Disorder in the 1D spinless Holstein model.

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Abstract: We investigate a spinless fermion system on a one dimensional lattice interacting locally with the optical modes of a quantized phonon field: the Holstein model. The system is shown to have a disordered ground state, for small enough coupling, at any density. This is in contrast to the non quantized phonon case, the static Holstein model, which at half filling has an ordered ground state for all couplings.

§1 *Model and results.*

The hamiltonian is the sum of a free lattice fermion hamiltonian H_F, a free phonon field hamiltonian H_B and a local fermion phonon interaction V:

$$\begin{aligned}H=&\Big\{-\frac{\hbar^2}{2ml}\sum_{\vec x}[a^+_{\vec x}a^-_{\vec x+l}+a^+_{\vec x}a^-_{\vec x-l}+(\mu_0-2)n_{\vec x}]\Big\}+\\&+\Big\{-\frac{\hbar^2}{2\sigma l}\sum_{\vec x}\frac{\partial^2}{\partial\varphi^2_{\vec x}}+\frac{l\sigma}{2}\sum_{\vec x}[\omega^2\varphi^2_{\vec x}+\frac{c^2}{l^2}(\varphi_{\vec x}-\varphi_{\vec x+l})^2]\Big\}+\\&+\Big\{\frac{\hbar^2}{2ml}\sum_{\vec x}(-\nu+\lambda\varphi_{\vec x})(n_{\vec x}-\frac12)\Big\}\end{aligned}\tag{1.1}$$

where $\vec x$ is a point on the lattice with spacing l, $\vec x=jl$, $j\in(-\frac{L}{2l},\frac{L}{2l}]$, j integer. Periodic boundary conditions are imposed at $\pm\frac{L}{2}$ (by identifying such points). The $a^\pm_{\vec x}$ are creation and annihilation operators for a spinless fermion at $\vec x$, $n_{\vec x}=a^+_{\vec x}a^-_{\vec x}$ and $a^+_x a^-_y+a^-_y a^+_x=\delta_{xy}$.

The field $\varphi_{\vec x}$ represents a quantized bosonic field, corresponding to a discretized vibrating string with linear density σ, optical frequency ω and maximum wave propagation speed c. The physical meaning of $\varphi_{\vec x}$ is that of deformation of the crystal cell sitting at $\vec x$. The parameter m is a scale parameter fixing the bare fermion mass. The fermions chemical potential is $\mu=\mu_0+\nu$.

The reason for writing μ in this form is that, since the early works on the theory of Fermi systems, [LW], it has been realized that it is more natural to study the properties of such systems when the interaction strength λ is varied at fixed Fermi momentum rather than at fixed μ. For the free particle system, corresponding to $\lambda=0$, the Fermi momentum p_F is defined as the value of the momentum where the momentum distribution, which is the Fourier transform of the one particle reduced density matrix or equal time pair Schwinger function, has a discontinuity. This manifests itself in the one particle density matrix having an oscillatory decay $G_0(\vec x)\simeq|\vec x|^{-1}\sin p_F|\vec x|$ and p_F is obtained from μ by the relation $\mu=2(1-\cos lp_F)$. For $\lambda\neq0$ a Fermi momentum $P(\mu,\lambda)$ can still be defined via the position of the singularity of the Fourier transform of the one particle reduced density matrix. In particular, for λ small we shall prove that the one point reduced density

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matrix has an oscillating leading asymptotic behaviour $G(\vec{x})$ proportional to $|\vec{x}|^{-1-2\eta}(O(\lambda)+\sin P\vec{|x|})$ for some $\eta>0$. We now define $\mu_0 \equiv 2(1-\cos lp_F)$ and choose ν such that $P(\mu,\lambda)=p_F$. This defines ν as a function of p_F and λ with $\nu=0$ when $\lambda=0$.

It follows from our analysis (see [BGPS] and below) that P is smooth in μ,λ for μ near any prefixed $\mu_0\in(0,2)$ and λ small enough (how small depends on the value of μ_0). Therefore we can write $P=p_F+c_1(\mu-\mu_0)+b_1\lambda+\ldots$ and $c_1\neq 0$ (in fact $c_1=(2l\sin lp_F)^{-1}$). Setting $P=p_F$ then yields $\nu=d_1\lambda+\ldots$ with $d_1=b_1/c_1$.

We note that, according to the (formal) Luttinger theorem [L], fixing $P(\mu,\lambda)$ is equivalent to fixing the physical density $\rho=\rho(\mu_0+\nu,\lambda)$, *i.e.* P is independent of λ if ρ is fixed: in fact $P=\pi\rho$. The (formal) extension of this theorem (proved formally in [LW], [L] and formulated there as the "conservation of the Fermi surface volume" at constant density) to cover the present case is discussed, for completeness, in the appendix.

In general the value of ν is a complicated function of λ, which can only be determined order by order in perturbation theory. There is however an exception; in fact, setting $\varepsilon_{\vec{x}}=(-1)^{\vec{x}/l}$, the unitary transformation $a^{\pm}_{\vec{x}}\to\varepsilon_{\vec{x}}a^{\mp}_{\vec{x}}$, $\varphi_{\vec{x}}\to-\varphi_{\vec{x}}$ maps the hamiltonian with chemical potential $\mu=2$ and $\nu=\nu_0$ into that with $\mu=2$ and $\nu=-\nu_0$ and a state with density ρ into one with density $1-\rho$. Hence we see that, if $\nu=0$, there must be a ground state with density $\frac{1}{2}$. Furthermore the hamiltonian has other symmetries; namely translation invariance and reflection (parity) symmetry. Thus, if we suppose that the ground state is unique (a property that we expect but do not prove) then by applying to it the above three symmetries we see that $G(\vec{x})=\langle a^-_{\vec{x}}a^+_0\rangle=0$ for $\vec{x}/l$ even. Hence if one can prove, as we claim here, the existence of P such that $G(\vec{x})$ is proportional to $|\vec{x}|^{-1-2\eta}(O(\lambda)+\sin P\vec{|x|})$, it follows that $P=\pi/2l$ (otherwise, for $\vec{x}/l$ even and large enough, $\sin P\vec{x}$ could be of order 1), so that the Luttinger theorem is automatically satisfied.

