A}} - \triangle\mathbf{A} &= \frac{4\pi}{c^2}\mathbf{j}_{tr} \ , \\ m_0\ddot{\mathbf{q}} &= -\frac{e}{c}\dot{\mathbf{A}}(0) - \alpha\mathbf{q} \ , \end{aligned} \tag{1.1}$$

where $\mathbf{j}_{tr}$ is the transversal part of the current $\mathbf{j}(\mathbf{x}) := e\dot{\mathbf{q}}\delta(\mathbf{x})$, with $\delta(\mathbf{x})$ the usual delta function, while e, and m_0 are respectively the particle's charge and bare mass; here we also added a linear restoring force $-\alpha\mathbf{q}$ (with $\alpha > 0$).

In order to regularize the system, following a long tradition (see for example Pauli and Fierz [6]), instead of the cutoffs indicated by Nelson we take a form factor corresponding to a rigid extended particle. Moreover, we concentrate our attention just on the particle's motion, which, due to the non trivial coupling with the field, clearly depends also on the initial data for the field variables.

The first result we prove is that the particle's motions corresponding to solutions of the Cauchy problem with regular data for the field (namely, in the Schwartz space) have a limit (in the C^0 topology) when the regularization is removed, i.e. that the particle performs regular motions when the "radius" of the form factor tends to zero. In this connection let us recall that, as is well known, a crucial point for the discussion of the point-like limit is mass renormalization, the need of which is expected from heuristic considerations [7, 8, 9, 10, 11]. We first show that an infinite mass renormalization, leading to a negative bare mass, is actually needed. Indeed we prove that if mass is not renormalized (i.e. the bare mass is taken positive and independent of the form factor), the limit particle's motion is trivial, in the sense that the particle remains forever at rest even if initially displaced from the equilibrium position of the spring. So, in order to obtain non trivial limiting particle's motions one has to take a negative bare mass (for any radius smaller than a certain critical radius coinciding with the so called "classical radius"), which moreover has to tend to infinity as the radius tends to zero. Therefore we study the limit of the particle's motion when the radius tends to zero, and simultaneously mass is renormalized, and prove the convergence result mentioned above. We also prove continuous dependence of the particle's motion on intial data. So it turns out that *classical electrodynamics of point particles, at least in the dipole approximation, is well defined*, even if the equations of motions might not have a well defined point–like limit (due to mass rinormalization).

A second kind of results concerns a description of the limiting particle's motions. We deal only with the simplest case, namely that of a particle initially at rest, displaced from the position of mechanical equilibrium, with vanishing initial data for the field. In such a case we prove that the limiting particle's motion satisfies exactly the well known Abraham–Lorentz–Dirac (or ALD) equation; in this sense we can say that we have here a rigorous proof of such an equation, which was up to now obtained only by heuristic arguments. The exact form of the solution is a damped oscillation superimposed to an exponentially increasing motion; thus we have here a *runaway motion*. We recall that runaway motions are well known to appear in the context of the Abraham–Lorentz–Dirac equation [12]. However, since the standard deductions of such an equation from the Maxwell–Lorentz system involve many unclear approximations, there could remain a doubt concerning the actual existence of runaways in classical electrodynamics. The present result shows that, at least in the linear approximation, in classical electrodynamics runaways appear for generic initial data (for a discussion of the analogous phenomenon in quantum mechanics see e.g. [3]). So the conclusion is that non trivial point particle's motions in the Maxwell–Lorentz system can be obtained only by paying the price of having generically runaway solutions, which are obviously unacceptable form a physical point of view. We postpone a discussion of this problem to future work. A further comment is that we expect that the results described above for the linearized system can be essentially extended to the complete nonlinear system. Work is in progress in this direction.

We come now to a discussion of the technical aspects of the paper. For the regularized system existence and uniqueness results are deduced by means of standard semigroup theory [13]. But such a theory fails in the point–like limit, essentially due to the difficulty of giving a meaning to the formal expression defining the Lorentz force in the that limit; indeed this is related to the old problem of defining the product of two singular distributions.

We are able to overcome the problem by providing directly a representation formula for the solution of the Cauchy problem for the regularized system, so that we can study explicitly the limit of the particle's motion. The representation formula is based on the explicit calculation of the spectral resolution of the linear operator describing the dipole approximation. The formulation of the spectral resolution theorem used here is essentially that of the book [14] by Gel'fand and Shilov.

The paper is divided into two parts. The first part consists of two sections which are devoted to the statement of the results. Precisely, in section 2 we present the model, discuss mass renormalization, and state the well posedness results for the dynamics of a point particle. In sect. 3 we give an explicit formula for the motion of a point particle, corresponding to vanishing initial field; we also compare these solutions with the solutions of the ALD equation. Part II consists of the proofs of the above results and is divided into three sections: in sect. 4 we give the representation formula for the solution of the Cauchy problem for an extended particle; in sect. 5 we prove the result on the dynamics of a rigid extended particle, and in sect. 6 those concerning the point–like limit.

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PART I: Statement of the results

2. Well posedness of the dynamics of a point particle

We consider a relativistic charged particle interacting with the electromagnetic field, and subjected to an external linear restoring force. We recall that, working in the Coulomb gauge, the complete nonlinear Maxwell–Lorentz system takes the form [4]

$$\begin{aligned} \frac{1}{c^2}\ddot{\mathbf{A}} - \triangle \mathbf{A} &= \frac{4\pi e}{c}\Pi(\dot{\mathbf{q}}\delta_q) \ , \\ \frac{d}{dt}\left(\frac{m_0\dot{\mathbf{q}}}{\sqrt{1-(\dot{q}/c)^2}}\right) &= -\frac{e}{c}\dot{\mathbf{A}}(\mathbf{q}) + e\frac{\dot{\mathbf{q}}}{c}\wedge \operatorname{rot}\mathbf{A}(\mathbf{q}) - \alpha\mathbf{q}, \end{aligned} \tag{2.1}$$

where the vector potential $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is subjected to the constraint $\operatorname{div}\mathbf{A} = 0$, and $\alpha > 0$ is a constant characterizing the external linear force, δ_q is the distribution translated with respect to the δ function centered at the origin, formally $\delta_q(\mathbf{x}) := \delta(\mathbf{x}-\mathbf{q})$. Finally Π is the projector on the subspace of vector fields with vanishing divergence, i.e. $\Pi(\mathbf{j})$ is the so called transversal part of the current $\mathbf{j}$, often denoted by $\mathbf{j}_t$. We take now the so called dipole approximation, namely linearize the system about the equilibrium point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{A} = \dot{\mathbf{A}} = 0$, obtaining system (1.1). We then regularize the system by substituting the δ function by a smooth normalized (in L^1) charge distribution ρ, obtaining the system

$$\begin{aligned} \frac{1}{c^2}\ddot{\mathbf{A}} - \triangle \mathbf{A} &= \frac{4\pi e}{c}\Pi(\dot{\mathbf{q}}\rho) \ , \\ m_0\ddot{\mathbf{q}} &= -\int_{\mathbb{R}^3}\frac{e}{c}\rho(x)\dot{\mathbf{A}}(x)d^3x - \alpha\mathbf{q} \ , \end{aligned} \tag{2.2}$$

which is the one that will be studied here. For the sake of brevity, *the system (2.2) will be called simply the Maxwell–Lorentz system,* omitting the qualification "in the dipole approximation" and also the qualification "regularized". It will be studied in the configuration space $\mathcal{Q}_0$ defined by

$$\mathcal{Q}_0 := \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3) \oplus \mathbb{R}^3 \ni (\mathbf{A}, \mathbf{q}) \ , \tag{2.3}$$

where $\mathcal{S}_*$ denotes the subset of the vector fields belonging to the Schwartz space $\mathcal{S}$ (C^∞ functions decaying at infinity faster than any power) having vanishing divergence.

Concerning ρ we will assume that (i) it is C^∞, (ii) it decays at least exponentially fast at infinity, (iii) it is spherically symmetric, and (iv) its Fourier transform $\hat{\rho}$, defined by

$$\rho(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\hat{\rho}(\mathbf{k})e^{i\mathbf{k}\mathbf{x}}d^3k \ ,$$

is everywhere non–vanishing (*i.e.* $\hat{\rho}(\mathbf{k}) \neq 0$, $\forall \mathbf{k} \in \mathbb{R}^3$); finally, in order to simplify the discussion of the point–like limit we will assume (v) that ρ has the form

$$\rho_a(x) := \frac{1}{a^3}\mathcal{D}\left(\frac{x}{a}\right) , \tag{2.4}$$

where $\mathcal{D}$ is a positive, normalized (in L^1) function. Notice that the pointlike limit is obtained by letting the "radius" a of the particle tend to zero (so that ρ_a tends to a delta concentrated at the origin), and that for $a > 0$ the Cauchy problem for system (2.2) is well posed in the phase space $\mathcal{Q}_0 \times \mathcal{Q}_0$. We discuss now the limit of the particle's motion as the "radius" a of the charge distribution tends to zero.

