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The Conformal Spin and Statistics Theorem

Daniele Guido[1*] and Roberto Longo[1,2*]

(1) Dipartimento di Matematica, Università di Roma "Tor Vergata"
via della Ricerca Scientifica, I–00133 Roma, Italia.

(2) Centro Linceo Interdisciplinare, Accademia Nazionale dei Lincei
via della Lungara 10, I–00165 Roma, Italia

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E-mail: guido@mat.utovrm.it, longo@mat.utovrm.it

Introduction

During the recent years Conformal Quantum Field Theory has become a widely studied topic, especially on a low dimensional space-time, because of physical motivations such as the desire of a better understanding of two-dimensional critical phenomena, and also for its rich mathematical structure providing remarkable connections with different areas such as Hopf algebras, low dimensional topology, knot invariants, subfactors among many others.

The Operator Algebra approach furnishes a powerful tool of investigation in this context, not only because it naturally leads to a model independent and intrinsic analysis, focusing on essential aspects such as the relative position of the local von Neumann algebras, but also because it makes visible otherwise hidden natural structures bringing to results inaccessible by different methods.

Two examples of this kind, the geometric description of the Tomita-Takesaki modular structure of the local von Neumann algebras [1,18,4], and the connection of the statistics of a superselection sector with the Jones index theory of subfactors [20], will play a fundamental role in this paper. These methods are present and important in general Quantum Field Theory, but provide an even richer structure in the low-dimensional case, conformal theories on S^1 in particular.

In the early seventies Doplicher, Haag and Roberts [7,8] developed a theory of superselection sectors, in the sense of [31], in the algebraic framework proposed by Haag and Kastler [17] starting from first principles. They described a superselection sector by a localized endomorphism ρ of the C^*-algebra generated by the local observable von Neumann algebras on the usual Minkowski space. In particular they showed that the statistics of ρ, a representation of the permutation group, is intrinsically encoded in ρ and classified it by an associated statistical parameter λ_ρ.

It was more recently realized that in the low dimensional case the statistics becomes a representation of Artin braid group. By applying generalized DHR methods, a first analysis in this case was given in [20,11]. In the simplest cases (small index or few channels) the statistics parameter classifies the braid group statistics by the Jones polynomial invariant for knots and links and its generalizations, see [21,23].

A key point in the analysis of superselection sectors is the index-statistics theorem [20] showing that, in any space-time dimension,

$$\mathrm{Ind}(\rho) = d(\rho)^2$$

where $\mathrm{Ind}(\rho)$ is the minimal index of ρ, an extension of the Jones index [19], and $d(\rho) := |\lambda_\rho|^{-1}$ is the DHR statistical dimension of ρ. We refer to [23] for a survey and for references on the index theory for infinite factors, but we recall that the square root of the minimal index of an endomorphism of a factor has the meaning of a dimension, that find an identification in this context by the above equation.

On the other hand important informations on the statistics are also contained in the statistics phase $\kappa_\rho := \lambda_\rho/|\lambda_\rho|$ of ρ: on the 4-dimensional space-time $\kappa_\rho = \pm 1$, a sign labeling the fundamental Fermi-Bose alternative. Therefore it is natural to look at a counterpart of the index-statistics relation for the statistics phase.

Based on the classical spin-statistics connection (see [28]), one may easily conjecture that in a conformal theory on S^1 the statistics phase has to agree with the univalence of the sector ρ

$$s_\rho = \kappa_\rho$$

where $s_\rho := e^{2\pi i L_\rho}$ (L_ρ the conformal spin, the lowest eigenvalue of the conformal Hamiltonian) is a label for the central extension associated with the occurring projective representation of the Möbius group $PSL(2, \mathbf{R})$.

Attempts to prove this relation have been made in particular by Fredenhagen, Rehren and Schroer [12] and, in the related $2+1$-dimensional context, by Fröhlich, Gabbiani and Marchetti [13]. Starting with assumptions on the existence of a global conjugate charge and of complete reducibility, they obtained a spin summation rule, which implies the equality up to a sign $s_\rho = \pm\kappa_\rho$. But the conformal spin-statistics theorem remained unproven unless adding ad hoc undesirable assumptions. Based on different ideas, this paper will show how the full strength of Operator Algebras provides the general and intrinsic spin and statistics relation, namely the equality $s_\rho = \kappa_\rho$. We deal with conformal theories on S^1 (one-dimensional components of two-dimensional chiral conformal theories) and base our analysis only on first principles: isotony of the local von Neumann algebras, locality, conformal invariance with positive energy, existence of the vacuum. We thus obtain the complete relation

$$\text{statistics parameter} = \frac{\text{univalence}}{\sqrt{\text{minimal index}}} .$$

Note that κ_ρ has a local nature while s_ρ is a global invariant. This is reminiscent of familiar situations in Geometry and suggests that extensions of our result to more general (curved) space-times should reveal further geometrical aspects. Our theorem is not only a prototype for further generalizations, but it already provides a number of immediate extensions or variants, like for the case of topological charges on a $2+1$-dimensional space-time [5]. This is due to the fact that we shall use the conformal invariance only indirectly, not in an essential way. For convenience we shall discuss these aspects together with related points and examples in a separate paper.

Our paper follows a previous work [15] where we reconsidered the classical spin and statistics theorem in Quantum Field Theory [28] and derived it in the algebraic setting assuming the "modular covariance property", namely the geometric meaning of the modular groups of the von Neumann algebras associated with wedge regions, consistently to the Bisognano-Wichmann theorem. That work, not directly extendible to the lower dimensional case due to the occurrence of the braid group statistics, focused however on the role played by the modular covariance property. The latter was shown to hold in conformal field theory on general grounds [4,14], and set thus the basis for the present analysis. Ultimalety only the geometric description of the modular conjugations is essential in our analysis.

We now pass to the description of the more specific content of this paper. In Section 1 we recall the basic properties shared by the local von Neumann algebras $\mathcal{A}(I)$ associated with intervals I of S^1.

Like in the classical case, the spin-statistics relation is strictly tied up with the PCT symmetry. Section 2 is indeed devoted to the construction of a global conjugate charge for a superselection sector ρ with finite statistics, a key point relevant in itself, previously an assumption in the related literature. As shown in [15], the sector

$$\bar{\rho} := j \cdot \rho \cdot j$$

is locally a conjugate of ρ in the sense that if ρ is an endomorphism localized in an interval I_0 and j is the adjoint (geometric) action given by the modular conjugation of an interval, one has the identity

$$\bar{\rho}|_{\mathcal{A}(I)} = \overline{\rho|_{\mathcal{A}(I)}}$$

where I is any interval containing I_0 and its reflection by j; the bar on the right hand side denotes the conjugate endomorphism in the sense of the sectors of the factor $\mathcal{A}(I)$ [21], a framework equivalent to the setting of the correspondences of Connes. In the irreducible case $\bar{\rho}$ is characterized by by the existence of an isometry $V_I \in \mathcal{A}(I)$ that intertwines the identity and $\bar{\rho}\rho|_{\mathcal{A}(I)}$. But the problem remained whether there is a global intertwiner V independent of I. We solve this problem positively by using an argument inspired by the "vanishing of the matrix coefficient theorem" for connected simple Lie groups, see the Appendix B.

We prove in fact the equivalence between the local and the global intertwiners for superselection sectors with finite index, namely the embedding into the sectors (endomorphisms modulo inners) of the factor $M := \mathcal{A}(I)$ determined, via the restriction map,

$$\text{Superselection sectors} \to \text{Sect}(M)$$

corresponds by the index-statistics theorem to a faithful functor of tensor C^*-categories with conjugates which is full (no new intertwiner arises in the range). This implies that the fusion rules of the superselection sectors are entirely described by the theory of subfactors.

As a first consequence we shall see in Section 3 that the (internal) intertwiner property of the above isometry V is equivalent to the (spatial) property of being the standard implementation of ρ, according to Araki, Connes and Haagerup, see Appendix A, with respect to the vacuum vector. To extract information from this fact we localize ρ in the upper-right quarter-circle and consider the standard implementations V_1 and V_2 of ρ as an endomorphism of the upper and of the right semicircle von Neumann algebra respectively and observe that

$$\mu_\rho := V_1^* V_2^* V_1 V_2 \tag{3.1}$$

is a scalar invariant for ρ that reflects by both analytic-algebraic and geometric aspects. It is indeed natural to look at μ_ρ as a generalized multiplicative commutator of local intertwiners, in the spirit of the statistics, and identify it with the statistics parameter λ_ρ, or as an invariant obtained by reversing the orientation, in the spirit of the spin, and identify μ_ρ with $\mathrm{Ind}(\rho)^{-\frac{1}{2}}$–times the univalence of ρ.

In more detail we shall obtain the spin-statistics relation by "squaring" a more primitive identity between operators (see eq. (3.8)) where a further invariant c_ρ enters. Our result is then completed by showing that c_ρ is a conjugate-invariant character on the semi-ring of the superselection sectors, so that it takes only the values ± 1. This reaches the goal of our paper, but leaves out the full understanding of the invariant c_ρ, in particular whether the value $c_\rho = -1$ might actually occur. We think this is the case, that reflects a cohomological obstruction, and plan to return to this point somewhere else.

Our work has been announced in [23].

1. General properties of conformal precosheaves on $\mathbf{S}^1$

In this section we recall the basic properties enjoyed by the family of the von Neumann algebras associated with a conformal Quantum Field Theory on S^1.

By an *interval* we shall always mean an open connected subset I of S^1 such that I and the interior I' of its complement are non-empty. We shall denote by $\mathcal{I}$ the set of intervals in S^1.

A precosheaf $\mathcal{A}$ of von Neumann algebras on the intervals of S^1 is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$ to the von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the following property:

A. Isotony. *If I_1, I_2 are intervals and $I_1 \subset I_2$, then*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \ .$$

$\mathcal{A}$ is a *conformal precosheaf* of von Neumann algebras if the following properties B–E hold too.

B. Conformal invariance. *There is a unitary representation U of $\mathbf{G}$ (the universal covering group of $PSL(2, \mathbf{R})$) on $\mathcal{H}$ such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \ , \qquad g \in \mathbf{G}, \ I \in \mathcal{I}.$$

The group $PSL(2, \mathbf{R})$ is identified with the Möbius group of S^1, i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore $\mathbf{G}$ has a natural action on S^1.

C. Positivity of the energy. *The generator of the rotation subgroup $U(R(\cdot))$ is positive.*

Here $R(\vartheta)$ denotes the (lifting to $\mathbf{G}$ of the) rotation by an angle ϑ. In the following we shall often write $U(\vartheta)$ instead of $U(R(\vartheta))$. We may associate two one-parameter groups with any interval I. Let I_1 be the upper semi-circle, i.e. the interval $\{e^{i\vartheta}, \vartheta \in (0, \pi)\}$. We identify this interval with the positive real line $\mathbf{R}_+$ via the Cayley transform $C : S^1 \to \mathbf{R} \cup \{\infty\}$ given by $z \to -i(z-1)(z+1)^{-1}$. Then we consider the one-parameter groups $\Lambda_{I_1}(s)$ and $T_{I_1}(t)$ of diffeomorphisms of S^1 (cf. Appendix B) such that

$$C\Lambda_{I_1}(s)C^{-1}x = e^s x, \qquad CT_{I_1}(t)C^{-1}x = x + t, \quad t, s, x \in \mathbf{R}.$$

We also associate with I_1 the reflection r_{I_1} given by

$$r_{I_1} z = \bar{z}$$

where $\bar{z}$ is the complex conjugate of z. We remark that Λ_{I_1} restricts to an orientation preserving diffeomorphisms of I_1, r_{I_1} restricts to an orientation reversing diffeomorphism of I_1 onto I_1' and $T_{I_1}(t)$ is an orientation preserving diffeomorphism of I_1 into itself if $t \geq 0$.

