An Example of a Quantum Ergodic System

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Abstract

We prove that the time average over the quantum evolution of a system of infinitely many harmonic oscillators whose classical flow is ergodic with respect to the Gibbs measure has a limit equal to the relevant classical phase average, and therefore independent of the initial state provided it belongs to the image of the quantum Gibbs measure.

1 Introduction

The ergodicity condition

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T a(S_tx)\,dt=\frac{1}{\mu(M)}\int_M a\,d\mu,\qquad \mu\quad \text{a.e.} \tag{1.1}$$

valid for any L^1 observable a, is a fundamental prerequisite for the chaotic behaviour of any (to fix the ideas, continuous and Hamiltonian) classical dynamical system (M,S_t,μ). In particular, ergodicity implies that the dependence of the evolution $S_t:M\to M$ on the initial condition $x\in M$ is averaged out in time. Now denote H the quantization of the Hamiltonian generating the flow S_t on the phase manifold M, A any suitable quantum observable, and consider the most interesting case in which the spectrum $\sigma(H)$ of H is discrete and simple. Denoting $A_H(t)=e^{iHt}Ae^{-iHt}$ the quantum evolution of A, it is well known (and easy to verify) that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle\psi,A_H(t)\psi\rangle=\sum_{n=0}^{\infty}|\lambda_n|^2\langle u_n,Au_n\rangle \tag{1.2}$$

Here $\psi=\sum_{n=0}^{\infty}\lambda_n u_n$ is any normalized quantum state expanded on the normalized eigenvector basis $\{u_n : n=0,1,\dots\}$ of H. This is the Von Neumannn definition of quantum ergodicity[VN]: it has been proved unsatisfactory because the time average cannot eliminate the dependence on the initial datum $\psi\equiv\{\lambda_n\}_{n=0}^{\infty}$, unless the scalar products $\langle u_n, Au_n\rangle$ turn out to be independent of n. This property cannot obviously hold in any sensible quantum system: however it may be valid at the classical limit, at which $\langle u_n, Au_n\rangle$ may tend to the phase average of the r.h.s. of (1.1), as verified in many instances (see e.g.[Sc, CdV, HMR, Ze, DEGI]), provided H is the quantization of a Hamiltonian generating an ergodic flow. This fact has even led some authors to assume the validity of this limiting property as the very definition of quantum ergodicity.

The persisting dependence on the initial datum after time average can be looked at as a manifestation of the so called "quantum suppression of classical chaos", which however, according to a deep remark by G.Jona-Lasinio [JL] (see also [JLPC]) can disappear when the number of particles tends to infinity. Hence the quantum counterparts of chaotic systems with infinitely many degrees of freedom are the best candidates to detect quantum chaotic behaviour: at least, in particular, of the disappearance of the dependence of the initial datum after time average.

Therefore we consider in this paper possibly the simplest among the above cases, namely the infinite linear harmonic system

$$\ddot{q}_i=-\sum_{i,j\in\mathbb{Z}}V_{ij}q_j \tag{1.3}$$

where the couplings V_{ij} generate an infinite dimensional dynamics ϕ_t ergodic *with respect to the (infinite dimensional) Gibbs measure* $d\mu_G(\beta)$ [LL, Ti, VH], and prove that the corresponding quantum evolution has a limit independent of any normalized initial state in the image of the quantum Gibbs measure (see below). Moreover, by the commutativity between time evolution and Weyl quantization, a characteristic property of the harmonic flow, the limit turns out to be given by the relevant classical phase average.

More precisely, denote

$$q_m(x,\xi)=\frac{1}{2}|\xi|^2+\langle V_m x,x\rangle,\qquad V_m=(V_{i,j})_{\substack{|i|\le m\\|j|\le m}} \tag{1.4}$$

$H_m=Op^W(q_m)$ the operator acting on $L^2(\mathbb{R}^{2m+1})$ generated by its canonical (Weyl) quantization, $Op^W(a)$ the operator in $L^2(\mathbb{R}^{2m+1})$ generated by the Weyl quantization of the symbols $a(x,\xi)$ where $a(x,\xi)$ is any smooth classical observable on $\mathbb{R}^m$.

Then a version of the present result (see Theorem 2.2 below) essentially states that for any $\beta > 0$ and $m_1 \in \mathbb{N}$ fixed, we can find a family of normalized functions $g_m \in L^2(\mathbb{R}^{2m+1})$ such that, at least on some sequences

$$\lim_{T\to\infty}\lim_{m\to\infty}\frac{1}{T}\int_0^T \left\langle e^{itH_m}Op^W(a\circ\Pi_{m_1})e^{-itH_m}g_m, g_m\right\rangle_{L^2} dt = \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\hat{\mu}_{2\beta} \tag{1.5}$$

Here the family of functions $\{g_m\}$ contains the image under the quantum Gibbs measure $e^{-\beta H_m}$ of "almost all" (in some natural meaning) coherent states on $\mathbb{R}^{2m+1}$, $a \circ \Pi_{m_1}$ denotes the restriction of the observable a to any set of m_1 fixed variables, and $d\hat{\mu}_{2\beta}$ is the infinite dimensional Gibbs measure defined by the limit of the Gibbs measure $d\hat{\mu}^N_{2\beta}$ on $\mathbb{R}^{2N+1}$ whose Weyl quantization yields $e^{-2\beta H_m}$: when β becomes small, $d\hat{\mu}_{2\beta}$ becomes asymptotically equal to $d\mu_G(2\beta)$.

