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The Trace Formula and the Distribution of Eigenvalues of Schrödinger Operators on Manifolds all of whose Geodesics are closed

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Abstract

We investigate the behaviour of the remainder term $R(E)$ in the Weyl formula

$$\#\{n|E_n \leq E\} = \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\,E^{d/2} + R(E)$$

for the eigenvalues E_n of a Schrödinger operator on a d-dimensional compact Riemannian manifold all of whose geodesics are closed. We show that $R(E)$ is of the form $E^{(d-1)/2}\,\Theta(\sqrt{E})$, where $\Theta(x)$ is an almost periodic function of Besicovitch class B^2 which has a limit distribution whose density is a box-shaped function. This is in agreement with a recent conjecture of Steiner [19, 1]. Furthermore we derive a trace formula and study higher order terms in the asymptotics of the coefficients related to the periodic orbits. The periodicity of the geodesic flow leads to a very simple structure of the trace formula which is the reason why the limit distribution can be computed explicitly.

1 Introduction

In the theory of quantum chaos [15] there has been intensive investigations of the statistical properties of the energy levels of quantum mechanical systems. Special emphasis has been on the relation to the classical limit $\hbar \to 0$. It has been found out that the statistical properties of the energy levels strongly depend on the ergodic properties of the associated classical system. Short range correlations in the spectrum are in case of classically integrable systems well described by a Poissonian distribution, whereas for classically chaotic system the distributions obtained from random matrix theory fit quite well. But these models fail to describe long range correlations, and also for short range correlation exceptions are known. The so called arithmetic systems, which are classically chaotic, possess some statistical features very similar to integrable systems [10]. A statistical measure which gives a clear distinction between classically chaotic and classically integrable systems was missing until now.

Inspired by number theoretical results [16] there has been recently a number of investigations of a global statistical measure in case of integrable systems [9, 5, 7, 6, 18, 8]. It is conjectured in [19], and numerically tested for some chaotic systems in [1], that this measure provide a classification of quantum mechanical systems according to their classical limit.

To describe it we introduce some notation. Let (M, g) be a compact, d-dimensional Riemannian manifold, with metric g, and Δ_g its Laplace-Beltrami operator, defined by $\Delta_g f = \mathrm{div}_g(\mathrm{grad}_g f)$. Denote by $E_0 \leq E_1 \leq E_2 \ldots$ the eigenvalues of $-\Delta_g$, counted with multiplicities, and by $N(E) = \#\{j | E_j \leq E\}$ the counting-function. The asymptotic behaviour of $N(E)$ for $E \to \infty$ is given by the famous Weyl law [17],

$$N(E) = \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\, E^{d/2} + O(E^{(d-1)/2}) \quad . \tag{1}$$

The remainder term is in general a very wild function and difficult to study, but for certain two-dimensional integrable systems the authors of the above cited articles were able to show that $N(E)$ can be decomposed as follows:

$$N(E) = \frac{1}{4\pi}\mathrm{Vol}(M)E + E^{1/4}\Theta(\sqrt{E}) \tag{2}$$

where $\Theta(x)$ is an almost periodic function of Besicovitch class B^1 (i.e., the mean value of $|\Theta(x)|$ exists, see below for more information on B^1)), whose frequencies are the lengths of the closed geodesics. The first term in (2) describes the mean behaviour of $N(E)$, while the second term generally oscillates around zero with an amplitude growing like $E^{1/4}$. So $\Theta(x)$ describes the normalized fluctuations of $N(E)$ around the mean behaviour, and to get a measure of this fluctuations one can ask for the limit distribution of $\Theta(x)$. The second main result of the above cited papers is that the function $\Theta(x)$ has a limit distribution, i.e., there exists a non-negative function $p(x)$, such that for every bounded continuous function $g(x)$, and any density of a probability distribution $\rho(x)$, concentrated on $[0, 1]$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\Theta(x))\rho(x)dx = \int_{-\infty}^{\infty} g(x)p(x)dx \quad . \tag{3}$$

The density $p(x)$ of the limit distribution is an entire function of x, possessing the following asymptotic behaviour on the real line,

$$\ln(p(x)) \sim -C_\pm x^4 \qquad x \to \pm\infty, \quad C_\pm > 0 \quad . \tag{4}$$

Several numerical studies have shown that $p(x)$ can look quite different for different integrable systems [6]. But for classically chaotic systems numerical tests suggest that there is a universal behavior, $p(x)$ was always found to be in excellent agreement with a normalized Gaussian distribution, in contrast to (4). The conjecture has been formulated [19, 1], that the limit distribution of the normalized remainder term of $N(E)$ is always a Gaussian distribution for chaotic systems, while for integrable systems it is non-universal and not Gaussian, and so provides a distinction between classically integrable and classically chaotic systems. This conjecture is well supported by the so far known analytical and numerical results [1, 2, 3, 10], including the results of this paper.