Then, if I is an interval and we chose $g \in \mathbf{G}$ such that $I = gI_1$ we may set (see also the Appendix B)

$$\Lambda_I = g\Lambda_{I_1}g^{-1}, \qquad r_I = gr_{I_1}g^{-1}, \qquad T_I = gT_{I_1}g^{-1}.$$

The elements $\Lambda_I(s)$, $s \in \mathbf{R}$ and r_I are well defined, while the one parameter group T_I is defined up to a scaling of the parameter. However, such a scaling plays no role in this paper. We note also that $T_{I'}(t)$ is an orientation preserving diffeomorphism of I into itself if $t \leq 0$.

Lemma B.4 in Appendix B states the equivalence between the positivity of the conformal Hamiltonian, i.e. the generator of the rotation group $U(R(\cdot))$, and the positivity of the usual Hamiltonian, i.e. the generator of the translations on the real line in the above specified identification of S^1 with $\mathbf{R} \cup \{\infty\}$.

D. Locality. *If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $A(I)$ commute.*

The lattice symbol $\vee$ will denote 'the von Neumann algebra generated by'.

E. Existence of the vacuum. *There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$-invariant and cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$.*

Let r be an orientation reversing isometry of S^1 with $r^2 = 1$ (e.g. r_{I_1}). The action of r on $PSL(2, \mathbf{R})$ by conjugation lifts to an action σ_r on $\mathbf{G}$, therefore we may consider the semidirect product of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$. Any involutive orientation reversing isometry has the form $R(\vartheta) r_{I_1} R(-\vartheta)$, thus $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ does not depend on the particular choice of the isometry r. Since $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ is a covering of the group generated by $PSL(2, \mathbf{R})$ and r, $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ acts on S^1. We call (anti-)unitary a representation U of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ by operators on $\mathcal{H}$ such that $U(g)$ is unitary, resp. antiunitary, when g is orientation preserving, resp. orientation reversing.

1.1 Proposition. *Let $\mathcal{A}$ be a conformal precosheaf. The following hold:*

(a) Reeh-Schlieder theorem [10]: Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.

(b) Bisognano-Wichmann property [4,14]: U extends to an (anti-)unitary representation of $\mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$ such that, for any $I \in \mathcal{I}$,

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it} \ , \tag{1.1}$$

$$U(r_I) = J_I \ , \tag{1.2}$$

where Δ_I, J_I are the modular operator and the modular conjugation associated with $(\mathcal{A}(I), \Omega)$ [29]. For each $g \in \mathbf{G} \times_{\sigma_r} \mathbf{Z}_2$

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \ .$$

(c) Additivity [10]: if a family of intervals I_i covers the interval I, then

$$\mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i) \ .$$

(d) Spin and statistics for the vacuum sector [16]: U is indeed a representation of $PSL(2, \mathbf{R})$, i.e. $U(2\pi) = 1$.

(*e*) *Haag duality [4,14]:*

$$\mathcal{A}(I') = \mathcal{A}(I)' \ , \quad I \in \mathcal{I}.$$

Proof. We sketch here only the proof of (d) and refer to the original literature for the rest. Note however that: the usual Reeh-Schlieder argument shows that (c) implies (a); (b) is proved by using a theorem of Borchers [2]; (e) is an immediate consequence of (b). To get (d) let I_1 and I_2 be the upper and the right semicircle respectively, then J_{I_1} fixes Ω and implements an anti-automorphism of $\mathcal{A}(I_{I_2})$, thus it commutes with J_{I_2}. By property (b) $J_{I_1}J_{I_2} = U(\pi)$, thus $U(\pi)$ is an involution. □

F. Uniqueness of the vacuum (or irreducibility). *The only $U(\mathbf{G})$-invariant vectors are the scalar multiples of Ω.*

The term irreducibility is due to the following.

1.2 Proposition. *The following are equivalent:*
(i) *$\mathbf{C}\Omega$ are the only $U(\mathbf{G})$-invariant vectors.*
(ii) *The algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, are factors. In this case they are type III_1 factors.*
(iii) *If a family of intervals I_i intersects at only one point ζ, then $\cap_i \mathcal{A}(I_i) = \mathbf{C}$.*
(iv) *The von Neumann algebra $\vee\mathcal{A}(I)$ generated by the local algebras coincides with $\mathcal{B}(\mathcal{H})$ ($\mathcal{A}$ is irreducible).*

Proof. (i) $\Rightarrow$ (ii). Indeed (i) implies (c) of Corollary B.2 in the Appendix B, hence the modular group of $\mathcal{A}(I)$ with respect to Ω is ergodic, showing that $\mathcal{A}(I)$ is a type III_1 factor.
(ii) $\Rightarrow$ (iii). If ζ is a boundary point of an interval I, then by additivity and duality $\cap_i \mathcal{A}(I_i)$ commutes both with $\mathcal{A}(I)$ and $\mathcal{A}(I')$, and is therefore trivial.
(iii) $\Rightarrow$ (iv). We have $\vee_{I\in\mathcal{I}}\mathcal{A}(I) \supset \vee_{\zeta\notin I}\mathcal{A}(I) = \mathcal{B}(\mathcal{H})$.
(iv) $\Rightarrow$ (i). Let I be an interval and $x \in \mathcal{A}(I)$ such that $U(g)x\Omega = x\Omega$ for all $g \in \mathbf{G}$. since Ω is locally separating, we have $x = U(g)xU(g)^{-1}$. Since $\mathbf{G}$ acts transitively on the intervals, x is in the commutant of $\cup_{I\in\mathcal{I}}\mathcal{A}(I)$, and is therefore a scalar. Since $\mathcal{A}(I)\Omega$ is dense in $\mathcal{H}$, by the Ergodic Theorem Ω is the only $U(\mathbf{G})$-invariant vector. □

By Corollary B.2 the irreducibility of $\mathcal{A}$ is also equivalent to Ω being unique invariant for any of the one-parameter subgroups of U corresponding to T_I,Λ_I or R.

Now any conformal precosheaf decomposes uniquely into a direct integral of irreducible conformal precosheaves. This can be seen as in Proposition 3.1 of [16]. We will therefore always assume that our precosheaves are irreducible.

2. Superselection structure. Constructing the global conjugate charge

2.1 Generalities on superselection sectors with finite index

In this section $\mathcal{A}$ is an irreducible conformal precosheaf of von Neumann algebras as defined in Section 1.

A covariant *representation* π of $\mathcal{A}$ is a family of representations π_I of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation U_π of the covering group $\mathbf{G}$ of $PSL(2,\mathbf{R})$, with *positive energy*, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$
\begin{aligned}
I \subset \tilde{I} \Rightarrow \pi_{\tilde{I}}\big|_{\mathcal{A}(I)} &= \pi_I \quad (\textit{isotony}) \\
\mathrm{ad}U_\pi(g) \cdot \pi_I &= \pi_{gI} \cdot \mathrm{ad}U(g) \quad (\textit{covariance}).
\end{aligned}
\tag{2.1}
$$

A unitary equivalence class of representations of $\mathcal{A}$ is called *superselection sector.*

Assuming $\mathcal{H}_\pi$ to be separable, the representations π_I are normal because the $\mathcal{A}(I)$'s are factors [30]. Therefore for any given I_0, $\pi_{I_0'}$ is unitarily equivalent $id_{\mathcal{A}(I_0')}$ because $\mathcal{A}(I_0')$ is a type III factor. By identifying $\mathcal{H}_\pi$ and $\mathcal{H}$, we can thus assume that π is localized in a given interval $I_0 \in \mathcal{I}$, i.e. $\pi_{I_0'} = id_{\mathcal{A}(I_0')}$ (cf. [6]). By Haag duality we then have $\pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ if $I \supset I_0$. In other words, given $I_0 \in \mathcal{I}$ we can choose in the same sector of π a *localized endomorphism* with localization support in I_0, namely a representation ρ equivalent to π such that

$$
I \in \mathcal{I},\ I \supset I_0 \Rightarrow \rho_I \in \mathrm{End}\mathcal{A}(I), \qquad \rho_{I_0'} = id_{I_0'}.
$$

In the following (with the exception of subsection 2.4) representations or endomorphisms are always assumed to be covariant with positive energy[1].

To capture the global point of view we may consider the *universal algebra* $\mathrm{C}^*(\mathcal{A})$. Recall that $\mathrm{C}^*(\mathcal{A})$ is a C^*-algebra canonically associated with the precosheaf $\mathcal{A}$ (see [9,15]). There are injective embeddings $\iota_I : \mathcal{A}(I) \to \mathrm{C}^*(\mathcal{A})$ so that the local von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, are identified with subalgebras of $\mathrm{C}^*(\mathcal{A})$ and generate

[1] Assuming strong additivity (i.e. Haag duality on the real line) the covariance property with positive energy follows automatically in the finite index case; in fact the weaker assumption of *3-regularity* is sufficient (cf. [15]). $\mathcal{A}$ is said to be n-regular if, after removing n points from S^1, the C^*-algebra generated by the local operators is irreducible. By Haag duality and factoriality any conformal precosheaf is 2-regular, but the validity of 3-regularity is not known in general. An example violating 4-regularity has been pointed out to us by H. Wiesbrock.

all together a dense *-subalgebra of $C^*(\mathcal{A})$, and every representation of the precosheaf $\mathcal{A}$ factors through a representation of $C^*(\mathcal{A})$. Conversely any representation of $C^*(\mathcal{A})$ restricts to a representation of $\mathcal{A}$. The vacuum representation π_0 of $C^*(\mathcal{A})$ corresponds to the identity representation of $\mathcal{A}$ on $\mathcal{H}$, thus π_0 acts identically on the local von Neumann algebras. We shall often drop the symbols ι_I and π_0 when no confusion arises.

By the universality property, for each $g \in PSL(2,\mathbf{R})$ the isomorphism $\mathrm{ad}U(g) : \mathcal{A}(I) \to \mathcal{A}(gI)$, $I \in \mathcal{I}$ lifts to an automorphism α_g of $C^*(\mathcal{A})$. It will be convenient to lift the map $g \to \alpha_g$ to a representation, still denoted by α, of the universal covering group $\mathbf{G}$ of $PSL(2,\mathbf{R})$ by automorphisms of $C^*(\mathcal{A})$.

The covariance property for an endomorphism ρ of $C^*(\mathcal{A})$ localized in I_0 means that $\alpha_g \cdot \rho \cdot \alpha_{g^{-1}}$ is equivalent to ρ for any $g \in \mathbf{G}$, i.e.

$$\mathrm{ad}z_\rho(g)^* \cdot \rho = \alpha_g \cdot \rho \cdot \alpha_{g^{-1}} \qquad g \in \mathbf{G}. \tag{2.2}$$

for a suitable unitary $z_\rho(g) \in C^*(\mathcal{A})$. The map $g \to z_\rho(g)$ can be chosen to be a localized α-cocycle, i.e.

$$\begin{aligned} z_\rho(g) &\in \mathcal{A}(I_0 \cup gI_0) \quad \forall g \in \mathbf{G} : I_0 \cup gI_0 \in \mathcal{I} \\ z_\rho(gh) &= z_\rho(g)\alpha_g(z_\rho(h)), \qquad g,h \in \mathbf{G}. \end{aligned} \tag{2.3}$$

The relations between (π, U_π) and (ρ, z_ρ) are

$$\begin{aligned} \pi &= \pi_0 \cdot \rho \\ \pi_0(z_\rho(g)) &= U_\pi(g)U(g)^* \ . \end{aligned} \tag{2.4}$$

As is known ([27], see also [15]) that the localized cocycle z_ρ reconstructs the endomorphism ρ via

$$\rho|_{\mathcal{A}(gI_0')} = \mathrm{ad}z_\rho(g)|_{\mathcal{A}(gI_0')} \tag{2.5}$$

A localized endomorphism of $C^*(\mathcal{A})$ is said *irreducible* if the associated representation π is irreducible.