To begin with, we consider the special class of initial data with the particle initially at rest in some position $q_0 \neq 0$, and vanishing initial field. First we show the need of mass renormalization. We have the following

Proposition 2.1. *Having fixed $m_0 > 0$, for each $a > 0$ denote by $\mathbf{q}_a(t)$ the solution of (2.2) corresponding to initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$, and $\mathbf{q}_0 \neq 0$; assume that $\mathbf{q}_a(.)$ converges weakly to a distribution $\mathbf{q}(.)$ as $a \to 0$. Then there exists a constant vector $\bar{\mathbf{q}}$ such that $\mathbf{q}(t) \equiv \bar{\mathbf{q}}$.*

This means that for positive bare mass, if the dynamics of the particle admits a point–like limit, such a limit dynamics is trivial. This could be expected since it is well known that the electromagnetic mass of a point particle is infinite; the above proposition proves exactly that a point particle is unaffected by the presence of a force $(-\alpha\mathbf{q}_0)$ no matter how large it is, so that it behaves as if its mass were infinite. ¿From the mathematical point of view this is seen as follows. As already anticipated in the introduction, the proof of all results of this section is based on the use of a representation formula for the solution of the Cauchy problem of (2.2). Now, in such a formula the bare mass m_0 appears only summed to the quantity

$$m_{em} := \frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty |\hat{\rho}(k)|^2 dk , \tag{2.5}$$

which is usually interpreted as the electromagnetic mass corresponding to the given charge distribution. So it can be expected that the particle behaves as if its experimental mass were the sum of m_0 and m_{em}.

So, we renormalize mass, i.e. we consider m_0 as a function of a, precisely we put

$$m_0 := m - m_{em} = m - \frac{1}{a}\left[\frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty \left|\hat{\mathcal{D}}(k)\right|^2 dk\right] , \tag{2.6}$$

where m is a fixed parameter to be identified with the physical mass of the particle. Notice that (2.6) requires to consider negative bare masses m_0, which, as a tends to zero, diverge to minus infinity.

Concerning the behaviour of the system in the limit $a \to 0$ we have

Proposition 2.2. *Consider the Cauchy problem for system (2.2) with form factor ρ given by (2.4), m_0 given by (2.6), and initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$, with $\mathbf{q}_0 \neq 0$. For each*

$a > 0$ let $\mathbf{q}^{(a)}(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ the function $\mathbf{q}^{(a)}(.)$ converges in $C^1([-T,T], \mathbb{R}^3)$ to a non constant function.

So, the particular solution corresponding to the above initial data has a point–like limit which is nontrivial, *provided mass is renormalized.*

We consider now the case where the initial particle's velocity too is different from zero; this is a nontrivial generalization. Indeed in such a case ($\dot{\mathbf{q}}_0 \neq 0$), if one takes a vanishing initial field, it turns out that the trajectory of the particle has no point–like limit; precisely one has

Proposition 2.3. *Consider the Cauchy problem for system (2.2) with form factor ρ given by (2.4), m_0 given by (2.6), and initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \mathbf{q}_0 = 0$, with $\dot{\mathbf{q}}_0 \neq 0$. For each $a > 0$ let $\mathbf{q}_{(a)}(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$, one has*

$$|\mathbf{q}_{(a)}(t)| \to \infty \ , \quad \forall t \in [-T,T] \backslash \mathcal{N} \ ,$$

where $\mathcal{N}$ is a finite (possibly empty) set.

It is not difficult to prove that the same happens also if one takes as initial data for the field any regular function (i.e. without singularities). On the other hand the result of proposition 2.3 is not astonishing, because it is known (at least in the case of a uniform motion) that a particle moving with some velocity carries a field [15, 16] which in the case of a point–like particle has a singularity at the particle's position. So, it seems natural to study the particular class of initial data such that a particle with non vanishing velocity is accompanied by "its own field". In order to give a precise statement we recall that [16] in the non–linear Maxwell–Lorentz system a free particle can move uniformly with velocity $\mathbf{w}$, only if accompanied by the field $\mathbf{X}$, which vanishes at infinity and solves the equation

$$\triangle \mathbf{X} - \frac{1}{c^2} \sum_{i,l=1}^{3} w_i w_l \frac{\partial^2}{\partial x_i \partial x_l} \mathbf{X} = -4\pi \frac{e}{c} \Pi(\rho \mathbf{w}) \ .$$

In the dipole approximation, i.e. after a linearization in the velocity and in the field, such an equation reduces to

$$\triangle \mathbf{X} = -4\pi \frac{e}{c} \Pi(\rho \mathbf{w}) \ . \tag{2.7}$$

We denote by $\mathbf{X}_w$ the unique solution of equation (2.7) vanishing at infinity, and study the point–like limit of the solutions of the Cauchy problem corresponding to initial data of the form $\dot{q}_0 \neq 0, \mathbf{A}_0 = \mathbf{X}_{\dot{q}_0}$. Initial data of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$ will be called of "congruent type".

Proposition 2.4. *Consider the Cauchy problem for system (2.2) with form factor ρ given by (2.4), m_0 given by (2.6), and initial data $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$. For each $a > 0$ let $\mathbf{q}^{(a)}(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ the function $\mathbf{q}^{(a)}(.)$ converges in $C^1([-T,T], \mathbb{R}^3)$ to a non constant function.*

We come now to the case of general initial data for the field. The following theorem holds.

Theorem 2.5. *Consider the Cauchy problem for system (2.2) with form factor ρ given by (2.4), m_0 given by (2.6), and initial data $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0} + \mathbf{A}'_0, \dot{\mathbf{A}}_0)$ with $(\mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathcal{S}_*(\mathrm{I\!R}^3, \mathrm{I\!R}^3) \times \mathcal{S}_*(\mathrm{I\!R}^3, \mathrm{I\!R}^3)$. For each $a > 0$ let $\mathbf{q}^{(a)}(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ the function $\mathbf{q}^{(a)}(.)$ converges in $C^0([-T,T], \mathrm{I\!R}^3)$. Moreover, the limiting particle's motion depends continuously on*

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathrm{I\!R}^3 \times \mathrm{I\!R}^3 \times \mathcal{S}_*(\mathrm{I\!R}^3, \mathrm{I\!R}^3) \times \mathcal{S}_*(\mathrm{I\!R}^3, \mathrm{I\!R}^3) \ .$$

So, even if (due to mass renormalization) the Maxwell– Lorentz system has no point–like limit, the above theorem shows that the dynamics of a point particle is well defined, at least for regular initial data for the field. Moreover, the Cauchy problem is well posed in the sense of Hadamard. By the way, our technique allows to generalize the existence result to the case of initial fields $(\mathbf{A}'_0, \dot{\mathbf{A}}_0)$ which are only C^0, and decay at infinity faster than $x^{-3/2}$; further generalizations are also possible, but we did not try to characterize the most general allowed initial fields.

3. On the solutions corresponding to congruent initial data, and comparison with the Abraham–Lorentz–Dirac equation.

In the case of congruent initial data it is possible to calculate explicitly the point–limit of the solution of the Maxwell-Lorentz system. In order to come to the corresponding formula we introduce some notations. We define

$$\omega_0^2 := \frac{\alpha}{m} \ , \qquad \tau_0 := \frac{2}{3}\frac{e^2}{mc^3} \ ,$$

then we consider the equation

$$\tau_0 \nu^3 - \nu^2 - \omega_0^2 = 0 \ , \tag{3.1}$$

and denote by ν_r, $\nu_+ = \nu_3 + i\nu_2$, $\nu_- = \nu_3 - i\nu_2$ its three solutions $(\nu_2, \nu_3 > 0)$.

Theorem 3.1. *The point–like limit of the particle's motion corresponding to the solution of the Cauchy problem for the Maxwell–Lorentz system with initial data*

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$$

is given by

$$\mathbf{q}(t) = \begin{cases} e^{-\nu_3 t}[\mathcal{A}_1^+ \cos(\nu_2 t) + \mathcal{A}_2^+ \sin(\nu_2 t)] + \mathcal{A}_3^+ e^{\nu_r t} \ , & \text{if } t > 0 \\ e^{\nu_3 t}[\mathcal{A}_1^- \cos(\nu_2 t) + \mathcal{A}_2^- \sin(\nu_2 t)] + \mathcal{A}_3^- e^{-\nu_r t} \ , & \text{if } t < 0 \end{cases} \ , \tag{3.2}$$

where $\mathcal{A}_1^{\pm}$, $\mathcal{A}_2^{\pm}$, $\mathcal{A}_3^{\pm}$ are real vector constants depending on the initial data, and on e, m, ω_0. Moreover, one has the following asymptotics

$$\begin{cases} \nu_r = \dfrac{\omega_0}{\epsilon} + O(\epsilon) \\ \nu_2 = \omega_0 + O(\epsilon^2) \\ \nu_3 = \omega_0\epsilon/2 + O(\epsilon^2) \end{cases} , \quad \begin{cases} \mathcal{A}_1^{\pm} = \mathbf{q}_0 + O(\epsilon^2) \\ \mathcal{A}_2^{\pm} = \dfrac{\dot{\mathbf{q}}_0}{\omega_0} + O(\epsilon^2) \\ \mathcal{A}_3^{\pm} = O(\epsilon) \end{cases} \tag{3.3}$$

as $\epsilon \to 0$, in terms of the dimensionless parameter

$$\epsilon := \omega_0\tau_0 \ . \tag{3.4}$$

We point out that, from the proof of this theorem, it is not difficult to deduce also a representation formula for the limit particle's motion, in the case of a nonvanishing initial field.