The units can be fixed so that:

$$\hbar=m=l=m\sigma\omega^2=1 \tag{1.2}$$

Setting $b=c\omega^{-1},\sigma_0^2=\sigma$, H becomes:

$$\begin{aligned}2H=&\left[\sum_{\vec{x}}(-a^+_{\vec{x}}a^-_{\vec{x}+1}-a^+_{\vec{x}}a^-_{\vec{x}-1}+(2-\mu)a^+_{\vec{x}}a^-_{\vec{x}})\right]+\\&+\left\{-\frac{1}{\sigma_0^2}\sum_{\vec{x}}\frac{\partial^2}{\partial\varphi^2_{\vec{x}}}+\sum_{\vec{x}}(\varphi^2_{\vec{x}}+b^2\,(\varphi_{\vec{x}}-\varphi_{\vec{x}+1})^2)\right\}+\\&+\left(\sum_{\vec{x}}(-\nu+\lambda\varphi_{\vec{x}})\,(a^+_{\vec{x}}a^-_{\vec{x}}-\frac{1}{2})\right)\end{aligned} \tag{1.3}$$

with the half filled band case corresponding to $\mu=2,\nu=0$.

§*1.1 The Static Holstein Model.*

If $\sigma_0=+\infty$, $b=0$ the model (1.3) is the static Holstein model with "magnetic field" equal to $\lambda/4$, [LM]. The ground state problem is now equivalent to the computation of the fermionic energy $E(\lambda,\nu;\varphi)$ in the presence of the external field $\varphi_{\vec{x}}$ and then finding the field φ minimizing:

$$E(\lambda,\nu;\varphi)+\frac{1}{2}\sum_{\vec{x}}\varphi^2_{\vec{x}} \tag{1.4}$$

Calling $\mathcal{E}(\lambda,\nu)$ the minimum value, the corresponding density is obtained by setting $\rho-\frac{1}{2}=-2\partial_\nu\mathcal{E}(\lambda,\nu)$ which, if ρ is given *a priori* , is an equation determining ν as a function of λ,ρ.

The case $\mu=2,\nu=0,\rho=\frac{1}{2}$, the so called "half filled band" case, has been solved by [LM], following the methods used by [KL] for the case in which the field $\varphi_{\vec{x}}$ can only take values ±1 (the Falikov–Kimball model, see also [MM]). It is shown in [LM] that the minimizing field is $\varphi_{\vec{x}} = (-1)^{\vec{x}} f$ where f is a suitable constant, which can be easily computed by remarking that $E(\lambda, \nu; (-1)^{\vec{x}} f)$ can be evaluated by the Bloch waves techniques.

One finds that $\lambda \neq 0$ implies $f \neq 0$. This means that, if $\lambda \neq 0$, there are two non translation invariant ground states, in which the field $\varphi_{\vec{x}}$ has a periodic structure with period 2. This is interpreted by saying that at half filling the atoms of the system acquire a crystalline ordering with a period 2, at any non zero coupling strength; *i.e.* the Peierls instability. They, thus, behave as if they were non interacting particles immersed in a periodic potential with period 2.

It follows from this that the one particle energies are split into two bands, the first corresponding to the momenta $|k| < \frac{\pi}{2}$, and the second to the larger $k's$. The two bands are separated by a gap $\kappa(\lambda)$ and, as a consequence, the fermions are in a filled band state (because the Fermi momentum is $p_F = \frac{\pi}{2}$ for $\rho = 1/2$, by the above symmetry). Consequently the two point equal time Schwinger function for the fermions, $G(\vec{x}) = \lim_{t \to 0^-} G(\vec{x}, t)$, decays at large separation as,

$$|G(\vec{x})| \simeq e^{-\kappa(\lambda)|\vec{x}|} \tag{1.5}$$

where $\kappa(\lambda)$ is the energy gap at momentum $p_F = \pi/2$. Note that the energy gap $\kappa(\lambda) \xrightarrow[\lambda \to 0]{} 0$, and that at $\lambda = 0$ one has instead: $G(\vec{x}) = G_0(\vec{x}) = -(\pi \vec{x})^{-1} \sin p_F \vec{x}$.

§*1.2 The Dynamic Holstein Model.*

The above picture holds *for all* values of $\lambda \neq 0$ at half filling ($\rho = \frac{1}{2}, \nu = 0$) when the phonon field is treated classically, by setting $\sigma_0^{-2} = 0$ in (1.3) from the beginning. We shall show that the results in [BGPS] imply that, if $\sigma_0 < \infty$, *i.e.* if quantum effects are not neglected, then for *any density* $\rho \in (0, 1)$, and λ small enough (depending on σ_0, b), the decay of $G(\vec{x}, 0)$ is given by:

$$G(\vec{x}, 0) \simeq \frac{G_0(\vec{x}, 0)}{|\vec{x}|^{2\eta(\lambda)}} \tag{1.6}$$

with $\eta(\lambda)$ analytic in λ^2; $\eta(\lambda) = a\lambda^4 + O(\lambda^6)$ if $b \neq 0$, or $\eta(\lambda) = a'\lambda^6 + O(\lambda^8)$ if $b = 0$, with $a, a' \neq 0$.

This is clearly incompatible with a periodic minimizing state of the field φ_x, showing that the Holstein model ground state for a spinless fermion system is "disordered" at small coupling, *i.e.* there is no long range order in $< \varphi_{\vec{x}} \varphi_{\vec{y}} >$ or $< n_{\vec{x}} n_{\vec{y}} >$ as $|\vec{x} - \vec{y}| \to \infty$. The situation could change at large coupling, where the ground state could again be ordered: but this is outside the domain of applicability of our techniques. The maximum value of λ, $|\lambda| < \lambda_m(\sigma_0)$, for which we can prove (1.6) goes to zero as $\sigma_0 \to \infty$.

The problem of fermions with spin is, of course, much more interesting. By the same argument given below this case can also be reduced to the problem of fermions with spin, interacting with a short range potential. However the fermionic interaction is attractive: and while this has no consequences in the spinless case it has profound consequences in the case with spin. Unfortunately the spinning case is not understood in the sense needed here to draw any conclusion, see [BM].

§2 Reduction to a continuum problem.