In this note we will study the limit distribution for a class of Hamilton operators on compact manifolds, which are characterized by the property that the flow generated by their principal symbol is periodic. To be more precise P should be a classical pseudo-differential operator of order two, positive, elliptic and self-adjoint, with the property that the Hamiltonian flow generated by its principal symbol is simply periodic.

To simplify the notation, we will state most of the results only for the case when (M, g) is a compact Riemannian manifold and $P = -\Delta_g + V(x)$ is a positive Schrödinger operator on M (in natural units $\hbar = 2m = 1$), where Δ_g is the Laplace Beltrami operator associated with g and $V(x)$ is a smooth (C^∞), and therefore bounded, potential on M. The principal symbol of P is just the Hamiltonian $H(x, \xi) = \sum g^{ij}(x)\xi_i\xi_j$ which generates the geodesic flow on M, and the periodicity of the flow means that all geodesics are closed. An example for such a manifold is the sphere S^2, and a potential system is for instance the spherical pendulum. Further examples will be given in section 2.

Note that the principal symbol is generally not the Hamiltonian given by the classical limit $\hbar \to 0$, but differs from it by the potential energy. This displays the fact that we will use high energy asymptotics, rather than the limit $\hbar \to 0$, and, because of the bounded potential, the high energy limit is ruled by the kinetic energy.

Our main result is:

Theorem 1.1 *Let (M, g) be a compact d-dimensional Riemannian manifold with simply periodic geodesic flow of period 2π. $P = -\Delta_g + V$ a positive Schrödinger operator on M, where Δ_g is the Laplacian associated with g, and V is a smooth function on M.*

Then $N(E)$ has the following representation

$$N(E) = \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\,E^{d/2} + E^{(d-1)/2}\,\Theta(\sqrt{E}). \tag{5}$$

Here $\Theta(x)$ is an almost periodic function of Besicovitch class B^2 with the Fourier series

$$\Theta(x) = \frac{ib_1}{2\pi} \sum_{\substack{k \neq 0 \\ k \in \mathbf{Z}}} \frac{e^{i\frac{\pi}{2}k\nu}}{k} e^{-2\pi ikx} \quad \mathrm{mod}\, B^2 , \tag{6}$$

where $b_1 = \frac{d\,\mathrm{Vol}\,(M)}{(4\pi)^{d/2}\Gamma(d/2+1)}$, and ν is the common Maslov index [20, 12] of the primitive periodic orbits. $\Theta(x)$ has a limit distribution whose density is a box-shaped function:

$$p(x) = \begin{cases} 0 & -\frac{b_1}{2} \geq x \\ \frac{1}{b_1} & -\frac{b_1}{2} \leq x \leq \frac{b_1}{2} \\ 0 & \frac{b_1}{2} \leq x \quad . \end{cases} \tag{7}$$

Some remarks are in order:

Remark 1. The space B^p is defined as follows. Denote by $||.||_{B^p}$ $(p \geq 1)$ the seminorm

$$||f||_{B^p} := \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(x)|^p dx \right)^{1/p} .$$

A function $f(x)$ belongs to a Besicovitch class B^p if it can be approximated, in the above seminorm, by trigonometric polynomials. One can show that this is equivalent to the existence of a formal Fourier series $\sum_{k\in\mathbf{Z}} a_k e^{i\lambda_k x}$, with $a_k \in \mathbf{C}$, $\lambda_k \in \mathbf{R}$, such that

$$\lim_{N\to\infty} ||f(x) - \sum_{|k|<N} a_k e^{i\lambda_k x}||_{B^p} = 0 \quad .$$

The coefficients a_k and the frequencies λ_k are uniquely determined by f. The space B^p is defined as the completion of the space of trigonometric polynomials with respect to $||.||_{B^p}$ modulo equivalence, where two functions are equivalent if the B^p norm of their difference vanishes. Note that the equivalence classes are very large in terms of pointwise defined functions, for instance if $f(x) \to 0$, for $x \to \infty$, then $||f||_{B^p} = 0$. Bleher has shown the existence of a limit distribution for every element of B^p [5], but an explicit expression is not known. More on this subject can be found in [5, 18].