Qualitatively we have thus that the Maxwell–Lorentz system in the dipole approximation predicts that the motion of a point–like charged harmonic oscillator is the superposition of a damped oscillation and a runaway motion, at least when the initial "free" field is vanishing. Moreover, it is possible to show that the particle's motion corresponding to generic initial deta is runaway. It turns out that it is possible to select initial field in such a way that the corresponding particle's motion is non runaway. A detailed discussion of this point is beyond the aim of this paper, and is deferred to forthcoming work.

We concentrate now on the comparison of (3.2) with the solutions of the Abraham–Lorentz–Dirac (ALD) equation

$$m\tau_0\dddot{\mathbf{q}} = m\ddot{\mathbf{q}} + \alpha\mathbf{q} \ . \tag{3.5}$$

Such a comparison is particularly interesting since all available deductions of the ALD equation appear to be just heuristic[7,2,17]. We have the following

Theorem 3.2. *The point limit of the particle's motion (3.2) in the ML system is also a solution of the following problem*

$$\begin{aligned} -m\tau_0\dddot{\mathbf{q}} &= m\ddot{\mathbf{q}} + \alpha\mathbf{q} \ , \quad t < 0 \ , \\ m\tau_0\dddot{\mathbf{q}} &= m\ddot{\mathbf{q}} + \alpha\mathbf{q} \ , \quad t > 0 \ . \end{aligned}$$

So, for positive times the particle's motion satisfies exactly the ALD equation, and a different one for negative times. In this connection we point out that a careful analysis of the usual deductions of Abraham–Lorentz–Dirac leads to the same conclusion. By the way the fact that different equations arise for negative and positive times is not a particular feature of the present model. A classical case where this happens is that of Boltzmann equation, and a simple and enlightening mechanical model where this phenomenon appears was given by Lamb at the beginning of the century (see [18]).

Now, an important problem remains open concerning the practical use of ALD equation. In order to determine a definite solution of the Abraham–Lorentz–Dirac equation, one has to assign initial data for position, velocity and acceleration of the particle. It is natural to choose the same initial particle's velocity and position as for the complete Maxwell–Lorentz system, while the problem of the determination of the initial acceleration is non trivial since the acceleration due to the Lorentz force is not defined in the pointlike limit. Usually the initial acceleration for the ALD equation is determined by imposing the so called nonrunaway condition, namely by choosing it in such a way that the corresponding solution has bounded acceleration as $t \to \infty$. We do not discuss here the problem of justifying such a procedure, since it would invove the discussion of a non runaway condition for the ML system.

PART II: Proof of the results

4. Solution of the Cauchy problem in the case of a rigid extended particle.

We state here our result concerning the solutions of the Cauchy problem in the case of a rigid extended particle. We will begin the study in the configuration space (2.3). We recall that, since ρ is spherically symmetric, $\hat{\rho}$ is spherically symmetric too, and moreover one has, $\hat{\rho}(\mathbf{k}) = \hat{\rho}(k) = \hat{\rho}^*(k)$, where $k = |\mathbf{k}|$, and the star denotes complex conjugation.

To begin with we remark that, due to the spherical symmetry of the form factor, there exists a subspace of $\mathcal{S}_*$ in which the dynamics of the field is decoupled from that of the particle. Indeed, the only interacting fields are the divergence free part of spherically symmetric fields. Precisely, consider in $\mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ the subset $\mathcal{F}_0$ of the fields depending only on the distance $x = |\mathbf{x}|$ of the point from the origin:

$$\mathcal{F}_0 := \left\{ \mathbf{A} \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3) \ : \ \mathbf{A}(\mathbf{x}) = \mathbf{A}(x) \right\} , \tag{4.1}$$

and define the set

$$\mathcal{F}_* := \Pi(\mathcal{F}_0) , \tag{4.2}$$

namely the set of the vector fields which are the divergence free part of some spherically symmetric (in the above sense) vector field. Then the following simple result, to be proven in sect. 5, holds

Proposition 4.1. *The configuration space $\mathcal{Q}_0$ (cf. eq. (2.3)) splits into two subspaces*

$$\mathcal{Q}_0 = \mathcal{Q} \oplus \mathcal{Q}_1 ,$$

with

$$\mathcal{Q} := \mathcal{F}_* \oplus \mathbb{R}^3 , \tag{4.3}$$

which are invariant for the dynamics of the Maxwell–Lorentz system (2.2). Moreover, $\mathcal{Q}$ and $\mathcal{Q}_1$ are mutually orthogonal with respect to the $L^2 \oplus \mathbb{R}^3$ metric, and the projection of (2.2) on $\mathcal{Q}_1$ is the free wave equation.

We thus confine our study of the Maxwell–Lorentz system to the configuration space $\mathcal{Q}$ where the nontrivial dynamics takes place.

In order to obtain a formula for the solution of the Cauchy problem we need to introduce a few more notations.

A fundamental role, is played by the electromagnetic mass m_{em} corresponding to a given charge distribution (which was defined by (2.5)). In addition, we introduce a function $m_1(\omega)$, having the dimension of a mass, by

$$m_1(\omega) := \frac{16}{3}\pi^2 \frac{e^2}{c^2}\omega \int_{-\infty}^{\infty} \frac{|\hat{\rho}(k)|^2}{\omega_k - \omega} dk \ ,$$

where $\omega_k = ck$, and the integral has to be interpreted as a principal value. Notice that $m_1(\omega)$ is proportional to the Hilbert transform of the real function $|\hat{\rho}(k)|^2$ (which is extended to $\mathbb{R}_-$ by symmetry) and that, in the case of a point–like particle, m_1 vanishes identically. We will denote by $m_{tot} = m_{tot}(\omega)$ the function

$$m_{tot}(\omega) := m_0 + m_{em} + m_1(\omega) \ . \tag{4.4}$$

which will play the role of total (renormalized) mass. Finally we introduce the following relevant function

$$C(\omega) := \frac{2}{\pi}\Omega^2 \tau 8\pi^3 \left|\hat{\rho}\left(\frac{\omega}{c}\right)\right|^2 \frac{\omega^2}{(\omega^2 - \Omega^2)^2 + \left(8\pi^3 \left|\hat{\rho}\left(\frac{\omega}{c}\right)\right|^2\right)^2 \omega^6 \tau^2} \ , \tag{4.5}$$

where

$$\Omega^2 = \Omega^2(\omega) := \frac{\alpha}{m_{tot}(\omega)} \ , \quad \tau := \frac{2}{3}\frac{e^2}{m_{tot}(\omega) c^3} \ .$$

Using these notations we can state the theorem concerning the solution of the Cauchy problem for the Maxwell–Lorentz system. In order to simplify its reading we point out that the solution of the equations of motion will be given as a superposition of simple harmonic oscillations corresponding to the normal modes of the system (for a previous application of this method to field theory see [19]); in the case of negative bare mass (which is needed in order to discuss the point–like limit) the solution contains a part with real exponentials which correspond to normal modes with imaginary frequencies (runaway modes). The distributions $\mathbf{A}^l_\omega$ and the functions $\mathbf{A}^l_r$ appearing in the forthcoming theorem are just the field components of the improper and of the proper eigenvectors of the linear operator describing the Maxwell–Lorentz system. Correspondingly $C(\omega)$ plays the role of a normalization constant for the improper eigenfunction corresponding to the frequency ω, while the constant C_r is the normalization constant for the runaway eigenvector.

Theorem 4.2. *Consider the Cauchy problem for system (2.2) with initial data $((\mathbf{A}_0, \mathbf{q}_0), (\dot{\mathbf{A}}_0, \dot{\mathbf{q}}_0)) \in \mathcal{Q} \times \mathcal{Q}$, and let $\mathbf{v}_l$, $l = 1, 2, 3$ be a fixed orthonormal basis of $\mathbb{R}^3$. Then there* *exists a family of distributions* $\{\mathbf{A}^l_\omega\}^{l=1,2,3}_{\omega\in[0,\infty)}$, $\mathbf{A}^l_\omega \in \mathcal{S}'$ *(the dual of* $\mathcal{S}$*) such that the solution of the Cauchy problem is given by*

$$
\begin{aligned}
\mathbf{q}(t) &:= \sum_{l=1,2,3} \mathbf{v}_l \int_0^\infty C(\omega)\left[-\xi^l_\omega \cos(\omega t) + \omega\eta^l_\omega \sin(\omega t)\right] d\omega + \theta(-m_0)\mathbf{q}_r(t) \ , \\
\mathbf{A}(t) &:= \sum_{l=1,2,3} \int_0^\infty \alpha C(\omega)\left[\eta^l_\omega \cos(\omega t) + \frac{\xi^l_\omega}{\omega}\sin(\omega t)\right]\mathbf{A}^l_\omega d\omega + \theta(-m_0)\mathbf{A}_r(t) \ ,
\end{aligned}
\tag{4.6}
$$

where θ *is the usual step function, while* $\mathbf{q}_r(t), \mathbf{A}_r(t)$*, which define the "runaway part" of the solution (which is present only in the case of negative bare mass), are given by*

$$
\begin{aligned}
\mathbf{q}_r(t) &:= \sum_{l=1,2,3} \mathbf{v}_l C_r\left[-\xi^l_r \mathrm{Ch}(\sqrt{-\lambda_r}t) - \sqrt{-\lambda_r}\eta^l_r \mathrm{Sh}(\sqrt{-\lambda_r}t)\right] \ , \\
\mathbf{A}_r(t) &:= \sum_{l=1,2,3} C_r\alpha\left[\eta^l_r \mathrm{Ch}(\sqrt{-\lambda_r}t) + \frac{\xi^l_r}{\sqrt{-\lambda_r}}\mathrm{Sh}(\sqrt{-\lambda_r}t)\right]\mathbf{A}^l_r \ ,
\end{aligned}
\tag{4.7}
$$