When $\lambda=\nu=0$ the phonon and fermion fields are independent. Their respective Schwinger functions (imaginary time Green functions, see [FW] for definitions) can be computed via the Wick rule from the two point functions $S_\Lambda(x)$ and $g_\Lambda(x)$, where $\Lambda\equiv[0,\beta]\times(-\frac{L}{2},\frac{L}{2}]$ and $x=(t,\vec{x})\in\Lambda$:

$$\begin{aligned}
S_\Lambda(x)=&\frac{1}{\beta L}\sum_{e^{ik_0\beta}=1}\ \sum_{e^{i\vec{k}L}=1,|\vec{k}|<\pi}\frac{e^{-ik_0t-i\vec{k}\vec{x}}}{\sigma_0^2k_0^2+1+b^2e_B(\vec{k})}\\
g_\Lambda(x)=&\frac{1}{\beta L}\sum_{e^{ik_0\beta}=-1}\ \sum_{e^{i\vec{k}L}=1,|\vec{k}|<\pi}\frac{e^{-ik_0t-i\vec{k}\vec{x}}}{-ik_0+e_F(\vec{k})}
\end{aligned}\tag{2.1}$$

where $k=(k_0,\vec{k})$, and:

$$e_B(\vec{k})=2(1-\cos\vec{k}),\qquad e_F(\vec{k})=(\cos p_F-\cos\vec{k})\tag{2.2}$$

The summation rule over k_0 is to take the limit of $\sum_{|k_0|<N}$ as $N\to\infty$, and we suppose that $p_F=\frac{2\pi}{L}(n_F+\frac{1}{2})$, with n_F integer, so that $e_F(\vec{k})\neq0$ for all $\vec{k}$.

An application of Trotter's formula allows us, as usual (see [BG1]), to write an expression for the ground state infinite volume Schwinger functions of the interacting fermions, $G(x)$, in terms of the gaussian integral $P_B(d\Phi)$ with propagator S_Λ in (2.1) and of the grassmannian integral $P_F(d\psi)$ with propagator g_Λ in (2.1); for example:

$$\begin{aligned}
G(x)=&\lim_{\substack{\beta\to\infty\\ L\to\infty}}\frac{\int P_B(d\Phi)P_F(d\psi)e^{-V_\Lambda(\Phi,\psi)}\psi_x^-\psi_0^+}{\int P_B(d\Phi)P_F(d\psi)e^{-V_\Lambda(\Phi,\psi)}}\\
V_\Lambda(\Phi,\psi)=&\int_0^\beta dt\sum_{\vec{x}\in(-L/2,L/2]}(-\bar{\nu}+\lambda\Phi_x)\psi_x^+\psi_x^-\\
\bar{\nu}=&\frac{1}{2}(\nu-\frac{\lambda^2}{4})\hat{S}_\Lambda(0)
\end{aligned}\tag{2.3}$$

where the fields Φ satisfy periodic boundary conditions in $\vec{x}$ and t, while the ψ fields (which are grassmannian variables) satisfy periodic boundary conditions in $\vec{x}$ and are antiperiodic in t.[1] The term $(\lambda^2/2)\hat{S}_\Lambda(0)\psi_x^+\psi_x^-$ in the definition of V_Λ is produced by the shift of the field Φ_x which makes it possible to write the measure $P_B(d\Phi)$ as a measure with zero mean.

The propagator $S(x)$ obtained from $S_\Lambda(x)$ in the limits $\beta\to\infty$, $L\to\infty$ is exponentially decreasing; in fact, if $|t|<\beta/2$ and $|\vec{x}|<L/2$, one can easily see that[2]:

$$|S_\Lambda(x)|\le\frac{C(b)}{\sigma_0}e^{-\kappa_1|t|\sigma_0^{-1}}e^{-\kappa_2(b)|\vec{x}|}\tag{2.4}$$

where $\kappa_1<1$ is independent of b, and:

$$\kappa_2(b)=\begin{cases}O(\log b^{-1}) & \text{for } b\to0\\ O(b^{-1}) & \text{for } b\to\infty\end{cases},\qquad C(b)=\begin{cases}O(1) & \text{for } b\to0\\ O(b^{-1}\log b) & \text{for } b\to\infty\end{cases}\tag{2.5}$$

In particular, for $b=0$ it is, in the limit $\beta\to\infty$, $S(x)=(2\sigma_0)^{-1}e^{-|t|\sigma_0^{-1}}\delta_{\vec{x}0}$.

[1] One should not confuse the fields Φ (which are real valued fields) with the operators φ_x; nor should one confuse the grassmannian variables ψ with the operators a_x.

[2] Recall that the boundary conditions are periodic.

The integral over the bosonic field Φ is a gaussian integral, which can be performed explicitly, yielding:

$$G(x)=\lim_{\substack{\beta\to\infty\\ L\to\infty}}\frac{\int P_F(d\psi)e^{-V_\Lambda(\psi)}\psi_x^-\psi_0^+}{\int P_F(d\psi)e^{-V_\Lambda(\psi)}} \tag{2.6}$$

and, denoting $\int_\Lambda dx\cdot\equiv\sum_{\vec x\in(-L/2,L/2]}\int_0^\beta dt\cdot$:

$$V_\Lambda(\psi)=-\bar\nu\int_\Lambda dx\psi_x^+\psi_x^- -\frac{\lambda^2}{8}\int_\Lambda dxdy\,S(x-y)\psi_x^+\psi_x^-\psi_y^+\psi_y^- \tag{2.7}$$

We see that the problem has the same formal structure as that of a fermion system on the continuous line interacting via a short range potential, considered in [BGPS], see [G1] for a summary. The attractivity is due to the positive definitness of $S(x)$, see (2.1).

Let us point out the main differences:

(1) In formula (2.3) the propagator $g(x)$ for the grassmannian integral, defined in (2.1) and (2.2), has a different "dispersion relation"; in[BGPS] the dispersion relation is $e_F(\vec k)=\frac{1}{2}(\vec k^2-p_F^2)$, while in the present case $e_F(\vec k)=(\cos p_F-\cos\vec k)$. The difference is due to the fact that in the Holstein model the fermions are on a lattice.
(2) The fermion potential (2.7) is non local in time, while in space it may even have zero range (in the case $b=0$, see (2.5)). In the continuum problem considered in [BGPS] the potential had the form (2.7) with $v(\vec x-\vec y)\delta(t-t')$ replacing $S(x-y)$, with $x=(t,\vec x),y=(t',\vec y)$.

These changes are of no consequence: in fact what was really used in [BGPS] was that $\frac{de_F}{d\vec k}(p_F)>0$ which is true in the present case as well, provided $0<p_F<\pi$, as we suppose ("positive density smaller than close packing").

Therefore we can perform the decomposition of the propagator in the same way as in [BGPS], see (13) in [G1], by writing:

$$\frac{1}{-ik_0+e_F(\vec k)}=\frac{1-e^{-k_0^2+e_F(\vec k)^2}}{-ik_0+e_F(\vec k)}+\frac{e^{-k_0^2-e_F(\vec k)^2}}{-ik_0-e_F(\vec k)} \tag{2.8}$$

(2.8) and (2.1) generate a decomposition of the propagator into a sum of an *ultraviolet part*, $g_{u.v.}(x)\equiv g^{(>0)}(x)$, and of an *infrared part*, $g_{i.r.}(x)\equiv g^{(\le0)}(x)$. The decomposition (2.8) allows us to represent $\psi_x^\pm$ as sums of two independent grassmannian variables, that we denote $\psi_x^{(>0)}$ and $\psi_x^{(\le0)}$ with respective propagators $g^{(>0)}$ and $g^{(\le0)}$.

