On the algebraic non-integrability of the Halphen system

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Abstract

It is proved that the Halphen system of ordinary differential equations has no non-trivial rational first integrals.

1 Introduction

different classes of systems of ordinary differential equations (in abbreviation ODE) is again of great actuality. In [14] an algebraic method of proving the non-integrability of polynomial systems of ODE with homogeneous right sides of the same degree was presented and tested on some non-trivial examples (see also [15] and [16] for another applications). This method, according

to the best of our knowledge, was presented for the first time in the fundamental book of J.-P. Jouanolou [10]. Indeed, on pages 193–195 of [10], M.H.A. Levelt, the referee of the book, proved using this method that the Jouanolou system of ODEs

d z$\overline{\scriptstyle dt=y^s; \qquad s\in\mathbb{N}, \quad s\geq 2,}$definition). In fact, the basic ideas of the method were already introduced by M.N. Lagutinskii in his pioneering, but, unfortunately, completely unknown works [12, 13]. See [4, 5], where one can find more details on M.N. Lagutinskii and his works on integrability which are direct continuation of the seminal Darboux paper [3].

In this note, we study the Halphen system of ODEs [8, 9] defined on $\mathbb{C}^3$ (or $\mathbb{R}^3$):

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2x_3 - x_1(x_2 + x_3), \\ \frac{dx_2}{dt} &= x_3x_1 - x_2(x_3 + x_1), \\ \frac{dx_3}{dt} &= x_1x_2 - x_3(x_1 + x_2), \end{aligned}\right\} \tag{1}$$

Theorem *The Halphen system (1) does not admit any non-trivial rational first integral.*

To do this we first apply the Lagutinskii-Levelt procedure that allows to state non-existence of a polynomial first integral. To finish the proof it was necessary to supplement the method by more subtle investigations where specific properties of the Halphen system are crucial.

It is well known that for the polynomial systems of ODE the non existence of a non-trivial rational first integral is equivalent to the non-existence of an algebraic first integral. This is a consequence of Bruns theorem [6, vol.III, chap. XVII], [11]. For
polynomial systems with the homogeneous right sides of the same degree,
called *the homogeneous* systems, the non-existence of a
rational first integral is equivalent to the non-existence of a meromorphic first integral [21]. Consequently,
the Halphen system (1) does not admit any non-trivial
algebraic or meromorphic first integral.

The Halphen system appears in different contexts, e.g., as a result of the self-dual Yang-Mills reduction [2], [18], [19], or, in general relativity,
in the study of $SU(2)$-invariant four metrics (Bianchi IX metrics) [17], [1].

It is worth noticing that, in spite of the lack of meromorphic first integrals, the Halphen system can be explicitly integrated—we can express its general solution in terms of elliptic integrals with 'variable modulus' (see [8], [18], [17]). Moreover, as it was mentioned in [19] and [20], the first integrals of the Halphen system do
indeed exist, although they are not global and are multi-valued non-algebraic functions. See also [7] for discussion of special properties of the Halphen system.

The paper is organized as follows. In Sec. 2 we gather all necessary algebraic facts and present the basic steps of the Lagutinskii-Levelt procedure. The proof of our theorem is given in Sec. 3.

2 Algebraic preliminaries

$$\frac{dx_i}{dt} = V_i(x_1, \ldots, x_n), \qquad 1 \leq i \leq n, \tag{2}$$

i$\leq$ n. Here, as usual, we denote by $\mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in n variables x_i, $1 \leq i \leq n$, with coefficients in a commutative field $\mathbb{K}$, and by $\mathbb{K}(x_1, \ldots, x_n)$ the field of rational fractions of n variables with coefficients from $\mathbb{K}$. Throughout this note we assume that $\mathbb{K} = \mathbb{C}$.

A function F is a *first integral* of system (2) if it satisfies the following equation

$$d_V(F) \stackrel{\text{def}}{=} \sum_{i=1}^{n} V_i \partial_i F = 0, \qquad \text{where} \quad \partial_i F = \frac{\partial F}{\partial x_i}.$$

notion of derivations is used. A derivation d is a linear mapping of $\mathbb{K}[x_1, \ldots, x_n]$ (or of $\mathbb{K}(x_1, \ldots, x_n)$) into itself satisfying the Leibnitz rule $d_V(FG) = Gd_V(F) + Fd_V(G)$. Thus, we can talk about derivation d_V instead of system (2). The derivation d_V is called *homogeneous* if the corresponding system of ODE (2) is homogeneous.