$\mathbf{A}^l_r$ *are functions of the space point* $\mathbf{x}$*, while* C_r *and* $\lambda_r < 0$ *are real constants depending on the form factor, on* m_0 *and on* e*. Moreover, the functions* $\xi^l_\omega, \eta^l_\omega$*, can be expressed in terms of the initial data by*

$$
\begin{aligned}
\xi^l_\omega &:= \frac{1}{4\pi c^2}\left\langle \mathbf{A}^l_\omega, \dot{\mathbf{A}}_0\right\rangle - \mathbf{q}_0\cdot\mathbf{v}_l \ , \\
\eta^l_\omega &:= \frac{1}{4\pi c^2}\left\langle \mathbf{A}^l_\omega, \mathbf{A}_0\right\rangle + \frac{m_0}{\alpha}\mathbf{v}_l\cdot\dot{\mathbf{q}}_0 + \frac{e}{c\alpha}\mathbf{v}_l\cdot\int_{\mathbb{R}^3}\rho(x)\mathbf{A}_0(x)d^3x \ ,
\end{aligned}
\tag{4.8}
$$

where $\langle .,.\rangle$ *denotes the pairing of a distribution with a smooth function, and* ξ^l_r, η^l_r *are given by the same expressions with* $\mathbf{A}^l_r$ *in place of* $\mathbf{A}^l_\omega$*. Explicitly* $\mathbf{A}^l_\omega$ *and* $\mathbf{A}^l_r$ *are given by eqs. (5.11) and (5.12) below.*

We point out that, in the case of negative bare mass ($m_0 < 0$), the above solution is the sum of a runaway part (4.7) and a "regular" part; in particular, the runaway part has a characteristic exponential dependence on time. This is compatible with energy conservation because the Hamiltonian is indefinite when $m_0 < 0$.

We remark that the field $\mathbf{X}_{\dot q}$ adapted to the velocity $\dot{\mathbf{q}}$ does not belong to our phase space, so in principle the formulas of the theorem do not apply to the case where $\mathbf{X}_{\dot q}$ is taken as initial an datum. However, it is easy to see that the above formulas define a unitary evolution operator on a larger space. Precisely, consider the family of Hilbert spaces $H^{\{s\}}_*$ defined as the completion of $\mathcal{F}_*$ (cf. (4.2)) in the norm $\left\| |\triangle|^{s/2} f\right\|_{L^2}$. By standard semigroup theory [13] it is immediate to show that the Cauchy problem for system (2.2) is well posed in each of the phase spaces

$$
\mathcal{Q}^{\{s\}} := \left(H^{\{s-1\}}_* \oplus \mathbb{R}^3\right) \oplus \left(H^{\{s\}}_* \oplus \mathbb{R}^3\right) \ni ((\dot{\mathbf{A}}_0, \dot{\mathbf{q}}_0), (\mathbf{A}_0, \mathbf{q}_0)) \ , \quad (s \geq 1)
$$

Moreover, remark that eqs. (4.6), (4.7), (4.8) describe the action of a linear continuous operator (the group action) on a dense subset of $\mathcal{Q}^{\{s\}}$, and therefore these formulas can be extended to the whole of $\mathcal{Q}^{\{s\}}$. The field $\mathbf{X}_{\dot q}$ belongs to $H^{\{1\}}_*$.

5. Proof of the results on the dynamics of an extended particle.

First we will prove proposition 4.1, and give a representation lemma for the fields of the "interacting subspace". Then we will come to the heart of the section, namely the proof of theorem 4.2, which is based on the application of the spectral resolution theorem for self–adjoint operators in Hilbert spaces. The first step consists in determining a suitable Hilbert space such that the Maxwell–Lorentz system turns out to be equivalent to the equation

$$\ddot{\zeta} + B\zeta = 0 \; , \tag{5.1}$$

where ζ is a point of the Hilbert space, and B a self–adjoint operator in it. Then we will calculate the spectrum and the eigenfunctions, of the operator B. Subsequently we will show that the singular continuous part of the spectrum of B is empty, and we will calculate the "normalization constants" for proper and improper eigenfunctions obtaining the spectral resolution of the identity. Finally we will write explicitly the solution of the Cauchy problem, obtaining theorem 4.2.

We recall that the Maxwell–Lorentz system (2.2) is Hamiltonian with Hamiltonian function given by

$$\begin{aligned} H = & \int_{\mathbb{R}^3} \left(2\pi c^2 \mathbf{E}^2(x) - \frac{1}{8\pi}\mathbf{A}(x)\cdot\triangle\mathbf{A}(x)\right) d^3x \\ & + \frac{1}{2m_0}\left(\mathbf{p} - \frac{e}{c}\int_{\mathbb{R}^3}\rho(x)\mathbf{A}(x)d^3x\right)^2 + \frac{1}{2}\alpha\mathbf{q}^2 \; , \end{aligned} \tag{5.2}$$

where $\mathbf{E} = \dot{\mathbf{A}}/(4\pi c^2)$ is the momentum conjugated to $\mathbf{A}$, and $\mathbf{p}$ is the momentum conjugated to $\mathbf{q}$. Using the above Hamiltonian formulation of the problem it is easy to obtain the

Proof of Proposition 4.1. The key point is that, denoting by

$$I(\mathbf{p},\mathbf{A}) := \left(\mathbf{p} - \frac{e}{c}\int_{\mathbb{R}^3}\rho(\mathbf{x})\mathbf{A}(\mathbf{x})d^3x\right)^2$$

the term of the Hamiltonian describing the interaction between field and particle, and by Π_* the orthogonal projector (in the L^2 metric) of $\mathcal{S}_*$ onto $\mathcal{F}_*$ it is easy to see that

$$I(\mathbf{p},\mathbf{A}) = I(\mathbf{p},\Pi_*(\mathbf{A})) \; . \tag{5.3}$$

The proof of the proposition follows from the remark that also the term in the first line of (5.2) does not mix the subspace $\mathcal{F}_*$ and its orthogonal complement. Thus the dynamics in the orthogonal complement of $\mathcal{F}_*$ is decoupled from the dynamics of the rest of the system. □

We give now some simple results which are useful in order to deal in practice with the space $\mathcal{F}_*$. First we fix the notation concerning the Fourier transform of the fields: we put

$$\mathbf{A}(\mathbf{x}) = \sum_{j=1,2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\hat{A}_j(\mathbf{k})\mathbf{e}_j(\mathbf{k})e^{i\mathbf{k}\mathbf{x}}d^3k \; ,$$

where, as usual $\mathbf{e}_j(\mathbf{k})$ are polarization vectors, which have the property of constituting together with the vector $\mathbf{k}$ an orthonormal basis of $\mathbb{R}^3$.

Lemma 5.1. *Let $\mathbf{A} \in \mathcal{F}_*$ be a field, and fix an orthonormal basis $\mathbf{v}_j$ of $\mathbb{R}^3$; correspondingly there exists a unique triple (A_c^1, A_c^2, A_c^3) of symmetric scalar functions such that*

$$\mathbf{A} = \sum_{i=1,2,3} \Pi(\mathbf{v}_i A_c^i) \ . \tag{5.4}$$

Proof. Since $\mathcal{F}_* = \Pi(\mathcal{F}_0)$, it is enough to show that Π is injective, *i.e.* that the only solution of the equation

$$\sum_i \Pi(\mathbf{v}_i A_c^i) = 0 \ , \tag{5.5}$$

is zero. Passing to the Fourier transform, and performing the angular integration the result is easily obtained.