The integration over $\psi^{(>0)}$ can be controlled by perturbation expansions, as in [BGPS], and we shall integrate over it. The remaining integral for the evaluation of the partition function, *i.e.* for the denominator of (2.6), then becomes:

$$\int P_F(d\psi^{(\le0)})e^{-V^{(0)}(\psi^{(\le0)})} \tag{2.9}$$

with:

$$\begin{aligned}V^{(0)}(\psi)=&-\frac{\lambda^2}{2}\int dx\;dy\,S(x-y)\,\psi_x^+\psi_y^+\psi_y^-\psi_x^-+\\&-\lambda^2\int dxdy\,S(x-y)g^{(>0)}(y-x)\psi_x^+\psi_y^-+\\&-(\bar\nu+\lambda^2g^{(>0)}(0))\int dx\,\psi_x^+\psi_x^-+\sum_{n=1}^\infty\int dx_1\ldots dx_{2n}\cdot\\&\cdot W_{2n}(x_1\ldots x_{2n})\psi_{x_1}^+\ldots\psi_{x_n}^+\psi_{x_{n+1}}^-\ldots\psi_{x_{2n}}^-\end{aligned} \tag{2.10}$$

where the kernels $W_{2n}(x_1\ldots)$ are analytic in $(\lambda^2,\bar\nu)=r$ for $|r|<\varepsilon$ (for some $\varepsilon>0$) and verify the short range property:

$$\int dx_1\ldots dx_{2n}\,|W_{2n}(x_1\ldots x_{2n})|\,e^{\kappa d(x_1,\ldots,x_{2n})}<|\Lambda|(D|r|)^{\max(2,n-1)} \tag{2.11}$$

where $\kappa=\frac{1}{2}\min(\kappa_1\sigma_0^{-1},\kappa_2(b))$, see (2.4), and $d(x_1,\ldots)$ is the length of the shortest path connecting all the points $x_1,\ldots$ (regarded as points on the torus Λ).

The main result achieved by (2.10) is that we have "disposed" of the ultaviolet part of the problem (in the evaluation of the partition function) and we can say that the problem is reduced to an essentially identical one with a purely infrared propagator and a new $V^{(0)}$ which, to lowest order in the couplings $r=(\lambda^2,\bar\nu)$ has the same form as the original one, as far as the quartic part is concerned, and a slightly different (non local) quadratic part; plus "higher order terms" of every degree in the ψ fields. All the terms in (2.10) are well defined, and have a convergent power series in $r=(\lambda^2,\bar\nu)$ if r is small enough.

The proof of the above statements is simpler than the corresponding one in [BGPS], see theorem 1, because in the present case there is a natural ultraviolet cut off, at least in the $\vec x$ direction, so that no multi-scale decomposition of $g_{u.v.}(x)$ is needed.

The non locality of the quartic part in $V^{(0)}$ is not important even for the infrared problem as it does not affect the notion of relevant or marginal and irrelevant operators, which is the notion on which the infrared integration is based when performed via the renormalization group method in [BGPS]. Hence we reach the same conclusions about the partition function and about the Schwinger functions (whose analysis can be performed once the partition function can be estimated in detail, see [BGPS], §5 and §6). The beta function is essentially 0 (as shown in §7 of [BGPS]) and this allows us to draw the "same" conclusions as in [BGPS].

In particular one can prove the anomalous asymptotic behaviour (1.6) of the two point Schwinger function with a coefficient $\eta(\lambda)$ which is analytic in λ^2 and in general is of order λ^4. However, if $b=0$, one can see by an explicit calculation that the leading term in the expansion of $\eta(\lambda)$ vanishes, while the following one seems different from zero.

Another interesting observation is that the formal "Luttinger Theorem" [LW] is valid in our case: it states that the density ρ is a function only of p_F, that is $\rho=p_F/\pi$ for any λ: see appendix for a discussion of this formal result (for which it would be nice to have a rigorus proof).

In the next section we give some details on the recursion procedure that we follow to obtain the estimates, and on how we define the relevant operators: this will follow closely [BGPS] but should be useful to the readers who wish to see where one is going, before plunging themselves into the analysis of our estimates. But neither the estimates nor the proof of the vanishing of the beta function will be reproduced here: they are identical to the corresponding proofs in [BGPS].

§3 The recursive evaluation of the partition function.

The following is the "standard" anomalous dimension renormalization procedure. It is illustrated in quite a general context, in which one is given *a priori* an arbitrary sequence of "wave function renormalization" constants: $Z_0\equiv 1, Z_{-1}, Z_{-2},\ldots$.

The infrared propagator $g^{(\le 0)}(x)$ is decomposed as:

$$g^{(\le 0)}(x)=\int e^{-ikx}\frac{dk}{(2\pi)^2}\frac{T_0(k)}{-ik_0+e_F(\vec k)}+\int e^{-ikx}\frac{dk}{(2\pi)^2}\frac{t_{-1}(k)}{-ik_0+e_F(\vec k)} \tag{3.1}$$

where $t_h(k)=\exp(-2^{-2h}(k_0^2+e_F(\vec k)^2))$ and $T_0(k)=t_0(k)-t_{-1}(k)$.

Eq. (3.1) allows us to represent $\psi^{(\le 0)}$ as $\psi^{(0)}+\psi^{(\le -1)}$ where $\psi^{(\le -1)}$ has propagator $Z_0^{-1}t_{-1}(k)(-ik_0+e_F(\vec k))^{-1}$ and $\psi^{(0)}$ has a propagator given by $Z_0^{-1}T_0(k)(-ik_0+e_F(\vec k))^{-1}$. Let $\overline P_{Z_0}(d\psi^{(0)})$ and $P_{Z_0}(d\psi^{(\le -1)})$

denote the corresponding integrations, the grassmannian integral in (2.9) thus becomes:

$$\overline{P}_{Z_0}(d\psi^{(0)})\,P_{Z_0}(d\psi^{(\le -1)}) \tag{3.2}$$

Using the sequence Z_j one can then write the identity:

$$(Z_0 t_{-1}^{-1} + Z_{-1} - Z_0)^{-1} = Z_{-1}^{-1} t_{-2} + Z_{-1}^{-1}\overline{\Gamma}^{(-1)} \tag{3.3}$$

defining $\overline{\Gamma}^{(-1)}(k)$.