Remark 2. The Fourier series (6) is L^2 convergent and gives

$$\frac{ib_1}{2\pi} \sum_{\substack{k\neq 0\\ k\in\mathbf{Z}}} \frac{e^{i\frac{\pi}{2}k\nu}}{k} e^{-2\pi ikx} = b_1 f(x - \frac{\nu}{4}), \tag{8}$$

with $f(x) = \frac{1}{2} - x$ for $0 < x < 1$, and periodically continued. So the statement above means $||\Theta(x) - b_1 f(x - \frac{\nu}{4})||_{B^2} = 0$.

Remark 3. In two dimensions we get $E^{1/2}$ in front of Θ rather than $E^{1/4}$ as in (2). This is due to the fact that the families of periodic orbits with the same period are submanifolds of T^*M of dimension three rather than two as in the generic integrable case.

In the next section we will collect some results about manifolds all whose geodesics are closed, especially about their spectra. In the final section we use this information to derive a trace formula for such manifolds and prove theorem 1.1 . Furthermore we investigate higher order terms in the asymptotics of the trace formula.

2 Some properties of manifolds all of whose geodesics are closed

We call a Riemannian manifold all of whose geodesics are closed with common length l a SC_l manifold, and its metric a SC_l metric (S for simple and C for closed). A thorough account on manifolds all of whose geodesics are closed can be found in the book by Besse [4]. In the following we will assume that $l = 2\pi$, this can always be achieved by a suitable rescaling of the metric.

The basic examples are the compact rank one symmetric spaces (CROSSes), S^n, $P^n\mathbf{R}$, $P^n\mathbf{C}$, $P^n\mathbf{H}$, $P^2\mathbf{Ca}$, equipped with their canonical metrics. For these spaces the eigenvalues of the Laplace-Beltrami operator are explicitly known (see table 1). On the spheres S^n there exists a large class of $SC_{2\pi}$ metrics different from the canonical one (cf. [13]), the so called Zoll-metrics, which are deformations of the canonical one. But the other CROSSes are rigid in the sense that they admit no more SC_l metrics beside the canonical ones. It is known [4] that for two dimensions the above examples exhaust all SC_l manifolds.

A subclass of the two dimensional Zoll metrics, the Zoll-surfaces of revolution, can be described very explicit (cf. [4]). In polar coordinates $(0 \leq \theta < \pi, 0 \leq \phi < 2\pi)$ their metrics are of the form

$$g = [1 + h(\cos\theta)]^2 d\theta^2 + \sin^2\theta \, d\phi^2 \quad ,$$
$$\text{where} \quad h :]-1,1[\quad \to \quad]-1,1[\quad \text{is odd, and} \quad h(-1) = 0 \quad .$$

No other restrictions on h are necessary (apart from the order of differentiability, which determines the smoothness of g; we will assume C^∞). An interesting feature of these metrics is that the Hamiltonians defined by them are all symplectically equivalent [22], i.e., they define the same classical system, and the different Laplace-Beltrami operators can be seen as different quantizations of the same classical system. This leads to the expectation that their quantum mechanical properties are similar. This is indeed true for the eigenvalues as Duistermaat and Guillemin have shown;

Theorem 2.1 [12] *Let $P = -\Delta_g + V$, and g a $SC_{2\pi}$ metric, then there is a positive constant K such that all eigenvalues of P lie in intervals $I_k = [\lambda_k^2 - K, \lambda_k^2 + K]$, where*

$$\lambda_k = k + \frac{\nu}{4} \quad , \quad k = 0, 1, 2, \dots$$

and ν is the common Maslov-index of the primitive periodic orbits. Conversely, if there is a K and an arithmetic sequence $\lambda_k = k + \nu/4$, such that the eigenvalues lie in the intervals I_k, then P has a principal symbol with 2π-periodic flow, and ν is the Maslov index of the primitive orbits.

To illustrate the result we compare it with the nonperiodic case. Generally the following Weyl law for the square roots of the eigenvalues $p_k = \sqrt{E_k}$ is valid ([17])

$$N(p) = \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\, p^d + O(p^{d-1}).$$

When the set of periodic orbits in phase space has measure zero, one can improve the remainder term ([17, 12])

$$N(p) = \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\, p^d + o(p^{d-1}),$$

and one gets for the number of p_k in an interval $[p - \epsilon/2, p + \epsilon/2]$

$$N(p+\epsilon/2) - N(p-\epsilon/2) = \frac{\epsilon\, d\, \mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\, p^{d-1} + o(p^{d-1}).$$

So the property that the periodic orbits form a set of measure zero leads to a rather uniform distribution of the p_k's, in sharp contrast to the case where all orbits are periodic, where theorem 2.1 has the consequence that the p_k's lie in decreasing intervals of length $O(1/k)$ around the arithmetic sequence λ_k.