The main object in our investigation are the so called *Darboux polynomials* of a derivation d_V (or *partial first integrals* of a system of ODEs (2)), i.e., polynomials $F \in \mathbb{C}[x_1, \ldots, x_n] \setminus \mathbb{C}$ such that

$$d_V(F) = \sum_{i=1}^{n} V_i \partial_i F = PF,$$

Darboux polynomial F is nothing else but a first integral of the system of ODEs called also a *constant of the derivation* d_V. These polynomials, as an investigation tool of the integrability of the system (2), were introduced for the first time by Darboux in [3].

It is easy to prove the following facts (see [14]).

1. An element $F = A/B \in \mathbb{C}(x_1, \ldots, x_n)$, with relatively prime polynomials $A, B \in \mathbb{C}[x_1, \ldots, x_n]$ is a first integral of (2) if and only if A and B are Darboux polynomials with the same 'eigenvalue' P, i.e., $d_V(A) = PA$, and $d_V(B) = PB$.

2. If $F \in \mathbb{C}[x_1, \ldots, x_n]$ is a Darboux polynomial of (2), then all its irreducible factors are also Darboux polynomials.

3. The finite product of Darboux polynomials is also a Darboux polynomial. More precisely,

$$\text{if} \quad d_V(F_i) = P_i F_i, \quad 1 \leq i \leq s, \qquad \text{then} \quad d_V(\prod_{i=1}^{s} F_i) = (\sum_{i=1}^{s} P_i)(\prod_{i=1}^{s} F_i).$$

4. For a homogeneous derivation d_V (or a homogeneous system) if $d_V(F) = PF$ for some $F, P \in \mathbb{C}[x_1, \dots, x_n]$ then also $d_V(F^+) = P^+F^+$, where by G^+ we denote the homogeneous component of the highest degree of a polynomial G.

5. For homogeneous derivations d_V (or homogeneous systems), the following assertion holds [14, Lemma 2.1]. If F is a Darboux polynomial, with eigenvalue P such that $d_V(F) = PF$, then P is a homogeneous polynomial, and all homogeneous components of F are also Darboux polynomials corresponding to the same P.

6. Let us assume that F_i, for $1 \leq i \leq s$, are all (up to a multiplicative constant) irreducible homogeneous Darboux polynomials of a homogeneous derivation d_V, and let $d_V(F_i) = P_i F_i$, for $1 \leq i \leq s$. In such a case if the polynomials P_i, for $1 \leq i \leq s$, are linearly independent over $\mathbb{Z}$, then any Darboux polynomial of d_V is (up to a multiplicative constant) of the form $\prod_{i=1}^{s} F_i^{\alpha_i}$, where $\alpha_i, 1 \leq i \leq s$, are non negative integers. This assertion easily follows from properties 2, 3 and 5 given above.

Thus, the non-existence of non constant Darboux polynomials implies the non-existence of rational first of integrals of the corresponding system of ODEs. However, there exist systems with Darboux polynomials that have no rational first integral. We will show that the Halphen system belongs to this class of systems.

Now, let us present the basic steps of the Lagutinskii-Levelt procedure. Let us consider the system (2) with homogeneous right hand sides of the same degree k. To prove the non-existence of a polynomial first integral or Darboux polynomial we make use of *Darboux points*, i.e., the points $z = (z_1, \dots, z_n) \neq (0, \dots, 0)$ satisfying the following equations

$$V_i(z) = \lambda z_i, \qquad 1 \leq i \leq n, \qquad \text{for some} \quad \lambda \in \mathbb{C}.$$

Note that the existence of a Darboux point z for the system is equivalent to the existence of the straight line solution of the form

$$x_i(t) = z_i \phi(t), \quad 1 \leq i \leq n, \qquad \text{where} \quad \frac{d\phi}{dt} = \lambda \phi^k.$$