Hence if $\psi^{(-1)}$ is a grassmannian field with propagator:

$$\frac{1}{Z_{-1}}\frac{\overline{\Gamma}^{(-1)}(k)}{-ik_0 + e_F(\vec{k})} \tag{3.4}$$

and $\psi^{(\le -2)}$ is a grassmannian field with propagator $Z_{-1} t_{-2}(k)(-ik_0 + e_F(\vec{k}))^{-1}$, independent of $\psi^{(-1)}$ and of $\psi^{(0)}$, then the grassmannian integral (3.2) becomes (formally), up to normalization constants:

$$\begin{aligned}&\overline{P}_{Z_0}(d\psi^{(0)}\overline{P}_{Z_{-1}}(d\psi^{(-1)})P_{Z_{-1}}(d\psi^{(\le -2)})\cdot\\ &\cdot e^{(Z_{-1}-Z_0)\int \psi_x^{(\le -1)+}(\partial_t + e(i\partial_{\vec{x}}))\psi_x^{(\le -1)-}}\end{aligned} \tag{3.5}$$

as is easily checked with some algebra, if $\psi^{(\le -1)} = \psi^{(-1)} + \psi^{(\le -2)}$.

By iteration:

$$\begin{aligned}P(d\psi^{(\le 0)}) = &\left(\prod_{j=0}^{h}\overline{P}_{Z_j}(d\psi^{(j)})\right)\cdot P_{Z_h}(d\psi^{(\le h-1)})\cdot\\ &\cdot \exp\sum_{j=0}^{h+1}(Z_{j-1} - Z_j)\int \psi_x^{(\le j-1)+}(\partial_t + e(i\partial_{\vec{x}}))\psi_x^{(\le j-1)-}\,dx\end{aligned} \tag{3.6}$$

and $\psi^{(\le 0)} = \psi^{(0)} + \psi^{(-1)} + \ldots + \psi^{(h)} + \psi^{(\le h-1)}$, with $\psi^{(j)}$ having propagator $Z_j^{-1}\overline{\Gamma}^{(j)}(k)\;(-ik_0 + e_F(\vec{k}))^{-1} \equiv Z_j^{-1}\overline{g}^{(j)}$ with:

$$\overline{\Gamma}^{(j)}(k) = t_j - t_{j-1} + (1 - t_j)t_j\,\frac{z_j}{1 + z_j t_j} \tag{3.7}$$

if z_j is defined by: $Z_j \equiv (1 + z_j)Z_{j+1}$.

The integration (2.9) can therefore be performed recursively by setting:

$$\begin{aligned}&e^{-V^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} = \int\prod_{h'=h+1}^{0}\overline{P}_{Z_{h'}}(d\psi^{(h')})\\ &e^{-V^{(0)}(\sqrt{Z_0}\psi^{(\le 0)} + \sum_{h'=h}^{-1}(Z_{h'} - Z_{h'+1})\int dx\,\psi_x^{(\le h')+}(\partial_t + e(i\partial_{\vec{x}}))\psi_x^{(\le h')-}}\end{aligned} \tag{3.8}$$

so that:

$$\begin{aligned}&\int e^{-V^{(0)}(\psi^{(\le 0)})}P(d\psi^{(\le 0)}) = \int e^{-V^{(0)}(\sqrt{Z_0}\psi^{(\le 0)})}\,P_{Z_0}(d\psi^{(\le 0)}) =\\ &= \int \overline{P}_{Z_h}(d\psi^{(h)})P_{Z_h}(d\psi^{(<h)})e^{-V^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\end{aligned} \tag{3.9}$$

where $\psi^{(\le h)} = \psi^{(h)} + \psi^{(\le h-1)}$.

The (independent) "fluctuation fields" $\psi^{(h)}$ have propagators with good scaling properties if $|z_h| < \frac{1}{2}$: they can be represented via quasi particle fields (see below) with propagators bounded, unformly in h, by $2^h\gamma(2^h x)$ for some function $\gamma(x)$ which decays exponentially fast as $x \to \infty$.

The idea is to select the sequence Z_h so that $|z_h| < \frac{1}{2}$ and so that the potential $V^{(h)}$ does not contain certain terms which would otherwise be difficult to control.

To understand the choice of Z_h one has to define the *relevant* and *marginal* terms in $V^{(h)}$. Such notions are not naturally defined from the $V^{(h)}$ considered as functions of the *particle fields* $\psi^{(\le h)}$. They are very natural if $V^{(h)}$ is regarded as a function of certain auxiliary fields that we call *the quasi particle fields.*

The infrared propagators $g^{(\le h)}(x)$ can be decomposed as:

$$\begin{aligned} g^{(\le h)}(x) &= \sum_{\vec\omega=\pm1} e^{-ip_F\vec\omega\vec x} g^{(\le h)}(x,\vec\omega) \\ g^{(\le h)}(x,\vec\omega) &= e^{ip_F\vec\omega\vec x}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{t_h(k)\chi(2^{-h}\vec\omega\vec k)}{-ik_0+e_F(\vec k)}e^{-ikx} \end{aligned} \tag{3.10}$$

where $\chi(r)$ may be chosen as $\pi^{-1/2}\int_{-\infty}^{r} e^{-s^2} ds$, so that χ is a smooth version of the step function, with the property that $\chi(r)+\chi(-r)\equiv1$.

It is easy to check that, in the infinite (space-time) volume limit, $g^{(\le h)}(x,\vec\omega)$ is essentially scale independent, *i.e.* h independent, as $h\to-\infty$. In fact:

$$\begin{aligned} g^{(\le h)}(x,\vec\omega) &\simeq_{h\to-\infty} 2^h\hat g(2^h x,\vec\omega) \\ \hat g(x,\vec\omega) &= \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\int_{-\infty}^{\infty}\frac{d\vec k}{2\pi}\frac{t(k_0^2+(\sin p_F)^2\vec k)^2)}{-ik_0+\vec\omega\vec k\sin p_F}e^{-ikx} \end{aligned} \tag{3.11}$$

The decomposition (3.10) of the covariance allows to perform a related decomposition of the fields $\psi_x^{\pm(\le h)}$:

$$\psi_x^{\pm(\le h)} = \sum_{\vec\omega=\pm1} e^{\pm ip_F\vec\omega\vec x}\psi_{x\vec\omega}^{\pm(\le h)} \tag{3.12}$$