Note that the representations $\pi_0 \cdot \rho_1$ and $\pi_0 \cdot \rho_2$ associated with the endomorphisms ρ_1, ρ_2 of $C^*(\mathcal{A})$ are unitarily equivalent if and only if ρ_1 and ρ_2 are equivalent endomorphisms of $\mathcal{A}$, i.e. ρ_2 is a perturbation of ρ_1 by an inner automorphism of $\mathcal{A}$.

An endomorphism of $C^*(\mathcal{A})$ localized in an interval I_0 is said to have *finite index* if ρ_I $(= \rho|_{\mathcal{A}(I)})$ has finite index, $I_0 \subset I$ (see [20,23]). The index is indeed well defined due to the following.

2.1 Proposition. *Let ρ be an endomorphism localized in the interval I_0. Then the index* $\mathrm{Ind}(\rho) := \mathrm{Ind}(\rho_I)$*, the minimal index of ρ_I, does not depend on the interval $I \supset I_0$.*

Proof. We show indeed that all the inclusions $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$ are isomorphic if $I \supset I_0$ (they are isomorphic to the inclusion $\pi(\mathcal{A}(I)) \subset \pi(\mathcal{A}(I'))'$ for all $I \in \mathcal{I}$). This follows because, if $g \in \mathbf{G}$ and $z_\rho(g)$ are chosen as in (2.2), (2.3) with $I \supset I_0$ and $gI = I_0$, then

$$\begin{aligned}\{\rho(\mathcal{A}(I_0)) \subset \mathcal{A}(I_0)\} &= \{U_\rho(g)\rho(\mathcal{A}(I))U_\rho(g)^* \subset U(g)(\mathcal{A}(I))U(g)^*\} \\ &\simeq \{\rho(\mathcal{A}(I)) \subset z_\rho(g^{-1})\mathcal{A}(I)z_\rho(g^{-1})^*\}\end{aligned}$$

and $z_\rho(g) \in \mathcal{A}(I)$. □

2.2 Proposition. *Let ρ be a covariant (not necessarily irreducible) endomorphism with finite index. Then the representation U_ρ described before is unique. In particular, any irreducible component of ρ is a covariant endomorphism.*

Proof. If ρ is localized in I_0 and has finite index the following inclusion shows that $\pi(\mathrm{C}^*(\mathcal{A}))'$ is finite-dimensional, $\pi := \pi_0 \cdot \rho$:

$$\pi(\mathrm{C}^*(\mathcal{A}))' \subset (\pi(\mathcal{A}(I)) \cup \pi(\mathcal{A}(I')))' = \pi(\mathcal{A}(I))' \cap \mathcal{A}(I) \ , \qquad I_0 \subset I. \tag{2.6}$$

Since U_π implements automorphisms of $\pi(\mathcal{A})$, it implements an action of $\mathbf{G}$ by automorphisms of $\pi(\mathcal{A})'$, that must be trivial because $\mathbf{G}$ has no non-trivial action by automorphisms of a finite-dimensional C^*-algebra. Indeed such an action should be trivial on the center because $\mathbf{G}$ is connected, thus it admits a faithful invariant trace that defines a scalar product unitarizing the representation, but the only finite-dimensional unitary representation of $\mathbf{G}$ is the identity. Therefore we proved that $U_\pi \in \pi(\mathrm{C}^*(\mathcal{A}))''$, and this fact implies that any irreducible subsector of ρ is covariant.
Let U'_π be another representation of $\mathbf{G}$ as in (2.1). Then, for each $x \in \mathcal{A}(I)$, $I \in \mathcal{I}$,

$$U'_\pi(g)U_\pi(g)^*\pi(x) = U'_\pi\pi(\alpha_g(x))U_\pi(g)^* = \pi(x)U'_\pi(g)U_\pi(g)^*,$$

which implies $U'_\pi(g)U_\pi(g)^*$ to belong to the center of $\pi(\mathrm{C}^*(\mathcal{A}))''$. Therefore

$$U_\pi(g)U'_\pi(g)^*U'_\pi(h)U_\pi(h)^* = U'_\pi(g)U'_\pi(h)U_\pi(h)^*U_\pi(g)^* = U'_\pi(gh)U_\pi(gh)^* \ ,$$

i.e. $g \to U'_\pi(g)U_\pi(g)^*$ is a representation of $\mathbf{G}$. Since $\mathbf{G}$ is perfect, any abelian representation is trivial, i.e. $U_\pi = U'_\pi$ □

By the above proposition the *univalence* of an endomorphism ρ is well defined by

$$s_\rho = U_\rho(2\pi) \ .$$

By definition s_ρ belongs to $\pi(\mathrm{C}^*(\mathcal{A}))'$ therefore, when ρ is irreducible, s_ρ is a complex number of modulus one

$$s_\rho = e^{2\pi i L_\rho}$$

with L_ρ the lowest weight of U_ρ. In this case, since $U_{\rho'}(g) := \pi_0(u)U_\rho(g)\pi_0(u)^*$, where $\rho'(\cdot) := u\rho(\cdot)u^*$, $u \in \mathrm{C}^*(\mathcal{A})$, then s_ρ depends only on the superselection class of ρ.

Let ρ_1, ρ_2 be endomorphisms of an algebra $\mathcal{B}$. Their intertwiner space is defined by

$$(\rho_1, \rho_2) = \{T \in \mathcal{B} : \rho_2(x)T = T\rho_1(x),\ x \in \mathcal{B}\}. \tag{2.7}$$

In case $\mathcal{B} = \mathrm{C}^*(\mathcal{A})$, ρ_i localized in the interval I_i and $T \in (\rho_1, \rho_2)$, then $\pi_0(T)$ is an intertwiner between the representations $\pi_0 \cdot \rho_i$. If $I \supset I_1 \cup I_2$, then by Haag duality its embedding $\iota_I \cdot \pi_0(T)$ is still an intertwiner in (ρ_1, ρ_2) and a local operator. We shall denote by $(\rho_1, \rho_2)_I$ the space of such local intertwiners

$$(\rho_1, \rho_2)_I = (\rho_1, \rho_2) \cap \mathcal{A}(I).$$

If I_1 and I_2 are disjoint, we may cover $I_1 \cup I_2$ by an interval I in two ways: we adopt the convention that, unless otherwise specified, a *local intertwiner* is an element of $(\rho_1, \rho_2)_I$ where I_2 follows I_1 inside I in the clockwise sense.

We now define the statistics. Given the endomorphism ρ of $\mathcal{A}$ localized in $I \in \mathcal{I}$, choose an equivalent endomorphism ρ_0 localized in an interval $I_0 \in \mathcal{I}$ with $\bar{I}_0 \cap \bar{I} = \emptyset$ and let u be a local intertwiner in (ρ, ρ_0) as above, namely $u \in (\rho, \rho_0)_{\tilde{I}}$ with I_0 following clockwise I inside $\tilde{I}$.

The *statistics operator* $\varepsilon := u^*\rho(u) = u^*\rho_{\tilde{I}}(u)$ belongs to $(\rho^2_{\tilde{I}}, \rho^2_{\tilde{I}})$. An elementary computation shows that it gives rise to a presentation of the Artin braid group

$$\varepsilon_i\varepsilon_{i+1}\varepsilon_i = \varepsilon_{i+1}\varepsilon_i\varepsilon_{i+1}, \qquad \varepsilon_i\varepsilon_{i'} = \varepsilon_{i'}\varepsilon_i \quad \text{if } |i - i'| \geq 2,$$

where $\varepsilon_i = \rho^{i-1}(\varepsilon)$. The (unitary equivalence class of the) representation of the braid group thus obtained is the *statistics* of the superselection sector ρ.

Recall that if ρ is an endomorphism of a C^*-algebra $\mathcal{B}$, a *left inverse* of ρ is a completely positive map Φ from $\mathcal{B}$ to itself such that $\Phi \cdot \rho = id$.

We shall see in Corollary 2.12 that if ρ is irreducible there exists a unique left inverse Φ of ρ and that the *statistics parameter*

$$\lambda_\rho := \Phi(\varepsilon) \tag{2.8}$$

depends only on the sector of ρ.

The *statistical dimension* $d(\rho)$ and the *statistics phase* κ_ρ are then defined by

$$d(\rho) = |\lambda_\rho|^{-1}\,, \qquad \kappa_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}\,.$$

We shall indeed prove the equality between the statistics phase and the univalence while the statistical dimension equals the square root of the index [20] (see Corollary 3.7).

2.2 Equivalence between local and global intertwiners

If ρ, σ are endomorphisms of $\mathrm{C}^*(\mathcal{A})$ localized in the interval I, we may consider their intertwiner space $(\rho_I, \sigma_I) := \{T \in \mathcal{A}(I) : \sigma(x)T = T\rho(x), \forall x \in \mathrm{C}^*(\mathcal{A})\}$. We always have $(\rho, \sigma)_I \subset (\rho_I, \sigma_I)$.

2.3 Theorem. *Let ρ, σ be endomorphisms with finite index localized in I_0. Then*

$$(\rho_I, \sigma_I) = (\rho, \sigma)_I$$

for any $I \in \mathcal{I}$ that contains I_0. In other words if $T \in (\rho_I, \sigma_I)$ then $\iota_I(T)$ intertwines ρ and σ in $\mathrm{C}^*(\mathcal{A})$

The proof of this theorem will be carried on in a few steps. In the following ρ denotes an endomorphism of $\mathrm{C}^*(\mathcal{A})$ with finite index localized in an interval I_0. Let $\zeta \in I_0'$ and identify $S^1 \backslash \zeta$ with $\mathbf{R}$. Then ρ restricts to an endomorphism of each von Neumann algebra $\mathcal{A}(-\infty, \ell)$, for sufficiently large $\ell \in \mathbf{R}$, hence it gives rise to an endomorphism ρ_ζ the C^*-algebra $\mathcal{A}_\zeta$, the norm closure of $\cup_{\ell \in \mathbf{R}} \mathcal{A}(-\infty, \ell)$. Let $\mathbf{P}$ be the stabilizer of the point ζ for the $PSL(2, \mathbf{R})$ action, namely the semidirect product of the translations $T(t)$ and dilations $\Lambda(s)$ on $\mathbf{R}$: each $g \in \mathbf{P}$ is written uniquely as a product $g = T(t)\Lambda(s)$. Notice that $\mathbf{P}$ is canonically embedded in $\mathbf{G}$ since $\mathbf{P}$ is simply connected and its Lie algebra is a subalgebra of the Lie algebra of $PSL(2, \mathbf{R})$ that coincides with the Lie algebra of $\mathbf{G}$. It follows that U_ρ restricts to a representation of $\mathbf{P}$ and we set

$$\beta_g(x) = U_\rho(g)xU_\rho(g)^* = z_\rho(g)U(g)xU(g)^*z_\rho(g)^*, \quad x \in \mathcal{A}_\zeta,\ g \in \mathbf{P},$$

so that β is an action of $\mathbf{P}$ by automorphisms of $\mathcal{A}_\zeta$, due to the fact that the cocycle z_ρ is a local operator.

We consider now the semigroup $\mathbf{P}_0$, the semidirect product of negative dilations with positive translations. $\mathbf{P}_0$ is an amenable semigroup and we need an invariant mean m constructed as follows: first we average (with an invariant mean) on positive translations and then over negative dilations. Observe that $f \to \int_{\mathbf{P}_0} f(g)dm(g)$ gives an invariant mean on all $\mathbf{P}$ vanishing on f if, for any given $s \in \mathbf{R}$, the map $t \to f(T(t)\Lambda(s))$ has support in a left half line.