Now, let us assume that a homogeneous polynomial F of degree $m \geq 1$ is a Darboux polynomial of our system:

$$\sum_{i=1}^{n} V_i \partial_i F = PF.$$

Combining the above equation with the Euler identity

$$\sum_{i=1}^{n} x_i \partial_i F = mF,$$

we can eliminate one partial derivative of F, e.g., $\partial_n F$, and we obtain the equation

$$\sum_{i=1}^{n-1} (V_i x_n - V_n x_i) \partial_i F = (x_n P - mV_n) F. \tag{3}$$

Without any loss of generality, we can assume that the last component of a chosen Darboux point z does not vanish, and we set $z_n = 1$. Putting $x_n = 1$ in the equation (3) and shifting the origin to the Darboux point by the transformation

x$_i = y_i + z_i, \qquad 1 \leq i \leq n-1, weobtainthefollowingequationwhere$
-1000

and where for an arbitrary polynomial $G(x_1, \ldots, x_n)$ we define

$$g(y_1, \ldots, y_{n-1}) \stackrel{\text{def}}{=} G(y_1 + z_1, \ldots, y_{n-1} + z_{n-1}, 1).$$

As z is a Darboux point, then all the polynomials w_i, $1 \leq i \leq n-1$, vanish at the origin $y = 0$. Comparing the minimal degree terms of both sides in (**??**) we obtained that

$$\sum_{i=1}^{n-1} l_i \frac{\partial h}{\partial y_i} = \chi h,$$

where

l$_i = \sum_{j=1}^{n-1} l_{ij} y_j, \qquad 1 \leq i \leq n-1, arehomogeneouslineartermsof$w$_i$, χ is the zero order term of q and h is the non-trivial homogeneous component of f of the lowest degree. Let us denote by ρ_i, $1 \leq i \leq n-1$ the eigenvalues of the matrix $L = [l_{ij}]_{1 \leq i,j \leq n-1}$. Then, it can be shown (see [14, Lemma 2.3]) that if F is a Darboux polynomial of (2) then there exist non-negative integers i_k, $1 \leq k \leq n-1$, such that

$$\sum_{k=1}^{n-1} i_k \rho_k = \chi, \qquad \sum_{k=1}^{n-1} i_k = \deg h \leq m. \tag{3}$$

The eigenvalues of the matrix L we will call *Lagutinskii-Levelt exponents.* The proof of the non-integrability presented here uses essentially the relations (3).

In the proof of our theorem we will also use the following well known fact. Let $F \in \mathbb{C}[y]\backslash\mathbb{C}$ be a polynomial of a degree s in one variable y. Let us denote by $y_i, 1 \leq i \leq s$, the roots of F and by $\alpha_i, 1 \leq i \leq s$, their multiplicities respectively. As it is well known, in this case

$$\frac{F'}{F} = \sum_{i=1}^{s} \frac{\alpha_i}{y - y_i}, \tag{4}$$

where F' is the derivative of F with respect to y. Moreover, such decomposition is unique. Indeed, if

$$\frac{F'}{F} = \sum_{i=1}^{p} \frac{\beta_i}{y - y_i'},$$

then $s = p$, $y_i = y_i', 1 \leq i \leq s$, and $\alpha_i = \beta_i, 1 \leq i \leq s$.

3 Proof of the theorem

The Halphen system is invariant with respect to permutations of variables. Let us denote by τ an automorphism of $\mathbb{C}[x_1, x_2, x_3]$ induced by a permutation of $\{x_1, x_2, x_3\}$. If by d_H we denote the derivation defined by the Halphen system then

$$\tau \circ d_H \circ \tau^{-1} = d_H.$$

¿From this invariance property of d_H we have immediately that

$$\text{if} \quad d_H(F) = PF \quad \text{then} \quad d_H(\tau(F)) = \tau(P)\tau(F). \tag{5}$$

First we show the following

Lemma 1 *The Halphen system (1) does not have any polynomial first integral.*

Proof. Let us consider a Darboux point $z^0 = (1, 1, 1)$ of the Halphen system and let us assume that $F \in \mathbb{C}[x_1, x_2, x_3]\backslash\mathbb{C}$ is a homogeneous first integral of degree m and $F \neq 0$ (see point (4) of sec. 2). Direct calculations show that for the chosen Darboux point we have (see (**??**))

$$w_1 = -y_1(1 + 2y_2 + y_1y_2), \quad w_2 = -y_2(1 + 2y_2 + y_1y_2), \quad q = m(1 - y_1y_2).$$