The fields $\psi_{x\vec\omega}^{\pm(\le h)}$, which will be called the *quasi particle fields*, are independent fields with propagator $g^{(\le h)}(x,\vec\omega)$; moreover they are antiperiodic also in the $\vec x$ variable, so that we can write:

$$\psi^{\pm}_{x\vec\omega} = \sum_{e^{ik_0\beta}=-1,e^{i\vec kL}=-1} e^{\pm ikx}\psi^{\pm}_{k\vec\omega} \tag{3.13}$$

One can think that the $\psi^{(\le h)}_{x,\vec\omega}$ have a distribution which, "up to scaling", is h independent. This means that the distribution of $\psi^{(\le h)}$ is the same as that of $2^{h/2}\hat\psi_{2^h x,\vec\omega}$ where $\hat\psi_{x,\vec\omega}$ has a propagator $\hat g(x,\vec\omega)\delta_{\vec\omega\vec\omega'}$, see (3.11) (up to corrections vanishing fast as $h\to-\infty$). The fluctuation fields can also be expressed in terms of quasi particle fields in the same way: their propagators have the good scaling properties mentioned above.

Thus if we think of the infrared problem as that of integrating over the quasi particle fields $\psi^{(\le h)}_{x\vec\omega}$, we have a natural way of introducing the notions of relevant, marginal and irrelevant operators. Precisely we define the relevant operator to be:

$$F_2 = \sum_{\vec\omega,\vec\omega'}\int \psi^+_{x\vec\omega}\psi^-_{x\vec\omega'}e^{i(\vec\omega-\vec\omega')p_F\vec x}dx = \int\psi^+_x\psi^-_x\,dx \tag{3.14}$$

and the marginal operators as:

$$\begin{aligned} F_4 &= \int \psi^+_{x+}\psi^+_{x-}\psi^-_{x-}\psi^-_{x+}\,dx \\ F_{2,\tau} &= \sum_{\vec\omega,\vec\omega'}\int e^{i(\vec\omega-\vec\omega')p_F\vec x}\psi^+_{x\vec\omega}(\partial_t+i\vec\omega'\sin p_F\mathcal{D}_{\vec\omega'})\psi^-_{x\vec\omega'}\,dx = \int\psi^+_x(\partial_t+e_F(i\partial_{\vec x}))\psi^-_x\,dx \\ F_{2,\sigma} &= \sum_{\vec\omega,\vec\omega'}\int e^{i(\vec\omega-\vec\omega')p_F\vec x}\psi^+_{x\vec\omega}\;i\vec\omega'\sin p_F\mathcal{D}_{\vec\omega'}\psi^-_{x\vec\omega'}\,dx = \int\psi^+_x e_F(i\partial_{\vec x})\psi^-_x\,dx \end{aligned} \tag{3.15}$$

where $\mathcal{D}_{\vec{\omega}}$ is a suitable operator, which acts by multiplication by $M_{\vec{\omega}}(\vec{k})$ on the Fourier transforms, with

$$M_{\vec{\omega}}(\vec{k}) = -2i \sin \vec{k}/2 \frac{\sin(p_F + \vec{\omega}\vec{k}/2)}{\sin p_F} \simeq_{\vec{k}\to 0} -i\vec{k} \tag{3.16}$$

Note that, while the second degree operators can be expressed easily in terms of the particle fields, the same is not true for the F_4.

The "usual" power counting attributes a size to the above operators evaluated by extending the integral over a box of size 2^{-h}, *i.e.* of volume 2^{-2h}, and by attributing to each field a size $2^{h/2}$ as suggested by the scaling properties discussed above; furthermore each derivative contributes to the size a factor 2^h.

Hence the conventional power counting attributes to F_2 a size that is evaluated as $2^{-2h}(2^{h/2})^2 = 2^{-h}$ (hence F_2 is relevant). The size of F_4 is $2^{-2h}(2^{h/2})^4 = 1$ and $F_{2,\tau}, F_{2,\sigma}$ have size $2^{-2h}2^h(2^{h/2})^2 = 1$ (hence they are marginal). All the other local operators are irrelevant (*i.e.* they have sizes $2^{h/2}$ or less).

Note that the size of the various operators is clear if they are regarded as functions of the quasi particle fields.

Given $V^{(h)}(\psi)$, we must identify the relevant and marginal parts of $V^{(h)}$. This is done by introducing the *localization operator* $\mathcal{L}$: it is a linear projection operator which is 0 unless acting on a fourth degree or on a second degree monomial in the fields. Imagining that $V^{(h)}$ is expressed in terms of quasi particle fields, hence as a sum (or integral) of monomials in the quasi particle fields, the action of $\mathcal{L}$ on the fourth degree monomials is described by:

$$\mathcal{L}\,\psi^+_{x_1\vec{\omega}_1}\psi^+_{x_2\vec{\omega}_2}\psi^-_{x_3\vec{\omega}_3}\psi^-_{x_4\vec{\omega}_4} = \frac{1}{2}\sum_{i=1}^{2}\psi^+_{x_i\vec{\omega}_1}\psi^+_{x_i\vec{\omega}_2}\psi^-_{x_i\vec{\omega}_3}\psi^-_{x_i\vec{\omega}_4} \tag{3.17}$$

while the action of $\mathcal{L}$ on the second degree monomials is:

$$\mathcal{L}\psi^+_{x\vec{\omega}_1}\psi^-_{y\vec{\omega}_2} = \psi^+_{x\vec{\omega}_1}[\psi^-_{x\vec{\omega}_2} + h_\beta(y_0 - x_0)\partial_{x_0}\psi^-_{x\vec{\omega}_2}\psi^-_{x\vec{\omega}_1}h_L(\vec{y}-\vec{x})\mathcal{D}_{\vec{\omega}_2}\psi^-_{x\omega_2}] \tag{3.18}$$

where

$$h_M(s-t) \equiv \frac{M}{2i\pi}[e^{i\pi(s-t)/M} - e^{-i\pi(s-t)/M}] \tag{3.19}$$

is an antiperiodic approximation of $(s-t)$, which converges to it as $M \to \infty$.

The operator $\mathcal{D}_{\vec{\omega}}$ differs from the analogous operator defined in [BGPS], where $\vec{x}$ was a continuum variable. However, it is easy to see that one has to change the bounds obtained in [BGPS] only in minor points, without affecting the final results. In (3.18) we have also taken explicitely into account the boundary conditions, while in [BGPS] this problem was neglected. However, as explicitely shown in the analysis of the spinning case in [BM], this approximation does not play any relevant role.