Then we associate to m the completely positive map Φ_ζ of $\mathcal{A}_\zeta$ to $\mathcal{B}(\mathcal{H})$ given by

$$\Phi_\zeta(x) := \int_{\mathbf{P}_0} z_\rho(g)^* x z_\rho(g) dm(g), \quad x \in \mathcal{A}_\zeta. \tag{2.9}$$

2.4 Lemma. *Φ_ζ is a left inverse of ρ_ζ. Moreover Φ_ζ is locally normal, i.e. has normal restriction to $\mathcal{A}(-\infty, \ell)$, $\ell \in \mathbf{R}$, and $\mathbf{P}$-invariant, namely*

$$\Phi_\zeta = \alpha_g^{-1} \Phi_\zeta \beta_g, \quad g \in \mathbf{P}.$$

Proof. Let x belong to $\mathcal{A}(-\infty,\ell)$, $\ell\in\mathbf{R}$. By formula (2.2)

$$\Phi_\zeta(\rho_\zeta(x)) = \int_{\mathbf{P}_0} \alpha_g(\rho_\zeta(\alpha_{g^{-1}}(x)))dm(g) = x$$

because of the above property of m since the integrand is constantly equal to x on the set $g\in\mathbf{P}_0 : g^{-1}(-\infty,a)\cap I_0=\emptyset$. Then the localization of ρ_ζ and Haag duality imply that the range of Φ_ζ is contained in $\mathcal{A}_\zeta$.

Setting $E=\rho_\zeta\cdot\Phi_\zeta$ we have a conditional expectation of $\mathcal{A}_\zeta$ onto the range of ρ_ζ that restricts to a conditional expectation $E_{(-\infty,\ell)}$ of $\mathcal{A}(-\infty,\ell)$ onto $\rho(\mathcal{A}(-\infty,\ell))$ if $(-\infty,\ell)\supset I_0$. Since $\rho_{(-\infty,\ell)}$ is assumed to have finite index, $E_{(-\infty,\ell)}$ is automatically normal [21]. Therefore $\Phi_\zeta|_{\mathcal{A}(-\infty,\ell)} = \rho^{-1}_{(-\infty,\ell)}E_{(-\infty,\ell)}$ is normal for ℓ large enough, hence for any ℓ.
Concerning the $\mathbf{P}$-invariance of Φ_ζ we have, making use of the cocycle condition,

$$\begin{aligned}\alpha_g^{-1}\Phi_\zeta\beta_g(x) &= \alpha_g^{-1}(\int_{\mathbf{P}_0} z_\rho(h)^*\beta_g(x)z_\rho(h)dm(h))\\ &= \alpha_g^{-1}(\int_{\mathbf{P}_0} z_\rho(h)^*z_\rho(g)\alpha_g(x)z_\rho(g)^*z_\rho(h)dm(h))\\ &= \int_{\mathbf{P}_0} z_\rho(hg^{-1})^*xz_\rho(hg^{-1})dm(h) = \Phi_\zeta(x)\end{aligned}$$

□

2.5 Corollary. *$\varphi=\omega\Phi_\zeta$ is a locally normal β-invariant state on $\mathcal{A}_\zeta$, where $\omega = (\,\cdot\,\Omega,,\Omega)$.*

Proof. We have $\varphi\beta_g=\omega\Phi_\zeta\beta_g=\omega\alpha_g\Phi_\zeta=\omega\Phi_\zeta=\varphi$ and φ is locally normal because both ω and Φ_ζ are locally normal. □

Let $\{\pi_\varphi,\xi_\varphi,\mathcal{H}_\varphi\}$ be the GNS triple associated with the above state φ and V be the unitary representation of $\mathbf{P}$ on $\mathcal{H}_\varphi$ given by $V_gx\xi_\varphi=\beta_g(x)\xi_\varphi$ for $x\in\mathcal{A}_\zeta$. Notice that V is strongly continuous because φ is locally normal. We now need a variation of known results, see [8,5].

2.6 Lemma. *If ρ_ζ is irreducible then*

$$\varphi(x) = \int_{\mathbf{P}_0}\beta_g(x)dm(g), \quad x\in\mathcal{A}_\zeta$$

Proof. If $x\in\mathcal{A}(-\infty,\ell)$ and $y\in\mathcal{A}_\zeta$ is localized in a bounded interval, the commutator function $t\to[\beta_{T(t)\Lambda(s)}(x),\rho(y)]=\beta_{T(t)\Lambda(s)}([x,\rho(\alpha^{-1}_{T(t)\Lambda(s)}(y)])$ vanishes on a right half line, hence $[\int_{\mathbf{P}_0}\beta_g(x),\rho(y)dm(g)]=\int_{\mathbf{P}_0}[\beta_g(x),\rho(y)]dm(g)=0$.

Since ρ_ζ is locally normal, $\int_{\mathbf{P}_0} \beta_g(x)dm(g)$ commutes with every $\rho(\mathcal{A}(-\infty,\ell))$, thus with $\rho_\zeta(\mathcal{A}_\zeta)$; since ρ_ζ is irreducible it therefore a scalar equal to its vacuum expectation value:

$$\int_{\mathbf{P}_0} \beta_g(x)dm(g) = \int_{\mathbf{P}_0} \omega(\beta_g(x))dm(g) = \int_{\mathbf{P}_0} \omega(z_g^* x z_g)dm(g) = \omega\Phi_\zeta(x) = \varphi(x),$$

due to the fact that ω is normal and α-invariant. $\square$

2.7 Corollary. *If ρ_ζ is irreducible, the one-parameter (translation) unitary group $V(T(t))$ has positive generator.*

Proof. If $f \in L^1(\mathbf{R})$ has Fourier transform $\hat{f}$ with support in $(-\infty,0)$, we have to show that $V_f := \int_{\mathbf{R}} f(t)V(T(t))dt = 0$. Choose by Lemma B.4 a non zero $\psi \in \mathcal{H}$ such that $\mathrm{sp}_{U_\rho}(\psi) + \mathrm{supp}\hat{f} \subset (-\infty,0)$, where sp_{U_ρ} denotes the spectrum relative to $U_\rho(T(\cdot))$. Setting $\beta_f := \int_{\mathbf{R}} f(t)\beta_{T(t)}dt$, for any $x \in \mathcal{A}_\zeta$ the vector $\beta_g(\beta_f(x))\psi = 0$, for all $g \in \mathbf{P}_0$, since it has negative spectrum relative to $U_\rho(T(\cdot))$. By averaging over $\mathbf{P}_0$ the vector $\beta_g(\beta_f(x^*)\beta_f(x))\psi$, Lemma 2.6 implies $\|V_f x\xi_\varphi\|^2 = \varphi(\beta_f(x)^*\beta_f(x)) = 0$. $\square$

2.8 Corollary. *If ρ_ζ is irreducible, φ is faithful on $\cup\rho(\mathcal{A}(-\infty,\ell))$.*

Proof. $\mathcal{A}_\zeta$ is a simple C^*-algebra since it is the inductive limit of type III factors (that are simple C^*-algebras). Therefore π_φ is one-to-one and the statement will follow if we show that ξ_φ if cyclic for $\mathcal{B}_\ell := \rho(\mathcal{A}(-\infty,\ell))'$, $\ell > 0$. To this end we may use a classical Reeh-Schlieder argument. If $\psi \in \mathcal{H}$ is orthogonal to $\mathcal{B}_\ell\xi_\varphi$, and $\ell_0 > \ell$, then for all $x \in \mathcal{B}_{\ell_0}$ we have $(x\xi_\varphi, V(T(t))\psi) = 0$ for t in a neighborhood of 0, thus for all $t \in \mathbf{R}$ by positivity of the generator shown by Corollary 2.7. Hence, setting $\alpha_t \equiv \alpha_{T(t)}$ and $\beta_t \equiv \beta_{T(t)}$, ψ is orthogonal to $(\cup_t\beta_t(\mathcal{B}_{\ell_0}))\xi_\varphi$, thus $\psi = 0$ because $\cup_t\beta_t(\mathcal{B}_{\ell_0})$ is irreducible since

$$\begin{aligned}(\bigcup_t \beta_t(\mathcal{B}_{\ell_0}))' &= \bigcap_t \beta_t(\rho(\mathcal{A}(-\infty,\ell_0)) = \bigcap_t \rho(\alpha_t(\mathcal{A}(-\infty,\ell_0))) \\ &= \rho(\bigcap_t \alpha_t(\mathcal{A}(-\infty,\ell_0))) = \bigcap_\ell \mathcal{A}(-\infty,\ell) = \mathbf{C}\end{aligned}$$

by the local normality of ρ. $\square$

2.9 Proposition. *(ρ_I,ρ_I) does not depend on the interval $I \supset I_0$.*

Proof. We begin with the case in which ρ_ζ is irreducible and assume for convenience that $\bar{I}_0 \subset (-\infty,0)$. Notice then that $(\rho_{(-\infty,0)},\rho_{(-\infty,0)})$ is finite-dimensional and, by covariance, globally β_g-invariant with g in the subgroup of dilations because these transformations preserve $(-\infty,0)$. Therefore $(\rho_{(-\infty,0)},\rho_{(-\infty,0)})\xi_\varphi$ is a finite-dimensional subspace of $\mathcal{H}_\varphi$ globally invariant for $V(\Lambda(s))$, $s \in \mathbf{R}$. By Proposition B.3 of the appendix B we thus have $V(T(t))x\xi_\varphi = x\xi_\varphi$ for every element $x \in (\rho_{(-\infty,0)}, \rho_{(-\infty,0)})$, thus $\beta_{T(t)}(x) = x$ because ξ_φ is separating. It follows that if $x \in (\rho_{(-\infty,0)}, \rho_{(-\infty,0)})$ and $y \in \mathcal{A}(-\infty, 0)$

$$[x, \rho(\alpha_g(y))] = \beta_g([\beta_g^{-1}(x), \rho(y)]) = \beta_g([x, \rho(y)]) = 0$$

namely

$$x \in (\rho_{(-\infty,0)}, \rho_{(-\infty,0)}) \Rightarrow x \in (\rho_\zeta, \rho_\zeta) = \mathbf{C}I$$

Since the converse implication is obvious by Haag duality we have the equality of the two intertwiner spaces.

Now if ρ is any endomorphism with finite index, (ρ_ζ, ρ_ζ) is finite-dimensional by the inclusion (2.6), and ρ_ζ decomposes into a direct sum of irreducible endomorphisms of $\mathcal{A}_\zeta$ which are covariant by Proposition 2.2, therefore the preceding analysis shows that also in this case $(\rho_{(-\infty,0)}, \rho_{(-\infty,0)}) = (\rho_\zeta, \rho_\zeta)$. Since (ρ_ζ, ρ_ζ) is translation invariant, we get $(\rho_{(-\infty,\ell)}, \rho_{(-\infty,\ell)}) = (\rho_\zeta, \rho_\zeta)$ whenever $I_0 \subset (-\infty, \ell)$ and, since ζ was arbitrary, we get the thesis. □

Proof of Theorem 2.3. The case $\sigma = \rho$ follows immediately by Proposition 2.6: if $T \in (\rho_I, \rho_I)$ then T also belongs to $(\rho_{\tilde{I}}, \rho_{\tilde{I}})$ for any interval $\tilde{I} \supset I$ hence by additivity T is a self-intertwiner of ρ on the whole algebra $\mathrm{C}^*(\mathcal{A})$.

To handle the general case consider a direct sum endomorphism $\eta := \rho \oplus \sigma$ localized in I, then

$$\dim(\eta_I, \eta_I) = \dim(\rho_I, \rho_I) + \dim(\sigma_I, \sigma_I) + 2\dim(\rho_I, \sigma_I)$$

while

$$\dim(\eta, \eta)_I = \dim(\rho, \rho)_I + \dim(\sigma, \sigma)_I + 2\dim(\rho, \sigma)_I$$

therefore $\dim(\rho_I, \sigma_I) = \dim(\rho, \sigma)_I$ and since we always have $(\rho, \sigma)_I \subset (\rho_I, \sigma_I)$ these two intertwiner spaces coincide. □

In particular we have proved the following.