We call the eigenvalues which lie in the k'th interval I_k the k'th eigenvalue cluster, and naturally the questions arise, how many eigenvalues lie in such a cluster, and how they are distributed. The answers are given in the next two theorems by Colin de Verdiere and Weinstein.

Theorem 2.2 [11] *There exists a polynomial* $R(t) = b_1 t^{d-1} + b_3 t^{d-3} + \cdots$ *with*

$$b_1 = \frac{d\,\mathrm{Vol}\,(M)}{(4\pi)^{d/2}\Gamma(d/2+1)} \quad ,$$

and a natural number k_0*, such that for* $k > k_0$ *the number of eigenvalues in* I_k *is* $m_k = R(k+\frac{\nu}{4})$.

Denote by

$$\mu_k(E) = \frac{1}{m_k} \sum_{E_i \in I_k} \delta(E - (E_i - \lambda_k^2)) \tag{9}$$

the normalized spectral-measure of the k'th eigenvalue-cluster. Weinstein has shown [23] that for a Zoll Laplacian there is a unitary operator U such that

$$U \Delta_g U^{-1} = \Delta_0 + A \quad ,$$

where Δ_0 is the canonical Laplacian on the sphere, and A is a pseudo-differential operator of order zero. Let $a_V(x,\xi)$ be the principal symbol of $A + UVU^{-1}$ and denote by

$$\overline{a_V}(x,\xi) = \frac{1}{2\pi} \int_0^{2\pi} a_V(\Phi^t(x,\xi))dt$$

the average of $a_V(x,\xi)$ over the geodesic flow Φ^t, generated by the canonical metric of S^d. In case of $\Delta_g = \Delta_0$, a_V clearly reduces to $a_V = V$, and this construction applies also to any other CROSS. The asymptotic distribution of the eigenvalues in the cluster can be described now.

Theorem 2.3 [23, 11] *There exists a sequence of distributions* $\{\beta_i\}_{i \in \mathbf{N}}$*, supported in* $[-K, K]$*, such that for every* $\varphi \in C^\infty(\mathbf{R})$*;*

$$\langle \mu_k, \varphi \rangle \sim \langle \beta_0, \varphi \rangle + \langle \beta_1, \varphi \rangle \frac{1}{k^2} + \langle \beta_2, \varphi \rangle \frac{1}{k^4} + \cdots \quad k \to \infty \,, \tag{10}$$

and

$$\langle \beta_0, \varphi \rangle = \frac{1}{\iint_{H(x,\xi)<1} dx\, d\xi} \iint_{H(x,\xi)<1} \varphi(\overline{a_V}(x,\xi)) dx\, d\xi \quad . \tag{11}$$

Here $H(x,\xi) = \sum g^{ij}(x)\xi_i\xi_j$ *is the Hamilton function which generates the canonical geodesic flow, and by* $\langle \beta, \varphi \rangle$ *we denote the application of the distribution* β *to the function* φ*.*

Remark. The above theorems are, with slight modifications, also valid in the more general pseudo-differential operator setting (see [17], chapter 29.2). One can show that for every pseudo-differential operator P, satisfying our assumptions, there is a pseudo-differential operator Q of order zero, such that the square-roots of the eigenvalues of $P + Q$ are an arithmetic sequence λ. Then (11) is valid with a_V replaced by the principal symbol q of Q. b_1 is given by $\frac{d}{(2\pi)^d} \iint_{p<1} dx d\xi$, where p is the principal symbol of P, and $dx d\xi$ is the canonical measure in phase space. In Theorem (2.3) additionally terms of odd order can occur in the asymptotic expansion.

M	$\dim M$	ν	E_k	λ_k	b_1
S^n	n	$2(n-1)$	$k(k+n-1)$	$k+\frac{n-1}{2}$	$\frac{2}{(n-1)!}$
$P^n\mathbf{C}$	$2n$	$2n$	$k(k+n)$	$k+\frac{n}{2}$	$\frac{2}{n!(n-1)!}$
$P^n\mathbf{H}$	$4n$	$4n+2$	$k(k+2n+1)$	$k+\frac{2n+1}{2}$	$\frac{2}{(2n-1)!(2n+1)!}$
$P^2\mathbf{Ca}$	16	22	$k(k+11)$	$k+\frac{11}{2}$	$\frac{2*39}{15!}$

Table 1: The relevant data for some CROSSes, from [4, 11].
The Maslov indices, the eigenvalues, the corresponding arithmetic sequence, and the coefficient of the leading order term in the polynomial which describes the multiplicities.