□

Notice also that the Fourier components $\hat{A}_j$ of $\mathbf{A}$ are given by

$$\hat{A}_j(\mathbf{k}) = \sum_{i=1,2,3} [\mathbf{v}_i \cdot \mathbf{e}_j(\mathbf{k})] \hat{A}_c^i(k) \ , \tag{5.6}$$

where A_c^i are defined by (5.4). In the following we will always use the above notation: *i.e.* *A_c^i will always denote the unique functions such that (5.4) holds*; the symmetrical functions corresponding to a given field will always be denoted by eliminating the boldface from the symbol of the field, by adding a lower index c and by adding the upper index of the corresponding direction. Moreover, it will be often useful to extend the functions A_c^i to the whole real axis, which will be done by symmetry. We also fix the vectors $\mathbf{v}_l$. Finally we remark that one has

$$\langle \mathbf{A}; \mathbf{A}' \rangle_{L^2} = \frac{8}{3}\pi \sum_l \int_0^\infty k^2 \hat{A}_c^l(k) \hat{A}_c'^l(k) dk \ . \tag{5.7}$$

We turn now to the determination of a self adjoint operator B and of a Hilbert space such that (2.2) turns out to be equivalent to (5.1). To this end we exploit a Lagrangian structure of system (2.2), although not the trivial one. In fact Hamiltonian (5.2) comes (via Legendre transform) from a Lagrangian which is not suitable for our purposes; indeed, its kinetic energy is indefinite when $m_0 < 0$, and so it cannot be used properly as a metric for the configuration space (one might use the theory of spaces with indefinite metric, but this is quite cumbersome). So, we first perform the canonical coordinate transformation $\mathbf{q} = -\mathbf{P}$, $\mathbf{p} = \mathbf{Q}$, and then a Legendre transform. We thus obtain the Lagrangian

$$\mathcal{L}(\zeta, \dot{\zeta}) = \frac{1}{2} \left\langle \dot{\zeta}; \dot{\zeta} \right\rangle_{\mathcal{Q}} - \frac{1}{2} \langle \zeta; B\zeta \rangle_{\mathcal{Q}} \ ,$$

where $\zeta = (\mathbf{A}, \mathbf{Q})$, the scalar product $\langle .; . \rangle_{\mathcal{Q}}$ is defined by

$$\langle (\mathbf{A}, \mathbf{Q}); (\mathbf{A}', \mathbf{Q}') \rangle_{\mathcal{Q}} := \frac{1}{4\pi c^2} \int_{\mathbb{R}^3} \mathbf{A}(x) \cdot \mathbf{A}'(x) d^3x + \frac{1}{\alpha} \mathbf{Q} \cdot \mathbf{Q}' \ , \tag{5.8}$$

and the operator B is explicitly given by

$$B\zeta = \begin{cases} 4\pi c^2 \left\{ -\frac{e}{m_0 c}\Pi \left[\rho \left(\mathbf{Q} - \frac{e}{c} \int_{\mathbb{R}^3} \rho(x)\mathbf{A}(x) d^3x \right) \right] - \frac{1}{4\pi} \triangle \mathbf{A} \right\} \\ \frac{\alpha}{m_0} \left(\mathbf{Q} - \frac{e}{c} \int_{\mathbb{R}^3} \rho(x)\mathbf{A}(x) d^3x \right) . \end{cases} \tag{5.9}$$

We now complete the configuration space $\mathcal{Q}$ in the topology induced by the metric (5.8) obtaining a Hilbert space that will be denoted by $\bar{\mathcal{Q}}$, and we complexify it. It is now easy to see that the operator B is self adjoint and that the equations of motions are of the form (5.1). Moreover, such an operator satisfies the hypotheses of the spectral resolution theorem in the form by Gel'fand, namely in the form ensuring the completeness of the generalized eigenfunction expansion. Indeed, we have that $\mathcal{Q}$ is a nuclear space dense in $\bar{\mathcal{Q}}$, which is left invariant by B.

Concerning the spectrum of B we have the following

Lemma 5.2. *The continuous spectrum of the operator B coincides with $[0, \infty)$; and, for $\lambda = \omega^2$ in the continuous spectrum, the corresponding generalized eigenfunctions are given by $\zeta^l_\omega := (\mathbf{A}^l_\omega, \mathbf{v}_l)$. If $m_0 > 0$, this is the whole spectrum. If $m_0 < 0$, the spectrum of B contains also a proper eigenvalue λ_r, given by the unique negative solution of the equation*

$$m_0 + \frac{32}{3}\pi^2 \int_0^\infty e^2 |\hat{\rho}(k)|^2 \frac{k^2}{c^2k^2 - \lambda_r} dk = \frac{\alpha}{\lambda_r} \ ; \tag{5.10}$$

the corresponding eigenvectors are given by $\zeta^l_r := (\mathbf{A}^l_r, \mathbf{v}_l)$. Explicitely $\mathbf{A}^l_\omega$ and $\mathbf{A}^l_r$ are given by $\mathbf{A}^l_\omega = \mathbf{A}^{o,l}_\omega + \mathbf{A}^{e,l}_\omega$, where $\mathbf{A}^{e,l}_\omega$, $\mathbf{A}^{o,l}_\omega$ are distributions belonging to the dual of $\mathcal{F}_*$ defined by

$$\begin{aligned} \langle \mathbf{A}^{e,l}_\omega ; \mathbf{A} \rangle &:= \frac{16}{3}\pi^2 \frac{ec}{\alpha}\omega \int_{-\infty}^{\infty} \frac{k^2 \hat{A}^l_c(k)\hat{\rho}(k)}{\omega_k - \omega} dk \\ \langle \mathbf{A}^{o,l}_\omega ; \mathbf{A} \rangle &:= -\frac{c}{e\alpha\hat{\rho}\,(\omega/c)} (m_{tot}(\omega)\omega^2 - \alpha)\hat{A}^l_c \left(\frac{\omega}{c}\right) \ , \end{aligned} \tag{5.11}$$

(the integral is just a symbol for the Hilbert transform), and $\mathbf{A}^l_r$ is given, in terms of Fourier components, by

$$\hat{A}^l_{r,j}(\mathbf{k}) = \frac{4\pi ec\lambda_r}{\alpha} \frac{\hat{\rho}(k)[\mathbf{v}_l \cdot \mathbf{e}_j(\mathbf{k})]}{c^2k^2 - \lambda_r} \ . \tag{5.12}$$

The spectrum contains no other points.

Proof. Consider the eigenvalue equation for B

$$\begin{pmatrix} \lambda \mathbf{A} \\ \lambda \mathbf{Q} \end{pmatrix} = B \begin{pmatrix} \mathbf{A} \\ \mathbf{Q} \end{pmatrix} \ . \tag{5.13}$$

By using the second of these equations ($\lambda Q = ...$), passing to the Fourier transform, and to the symmetric fields $\hat{A}^i_c$ as in (5.6), one obtains that the first equation ($\lambda A = ...$) can be given the form

$$(\lambda - c^2k^2)\hat{A}^i_c(k) = -\frac{4\pi ec}{\alpha}\lambda\hat{\rho}(k)[\mathbf{Q} \cdot \mathbf{v}_i] \ ,$$

whose general solution in the distribution space is

$$\hat{A}^i_c(k) = 4\pi \frac{ec\lambda}{\alpha} \frac{\hat{\rho}(k)[\mathbf{Q}\cdot\mathbf{v}_i]}{\omega_k^2 - \lambda} + K^i(k)\delta(ck - \sqrt{\lambda}) \ , \tag{5.14}$$

where K^i are functions. We look now for negative eigenvalues corresponding to which the above δ function is always zero. Substituting (5.14) in the second equation (5.13), and performing the angular integration, one obtains equation (5.10), which has a solution only if m_0 is negative. So, if this is the case one has that λ_r is a proper eigenvalue, and $(\mathbf{A}^l_r, \mathbf{v}_l) \in \mathcal{Q}$ are the corresponding eigenfunctions. We turn now to the case $\lambda > 0$. In this case the r.h.s. of (5.14) has a singularity; so, in order to use it to define a distribution, we have to choose a prescription for the calculation of its integral with a regular function. In fact different prescriptions would lead to different values of the functions K^i. We choose here the prescription "calculate the integrals as Cauchy principal values". Substituting in the second of equations (5.13), performing the angular integration, and where possible performing the radial integration, one gets the equation

$$m_0 \frac{\omega^2}{\alpha}\mathbf{Q} = \mathbf{Q} - \mathbf{Q}\frac{32}{3}\pi^2 e^2 \frac{\omega^2}{\alpha}\int_0^\infty \frac{k^2 \left|\hat{\rho}\left(\frac{\omega}{c}\right)\right|^2}{\omega_k^2 - \omega^2} dk - \frac{8}{3}\pi \frac{\omega^2}{c^4} e \hat{\rho}^*\left(\frac{\omega}{c}\right) \sum_l K^l\left(\frac{\omega}{c}\right)\mathbf{v}_l \ ,$$

where we have put $\omega := \sqrt{\lambda}$. For any choice of $\mathbf{Q}$ this equation (for K^l) has a solution. Three independent solutions are obtained by choosing $\mathbf{Q} = \mathbf{v}_l$, $l = 1, 2, 3$, from which

$$K^l\left(\frac{\omega}{c}\right) = -\frac{m_{tot}(\omega)\omega^2 - \alpha}{\frac{8}{3}\pi \frac{\omega^2 \alpha}{c^4} \hat{\rho}^*\left(\frac{\omega}{c}\right) e} \ , \quad l = 1, 2, 3 \ ,$$

where $m_{tot}(\omega)$ is defined by (4.4). The thesis immediately follows.