2.10 Corollary. *Let ρ be an endomorphism of $\mathrm{C}^*(\mathcal{A})$ with finite index localized in I_0. The following are equivalent:*

(i) *$\pi_0 \cdot \rho$ is an irreducible representation of $\mathrm{C}^*(\mathcal{A})$*

(ii) *$\rho(\mathcal{A}(I))' \cap \mathcal{A}(I) = \mathbf{C}$ for some, hence for all, $I \supset I_0$*

(iii) *$\rho_\zeta(\mathcal{A}_\zeta)' \cap \mathcal{A}_\zeta = \mathbf{C}$*

(iv) *ρ_ζ is an irreducible representation of $\mathcal{A}_\zeta$.*

Moreover any finite index representation π of $\mathrm{C}^*(\mathcal{A})$ is the direct sum of irreducible representations.

2.3 The conjugate sector

Let ρ be an endomorphism of $\mathrm{C}^*(\mathcal{A})$ with finite index and localized in the interval I_0 as before. We shall say that the endomorphism $\bar\rho$ is a conjugate of ρ if there exist isometries $V \in (id, \bar\rho\rho)$ and $\bar V \in (id, \rho\bar\rho)$ such that

$$\bar V^* \bar\rho(V) = \frac{1}{d}, \quad V^*\rho(\bar V) = \frac{1}{d} \tag{2.10}$$

where d is a positive scalar. In this case one can in fact choose $V, \bar V$ so that d is the square root of the minimal index of ρ.

Denote by j_I the lifting to an anti-automorphism of $\mathrm{C}^*(\mathcal{A})$ of the adjoint action of the modular conjugation J_I on the precosheaf $\mathcal{A}$.

2.11 Theorem. *Let ρ be a covariant endomorphism with finite index. There exists a conjugate endomorphism $\bar\rho$, unique as superselection sector. $\bar\rho$ is covariant with positive energy and is given by the formula*

$$\bar\rho = j \cdot \rho \cdot j \tag{2.11}$$

where $j = j_I$. If both ρ and $\bar\rho$ are localized in the interval I, then there exist isometries $V \in (id, \bar\rho\rho)_I$ and $\bar V \in (id, \rho\bar\rho)_I$ such that the conjugate equations (2.10) holds with $d = \sqrt{\mathrm{Ind}(\rho)}$.

If moreover ρ is irreducible, then $\bar\rho$ is the unique irreducible endomorphism of $\mathrm{C}^*(\mathcal{A})$, up to inner automorphisms, such that $\rho\bar\rho$ contains the identity and in this case there exists a unique (up to a phase) isometry $V \in (id, \bar\rho\rho)_I$.

Proof. As shown in [15], $\bar\rho := j_I \cdot \rho \cdot j_I$ is an endomorphism of $\mathrm{C}^*(\mathcal{A})$ locally conjugate to ρ, namely $\bar\rho_{\tilde I}$ is a conjugate endomorphism of $\rho_{\tilde I}$ according to [21], for any interval $\tilde I$ such that both ρ and $\bar\rho$ are localized in $\tilde I$. Fixing such an interval $\tilde I$, since ρ_I has finite index, there exist isometries $V \in (id_{\tilde I}, \bar\rho_{\tilde I}\rho_{\tilde I})$, $\bar V \in (id_{\tilde I}, \rho_{\tilde I}\bar\rho_{\tilde I})$ such that $\bar V^*\bar\rho(V) = \frac{1}{d}$, $V^*\rho(\bar V) = \frac{1}{d}$, with $d = \sqrt{\mathrm{Ind}(\rho_{\tilde I})}$ [21]. By Theorem 2.3 V and $\bar V$ are global intertwiners, namely $\bar\rho$ is a global conjugate. The uniqueness of $\bar\rho$, the characterization of $\bar\rho$ in the irreducible case and the uniqueness of V follow again by the corresponding statements for sectors of factors [21] because of Theorem 2.3. The covariance of $\bar\rho$ follows by the formula $\bar\rho = j \cdot \rho \cdot j$, see [15]. □

2.12 Corollary. *If ρ is a an endomorphism of $\mathrm{C}^*(\mathcal{A})$ with finite index, there exists a (global) faithful left inverse Φ of ρ which is given by the formula*

$$\Phi = V^*\bar\rho(\cdot)V \tag{2.12}$$

where $V \in (id, \bar\rho\rho)$ verifies the conjugate equations (2.10) and all faithful left inverses have this form. If ρ is localized in I, also Φ is localized in I and $\Phi|_{\mathcal{A}(\tilde I)}$ is normal if $\tilde I \supset I$.

If V $\bar{V}$ are chosen so that the constant d in (2.10) *is equal to* $\sqrt{\mathrm{Ind}(\rho_I)}$*, then Φ is uniquely determined. In particular if ρ is irreducible then Φ is the unique left inverse of ρ.*

Proof. Only the uniqueness of Φ needs still to be proved. We assume that ρ is localized in I and $V \in \mathcal{A}(I)$. By the same argument as in Corollary 5.7 of [20], essentially the push-down lemma in [25], every element $x \in \mathrm{C}^*(\mathcal{A})$ can be written as

$$x = \mathrm{Ind}(\rho)\rho\Phi(x\bar{V}^*)\bar{V}. \tag{2.13}$$

If Ψ is a left inverse of ρ and satisfies the conjugate equations with $d = \sqrt{\mathrm{Ind}(\rho)}$, then Ψ and Φ have the same restriction to $\mathcal{A}(I)$ because the corresponding statement is true for endomorphisms of factors [21] and, by Corollary 2.10, $\Psi(\bar{V}) = \Phi(\bar{V})$. Thus, by formula (2.13),

$$\Psi(x) = \mathrm{Ind}(\rho)\Phi(x\bar{V}^*)\Psi(\bar{V}) = \mathrm{Ind}(\rho)\Phi(x\bar{V}^*)\Phi(\bar{V}) = \Phi(x).$$

$\square$

If ρ is a finite index endomorphism of $\mathrm{C}^*(\mathcal{A})$, we define $\lambda_\rho = \Phi(\varepsilon)$ where Φ is the unique "minimal" left inverse provided by Corollary 2.12. As shown in [20], Φ is a standard left inverse in the sense of [8], namely λ_ρ is a positive scalar multiple of a unitary $\kappa_\rho \in (\rho, \rho)_I$ and the statistical dimension is then defined by $d(\rho) = \|\lambda_\rho\|^{-1}$. By the index-statistics theorem (see Corollary 3.7) if ρ has finite index, then also $d(\rho)$ is finite.

2.13 Corollary. *If ρ is irreducible with finite index, the statistics parameter λ_ρ in formula* (2.8) *is a non-zero scalar.*

Proof. $\lambda_\rho = \Phi(\varepsilon) = \Phi_\zeta(\varepsilon)$ belongs to (ρ_I, ρ_I) thus is a scalar by Corollary 2.10. λ_ρ does not vanish as mentioned above. $\square$

2.4 Equivalence between finite index and finite statistics.

If a covariant, positive energy superselection sector ρ has finite index, then also the statistical dimension is finite. In fact Corollary 3.7 will relate the two quantities in the general reducible case. For completeness, in this subsection we will outline an argument showing a converse of this assertion. We shall say that a localized endomorphism ρ of $\mathrm{C}^*(\mathcal{A})$ has *finite statistics* if there exists a left inverse Φ of ρ such that the statistical parameter $\lambda_\rho := \Phi(\varepsilon)$ is an invertible operator; even in the irreducible case we do not know a priori that λ_ρ is a scalar since Corollary 2.10 has not been proved.

In the following proposition ρ is a covariant endomorphism of $\mathrm{C}^*(\mathcal{A})$, but positive energy is not assumed.

2.14 Proposition. *If ρ is covariant with finite statistics, then ρ has finite index and positive energy.*

Proof. Let ρ be localized in I_0, $\zeta \in I_0'$ and $\Phi_\zeta := \Phi|_{\mathcal{A}_\zeta}$. Because of finite statistics the DHR inequality holds:

$$\|\Phi_\zeta(x)\| \geq c\|x\|, \quad x \in \mathcal{A}_\zeta^+ \tag{2.14}$$

where $c = \|\lambda_\rho^{-1}\|^{-2} > 0$, by reasoning as in [8]. Indeed if $x = x^* \in \mathcal{A}(-\infty, \ell)$ with $I_0 \subset (-\infty, \ell)$ and u is a unitary such that $u\rho(\cdot)u^*$ is localized in (ℓ, ∞), so that $\rho(x) = u^*xu$ and $\varepsilon = u^*\rho(u)$, we have $\Phi(u^*x) = \Phi(\rho(x)u^*) = \Phi(\rho(x)\varepsilon\rho(u^*)) = x\lambda_\rho u^*$ and therefore $\|\Phi(x^2)\| \geq \|\Phi(xu)\Phi(u^*x)\| = \|\lambda_\rho^* x^2 \lambda_\rho\| = \|x\lambda_\rho^*\lambda_\rho x\| = \geq c\|x^2\|$.

As ρ_ζ is isometric, the inequality (2.14) is clearly equivalent to the Pimsner-Popa inequality [25]

$$\|E(x)\| \geq c\|x\|, \quad x \in \mathcal{A}_\zeta^+,$$

with $E = \rho_\zeta \cdot \Phi_\zeta$ the associated conditional expectation onto the range of ρ_ζ, and it is also equivalent to

$$E(x) \geq cx, \quad x \in \mathcal{A}_\zeta^+, \tag{2.15}$$

(see [20] for the version of these inequalities on infinite factors). In particular $E|_{\mathcal{A}(I)}$ is normal and ρ_I has finite index $I \supset I_0$.

We can now replace Φ_ζ by its average Φ'_ζ over $\mathbf{P}$ with respect to an invariant mean, e.g. the m in the previous section, $\Phi'_\zeta := \int_{\mathbf{P}_0} \alpha_g^{-1}\Phi_\zeta\beta_g dm(g)$. Since ρ is locally normal $\rho\Phi'_\zeta$ still satisfies the inequality (2.15) and hence Φ'_ζ the inequality (2.14).

At this point the state $\varphi = \omega\Phi'_\zeta$ in Corollary 2.5 is again locally normal and faithful, thus Proposition 2.9 applies and provides the global conjugate in Theorem 2.11. The usual additivity of the spectrum argument then shows that ρ is a positive energy representation. □

3. The conformal spin-statistics theorem

3.1 A first relation between spin and statistics

In this subsection we prove a first relation between spin and statistics. We shall not use the full conformal invariance, but only the covariance with respect to the rotation subgroup and the geometric interpretation of the modular conjugations.

In the following I_1 and I_2 always denote the upper semicircle $\{e^{i\theta}, \vartheta \in (0,\pi)\}$ and the right semicircle $\{e^{i\vartheta}, \vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ respectively and ρ is an irreducible covariant with positive energy endomorphism of $\mathrm{C}^*(\mathcal{A})$ with finite index localized in an interval whose closure is contained in $I_1 \cap I_2$. Then $\rho_{I_i} = \rho|_{\mathcal{A}(I_i)}$ is an irreducible finite-index endomorphism of $\mathcal{A}(I_i)$ and we denote by V_i the standard implementation of ρ_{I_i}, $i = 1, 2$, with respect to the vacuum vector, see Appendix A. We also shorten the notations: J_i stands for the modular conjugation J_{I_i}, $\overline{\rho}_i$ for the conjugate $j_i \rho j_i$ of ρ where j_i is the promotion to an anti-automorphism of $\mathrm{C}^*(\mathcal{A})$ of the precosheaf anti-automorphism $J_i \cdot J_i$. The symbol $\mathrm{ad}U$ denotes the automorphism of $\mathrm{C}^*(\mathcal{A})$ corresponding to a unitary U (e.g. $\mathrm{ad}U_\rho(g) := \mathrm{ad}z_\rho(g) \cdot \alpha_g$).