3 The trace formula

The trace formula is a relation between the quantum mechanical energy-spectrum and the periodic orbits of the classical system. From [12, 20] follows that for $SC_{2\pi}$ manifolds the trace formula reads:

$$\sum_{n=0}^{\infty}\delta(p-\sqrt{E_n})=\sum_{k\in\mathbf{Z}}\alpha_k(p)e^{i\frac{\pi}{2}\nu k}e^{-2\pi ikp}.$$

The left hand side is the spectral momentum density $d(p)$. The sum on the right hand side is a sum over all families of periodic orbits, ν is the Maslov index of the primitive orbits, and for the functions $\alpha_k(p)$ only the asymptotic behavior is known:

$$\alpha_k(p)=\frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\,p^{d-1}+O(p^{d-2}),\quad p\to+\infty,$$

and

$$\alpha_k(p)=O(p^{-\infty}),\quad p\to-\infty\quad .$$

There is no dependence on the potential in the leading order, but this is not very surprising because the potential is bounded and therefore the high energy limit is dominated by the kinetic energy, i.e., by the metric. But what is more surprising is that, because the volume is invariant under Zoll deformations [21], the leading order in the trace formula does not distinguish between Zoll metrics on S^n.

We will choose a different approach to the trace formula, based on the information from the last section, which will give us also the higher order aymptotics for $\alpha_k(p)$, where the differences between different Zoll metrics, and the influence of a potential will be visible. Our starting point is an exact trace formula for the CROSSes.

Proposition 3.1 *Let (M,g) be a CROSS and $P_g=-\Delta_g+c$ the Laplacian shifted by a constant c such that the eigenvalues become $\lambda_n^2=(n+\nu/4)^2$, $n=1,2,3,\ldots$, with multiplicities $R(\lambda_n)$. $R(t)$ is the polynomial from theorem 2.2, and ν is the common Maslov index of the primitive periodic orbits. Then there is the following periodic orbit sum, valid in $\mathcal{S}'(\mathbf{R})$, for the spectral momentum density:*

$$\sum_{n=1}^{\infty}R(\lambda_n)\delta(p-\lambda_n)=R^+(p)\sum_{k\in\mathbf{Z}}e^{i\frac{\pi}{2}\nu k}e^{-2\pi ikp},\tag{12}$$

where $R^+(p)$ denotes a smooth (C^∞) function with $R^+(p) = R(p)$ for $p \geq \nu/4 + 1 = \lambda_1$, and $R^+(p) = 0$ for $p \leq \nu/4$.

Proof. With the Poisson identity [17]

$$\sum_{n\in\mathbf{Z}} \delta(p-n) = \sum_{k\in\mathbf{Z}} e^{-2\pi ikp}$$

we get

$$\begin{aligned}
\sum_{n=1}^{\infty} R(\lambda_n)\delta(p-\lambda_n) &= R(p)\sum_{n=1}^{\infty}\delta(p-(n+\frac{\nu}{4})) \\
&= R^+(p)\sum_{n\in\mathbf{Z}}\delta(p-\frac{\nu}{4}-n) = R^+(p)\sum_{k\in\mathbf{Z}} e^{i\frac{\pi}{2}\nu k}e^{-2\pi ikp} \quad .
\end{aligned}$$

□

Remark. In the following calculations it will be important that $R^+(p)$ is smooth and vanishes for negative p. To make such a choice for $R^+(p)$ possible is the reason that we let n start with 1 instead of 0, so that the eigenvalues of P_g are strictly positive.

Using the trace formula for CROSSes, and the information from the preceding section we will now derive a trace formula for general $SC_{2\pi}$ manifolds. The idea is to construct, for a given $SC_{2\pi}$ manifold, an operator $\mathbf{A}$ which maps $\delta(p-\lambda_n)$ to $\frac{1}{R(\lambda_n)}\sum_{E_i\in I_n}\delta(p-\sqrt{E_n})$, and so the sum $d_0(p) = \sum R(\lambda_n)\delta(p-\lambda_n)$ to the spectral momentum density $d(p) = \sum\delta(p-\sqrt{E_n})$. Applying this operator to (12) will lead to the trace formula.