□

Lemma 5.3. *The operator B describing the Maxwell–Lorentz system has no singular continuous spectrum.*

Proof. We will apply theorem XIII.20 of ref. [20], according to which, if there exists a $p > 1$, a positive δ and a dense subset D of $\bar{\mathcal{Q}}$ such that $\forall \zeta \in D$ one has

$$\sup_{0<\epsilon<\delta} \int_a^b |\mathrm{Im}\ \langle \zeta; R(x + i\epsilon)\zeta\rangle_{\mathcal{Q}}\,|^p dx < \infty \ , \tag{5.15}$$

where $R(\lambda) := (B - \lambda)^{-1}$ is the resolvent of B, then B has a purely absolutely continuous spectrum on (a, b).

So we calculate the resolvent of B by solving the equation $\zeta = (B - \lambda)\zeta_1$. Denoting $\zeta = (\mathbf{A}, \mathbf{Q})$, one obtains, for λ with non vanishing imaginary part,

$$\begin{aligned} \left\langle R(\lambda)\begin{pmatrix}\mathbf{A}\\ \mathbf{Q}\end{pmatrix}; \begin{pmatrix}\mathbf{A}\\ \mathbf{Q}\end{pmatrix}\right\rangle_{\mathcal{Q}} &= \frac{2}{3c^2}\sum_l \int_0^\infty \frac{k^2 \hat{A}^l_c(k)^2}{\omega_k^2 - \lambda} dk \\ &- \frac{64\lambda\pi^2 e^2}{9c^2(m_{tot}\lambda - \alpha)} \sum_l \left[\int_0^\infty \frac{k^2 \hat{A}^l_c(k)\hat{\rho}(k)}{\omega_k^2 - \lambda} dk\right]^2 \\ &- \frac{16}{3}\sum_l \frac{\pi e}{c}[\mathbf{Q}\cdot\mathbf{v}_l]\frac{1}{m_{tot}\lambda - \alpha}\int_0^\infty \frac{k^2 \hat{A}^l_c(k)\hat{\rho}(k)}{\omega_k^2 - \lambda} dk - \frac{1}{\alpha}\frac{m_{tot}\mathbf{Q}\cdot\mathbf{Q}}{m_{tot}\lambda - \alpha} \ , \end{aligned} \tag{5.16}$$

where

$$m_{tot} := m_0 + \frac{32}{3}\pi^2 e^2 \int_0^\infty \frac{k^2 \hat{\rho}(k)}{\omega_k^2 - \lambda} dk \ .$$

Let us analyze (5.16). We begin by the first term at the r.h.s.; it is an expression of the form

$$\int_0^\infty \frac{k^2 g(k)}{\omega_k^2 - \lambda} dk \ , \tag{5.17}$$

where g is a function of class $\mathcal{S}$, provided $A \in \mathcal{S}$, i.e. $\zeta \in \mathcal{Q}$ (which is dense in $\bar{\mathcal{Q}}$). Extending g to $(-\infty, 0)$ by symmetry, we have that (5.17) is proportional to

$$\int_{-\infty}^\infty \frac{k g(k)}{\omega_k - \sqrt{\lambda}} dk \ . \tag{5.18}$$

The above function of λ is exactly the composition of the function

$$\tilde{g}(z) := \int_{-\infty}^\infty \frac{k g(k)}{\omega_k - z} dk \ , \tag{5.19}$$

with the function square root. Moreover, (5.19) is just the complex extension of the Hilbert transform of $kg(k)$, and it is well known that, if $\|kg(k)\|_{L^p} < \infty$, then one has (see e.g. [21])

$$\sup_{0 \le \epsilon < \delta} \int_{-\infty}^\infty |\tilde{g}(x + i\epsilon)|^p dx \le C \, \|kg(k)\|_{L^p}^p \ , \quad \forall p > 1 \ ,$$

for some positive C (recall that the Hilbert transform is an operator of type (p,p) for any $p > 1$). From this, it follows immediately that the (supremum for $0 < \operatorname{Im}\lambda < \delta$ of the) integral of the $p-$th power ($p > 1$) of this term over any finite interval is finite.

We come to the other terms of (5.16). We have to show that, as functions of λ, also all these terms satisfy the above property. First we consider the quantity $m_{tot}\lambda - \alpha$ appearing at the denominator of all these terms, and prove that it is an analytic function of λ. We recall that the Fourier transform of an L^1 function which decays exponentially at infinity is analytical; viceversa, if a real analytic function (over the whole real space) is L^1 then its Fourier transform decays exponentially at infinity. From this it is easy to conclude that $\hat{\rho}^2$ is an analytic function, and therefore also its Hilbert transform is. It follows that $m_1(\omega)$ is analytic, and so that also $m_{tot}\lambda - \alpha$ is analytic as a function of λ, provided $|\operatorname{Im}\lambda|$ is small enough.

¿From this we have that the zeroes of the above denominator cannot have accumulation points. Denote such zeros by $\lambda_1, ..., \lambda_n, ...$, fix a positive ϵ, denote by B_i the closed ball of radius ϵ and center λ_i. In the complementary of the union of these balls the above denominator is bounded. So, in this set we can repeat the argument used for the first term of (5.16) and deduce that if the operator B has a singular continuous spectrum, then such a singular continuous spectrum is contained in the union of the above balls. However, this has to hold for any ϵ, so the singular continuous spectrum has to be concentrated in the points λ_i. Moreover, these points are isolated, so the singular continuous spectrum is empty. By the way, we remark that the appearance of the quantity $m_{tot}\lambda - \alpha$ at the denominator of the resolvent could induce to think that the points λ_i belong to the point spectrum of B. This is not true. In fact a careful analysis of the calculation of the resolvent shows that in correspondence to these points the resolvent exists, but has a formal expression different from that used in (5.16); moreover, its domain is dense, but does not contain $\mathcal{Q}$.

□

Next we calculate the functions $C^l(\omega)$ such that, for regular enough f one has

$$\sum_l \langle \zeta_\nu^n, \int_0^\infty f^l(\omega)\zeta_\omega^l d\omega\rangle_{\mathcal{Q}} = \frac{f^n(\nu)}{C^n(\nu)} \ . \tag{5.20}$$

These quantities exist due to the spectral resolution theorem. It is very useful to remark that the l.h.s. of this equation can be interpreted as the definition of the family of distributions (labelled by ν) $\left\langle \zeta_\nu^n ; \zeta_\omega^l \right\rangle_{\mathcal{Q}}$:

$$\int_0^\infty \left\langle \zeta_\nu^n ; \zeta_\omega^l \right\rangle_{\mathcal{Q}} f(\omega) d\omega := \left\langle \zeta_\nu^n ; \int_0^\infty \zeta_\omega^l f(\omega) d\omega \right\rangle_{\mathcal{Q}} \ ,$$

and, with this definition eq. (5.20) can be rewritten in the form

$$\left\langle \zeta_\nu^n ; \zeta_\omega^l \right\rangle_{\mathcal{Q}} = \frac{\delta(\nu - \omega)}{C^n(\omega)} \delta_{l,n} \ . \tag{5.21}$$

In particular, this relation states that the distribution on the l.h.s. is singular.

Lemma 5.4. *The functions $C^n(\omega)$ of equation (5.20) coincide with $\alpha C(\omega)$ where $C(\omega)$ is the function defined by equation (4.5).*

Proof. We take $f^l(\omega) = f(\omega)\delta_{l,n}$, with n fixed, so that, taking into account the structure of the pairing $\langle .; . \rangle_{\mathcal{Q}}$ and of the eigenfunctions A_ω^l, the l.h.s. of (5.20) can be given the form

$$\begin{aligned}
&\frac{\delta_{m,n}}{\alpha} \int_0^\infty f(\omega) d\omega + \frac{1}{4\pi c^2} \left\langle \mathbf{A}_\nu^{e,n} ; \int_0^\infty f(\omega) \mathbf{A}_\omega^{e,m} d\omega \right\rangle_{L^2} \\
&+ \frac{1}{4\pi c^2} \left\langle \mathbf{A}_\nu^{e,n} ; \int_0^\infty f(\omega) \mathbf{A}_\omega^{o,m} d\omega \right\rangle_{L^2} + \frac{1}{4\pi c^2} \left\langle \mathbf{A}_\nu^{o,n} ; \int_0^\infty f(\omega) \mathbf{A}_\omega^{e,m} d\omega \right\rangle_{L^2} \\
&+ \frac{1}{4\pi c^2} \left\langle \mathbf{A}_\nu^{o,n} ; \int_0^\infty f(\omega) \mathbf{A}_\omega^{o,m} d\omega \right\rangle_{L^2} \ .
\end{aligned} \tag{5.22}$$

In the same way as in (5.21) one can interpret all integrals in (5.22) as defining some distributions. We know by the general theorem stating eq. (5.21) (but this can also be verified explicitly) that the regular parts of these distributions cancel each other. So we will calculate just their singular part, and in the whole following calculations we shall retain only terms contributing to this singular part.