3.1 Lemma. *We have $V_1 \in \mathcal{A}(I_2)$, $V_2 \in \mathcal{A}(I_1)$ and V_i is the unique isometry (up to a phase) with this localization support that intertwines the identity and $\rho\overline{\rho}_i$, $i = 1, 2$.*

Proof. By the geometric meaning of J_1, both ρ and $\overline{\rho}_1$ are localized in I_2, thus by Theorem 2.11 we can take an isometry $v \in (id, \overline{\rho}_1 \rho)_{I_2}$, in fact v belongs to $\mathcal{A}(I)$ if I is any subinterval of I_2 that contains both the localization support of ρ and of $\overline{\rho}_1$. Since ρ_{I_2} is irreducible, v is uniquely determined (up to a phase) by such properties. Therefore we may choose v so that $j_1(v) = v$. By additivity v implements ρ_{I_1} and since it also commutes with J_1 we have $V_1 = \pm v$ by Lemma A.3. The argument for V_2 is similar. $\square$

Since $\rho, \bar{\rho}_1, \bar{\rho}_2$ are localized in disjoint intervals, they pairwise commute, thus $V_1 V_2$ and $V_2 V_1$ both belong to $(id, \rho^2 \overline{\rho}_1 \overline{\rho}_2)_{I_1 \cup I_2}$ hence

$$\mu_\rho = V_1^* V_2^* V_1 V_2$$

is a scalar. It is an invariant for ρ that, by construction, reflects algebraic, analytical and geometric aspects. By looking at μ_ρ from these different point of view we shall identify it, with different arguments, with the statistics parameter and with the univalence of ρ times $d(\rho)^{-1}$, proving the conformal spin-statistics theorem.

3.2 Lemma. *The following identities between endomorphisms of* $\mathrm{C}^*(\mathcal{A})$ *hold:*

(a) $\rho\overline{\rho}_1 = \mathrm{ad}U_\rho(\pi)\rho\overline{\rho}_2 \mathrm{ad}U(\pi)$, (*b*) $\rho\overline{\rho}_1 j_2 \rho\overline{\rho}_1 j_2 = \rho\overline{\rho}_2 j_1 \rho\overline{\rho}_2 j_1$.

Proof. By formula (1.2) we have $J_1 J_2 = U(\pi)$, hence $j_1 j_2 = j_2 j_1 = \mathrm{ad}U(\pi)$, therefore

$$\overline{\rho}_1 = \mathrm{ad}U(\pi)\overline{\rho}_2 \mathrm{ad}U(\pi).$$

Thus by covariance

$$\rho\overline{\rho}_1 = \rho \mathrm{ad}U(\pi)\overline{\rho}_2 \mathrm{ad}U(\pi) = \mathrm{ad}U_\rho(\pi)\rho\overline{\rho}_2 \mathrm{ad}U(\pi)$$

and, since $\overline{\rho}_1$ and $\overline{\rho}_2$ are localized in disjoint intervals and thus commute,

$$\rho\overline{\rho}_1 j_2 \rho\overline{\rho}_1 j_2 = \rho\overline{\rho}_1\overline{\rho}_2 \mathrm{ad}U(\pi)\rho \mathrm{ad}U(\pi) = \rho\overline{\rho}_2\overline{\rho}_1 \mathrm{ad}U(\pi)\rho \mathrm{ad}U(\pi) = \rho\overline{\rho}_2 j_1 \rho\overline{\rho}_2 j_1$$

□

3.3 Lemma. *We have*

$$U_\rho(\pi)V_2 U(\pi) = c_\rho V_1 \tag{3.2}$$

where c_ρ is a complex number of modulus one.

Proof. By Lemma 3.1 V_1 is the unique isometry (up to a phase) in $(id, \rho\overline{\rho}_1)_{I_2}$. By Lemma 3.2 (*a*), also $U_\rho(\pi)V_2U(\pi)$ belongs to $(id, \rho\overline{\rho}_1)$. Moreover, if $x \in \mathcal{A}(I_2')$, then $\mathrm{ad}U(\pi)(x) \in \mathcal{A}(I_2)$ hence

$$\mathrm{ad}U_\rho(\pi)V_2U(\pi)(x) = \mathrm{ad}U_\rho(\pi)\rho \mathrm{ad}U(\pi)(x) = \mathrm{ad}U_\rho(2\pi)\rho(x) = \rho(x) = x$$

showing that $U_\rho(\pi)V_2U(\pi)$ belongs to $\mathcal{A}(I_2)$ too, thus it coincides with V_1 up to a phase. □

3.4 Lemma. $\beta_\rho := (V_1J_2V_1J_2)^* V_2J_1V_2J_1$ *belongs to* $(0, 1]$.

Proof. According to Lemma 3.2 (*ii*), $V_1J_2V_1J_2$ and $V_2J_1V_2J_1$ are both isometries in $(id, \rho\overline{\rho}_1 j_2 \rho\overline{\rho}_1 j_2)$ and both belongs to the same local von Neumann algebra $\mathcal{A}(I)$ where $\rho\overline{\rho}_1 j_2 \rho\overline{\rho}_1 j_2$ is localized in $I \in \mathcal{I}$ therefore β_ρ is a complex scalar.

Setting $e_i := V_iV_i^*$ we deduce that

$$\beta_\rho V_1J_2V_1J_2 = e_1J_2e_1J_2\ V_2J_1V_2J_1. \tag{3.3}$$

Since V_i is the standard implementation of ρ_{I_i}, V_i preserves the positive cone $\mathcal{P}^\natural(\mathcal{A}(I_i), \Omega)$. Moreover J_1 preserves $\mathcal{P}^\natural(\mathcal{A}(I_2), \Omega)$ because it implements an antiautomorphism of $\mathcal{A}(I_2)$ and fixes Ω, and J_2 preserves $\mathcal{P}^\natural(\mathcal{A}(I_1), \Omega)$ analogously. By the definition of the natural positive cones and the relations $V_1, e_1 \in \mathcal{A}(I_2)$, $V_2 \in \mathcal{A}(I_1)$, we have that $V_2J_1V_2J_1\Omega$ and $V_1J_2V_1J_2\Omega$ belong to $\mathcal{P}^\natural(\mathcal{A}(I_1),\Omega)\cap\mathcal{P}^\natural(\mathcal{A}(I_2),\Omega)$ and $e_1J_2e_1J_2\Omega\in\mathcal{P}^\natural(\mathcal{A}(I_2),\Omega)$.

Since the scalar product of non-zero vectors in a natural cone is non-negative, and furthermore positive if one of the vectors is cyclic (equivalently separating), and since $(e_2J_1e_2J_1\Omega,\Omega)=\|\Delta_{I_1}^{\frac{1}{4}}e_2\Omega\|^2\neq 0$ we have

$$(V_1J_2V_1J_2\Omega,\Omega)>0,\quad (e_2J_1e_2J_1V_2J_1V_2J_1\Omega,\Omega)=(V_2J_1V_2J_1\Omega,e_2J_1e_2J_1\Omega)>0$$

that entails $\beta_\rho>0$ by comparing with (3.3), provided we show that $V_2J_1V_2J_1\Omega$ is separating for $\mathcal{A}(I_1)$. But this is true because if $x\in\mathcal{A}(I_1)$ and $xV_2J_1V_2J_1\Omega=0$ then

$$(j_1\Phi j_1\Phi(x^*x)\Omega,\Omega)=(J_1V_2^*J_1V_2^*x^*xV_2J_1V_2J_1\Omega,\Omega)=0$$

and this implies $x=0$ because the left inverse Φ of ρ is faithful. The rest is clear since by definition $||\beta_\rho||\leq 1$. □

3.5 Lemma. $\lambda_\rho=V_1^*V_2^*V_1V_2$.

Proof. As in [7] we get $\lambda_\rho=\rho(V_1^*)V_1$; indeed if ρ' is localized in $I_1\cap I_2'$ and u is a unitary in $(\rho,\rho')_{I_1}$, then $\mathrm{ad}u^*|_{\mathcal{A}(I_2)}=\rho_{I_2}$, thus $\rho(V_1^*)V_1=u^*V_1^*uV_1=u^*\Phi(u)=\Phi(\varepsilon_\rho)=\lambda_\rho$. Since $V_1\in\mathcal{A}(I_2)$ and V_2 implements ρ on $\mathcal{A}(I_2)$ we thus have

$$V_1^*V_2^*V_1V_2=V_1^*\Phi(V_1)=\Phi(\rho(V_1^*)V_1)=\Phi(\lambda_\rho)=\lambda_\rho. \tag{3.5}$$

□

3.6 Proposition. *The following relations hold:*

$$\beta_\rho=d(\rho)^{-1} \tag{3.6}$$

$$s_\rho=c_\rho^2\kappa_\rho \tag{3.7}$$

where κ_ρ is the phase of the statistical parameter.

Proof. Taking adjoints in (3.2), we have $U(\pi)V_1^*U_\rho(\pi)=c_\rho V_2^*$, and multiplying side by side this expression with formula (3.2) we have

$$c_\rho^2V_2^*V_1=s_\rho U(\pi)V_1^*V_2U(\pi) \tag{3.8}$$

because $s_\rho:=U_\rho(2\pi)$.

Since J_i commutes with V_i and $J_1J_2=J_2J_1=U(\pi)$ we have

$$\beta_\rho=J_1\beta_\rho J_1=J_1(V_1^*J_2V_1^*J_2V_2J_1V_2J_1)J_1=V_1^*U(\pi)V_1^*V_2U(\pi)V_2 \tag{3.9}$$

therefore, by inserting formula (3.8) in the expression for λ_ρ given by Lemma 3.5 and comparing with (3.9) we obtain

$$\lambda_\rho = V_1^* V_2^* V_1 V_2 = c_\rho^{-2} s_\rho V_1^* U(\pi) V_1^* V_2 U(\pi) V_2 = c_\rho^{-2} s_\rho \beta_\rho \tag{3.10}$$

and the thesis easily follows. □

3.7 Corollary. *(Index-statistics theorem) For every covariant endomorphism ρ of* $\mathrm{C}^*(\mathcal{A})$ *we have* $\mathrm{Ind}(\rho) = d(\rho)^2$.

Proof. If ρ is irreducible we have $\bar{V}_1^* \rho_{I_2}(V_1) = \frac{1}{d}$ with $d = \sqrt{\mathrm{Ind}(\rho)}$ by Corollary 2.12 and comparing with formula (3.5) we have the thesis since $\bar{V}_1$ and V_1 are equal up to a phase. The general case follows by additivity of both the statistical dimension and the square root of the minimal index (or by a direct argument). The case of infinite index is treated in subsection 2.4. □

3.2 The spin-statistics theorem.

We prove now that $c_\rho^2 = 1$, completing our result. In this step the role of the conformal invariance is to fix uniquely the representation of the rotation group $U_\rho(\vartheta)$, otherwise defined up to a one-dimensional representation, as the restriction of the unique representation of $\mathbf{G}$. We could nevertheless fix $U_\rho(\vartheta)$ by using the positivity of the conformal Hamiltonian.

It is convenient to extend the definition of c_ρ to the case of a reducible finite index ρ. To this end notice that, as in the proof Lemma 3.3, both $U_\rho(\pi)V_2U(\pi)$ and $c_\rho V_1$ belong to $(id, \rho\bar{\rho}_1)_{I_2}$, thus there exists $c_\rho \in (\rho\bar{\rho}_1, \rho\bar{\rho}_1)_{I_2}$ such that formula (3.2) holds. Replacing c_ρ by its push-down if necessary, we may further assume that $c_\rho \in (\rho, \rho)$ and this condition define it uniquely, see [24].