Let $\chi(s)$ be a function with $\chi \in C_0^\infty(\mathbf{R})$, $\operatorname{supp}\chi \in [-1,1]$, $\chi(0) = 1$, and $\sum_{l=0}^{\infty}\chi(s-l) = 1$ for $s > 0$. We define a distribution $\mu(s)$, depending on a parameter s, by

$$\mu(s,E) := \sum_{l=0}^{\infty}\chi(s-l)\mu_l(E) \quad , \tag{13}$$

where μ_l is the normalized spectral measure of the l'th eigenvalue cluster (9). Then we have $\mu(k,E) = \mu_k(E)$, and from theorem 2.3 follows that for every $\varphi \in C^\infty$

$$\langle\mu(s),\varphi\rangle \sim \sum_{i=0}^{\infty}\langle\beta_i,\varphi\rangle\frac{1}{s^{2i}} \quad s\to\infty \quad . \tag{14}$$

This allows us to write for the spectral density $D(E) = \sum R(\lambda_n)\mu_n(E-\lambda_n^2)$ of any $SC_{2\pi}$ manifold

$$D(E) = \int_{-\infty}^{\infty} D_0(s)\mu(\sqrt{s}-\nu/4, E-s)ds \quad ,$$

with $D_0(E) = \sum R(\lambda_n)\delta(E-\lambda_n^2)$, as a short calculation shows. If we use $d(p) = 2p^+D(p^2)$, where p^+ is defined similar to $R^+(p)$, we get an expression for $d(p)$

$$d(p) = 2p^+\int d_0(t)\mu(t-\nu/4, p^2-t^2)dt \quad .$$

So we have constructed the operator $\mathbf{A}$ with Schwartz kernel $K_{\mathbf{A}}(p,t) = 2p^+\mu(t-\nu/4,p^2-t^2)$. Applying $\mathbf{A}$ to both sides of equation (12) leads to

$$d(p) = 2p^+\sum_{k\in\mathbf{Z}} e^{i\frac{\pi}{2}\nu k}\int R^+(t)\mu(t-\nu/4,p^2-t^2)e^{-2\pi ikt}dt \quad , \tag{15}$$

and for $\alpha_k(p)$ we get

$$\alpha_k(p) = e^{2\pi ikp}\mathbf{A}(R^+e^{-2\pi ik(.)}) \quad ,$$

so we have arrived at the trace formula. Note that $d_0(p)$ must not come from a CROSS, here we use (12) just as a summation formula for certain distributions.

Proposition 3.2 *Let (M,g) be a $SC_{2\pi}$ manifold, and $P = -\Delta_g + V$ a positive Schrödinger operator with spectral momentum measure $d(p)$, then*

$$d(p) = \sum_{k\in\mathbf{Z}} \alpha_k(p)\, e^{i\frac{\pi}{2}\nu k}\, e^{-2\pi ikp} \quad , \tag{16}$$

as elements of $\mathcal{S}'(\mathbf{R})$, with

$$\alpha_k(p) = p^+\int_{-\infty}^{p^2} \frac{R^+(\sqrt{p^2-s})}{\sqrt{p^2-s}}\,\mu(\sqrt{p^2-s}-\nu/4,s)\, e^{-2\pi ik(\sqrt{p^2-s}-p)}\, ds \quad . \tag{17}$$

Proof. From (15) we get

$$\alpha_k(p) = 2p^+\int_0^\infty R^+(t)\,\mu(t-\nu/4,p^2-t^2)\, e^{-2\pi ik(t-p)}\, dt \tag{18}$$

and (17) follows by a change of variables. There only remains to justify the interchange of summation and integration, i.e., we have to show that the right hand side is well defined as a distribution. Let $\varphi\in\mathcal{S}(\mathbf{R})$, then

$$\int d(p)\varphi(p)dp = \sum_{k\in\mathbf{Z}} e^{i\frac{\pi}{2}\nu k}\int \alpha_k(p)\varphi(p)e^{-2\pi ikp}dp$$

but

$$\begin{aligned}
\int \alpha_k(p)\varphi(p)e^{-2\pi ikp}dp &= \iint \varphi(p)\, 2p^+\mu(t-\nu/4,p^2-t^2)\, dp\ R^+(t)\, e^{-2\pi ikt}\, dt \\
&= \int\left\{\int \varphi(\sqrt{q+t^2}\,)\,\frac{(\sqrt{q+t^2})^+}{\sqrt{q+t^2}}\,\mu(t-\nu/4,q)\, R^+(t)\, dq\right\} e^{-2\pi ikt}\, dt \\
&= O(k^{-\infty})
\end{aligned}$$

because the inner integral is a Schwartz function of t (here it enters that R^+ is smooth) and so its Fourier transform is again a Schwartz function. So the sum exists.

□

We will now study $\alpha_k(p)$ more closely, but first we look at an example which displays already typical properties.