Appendix

In this appendix we give a heuristic derivation of the so called Luttinger Theorem, following the analysis presented in [LW]. We believe that the techniques used in [BGPS] to study the Schwinger functions could be used also to prove rigorously the Luttinger Theorem, but this has not been done yet.

We consider a system in finite volume L at zero temperature. If we define the hamiltonian H so that the ground state $|0\rangle$ has eigenvalue 0, the two point Schwinger function is defined in the following way:

$$G(\vec{x},t) = \chi(t>0)\langle 0|a^-_{\vec{x}}e^{-tH}a^+_0|0\rangle - \chi(t\le 0)\langle 0|a^+_0 e^{+tH}a^-_{\vec{x}}|0\rangle \tag{A1}$$

where $\chi(A)$ is the characteristic function of the set of points where the condition A holds.

Hence its Fourier tranform $\hat{G}(\vec{k},k_0)$ is given by:

$$\begin{aligned}\hat{G}(\vec{k},k_0) &= \int_0^\infty dt\Big[e^{ik_0t}\langle 0|a^-_{\vec{k}}e^{-tH}a^+_{\vec{k}}|0\rangle - e^{-ik_0t}\langle 0|a^+_{\vec{k}}e^{-tH}a^-_{\vec{k}}|0\rangle\Big]\\ &= \sum_{n:E_n>0}\int_0^\infty dt\Big[e^{-t(E_n-ik_0)}|\langle 0|a^-_{\vec{k}}|n\rangle|^2 - e^{-t(E_n+ik_0)}|\langle 0|a^+_{\vec{k}}|n\rangle|^2\Big] = \\ &= \sum_{n:E_n>0}\Big[\frac{|\langle 0|a^-_{\vec{k}}|n\rangle|^2}{-ik_0+E_n} + \frac{|\langle 0|a^+_{\vec{k}}|n\rangle|^2}{-ik_0-E_n}\Big]\end{aligned} \tag{A2}$$

where $E_n, n=0,\ldots$, are the eigenvalues of H and $|n\rangle$ the corresponding eigenstates. The ground state gives no contribution to the sum in (A2) (so allowing the integration over t) because of the conservation of the total momentum, which implies that $\langle 0|a^\pm_{\vec{k}}|0\rangle = 0$, when $\vec{k}\neq 0$, and because of a cancellation between the two different terms in the r.h.s. of (A2), when $\vec{k}=0$.

The function $\hat{G}(k)$, $k=(\vec{k},k_0)$, is often written also in the following form:

$$\hat{G}(k) = \frac{1}{-ik_0 + e_F(\vec{k}) + \Sigma(k)} \tag{A3}$$

which defines the *self-energy function* $\Sigma(k)$. As is well known, [LW], $\Sigma(k)$ can be expressed in perturbation theory as the sum of all connected graphs with two external lines, which are irreducible, that is which do not become disconnected by erasing any internal line.

By using (A1), one sees immediately that the density ρ is given by:

$$\rho = -\lim_{t\to 0^-} G(0,t) = \frac{1}{L}\sum_{\vec{k}}\lim_{\varepsilon\to 0^+}\int\frac{dk_0}{2\pi}\frac{e^{ik_0\varepsilon}}{ik_0 - e_F(\vec{k}) - \Sigma(k)} \tag{A4}$$

Let us write

$$\frac{1}{\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)} = \frac{\partial}{\partial\zeta}\log[\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)] + \frac{\partial\Sigma(\vec{k},-i\zeta)/\partial\zeta}{\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)} \tag{A5}$$

by choosing the branch of $\log z$ so that $\log z = \log|z| + i\arg z$, $0<\arg z<2\pi$.

It was argued in [LW], if we insert the r.h.s of (A5) in the r.h.s. of (A4), the second term gives no contribution, that is:

$$\frac{1}{L}\sum_{\vec{k}}\lim_{\varepsilon\to 0^+}\int\frac{dk_0}{2\pi}e^{ik_0\varepsilon}\frac{\partial\Sigma(k)/\partial k_0}{ik_0 - e_F(\vec{k}) - \Sigma(k)} = -\frac{1}{L}\sum_{\vec{k}}\int\frac{dk_0}{2\pi}\Sigma(k)\frac{\partial\hat{G}(k)}{\partial k_0} = 0 \tag{A6}$$

We recall briefly the arguments given in [LW] to justify (A6). Indeed, our problem is not explicitly considered in [LW], where the interaction between the fermions is local in time; hence only the case $b=0$ is covered by [LW]. However, their arguments extend in a trivial way to our general case.

We start from the observation (not easy to prove rigorously) that $\Sigma(k)$ can be thought as the sum of all graphs (connected and irreducible) with two external lines, such that the internal lines carry the complete propagator $\hat{G}(k)$ and the following condition is satisfied: there is no proper subgraph with two external lines. We shall call such graphs, as usual, *skeleton graphs* and say that a graph is of order n, if it contains n four-fermions interacting terms (hence it has $2n$ vertices).

Let $\Sigma_n(k)$ be the sum of all skeleton graphs of order n and let Y_n be the sum of all vacuum (that is with no external lines) skeleton graphs of order n, which can be obtained by closing the two external lines of a graph contributing to $\Sigma(k)$ with a propagator $\hat{G}(k)$. Since each graph contributing to Y_n can be obtained in $2n$ different ways from a graph contributing to $\Sigma(k)$, we have:

$$Y_n = \frac{1}{2n}\frac{1}{L}\sum_{\vec{k}}\int \frac{dk_0}{2\pi}\Sigma(k)\hat{G}(k) \tag{A7}$$