In the following ρ, σ are finite index endomorphisms of $\mathrm{C}^*(\mathcal{A})$.

3.8 Lemma. *Let ρ and σ be localized in $I_1 \cap I_2$, with ρ an irreducible subsector of σ and $p_\rho \in \mathcal{A}(I_1 \cap I_2)$ is the minimal idempotent in $(\sigma, \sigma)_{I_1 \cap I_2}$ corresponding to ρ, then $c_\sigma p_\rho = c_\rho p_\rho$. In particular, if c_σ is a scalar, then $c_\rho = c_\sigma$.*

Proof. With $w \in (\rho, \sigma)$ an isometry in $\mathcal{A}(I_1 \cap I_2)$, we have by Lemma A.4 of the appendix

$$\sqrt{d(\rho)} V_i^\rho = \sqrt{d(\sigma)} w^* J w^* J V_i^\sigma . \tag{3.11}$$

The projection $p_\rho = w^* w \in \mathcal{A}(I_1 \cap I_2)$ commutes with the range of σ, hence it commutes with U_σ (see the proof of Proposition 2.2), therefore

$$(w^* U_\sigma(g) w)(w^* U_\sigma(h) w) = w^* U_\sigma(gh) w, \quad g, h \in \mathbf{G}, \tag{3.12}$$

namely $g \to w^* U_\sigma(g) w$ is a unitary representation. Since for every $x \in \mathrm{C}^*(\mathcal{A})$ we have

$$\begin{aligned}(w^* U_\sigma(g) w)\rho(x)(w^* U_\sigma(g)^* w) &= w^* U_\sigma(g)\sigma(x) U_\rho(g)^* w \\ &= w^*\sigma(U(g)xU(g)^*)w = \rho(U(g)xU(g)^*),\end{aligned}$$

we get by the uniqueness of the representation in Proposition 2.2

$$U_\rho(g) = w^* U_\sigma(g) w. \tag{3.13}$$

Since c_σ leaves in a finite-dimensional algebra, we may assume that p_ρ is an eigen-projection of c_σ namely $c_\sigma p_\rho = \ell p_\rho$ with $\ell \in \mathbf{C}$. Making substitutions in the formula (3.2) according to the equations (3.12), (3.13), we then get

$$\begin{aligned}\sqrt{d(\rho)} c_\rho V_1^\rho &= \sqrt{d(\sigma)} w^* U_\sigma(\pi) w w^* J_2 w^* J_2 V_2^\sigma U(\pi) \\ &= \sqrt{d(\sigma)} w^* U_\sigma(\pi) J_2 w^* J_2 V_2^\sigma U(\pi) \\ &= \sqrt{d(\sigma)} w^* U_\sigma(\pi) J_2 w^* J_2 U_\sigma(-\pi) c_\sigma V_1^\sigma \\ &= \sqrt{d(\sigma)} w^* U(\pi) z_\sigma(\pi) J_2 w^* J_2 z_\sigma^*(\pi) U(\pi) c_\sigma V_1^\sigma \\ &= \sqrt{d(\sigma)} w^* U(\pi) J_2 w^* J_2 U(\pi) c_\sigma V_1^\sigma \\ &= \sqrt{d(\sigma)} w^* J_1 w^* J_1 c_\sigma V_1^\sigma \\ &= \sqrt{d(\sigma)} w^* c_\sigma J_1 w^* J_1 V_1^\sigma = \sqrt{d(\rho)} \ell V_1^\rho\end{aligned}$$

where we have used that $[J_2 w^* J_2, z_\sigma(\pi)] = 0$ due to the localization in disjoint intervals of w^* and $z_\sigma((\pi))$ and again of the identity $J_1 J_2 = U(\pi)$, and this concludes the proof. □

Our choice of the intervals I_1 and I_2 is, of course, conventional. If we replace them by their rotates $R(\vartheta)I_1$, $R(\vartheta)I_2$, we would get a priori another invariant $c_\rho(\vartheta)$ for a ρ localized in their intersection. But this is soon seen to be equal to c_{ρ_ϑ}, the old invariant for $\rho_\vartheta := \mathrm{ad}U(-\vartheta)\rho\,\mathrm{ad}U(\vartheta) = \mathrm{ad}z_\rho(-\vartheta)\rho$ (because $U(\vartheta)$ establishes an isomorphism between the old and the rotated structures). Next lemma implies that $c_{\rho_\vartheta} = c_\rho$ if also ρ_ϑ is localized in $I_1 \cap I_2$.

3.9 Lemma. *c_ρ depends only on the superselection class of ρ and not on its representative ρ nor on the choice of I_1 and I_2 as above.*

Proof. If ρ is localized in $I_1 \cap I_2$ and $\sigma = \mathrm{ad}W^* \cdot \rho$ for some unitary $W \in \mathcal{A}(I_1 \cap I_2)$ then $V_i^\rho = W^* J W^* J V_i^\sigma$ and by a computation similar to the one in the Lemma 3.7 we see that $c_\sigma = c_\rho$. By the comment preceding the this lemma it thus follows that c_ρ remains unchanged if we rotate the I_i's provided ρ stays localized in the intersection of the intervals. Thus, in finitely many steps, replacing ρ by an equivalent endomorphism and making small rotations of the intervals, we see that c_ρ does not vary in its superselection class. $\square$

3.10 Lemma. $c_\rho = c_{\bar\rho}$.

Proof. By Lemma 3.9 we may choose $\bar\rho = \bar\rho_1 = j_1\rho j_1$. Thus $\bar\rho$ is localized in $I_1' \cap I_2$ and $c_{\bar\rho}$ is definable with respects to the intervals $I_2 = R(-\frac{\pi}{2})I_1, I_1' = R(-\frac{\pi}{2})I_2$. The standard implementations of $\bar\rho$ relative to these intervals are respectively given by $J_1V_2^\rho J_1$ and $J_1V_1^\rho J_1 = V_1^\rho$, moreover $U_{\bar\rho}(\vartheta) = J_1U_\rho(-\vartheta)J_1$, see [15]. Inserting these identities in the defining expression (3.2) for $c_{\bar\rho}$ we thus have $J_1U_\rho(-\pi)J_1V_1^\rho U(\pi) = c_{\bar\rho}J_1V_2^\rho J_1$ and after cancellations this gives the stated equality. $\square$

3.11 Lemma. *Let ρ, σ be irreducible and localized in $I_1 \cap I_2$. Then $c_{\rho\sigma} = c_\rho c_\sigma$.*

Proof. By the cocycle equation $z_{\rho\sigma}(g) = z_\rho(g)\rho(z_\sigma(g))$ and the multiplicativity of the standard implementations $V_i^{\rho\sigma} = V_i^\rho V_i^\sigma$ the equation (3.2) for $\rho\sigma$ gives

$$\begin{aligned} c_\rho c_\sigma U(\pi)V_1^\rho V_1^\sigma U(\pi) &= z_\rho(\pi)V_2^\rho z_\sigma(\pi)V_2^\sigma = z_\rho(\pi)\rho(z_\sigma(\pi))V_2^\rho V_2^\sigma \\ &= z_{\rho\sigma}(\pi)V_2^{\rho\sigma} = c_{\rho\sigma}U(\pi)V_1^{\rho\sigma}U(\pi) \end{aligned} \tag{3.14}$$

where we used that $z_\sigma(\pi) \in \mathcal{A}(I_2 \cup I_1')$ and that V_2^ρ implements ρ on $\mathcal{A}(I_2 \cup I_1')$. Since $V_1^{\rho\sigma} = V_1^\rho V_1^\sigma$ we have the thesis. $\square$

3.12 Corollary. $c_\rho^2 = 1$.

Proof. If ρ is irreducible, then by the Lemmas 3.8 and 3.9 we have $c_\rho^2 = c_\rho c_{\bar\rho} = c_{\rho\bar\rho} = 1$. The general case follows by Lemma 3.8. $\square$

Now the spin and statistics relation immediately follows immediately by Proposition 3.6.

3.13 Theorem. *(Spin and Statistics) Let ρ be a superselection sector with finite statistics. Then $\kappa_\rho = s_\rho$.*

Appendix A. Standard implementation of left inverses

We will deal here with the the notion of standard implementation (see e.g. [3]) in the endomorphism case.

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and ρ a unital injective endomorphism of M. The left inverses Φ of ρ correspond bijectively to the conditional expectations E of M onto $\rho(M)$:

$$\begin{aligned} &\Phi \to E = \rho \cdot \Phi \\ &E \to \Phi = \rho^{-1} \cdot E\,. \end{aligned} \tag{A.1}$$

We shall say that an isometry $V \in \mathcal{B}(\mathcal{H})$ *implements the left inverse* Φ if

$$V^* x V = \Phi(x)\,, \qquad x \in M \tag{A.2}$$

A.1 Lemma. *Let the isometry* V *implement* Φ. *Then*
(a) $Vx = \rho(x)V, \quad x \in M,$
(b) $exe = E(x)e, \quad x \in M,$
where $e = VV^*$ *and* $E = \rho\Phi$. *Conversely if* (a) *and* (b) *hold then* V *implements* Φ.

Proof. If we set $x \equiv \rho(y)$ in $(A.2)$ we have $V^*\rho(y)V = y$ for all $y \in M$ hence

$$e\rho(y)V = Vy. \tag{A.3}$$

In particular, if y is unitary, $\|e\rho(y)V\xi\| = \|Vy\xi\| = \|\xi\| = \|\rho(y)V\xi\|$, $\xi \in \mathcal{H}$, showing that $e\rho(y)V\xi = \rho(y)V\xi$, hence $e\rho(y)e = \rho(y)e$, so we have

$$e\rho(y) = (\rho(y^*)e)^* = (e\rho(y^*)e)^* = e\rho(y)e = \rho(y)e$$

which implies $e \in \rho(M)'$ because M is generated by its unitaries. Formula $(A.3)$ then entails (a). To check (b) notice that

$$exe = VV^*xVV^* = V\Phi(x)V^* == \rho(\Phi(x))VV^* = E(x)e.$$

Conversely, assuming (a) and (b), we have

$$V^*xV = V^*exeV = V^*E(x)V = V^*\rho(\Phi(x))V = \Phi(x), \quad x \in M\ .$$

$\square$

We shall say that an isometry V *implements the endomorphism* ρ and that the projection e *implements the conditional expectation* E if the equations (a) and (b) of Lemma A.1 are respectively satisfied.

We now fix a unit cyclic and separating vector $\Omega \in \mathcal{H}$ for M and its corresponding natural cone $\mathcal{P}^\natural(M,\Omega)$.

If Φ is a normal left inverse of ρ let us consider the state

$$\varphi = \omega \cdot \Phi$$

where $\omega = (\cdot\Omega,\Omega)$ and the corresponding vector $\xi \in \mathcal{P}^\natural(M,\Omega)$ such that $\varphi = (\cdot\xi,\xi)$.

Let $e := [\rho(M)\xi] \in \rho(M)'$ and let V_Φ be the isometry of $\mathcal{H}$ with final projection e such that $V_\Phi : \mathcal{H} \to e\mathcal{H}$ is the Araki-Connes-Haagerup standard implementation of ρ as an isomorphism of M with $\rho(M)$ with respect to the positive cones $\mathcal{P}^\natural(M,\Omega)$ and $\mathcal{P}^\natural(\rho(M),\xi)$. Then V_Φ is given by

$$V_\Phi x\Omega = \rho(x)\xi\,, \qquad x \in M$$

We check that V_Φ implements Φ. To this end note first that $E = \rho\Phi$ is φ-invariant since

$$\varphi \cdot E = \omega \cdot \Phi \cdot \rho \cdot \Phi = \omega \cdot \Phi = \varphi$$

Then

$$\begin{aligned}(x\rho(b)\xi, \rho(a)\xi) = \varphi(\rho(a^*)x\rho(b)) = \varphi \cdot E(\rho(a^*)x\rho(b)) = \\ = \varphi(\rho(a^*)E(x)\rho(b)) = (E(x)\rho(b)\xi, \rho(a)\xi) \quad a, b, x \in M\end{aligned}$$

i.e. $eE(x)e = exe$, $x \in M$, but $e \in \rho(M)'$, hence e implements E; in particular, if Φ is faithful, e is the Takesaki projection for E.