Example. Let M be a CROSS, and $P = P_g + C$ where C is a constant, and P_g is the shifted Laplacian defined in proposition 1. Then $E_n = \lambda_n^2 + C$ and $\mu_k = \delta(E-C)$, so $\mu(s,E) = \delta(E-C)$ and this leads to

$$\alpha_k(p) = p^+\frac{R^+(\sqrt{p^2-C})}{\sqrt{p^2-C}}e^{-2\pi ik(\sqrt{p^2-C}-p)}$$

for $p^2 > C$, and 0 else. If one fixes k and let $p\to\infty$ then $\alpha_k(p)$ has a regular asymptotics

$$\alpha_k(p) = b_1p^{d-1} + O(p^{d-2}) \quad ,$$

if one takes the limit $p, k \to \infty$ with $k/p \to \gamma$ =const., then

$$\alpha_k(p) = (b_1 p^{d-1} + O(p^{d-2}))\, e^{\pi i \gamma c} \quad ,$$

and for p fixed and $k \to \infty$ $\alpha_k(p)$ is oscillating. But notice that

$$|\alpha_k(p)| = b_1 p^{d-1} + O(p^{d-2}) \quad ,$$

independent of k.

We will now show that the asymptotic behavior of general $\alpha_k(p)$ is similar.

Lemma 3.1 *$\alpha_k(p)$ has the following asymptotic properties:*

i) Generally, for $p \to \infty$,

$$\alpha_k(p) = b_1 \langle \beta_0, e^{-2\pi i k(\sqrt{p^2-(.)}-p}\rangle\, p^{d-1} + O(p^{d-2}),$$

depending on how fast k growth compared to p.

ii) For $p \to \infty$ and k fixed

$$\begin{aligned}\alpha_k(p) = b_1\, p^{d-1} \quad &+ \quad i\pi k b_1 \int s\beta_0(s) ds\, p^{d-2} \\ &- \quad \left[\frac{1}{2} b_1 \pi^2 k^2 \int \beta_0 s^2 ds + \frac{1}{2} b_1 (d-2) \int \beta_0 s ds - b_3\right] p^{d-3} + O(p^{d-4}).\end{aligned}$$

iii) For $p, k \to \infty$, with $k/p \to \gamma$ = const.

$$\alpha_k(p) = b_1 \langle \beta_0, e^{i\pi\gamma(.)}\rangle p^{d-1} + O(p^{d-3}).$$

Here all terms in ii) and iii) have been written down which depend on β_0 only, the next order terms contain contributions from β_1.

Proof. Inserting (13) in (17) gives

$$\begin{aligned}\alpha_k(p) &= \sum_{l=0}^{\infty} \frac{1}{m_l} \sum_{E_j \in I_l} p^+ \int \frac{R^+(\sqrt{p^2-s})}{\sqrt{p^2-s}} \chi(\sqrt{p^2-s} - \nu/4 - l)\delta(s - (E_j - \lambda_l^2)) e^{-2\pi i k(\sqrt{p^2-s}-p)} ds \\ &= \sum_{l=0}^{\infty} \frac{1}{m_l} \sum_{E_j \in I_l} p^+ \frac{R^+(\sqrt{p^2-(E_j-\lambda_l^2)})}{\sqrt{p^2-(E_j-\lambda_l^2)}} \chi(\sqrt{p^2-(E_j-\lambda_l^2)} - \nu/4 - l) e^{-2\pi i k(\sqrt{p^2-(E_j-\lambda_l^2)}-p)}\end{aligned}$$

but $|E_j - \lambda_l^2| \leq K$ for $E_j \in I_l$, so for p, l large

$$p^+ \frac{R^+(\sqrt{p^2-(E_j-\lambda_l^2)})}{\sqrt{p^2-(E_j-\lambda_l^2)}} \chi(\sqrt{p^2-(E_j-\lambda_l^2)} - \nu/4 - l) \sim b_1 p^{d-1} \chi(p - \nu/4 - l) + O(p^{d-2})$$

and since, for $l \to \infty$,

$$\frac{1}{m_l} \sum_{E_j \in I_l} e^{-2\pi i k(\sqrt{p^2-(E_j-\lambda_l^2)}-p)} = \langle \mu_l, e^{-2\pi i k(\sqrt{p^2-(.)}-p)}\rangle = \langle \beta_0, e^{-2\pi i k(\sqrt{p^2-(.)}-p)}\rangle + O(\frac{1}{l^2})$$

we get

$$\alpha_k(p) = b_1 p^{d-1} \langle \beta_0, e^{-2\pi i k(\sqrt{p^2-(.)}-p)} \rangle + O(p^{d-2}).$$

This proves i). The behaviour of $\alpha_k(p)$ depends now on how k grows with p, and inserting the asymptotic expansion of $e^{-2\pi i k(\sqrt{p^2-s}-p)}$ leads to the leading order terms in ii) and iii).
For the higher order terms in ii) and iii) one inserts the asymptotic development of the integrand in (17) (notice that it has compact support in s), and uses furthermore that $\int \beta_i dE = 0$ for $i > 0$.