Thus, the Lagutinskii-Levelt exponents are $\rho_1 = \rho_2 = -1$, and $\chi = m$. Then (see (3)) there exist two non-negative integers i_1 and i_2 such that $i_1\rho_1 + i_2\rho_2 = -(i_1 + i_2) = m$ but $m > 0$. Contradiction. $\square$

Although the Halphen system has no polynomial first integrals it has Darboux polynomials. If we denote

$$
\begin{aligned}
F_1 &= x_1 - x_2, \qquad & P_1 &= -2x_3,\\
F_2 &= x_2 - x_3, \qquad & P_2 &= -2x_1, \qquad & -1000\\
F_3 &= x_3 - x_1, \qquad & P_3 &= -2x_2,
\end{aligned}
$$

then we have

$$d_H(F_i) = P_i F_i, \qquad i = 1, 2, 3.$$

In order to prove the theorem, we show that (**??**) are all irreducible Darboux polynomials of the system.

For $f, g \in \mathbb{C}[x_1, \dots, x_n]$ we write $f \propto g$ if there exists $a \in \mathbb{C}$, $a \neq 0$, such that $f = ag$. Note that every 'eigenvalue' P for the Halphen system must be linear, i.e., it has the form $P = p_1 x_1 + p_2 x_2 + p_3 x_3$.

Now we show the following

Lemma 2 *If $F \in \mathbb{C}[x_1, x_2, x_3] \setminus \mathbb{C}$ is a irreducible homogeneous polynomial of degree m such that $d_H(F) = PF$, and $P = \alpha P_i$ for some $i \in \{1, 2, 3\}$, and $\alpha \in \mathbb{C}$ then $F \propto F_i$ and $\alpha = 1$.*

Proof.

First, assume that $i = 1$ and introduce new variables

$$z_1 = x_1 - x_2, \qquad z_2 = x_2 - x_3, \qquad z_3 = x_1 + x_3.$$

In the new variables the Halphen system has the form

$$\frac{dz_1}{dt} = z_1 l_1, \qquad \frac{dz_2}{dt} = -z_2 l_2, \qquad \frac{dz_3}{dt} = \frac{1}{2} l_1 l_2,$$

where

$$l_1 = z_1 + z_2 - z_3, \qquad l_2 = z_1 + z_2 + z_3. \tag{6}$$

Darboux polynomials and their corresponding 'eigenvalues' (**??**) in terms of the new variables have the form

$$\left.\begin{aligned}
F_1 &= z_1, \qquad & F_2 &= z_2, \qquad & F_3 &= -z_1 - z_2,\\
P_1 &= l_1, \qquad & P_2 &= -l_2, \qquad & P_3 &= l_3 = z_1 - z_2 - z_3.
\end{aligned}\right\} \tag{7}$$

Let us assume that there exists an irreducible homogeneous Darboux polynomial F of degree $m \geq 1$ with the 'eigenvalue' $P = p_1 z_1 + p_2 z_2 + p_3 z_3 = \alpha l_1$ for some $\alpha \in \mathbb{C}$, i.e.,

$$z_1 l_1 \partial_1 F - z_2 l_2 \partial_2 F + \tfrac{1}{2} l_1 l_2 \partial_3 F = PF; \qquad \partial_i = \tfrac{\partial}{\partial z_i}, \quad i = 1, 2, 3.-1000$$

Using the Euler identity for F, we can eliminate $\partial_3 F$ from the above equation and, taking into account (6), we obtain

$$z_1 l_1^2 \partial_1 F + z_2 l_2^2 \partial_2 F = (m l_1 l_2 - 2 z_3 P) F. \tag{8}$$