Moreover V_Φ implements Φ because

$$\rho(x)V_\Phi y\Omega = \rho(x)\rho(y)\xi = \rho(xy)\xi = V_\Phi xy\Omega \quad x, y \in M.$$

The isometry V_Φ will be called the *standard implementation* of Φ with respect to Ω. In case ρ has finite index, namely $\rho(M)$ is a finite index subfactor of M, and $\Phi = \Phi_{\min}$, the minimal left inverse of ρ, we shall denote $V_{\Phi_{\min}}$ by V_ρ and call it the standard implementation of ρ with respect to Ω.

We collect here some properties of the standard implementations.

A.2 Proposition.

(a) *V_Φ is the unique isometry that implements Φ and sends $\mathcal{P}^\natural(M,\Omega)$ into itself. In particular V_Φ depends on $\mathcal{P}^\natural(M,\Omega)$ but not on the particular vector Ω.*

(b) *V_Φ is the unique isometry that implements Φ and verifies $V_\Phi\Omega \in \mathcal{P}^\natural(M,\Omega)$.*

(c) *$V_{\Phi_1\Phi_2} = V_{\Phi_1}V_{\Phi_2}$, with Φ_1, Φ_2 normal left inverses of ρ_1, ρ_2. In particular, if ρ_1, ρ_2 have finite index, $V_{\rho_1\rho_2} = V_{\rho_1}V_{\rho_2}$.*

(*d*) $JV_\Phi J = V_\Phi$, *where J is the modular conjugation of (M,Ω).*

Proof. By construction V_Φ implements Φ and maps $\mathcal{P}^\natural(M,\Omega)$ into itself, in particular $V_\Phi\Omega \in \mathcal{P}^\natural(M,\Omega)$. Now suppose that an isometry V implements Φ and $V\Omega \in \mathcal{P}^\natural(M,\Omega)$. Then

$$(\,\cdot\, V\Omega, V\Omega) = (V^* \cdot V\Omega, \Omega) = \omega \cdot \Phi = \varphi$$

thus $V\Omega$ is the unique vector $\xi \in \mathcal{P}^\natural(M,\Omega)$ associated with φ and $Vx\Omega = \rho(x)V\Omega = \rho(x)\xi$, namely $V = V_\Phi$. This proves (a) and (b).
(c) is consequences of (a) and of the multiplicativity of the minimal index [22].
(d) J restricted to the range of $e = V_\Phi V_\Phi^*$ coincides with the modular conjugation of M_e because φ preserves the conditional expectation E, thus $V_\Phi^* J V_\Phi = J$ because V_Φ is the standard implementation of ρ as an isomorphism of M with $\rho(M)$. □

A.3 Lemma. *Let M be a factor and ρ a finite index endomorphism. If W is an isometry that implements ρ and commutes with J, then W implements a left inverse Φ of ρ and $W = mV_\rho$ for some $m \in (\rho,\rho)$, which is invertible iff Φ is faithful. In particular, if ρ is irreducible, then $W = \pm V_\rho$*

Proof. The partial isometry $Z = WV_\rho^*$ commutes with J and belongs to $\rho(M)'$, thus $Z \in N' \cap M_1$, where we set $N = \rho(M)$ and $M_1 = JN'J$ denotes the Jones basic extension of $N \subset M$. Clearly we have $W = ZV_\rho$. Let m be the Pimsner-Popa push-down of Z, namely the unique element $m \in M$ such that $mV_\rho = ZV_\rho$. We have $m = \mathrm{Ind}(\rho)E(Ze)$ with $E = \rho\Phi_{\min}$, thus $m \in \rho(M)' \cap M$ and $W = mV_\rho$ showing in particular that W implements a left inverse Φ of ρ. Clearly Φ is faithful if m is invertible. Conversely, if Φ is faithful, $p \in (\rho,\rho)$ is a projection and $pm = 0$ then $\Phi(p) = V_\rho^* m^* p m V_\rho = 0$, thus $p = 0$ so m is invertible.

If moreover ρ is irreducible, then $m \in \mathbf{C}$, thus $m = \pm 1$ because both W and V_ρ are isometries commuting with J. □

Recall now that the *dimension* $d(\rho)$ of ρ is defined as the square root of the minimal index of ρ.

A.4 Proposition. *Let σ be a finite index endomorphism of the factor M and ρ an irreducible subsector of σ. If w is an isometry in (ρ,σ), then*

$$V_\rho = \sqrt{\frac{d(\sigma)}{d(\rho)}}\, w^* J w^* J V_\sigma .$$

If $\sigma = \oplus_{i=1}^N n_i\rho_i$ is an irreducible decomposition of σ and for each i $\{w_k^{(i)},\ k = 1,\dots n_i\}$ is an orthonormal basis of isometries in (ρ_i,σ), then

$$V_\sigma = \sum_{i=1}^N \sum_{k=1}^{n_i} \sqrt{\frac{d(\rho_i)}{d(\sigma)}}\, w_k^{(i)} J w_k^{(i)} J V_{\rho_i} . \tag{A.4}$$

Proof. We prove the second assertion that implies the first one. Set W equal to the right hand side in $(A.4)$. The ranges of the $w_k^{(i)}$'s are pairwise orthogonal and the coefficients verifies $\sum_{i=1}^N n_i \frac{d(\rho_i)}{d(\sigma)} = 1$, thus W is an isometry and a direct verification shows that it implements σ. Moreover W commutes with J, thus Lemma A.3 shows that W implements a left inverse Φ of σ. But W also preserves the natural cone $\mathcal{P}^\natural(M,\Omega)$, because this is true for each of its terms, thus W is the standard implementation of Φ by Proposition A.2. It remains to show that Φ is the minimal left inverse. Now a left inverse is determined by the state obtained by restricting it to (σ,σ). The value of Φ on the minimal projection $w_k^{(i)} w_k^{(i)*}$ is $\frac{d(\rho_i)}{d(\sigma)}$, hence it is the minimal left inverse. □

Appendix B. Invariant vectors for representations of SL(2,R).

We start by recalling the "vanishing of the matrix coefficient theorem" for a connected simple Lie group $\mathbf{G}$ with finite center, see [32].

B.1 Theorem. *Let U be a unitary representation of $\mathbf{G}$ on a Hilbert space $\mathcal{H}$. If U does not contain the identity, then $(U(g)\xi,\eta) \to 0$ as $g \to \infty$ for all $\xi,\eta \in \mathcal{H}$.*

As a consequence, if U is a unitary representation of $\mathbf{G}$ and $\xi \in \mathcal{H}$ then the subgroup $\{g \in \mathbf{G},\ U(g)\xi = \xi\}$ is either compact or equals to $\mathbf{G}$.

In the following $\mathbf{G}$ always denotes the universal covering group of $SL(2,\mathbf{R})$ and we state an explicit corollary in this case. Let us consider the one-parameter subgroups of $\mathbf{G}$ of the translations, dilations and rotations defined as the lifting to $\mathbf{G}$ of the one-parameter subgroups of $SL(2,\mathbf{R})$

$$T(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \Lambda(s) = \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{pmatrix}, \quad R(\vartheta) = \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}, \qquad (B.1)$$

and we still denote them by the same symbols T, Λ, R (cf. the definitions in Section 1).

B.2 Corollary. *Let U be a unitary representation of $\mathbf{G}$ and Ω a vector of the Hilbert space $\mathcal{H}$. The following are equivalent:*
(i) $\mathbf{C}\Omega$ *are the only U invariant vectors.*
(ii) $\mathbf{C}\Omega$ *are the only $U(T(\cdot))$ invariant vectors.*
(iii) $\mathbf{C}\Omega$ *are the only $U(\Lambda(\cdot))$ invariant vectors.*
If moreover the generator of $U(R(\cdot))$ is positive then the former are also equivalent to
(iv) $\mathbf{C}\Omega$ *are the only $U(R(\cdot))$ invariant vectors.*

Proof. Although the cardinality of the center $\mathbf{Z}$ of $\mathbf{G}$ is infinite, we check that Theorem B.1 still applies. By decomposing U into a direct integral of irreducible representations, it is sufficient to consider the case in which U is irreducible. Since U

is infinite-dimensional, the tensor product with its conjugate representation $U \otimes \bar{U}$ does not contain the identity. Now $U \otimes \bar{U}$ is trivial on the center $\mathbf{Z}$, hence defines a representation of $PSL(2, \mathbf{R})$. If $\xi \in \mathcal{H}$ then by Theorem B.1

$$|(U(g)\xi, \xi)|^2 = (U(g) \otimes \bar{U}(g)\xi \otimes \bar{\xi}, \xi \otimes \bar{\xi}) \to 0 \quad \text{as} \quad g \to \infty.$$

Then the first set of equivalences is then clear. Furthermore (i) is equivalent to (iv) if the conformal Hamiltonian is positive because the identity is the only irreducible unitary representation of $\mathbf{G}$ with lowest weight 0. □

In this paper we need a result in the spirit of Theorem B.1 concerning representations of the subgroup $\mathbf{P}$ of the upper triangular matrices in $SL(2, \mathbf{R})$, namely the group generated by the translations and the dilations.

B.3 Proposition. *Let U be a unitary representation of $\mathbf{P}$ on a Hilbert space $\mathcal{H}$. If $F \subset \mathcal{H}$ is a finite-dimensional subspace which is globally $U(\Lambda(\cdot))$-invariant, then F is left pointwise fixed by $U(T(\cdot))$.*

Proof. Setting $u(t) := U(T(t))$ and $v(s) := V(\Lambda(s))$ we have two one-parameter unitary groups on $\mathcal{H}$ satisfying the commutation relations

$$v(s)u(t)v(-s) = u(e^s t) , \qquad t, s \in \mathbf{R}. \tag{$B.2$}$$

Since F is finite dimensional, we need to show that $u(t)\xi = \xi$ if ξ is a v-eigenvector, i.e. there exists a character $\chi \in \widehat{\mathbf{R}}$ such that

$$v(s)\xi = \chi(s)\xi , \quad s \in \mathbf{R} . \tag{$B.3$}$$

Indeed in this case by the formula $(B.2)$ implies

$$u(e^s t)\xi = v(s)u(t)v(-s)\xi = \overline{\chi(s)}v(s)u(t)\xi$$

hence

$$(u(e^s t)\xi, \xi) = (u(t)\xi, \xi), \quad t, s \in \mathbf{R}.$$

As $s \to -\infty$ we thus have

$$(\xi, \xi) = (u(t)\xi, \xi)$$

that implies $u(t)\xi = \xi$ by the limit case of the Schwartz inequality. □

Before concluding this appendix, we recall a known fact needed in the text.

B.4 Lemma. *Let U be a unitary representation of $\mathbf{G}$. The following are equivalent:*
(i) The generator of $U(R(\cdot))$ is positive.
(ii) The generator of $U(T(\cdot))$ is positive.
In this case, if U is non-trivial, the spectrum of the generator of $U(T(\cdot))$ is $[0, \infty)$.

Proof. For the equivalence $(i) \Leftrightarrow (ii)$ see e.g. [26]. The last statement follows because the spectrum of $U(T(\cdot))$ has to be dilation invariant because of the commutation relations $(B.2)$. □

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