□

Remark 1. In the asymptotic expansion ii) of $\alpha_k(p)$ there appear all moments of the measures $\{\beta_i\}$. So knowing these moments and the coefficients of the polynomials $R(t)$, one can recover the asymptotics of $\alpha_k(p)$ and vice versa. This is a relation between the spectrum and geometric properties of (M, g).

Remark 2. The asymptotics ii) have a similar structure, especially the dependence on k, to the one derived by Guillemin in [14] for an $\alpha_k(p)$ coming from a nondegenerate elliptic orbit,

$$\alpha_k(p) = c_1 + c_2 p^{-1} + O(p^{-2})$$

with

$$c_2 = 2\pi i k c_1(\cdots) \ .$$

Here k denotes the number of iterates of the primitive orbit, and the expression in brackets depends on the symbol of the Hamilton operator in a neighborhood of the orbit.

Now we come to the proof of theorem 1.1. $N(E)$ is given by

$$N(E) = \int_0^{\sqrt{E}} d(p)dp = \int_0^{\sqrt{E}} \alpha_0(p)dp + \int_0^{\sqrt{E}} \sum_{k\neq 0} \alpha_k(p) e^{i\frac{\pi}{2}\nu k} e^{-2\pi i k p} dp.$$

The first integral gives

$$\int_0^{\sqrt{E}} \alpha_0(p)dp = \frac{b_1}{d} E^{d/2} + O(E^{d/2-1}) \quad ,$$

so we get

$$N(E) = \frac{b_1}{d} E^{d/2} + E^{(d-1)/2} \Theta(\sqrt{E}),$$

with $\Theta(x)$ given by

$$\Theta(x) = \frac{1}{x^{d-1}} \sum_{k\neq 0} e^{i\frac{\pi}{2}\nu k} \int_0^x \alpha_k(p) e^{-2\pi i k p} dp + O(1/x) \quad .$$

The sum converges in the B^2 norm to the function

$$\sum_{k\neq 0} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi i k x} \quad .$$

To see this we compute $||\Theta_N(x) - \sum_{k\neq 0} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi i k x}||_{B^2}$ with

$$\Theta_N(x) = \sum_{\substack{k\neq 0 \\ |k|\leq N}} \frac{e^{i\frac{\pi}{2}\nu k} e^{-2\pi i k x}}{x^{d-1}} \int_0^x \alpha_k(p) e^{-2\pi i k(p-x)} dp + O(1/x) \quad .$$

The triangle inequality gives

$$\begin{aligned} ||\Theta_N(x) - \sum_{k\neq 0} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi ikx}||_{B^2} \ \leq\ & ||\Theta_N(x) - \sum_{\substack{k\neq 0\\ |k|\leq N}} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi ikx}||_{B^2} \\ & +||\sum_{|k|>N} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi ikx}||_{B^2} \quad , \end{aligned}$$

but

$$\frac{1}{x^{d-1}} \int_0^x \alpha_k(p) e^{-2\pi ik(p-x)} dp = \frac{ib_1}{2\pi k} + O(\frac{1}{x})$$

by lemma 3.1, so the first term on the right hand side vanishes. The remaining term can be estimated by

$$\sum_{|k|>N} \frac{b_1^2}{(2\pi k)^2} \sim \frac{b_1^2}{(2\pi)^2} \frac{1}{N} \quad ,$$

so

$$\lim_{N\to\infty} ||\Theta_N(x) - \sum_{k\neq 0} \frac{ib_1}{2\pi k} e^{i\frac{\pi}{2}\nu k} e^{-2\pi ikx}||_{B^2} = 0$$

and we have proven (5) and (6).

To prove (7) we notice that for a periodic function $f(x)$, with period 1 and for every continuous g,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(f(x))dx = \int_0^1 g(f(\theta))d\theta \quad , \tag{19}$$

because

$$\begin{aligned} \frac{1}{T} \int_0^T g(f(x))dx &= \frac{1}{T} \int_0^{[T]} g(f(x))dx + \frac{1}{T} \int_{[T]}^T g(f(x))dx \\ &= \frac{[T]}{T} \int_0^1 g(f(x))dx + \frac{1}{T} \int_{[T]}^T g(f(x))dx \quad . \end{aligned}$$

Inserting (8) in (19) leads to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(\Theta(x))dx = \int_0^1 g(b_1 f(x - \nu/4))dx = \frac{1}{b_1} \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} g(x)dx \quad ,$$

so we have proven (7) for ρ in (3) a step function which is 1 on $[0,1]$ and 0 elsewhere. The proof for general ρ works with an approximation of ρ by step functions and can be found in [5, 18].

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