We begin by calculating the singular part of $\left\langle \mathbf{A}_\omega^{o,l} ; \mathbf{A}_\nu^{o,n} \right\rangle_{L^2}$. It is easy to see that

$$\frac{1}{4\pi c^2} \left\langle \mathbf{A}_\omega^{o,l} ; \mathbf{A}_\nu^{o,n} \right\rangle_{L^2} = \frac{\left[m_{tot}(\omega)\omega^2 - \alpha \right]^2}{\left| \hat{\rho}\left(\frac{\omega}{c}\right) \right|^2 e^2 \frac{32}{3} \pi^2 \frac{\omega^2}{c^3} \alpha^2} \delta(\omega - \nu) \delta_{l,n} \ .$$

With a similar calculation it is easily proved that the singular part of $\left\langle \mathbf{A}_\omega^{e,l}; \mathbf{A}_\nu^{o,n} \right\rangle_{L^2}$ vanishes. We calculate now the singular part of $\left\langle \mathbf{A}_\omega^{e,l}; \mathbf{A}_\nu^{e,n} \right\rangle_{L^2}$. Extending the functions f and $\hat{\rho}$ to $(-\infty, 0)$ by symmetry, we have

$$\begin{aligned} &\frac{1}{4\pi c^2} \int_0^\infty \left\langle \mathbf{A}_\omega^{e,l}; \mathbf{A}_\nu^{e,l} \right\rangle_{L^2} f(\nu) d\nu = \\ &\frac{8}{3}\pi^2 \frac{e^2}{\alpha^2} \frac{\omega^2}{c^3} \int_{-\infty}^\infty \nu^2 f(\nu) \int_{-\infty}^\infty \omega_k^2 d\omega_k \frac{|\hat{\rho}(k)|^2}{(\omega_k^2 - \omega^2)(\omega_k^2 - \nu^2)} \ . \end{aligned} \tag{5.23}$$

We then calculate the integral. Splitting the denominator, exploiting the symmetries of the arguments, by simple manipulations one obtains that the integral at the r.h.s. of (5.23) coincides with

$$\begin{aligned} &\frac{1}{2} \int_{-\infty}^\infty d\nu \int_{-\infty}^\infty d\omega_k \frac{\omega_k}{\omega_k - \omega} f(\nu) \left| \hat{\rho}(k) \right|^2 \\ +&\frac{1}{2} \int_{-\infty}^\infty d\nu \int_{-\infty}^\infty d\omega_k \frac{\omega_k}{\omega_k - \omega} \frac{\omega_k}{\nu - \omega_k} f(\nu) \left| \hat{\rho}(k) \right|^2 \\ +&\frac{1}{2} \int_{-\infty}^\infty d\nu \int_{-\infty}^\infty d\omega_k \frac{\omega_k}{\omega_k + \omega} f(\nu) \left| \hat{\rho}(k) \right|^2 \\ +&\frac{1}{2} \int_{-\infty}^\infty d\nu \int_{-\infty}^\infty d\omega_k \frac{\omega_k}{\omega_k + \omega} \frac{\omega_k}{\nu - \omega_k} f(\nu) \left| \hat{\rho}(k) \right|^2 \ . \end{aligned} \tag{5.24}$$

The first and the third of these integrals have the form of a regular function of ω (the first one is proportional to the Hilbert transform of $\omega_k |\hat{\rho}(k)|^2$) multiplied by $f(\nu)$ and integrated over ν, so they do not contribute to the singular part of our distribution, and we just forget about them. The calculation of the remaining two integrals can be obtained by introducing the Hilbert transform f, exploiting the symmetries of the various terms, isolating the part of the integrals contributing to the singular part of the distributions, and exploiting the fact that the square of the Hilbert transform is the identity. One thus obtains

$$\frac{1}{4\pi c^2} \left\langle \mathbf{A}_\omega^{e,l}; \mathbf{A}_\nu^{e,l} \right\rangle_{L^2} = \frac{8}{3}\pi^4 \frac{e^2}{\alpha^2} \frac{\omega^4}{c^3} \left| \hat{\rho}\left(\frac{\omega}{c}\right) \right|^2 \delta(\omega - \nu) \ .$$

¿From this, using (5.22) and the definitions of Ω and τ, the thesis easily follows.

□

Proof of theorem 4.2 First notice that, by the results of lemmas 5.2, 5.3, 5.4, using also the spectral resolution theorem, any $\zeta \in \mathcal{Q}$ can be written as

$$\zeta = \sum_l \left[\int_0^\infty \left\langle \zeta; \zeta_\omega^l \right\rangle_{\mathcal{Q}} \zeta_\omega^l \alpha C(\omega) d\omega + \left\langle \zeta; \zeta_r^l \right\rangle_{\mathcal{Q}} \zeta_r^l \alpha C_r \right] \ ,$$

where ζ_r^l are the runaway eigenvectors, and

$$C_r := (\left\| \zeta_r^l \right\|^2 \alpha)^{-1} \tag{5.25}$$

(here and in the definition of $C(\omega)$ the division by α is introduced for future convenience).

Then, using Stone theorem and the relations

$$\mathbf{Q} = \mathbf{p} = m_0\dot{\mathbf{q}} + \frac{e}{c}\int_{\mathbb{R}^3} \rho(\mathbf{x})\mathbf{A}(\mathbf{x})d^3x \ ; \quad -\alpha\mathbf{q} = \alpha\mathbf{P} = \dot{\mathbf{Q}} \ .$$

in order to go back to the original coordinates one obtains theorem 4.2.

□

6. Proof of the Results Concerning the Point–like Limit.

We will denote by $C_a(\omega)$ the function $C(\omega)$ (cf. (4.5)) corresponding to the charge distribution (2.4), and we notice that

$$C_a(\omega) = \frac{2}{\pi}\frac{8\pi^3\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2\omega^2}{\frac{1}{\gamma}\left(\frac{m_{tot}}{\alpha}\omega^2 - 1\right)^2 + \gamma\left(8\pi^3\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2\right)^2\omega^6} \ , \tag{6.1}$$

where $\gamma = 2e^2/(3\alpha c^3)$.

The nonrunaway part of the particle's motion corresponding to $\dot{\mathbf{q}}_0 = \mathbf{A}_0 = \dot{\mathbf{A}}_0 = 0$ is given by

$$\mathbf{q}_0 \int_0^\infty C_a(\omega)\cos(\omega t)d\omega. \tag{6.2}$$

We begin by the

Proof of proposition 2.1. Notice that the Fourier transform of the motion $\mathbf{q}_a(t)$ is given by the function $C_a(\omega)$, and recall that weak convergence of $\mathbf{q}_a$ is equivalent to weak convergence of $C_a(\omega)$. From (6.1) one has that, as $a \to 0$ with fixed m_0, $C_a(\omega) \to 0$ pointwise, uniformly on $[\epsilon, \infty)$, and therefore it tends to a (possibly vanishing) singular distribution concentrated at the origin $\omega = 0$. So the limit, which is assumed to exist, is a finite linear combination of the delta and of its derivatives. Taking the Fourier transform one has

$$\mathbf{q}(t) = \mathbf{q}_0\sum_{k=0}^{n} C_k t^k \ .$$

By conservation of energy one has the uniform estimate

$$\frac{1}{2}m_0\dot{q_a}^2(t) + \frac{1}{2}\alpha q_a^2(t) \le \frac{1}{2}\alpha q_0^2 \ ,$$

from which $C_k = 0 \ \forall k \ge 1$.

□

We come now to the proof of proposition 2.2. First we prove the convergence of the runaway part of the solution. This follows from the following

Lemma 6.1. *When $a \to 0$ the negative eigenvalue λ_r of the Maxwell–Lorentz system converges to*

$$\lambda_r = -\nu_r^2 \ ,$$

where ν_r is the only real solution of equation (3.1). The constant C_r then converges to

$$C_r^0 := \left(\frac{3}{2} + \frac{\nu_r^2}{2\omega_0^2} \right)^{-1}$$

Proof. The equation defining λ_r can be given the form

$$m + \frac{32}{3}\pi^2 \frac{e^2}{c^3} \lambda_r \int_0^\infty \frac{|\hat{\mathcal{D}}(ak)|^2}{\omega_k^2 - \lambda_r} dk = \frac{\alpha}{\lambda_r} \ , \tag{6.3}$$

which is of the form

$$m + f(a, \lambda_r) = \frac{\alpha}{\lambda_r} \ ,$$

where f is a regular function of $a \in [0, \infty)$, and λ_r. Therefore, by the implicit function theorem λ_r depends regularly on a, so the limiting value of λ_r is just obtained by solving the limiting equation. So, one obtains eq. (3.1). The proof of the convergence of C_r is very similar and is omitted.

□

The convergence (as $a \to 0$) of the non–runaway part is an immediate consequence of the following

Lemma 6.2. *Consider the function $C_a(\omega)$ cf. (6.1), with m_0 given by (2.6). There exist positive constants $\bar{a}$, M, K_1, K_2, $\bar{\omega}$ such that the following estimate holds*

$$C_a(\omega) \le g(\omega) \ , \forall 0 \le a \le \bar{a} \ ,$$

where $g(\omega)$ is defined by

$$g(\omega) := \begin{cases} M & \omega \in [0, \bar{\omega}] \\ \max\left\{ \dfrac{1}{(K_1\omega^2 - \alpha)\omega}, \dfrac{1}{K_2\omega^4} \right\} & \omega > \bar{\omega} \end{cases} ,$$

Moreover one has $g \in L^1(\mathbb{R}^+)$

Proof. To obtain the estimate of $C_a(\omega)$ in the domain $\omega \le \bar{\omega}$ it is enough to show that its denominator is uniformly bounded away from zero in this domain. To this end notice that $\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2$ does not vanish in this domain, so that the second term in the denominator of $C_a(\omega)$ vanishes only at $\omega = 0$, while for ω in a fixed neighbourhood of the origin the first term of this denominator is uniformly bounded away from zero (notice that $m_1(\omega) \le K\omega$ for suitable K and ω in a neighbourhood of the origin).