We put $z_1 = 0$ in the above equation. The obtained equation has the form

$$z_2 \tilde{l}_2^2 \partial_2 \tilde{F} = (m \tilde{l}_1 \tilde{l}_2 - 2 z_3 \tilde{P}) \tilde{F}, \tag{9}$$

where we denote $\tilde{G}(z_2, z_3) = G(0, z_2, z_3)$ for an arbitrary polynomial G. Indeed, $\tilde{\partial_2 F} = \partial_2 \tilde{F}$. Now suppose that $F \not\propto z_1$. Under this assumption $\tilde{F} \neq 0$ because $F \neq 0$ is homogeneous and irreducible and thus $\deg \tilde{F} = \deg F = m$. From (9) we obtain

$$\frac{\partial_2 \tilde{F}}{\tilde{F}} = -\frac{m + 2p_3}{z_2} + \frac{2(m + p_3)}{z_2 + z_3} - \frac{2(p_2 - p_3) z_3}{(z_2 + z_3)^2}.$$

For fixed $z_3 \neq 0$ it represents decomposition (4), and, as a consequence, we obtain that

$$p_2 - p_3 = 2\alpha = 0, \tag{10}$$

thus $P = 0$. However, in such a case, from Lemma 1 we know that F is constant. This contradicts the assumption that $\deg F \geq 1$. Consequently, $F \propto z_1$ and thus $m = 1$. This ends the proof when $i = 1$. Permutational invariance of the Halphen system (see (5)) implies that the same is true when $i = 2$ or $i = 3$. $\square$

Almost in the same way we prove the following

Lemma 3 *If $F \in \mathbb{C}[x_1, x_2, x_3] \setminus \mathbb{C}$ is an irreducible homogeneous polynomial of degree m such that $d_H(F) = PF$, and $F \not\propto F_1$, $F \not\propto F_2$ then $P = \alpha P_3$, $\alpha \in \mathbb{C}$.*

Proof. Let us take $P = p_1 z_1 + p_2 z_2 + p_3 z_3$ in equation (8). ¿From the proof of the previous lemma we obtain that $p_2 = p_3$ (see (10)). Next, put $z_2 = 0$ in equation (8). We obtain that

$$z_1 \hat{l}_1^2 \partial_1 \hat{F} = (m \hat{l}_1 \hat{l}_2 - 2 z_3 \hat{P}) \hat{F}, \tag{11}$$

where now for an arbitrary polynomial G we denote $\hat{G} = G(z_1, 0, z_3)$. Under our assumption $\hat{F} \neq 0$ because $F \neq 0$ is homogeneous and irreducible and thus $\deg \hat{F} = \deg F = m$. From (11) we obtain

$$\frac{\partial_1 \hat{F}}{\hat{F}} = -\frac{m + 2p_3}{z_1} + \frac{2(m + p_3)}{z_1 - z_3} - \frac{2(p_1 + p_3) z_3}{(z_1 - z_3)^2}.$$

This implies that necessarily $p_1 = -p_3$ and, consequently,

$$P = -p_3(z_1 - z_2 - z_3) = -p_3 l_3.$$

But l_3 is equal to P_3 expressed in variables (z_1, z_2, z_3) (see (7)). $\square$

Now we can proceed to the proof of our theorem stated in Introduction.
Proof. Let us assume that the Halphen system (1) possesses a rational first integral $F = A/B$, with relatively prime $A, B \in \mathbb{C}[x_1, x_2, x_3]$. From point (1) in section 2 we know that polynomials A and B are Darboux polynomials corresponding to the same 'eigenvalue' $P \in \mathbb{C}[x_1, x_2, x_3]$. From Lemma 1 we deduce that $P \not\equiv 0$. From Lemmas 2 and 3 we deduce that the only irreducible homogeneous Darboux polynomials of the Halphen system are those given by (**??**). Let us observe that the linear forms P_i, $1 \leq i \leq 3$ are linearly independent, and thus, from the point (6) of the previous section, we deduce that all the Darboux polynomial of the Halphen system are monoms $\alpha F_1^{i_1} F_2^{i_2} F_3^{i_3}$ for some non-negative integers i_1, i_2, i_3 such that $i_1 + i_2 + i_3 > 0$, and $\alpha \in \mathbb{C}$. It follows that both A and B are of this form. As they corresponds to the same 'eigenvalue' P, by Lemmas 2 and 3 they proportional and thus they are not relatively prime as was assumed. Contradiction shows that our theorem is true. $\square$

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