Moreover, given a graph $\mathcal{G}$ contributing to Y_n, its value $Y_n^{\mathcal{G}}$ can be written in the following way:

$$\begin{aligned} Y_n^{\mathcal{G}} = \int dk^{(1)}\dots dk^{(2n)}dq^{(1)}\dots dq^{(n)}\hat{G}(k^{(1)})\cdots\hat{G}(k^{(2n)})\cdot \\ \cdot S(q^{(1)})\dots S(q^{(n)})\prod_{i=1}^{2n}\delta(k^{(s_i)}-k^{(r_i)}+q^{(b_i)}) \end{aligned} \tag{A8}$$

where $\int dk \equiv L^{-1}\sum_{\vec{k}}\int dk_0/(2\pi)$ and $k^{(1)},\dots,k^{(2n)}$ are the (space-time) momenta of the $2n$ fermion lines, $q^{(1)},\dots,q^{(n)}$ are the momenta of the n phonon lines and, finally, $k^{(s_i)}$ and $k^{(r_i)}$ are the momenta of the fermion entering and leaving the vertex i, respectively, while $q^{(b_i)}$ is the momentum of the phonon propagator entering vertex i. Moreover, if we consider the sum of all the quantities that we obtain by substituting, in the r.h.s. of (A8), one fermion propagator by its derivative with respect to k_0, we get:

$$\begin{aligned} &\sum_{j=1}^{2n}\int dk^{(1)}\dots dk^{(2n)}dq^{(1)}\dots dq^{(n)}\hat{G}(k^{(1)})\cdots\frac{\partial\hat{G}}{\partial k_0^{(j)}}(k^{(j)})\cdots\hat{G}(k^{(2n)})\cdot \\ &\cdot S(q^{(1)})\dots S(q^{(n)})\prod_{i=1}^{2n}\delta(k^{(s_i)}-k^{(r_i)}+q^{(b_i)}) = \\ &= \pm\sum_{j=1}^{2n}\int dk^{(1)}\dots dk^{(2n)}dq^{(1)}\dots dq^{(n)}\hat{G}(k^{(1)})\cdots\hat{G}(k^{(2n)})\cdot \\ &\cdot S(q^{(1)})\dots S(q^{(n)})\left(\frac{\partial}{\partial k_0^{(s_j)}}+\frac{\partial}{\partial k_0^{(r_j)}}\right)\prod_{i=1}^{2n}\delta(k^{(s_i)}-k^{(r_i)}+q^{(b_i)}) = 0 \end{aligned} \tag{A9}$$

If we now sum the l.h.s. of (A9) over $\mathcal{G}$, we get, by an argument similar to that used in order to prove (A7), that:

$$2n\int dk\ \Sigma(k)\frac{\partial\hat{G}}{\partial k_0}(k) = 0 \tag{A10}$$

so that (A6) is proved.

The formal identity (A6) implies that $\rho = \frac{1}{L}\sum_{\vec{k}}\rho_{\vec{k}}$, with

$$\rho_{\vec{k}} = \lim_{\varepsilon\to 0^+}\int_{-i\infty}^{+i\infty}\frac{d\zeta}{2\pi i}e^{\varepsilon\zeta}\frac{\partial}{\partial\zeta}\log[\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)] \tag{A11}$$

Moreover, (A2) implies that the integrand in the r.h.s. of (A11) is analytic in all the complex ζ plane, except on the real axis, where there are branch points at $\zeta = \pm E_n$, $n > 0$. Therefore, we can deform the integration contour into the contour $\mathcal{C}$ of Fig. 1

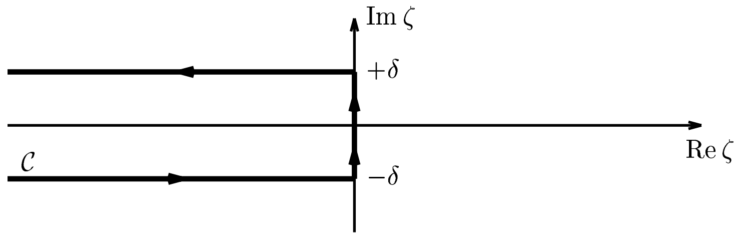

Fig. 1: Integration Contour for (A11).

and then we can integrate by parts. We get:

$$\begin{aligned}\rho_{\vec{k}} = & \lim_{\eta\to 0^+} \frac{\log[-e_F(\vec{k}) - \Sigma(\vec{k},-\eta)] - \log[-e_F(\vec{k}) - \Sigma(\vec{k},\eta)]}{2\pi i} \\ & - \lim_{\varepsilon\to 0^+} \varepsilon \int_{\mathcal{C}} \frac{d\zeta}{2\pi i} e^{\varepsilon\zeta} \log[\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)]\end{aligned} \tag{A12}$$

It is easy to see that the second term in (A12) vanishes, if the argument of the logarithm has a negative real part for $\mathrm{Re}\,\zeta \to -\infty$, by recalling that we have chosen the branch of $\log z$ with the cut along the positive real axis.

We want to show that this is certainly true, if $|\langle 0|a^+_{\vec{k}}|n\rangle| \to 0$ sufficiently fast, as $E_n \to \infty$. We note that $\mathrm{Re}\,[\zeta - e_F(\vec{k}) - \Sigma(\vec{k},-i\zeta)]$ has the same sign as $-\,\mathrm{Re}\,\hat{G}(\vec{k},-i\zeta)$; moreover, by (A2), if $\zeta = u + i\delta$:

$$-\,\mathrm{Re}\,\hat{G}(\vec{k},-i\zeta) = \int_{-\infty}^{+\infty} \sigma_{\vec{k}}(dE) \frac{u-E}{(u-E)^2+\delta^2} \tag{A13}$$

where $\sigma_{\vec{k}}(dE)$ is a probability measure, as it is easy to check. Moreover, if $u \le -u_0$, with u_0 large enough, there is an interval $[E_1, E_2]$, such that $|u - E| \ge |u|/2$ for $E \in [E_1, E_2]$ and $\int_{[E1,E_2]} \sigma_{\vec{k}}(dE) = 1/2$; hence it is easy to show that:

$$-\,\mathrm{Re}\,\hat{G}(\vec{k},-i\zeta) \le (2\delta)^{-1} \int_{-\infty}^{u} \sigma_{\vec{k}}(dE) - \frac{|u|}{u^2+4\delta^2} \tag{A14}$$

which immediately implies that $-\,\mathrm{Re}\,\hat{G}(\vec{k},-i\zeta)$ is definitely negative for $u \to -\infty$, if

$$\lim_{u\to-\infty} u \int_{-\infty}^{u} \sigma_{\vec{k}}(dE) = 0 \tag{A15}$$

Since, by (A2), the function $\Sigma(\vec{k},-i\zeta)$ is real at $\zeta = 0$, it follows from (A12) that:

$$\rho = \frac{1}{L} \sum_{\vec{k}} \chi[e_F(\vec{k}) + \Sigma(0,\vec{k}) < 0] \tag{A16}$$

Equation (A16) implies that, in the limit of infinite volume, there is a relation, independent of the strength of the interaction λ, between the density and the Fermi momentum p_F, defined as the value of $\vec{k}$, such that $e_F(\vec{k}) + \Sigma(\vec{k},0) = 0$ (that is the value of $\vec{k}$ where the interacting propagator is singular for $k_0 = 0$). For $\lambda = 0$ we have $\rho = p_F/\pi$, hence this relation has to be valid for any λ.

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