Consider now the set $\omega > \bar{\omega}$. Using the standard inequality $\gamma a^2 + b^2/\gamma \ge 2ab$, one obtains

$$C_a(\omega) \le \frac{2}{\pi} \frac{1}{(m_{tot}\omega^2 - \alpha)\omega} \ ,$$

and, provided m_{tot} is uniformly bounded away from zero,

$$C_a(\omega) \le \frac{2}{\pi} \frac{1}{(K_1\omega^2 - \alpha)\omega} \ . \tag{6.4}$$

Moreover, if $\bar{\omega}$ is large enough this is integrable. However, $m_{tot}(\omega)$ generally has zeroes which tend to infinity as $a \to 0$, so that the above inequality is not true in $(\bar{\omega}, \infty)$ and more work is needed. We will use (6.4) in a domain where m_{tot} is uniformly bounded away from zero, and we will show that in the complementary domain the quantity $\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2$ does not vanish, so that here one can use the inequality

$$C_a(\omega) \le \frac{2}{\pi} \frac{1}{\gamma 8\pi^3 \left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2 \omega^4} \ .$$

To make this precise we first make the change of variables $(\omega, a) \mapsto (b, a)$, with $b = \omega a/c$ and define the set

$$S := \left\{(a, b) \in [0, \bar{a}] \times \mathbb{R}^+ \ : \ m_{tot}(b, a) = 0\right\} \ ;$$

then we consider a closed neighbourhood depending on a small parameter ϵ of this set, defined by

$$S_\epsilon := \bigcup_{(a,b)\in S} B_\epsilon(a, b) \ ,$$

where

$$B_\epsilon(a, b) := \left\{\tilde{a}, \tilde{b} \ : \ |a - \tilde{a}| \le \epsilon \quad |\tilde{b} - b| \le \epsilon\right\} \ .$$

Since (as is easily seen) $m_1(b, a)$ is a regular function of its variables, in $(S_\epsilon)^c$ one has

$$m_{tot}(b, a) \ge k_1 > 0 \ ,$$

for some k_1, so that going back to the original variables one obtains

$$|m_{tot}(\omega)\omega^2 - \alpha| \ge \left||k_1\omega^2| - \alpha\right| \ ,$$

which, provided $\bar{\omega} > \sqrt{\alpha/k_1}$, is uniformly away from zero, so (6.4) holds on this set.

Consider now S_ϵ. We show that there exists a positive K_3 such that, in this set, one has

$$\hat{\rho}_a\left(\frac{\omega}{c}\right) = \hat{\mathcal{D}}(b) \ge K_3 \ .$$

To this end we have to discuss in some detail the structure of this set. The equation which defines S is

$$m_1(b,a) = -m \ . \tag{6.5}$$

Now, the quantity $m_1(b,a)$ is proportional (through inessential constants) to $b[H(|\hat{\mathcal{D}}|^2)](b)/a$, where $H(|\hat{\mathcal{D}}|^2)$ denotes the Hilbert transform of the function $|\hat{\mathcal{D}}^2|$. So, equation (6.5) has the form

$$K_4 b H(|\hat{\mathcal{D}}|^2)(b) = -\frac{ma}{c} \ , \tag{6.6}$$

where K_4 is a positive constant. Using the formula

$$[H(xf)](y) = y[H(f)](y) + \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)dx \ ,$$

which holds for any regular f, and the regularity of $|\hat{\mathcal{D}}|^2$, it is easy to see that the Hilbert transform of this function tends to zero exactly as $1/b$ when $b \to \infty$; it follows that the l.h.s. of (6.6) tends to a finite strictly negative limit (obviously independent of a) as $b \to \infty$. Then, there exists $\bar{b}$ and $\bar{a}$ such that for any $a < \bar{a}$ all solutions of (6.6) are in the region $b \leq \bar{b}$. It follows that $\hat{\rho}_a(\omega/c) = \hat{\mathcal{D}}(b)$, which is a function of b vanishing only at infinity, is uniformly and strictly positive in $\mathcal{S}$, and therefore also in S_ϵ, provided ϵ is small enough. □

¿From this lemma, exploiting Lebesgue dominated convergence theorem, one obtains that the particle's motion converges in $C^1[-T,T]$ (as $a \to 0$) to

$$\mathbf{q}(t) := \mathbf{q}_0 \frac{2}{\pi}\omega_0^2\tau_0 \int_0^\infty \frac{\omega^2\cos(\omega t)}{(\omega^2-\omega_0^2)^2+\omega^6\tau_0^2} d\omega + \mathbf{q}_0 C_r^0 \mathrm{Ch}(\nu_r t) \ . \tag{6.7}$$

Concerning the case of non–vanishing initial velocity, first we remark that proposition 2.3 is a simple consenquence of the fact that the coefficients η_ω^l diverge (due to mass renormalization) as $a \to 0$. For the proof of proposition 2.4 one has to calculate the coefficients η_ω^l and η_r^l corresponding to the initial data $\dot{\mathbf{q}}_0$, $\mathbf{A}_0 = \mathbf{X}_{\dot{q}_0}$. This gives

$$\eta_\omega^l = \frac{\dot{\mathbf{q}}_0 \cdot \mathbf{v}_l}{\omega^2} \ , \quad \eta_r^l = \frac{\dot{\mathbf{q}}_0 \cdot \mathbf{v}_l}{\lambda_r} \ . \tag{6.8}$$

It is then a simple variant of the proof of eq. (6.7) to show that corresponding to the initial data considered in proposition 2.4, the limit of the particle's motion is

$$\begin{aligned} \mathbf{q}(t) := {} & \frac{2}{\pi}\omega_0^2\tau_0 \int_0^\infty \omega^2 \frac{\mathbf{q}_0\cos(\omega t) + (\dot{\mathbf{q}}_0/\omega)\sin(\omega t)}{(\omega^2-\omega_0^2)^2+\omega^6\tau_0^2} d\omega \\ & + C_r^0[\mathbf{q}_0 \mathrm{Ch}(\nu_r t) + \frac{\dot{\mathbf{q}}_0}{\nu_r}\mathrm{Sh}(\nu_r t)] \ . \end{aligned} \tag{6.9}$$

Proof theorem 3.1. One has to calculate the integral (6.7). Denoting by $P(\omega)$ the denominator of the argument of the integral one has

$$P(\omega) = (\omega^2 - \omega_0^2 + i\tau_0\omega^3)(\omega^2 - \omega_0^2 - i\tau_0\omega^3) \ ,$$

so that the roots with positive imaginary part of $P(\omega)$ are given by $\omega_1 = i\nu_r$, $\omega_2 = \nu_2 + i\nu_3$, $\omega_3 = -\nu_2 + i\nu_3$. For the explicit calculation of the integrals one has to compute the residues of their arguments in the poles ω_1, ω_2, ω_3. A long but straightforward calculation allows to calculate all the residues and to put the various expressions in real form obtaining that the integral in (6.7) is given by

$$e^{-\nu_3|t|}[\mathcal{A}_1 \cos(\nu_2|t|) + \mathcal{A}_2 \sin(\nu_2|t|)] + \mathcal{A}_4 e^{-\nu_r|t|} \ . \tag{6.10}$$

The explicit form of the constants $\mathcal{A}_1$ and $\mathcal{A}_2$ is not important here (except for the asymptotics (3.3)), while $\mathcal{A}_4$ is given by

$$\mathcal{A}_4 = \frac{\omega_0^2}{\tau_0} \frac{1}{\nu^4 + 4\nu_2^2\nu_3^2} [\dot{\mathbf{q}}_0 - \mathbf{q}_0 \nu_r] \ ,$$

where $\nu^2 := \nu_2^2 - \nu_3^2 + \nu_r^2$. Then, one has to show that the term proportional to $\mathcal{A}_4$ in (6.10) exactly cancels the decreasing exponential of the hyperbolic sine and cosine present in (6.9). This is a quite complicated calculation which makes use of some known identities among the roots of an equation of third order. Precisely the identities needed are

$$\nu_2^2 + \nu_3^2 - 2\nu_r\nu_3 = 0 \ ,$$
$$(\nu_2^2 + \nu_3^2)\nu_r = \frac{\omega_0^2}{\tau_0} \ .$$

The explicit calculation is omitted.

□

Due to the linearity of the problem the proof of theorem 2.5 requires only the study of the limit of motions with initial data of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (0, 0, \mathbf{A}'_0, \dot{\mathbf{A}}_0)$ with $(\mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3)$. This is a simple variant of the proof of equation (6.7) and of theorem 3.1, and therefore is omitted.

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