2 Notation, Assumptions and Statement of the Main Results

We now turn to the precise formulation of our assumptions and results.

Following the notation of [LL], to which the reader is referred for any detail on the system of infinitely many oscillators not reproduced here, let $V = (V_{i,j})_{i,j\in Z}$ be an infinite real-symmetric matrix; define q_m and V_m as in (1.4) and denote $\Lambda_m = (\mathbb{R}^{2m+1})^2$.

We write $S_m(1)$ for the set of C^∞ functions on Λ_m which are bounded together with all their derivatives, and for $a \in S_m(1)$ we denote $Op^W(a)$ the Weyl quantization (with $\hbar = 1$) of the symbol (equivalently, classical observable) a, explicitly given by the oscillatory integral :

$$Op^W(a)u(x) = (2\pi)^{-m}\int_{\Lambda_m} e^{i\langle (x-y),\xi\rangle} a(\frac{x+y}{2},\xi)u(y)\,dy\,d\xi \tag{2.1}$$

for all $u \in \mathcal{S}(\mathbb{R}^{2m+1})$. In particular the Schrödinger operator H_m acting on $L^2(\mathbb{R}^{2m+1})$

$$H_m := Op^W(q_m) = \frac{1}{2}(\sum_{j=1}^{2m+1} D^2_{x_j}) + \langle V_m x, x\rangle, \qquad D_{x_j} = -i\frac{\partial}{\partial x_j} \tag{2.2}$$

is the quantization of the classical Hamiltonian q_m describing m harmonic oscillators coupled through V_m.

We assume from now on

(H1)

$$|V_{ij}| = \mathcal{O}(|i-j|^{-\infty}, \qquad |i-j| \to +\infty$$

and there exist constants $0 < \varepsilon < M < \infty$ such that $\forall m \geq 0$, the spectrum of V_m is included in $[\varepsilon, M]$.

In particular, V defines a bounded strictly positive operator on $\ell^2(\mathbb{Z})$ with spectrum in $[\varepsilon, M]$.

(H2) The operator V acting on $\ell^2(\mathbb{Z})$ has no point spectrum.

Denote

$$\Lambda_\infty := \bigcup_{k\in\mathbb{N}} \Big\{(x_j,\xi_j)\}; \qquad |x_j| + |\xi_j| = \mathcal{O}(|j|^k),\ |j| \to \infty\Big\} := \bigcup_{k\in\mathbb{N}} \mathcal{H}_k \tag{2.3}$$

It is proved in [LL] that under condition (H1), Λ_∞ is invariant under the classical evolution of infinitely many degrees of freedom defined as follows

$$\phi_t(x,\xi) \equiv \phi(t,x,\xi) = e^{tB}(x,\xi), \qquad \forall (x,\xi) \in \Lambda_\infty, \forall t \in \mathbb{R} \tag{2.4}$$

where $B(x,\xi)$ is the infinite-dimensional Hamiltonian vector field generated by q_m when $m \to \infty$

$$B(x,\xi) = \left(\xi_j, -\sum_{k\in\mathbb{Z}} V_{jk}x_k\right)_{j\in\mathbb{Z}} \tag{2.5}$$

Moreover, if $\Pi_m : \Lambda_\infty \to \Lambda_m$ denotes the projection

$$\Pi_m(x,\xi) = (x_j,\xi_j)_{|j|\leq m} \tag{2.6}$$

for any $(x,\xi) \in \mathcal{H}_k$ one has

$$\phi(t,x,\xi) = \lim_{m\to\infty} \exp t H_{q_m}(\Pi_m(x,\xi)) \in \mathcal{H}_k \tag{2.7}$$

where $H_{q_m} = \left(\frac{\partial q_m}{\partial \xi}, -\frac{\partial q_m}{\partial x}\right)$ is the Hamiltonian vector field generated by q_m, and the limit is taken with respect to the natural Banach-space topology of $\mathcal{H}_k$.

Now by (H1) the operator $V^{-\frac{1}{2}}$ exists and is continuous on $\ell^2(\mathbb{Z})$. This assumption and (H2) allow Lanford and Lebowitz[LL] to prove the existence of the infinite dimensional, ergodic Gibbs measure $d\mu_G(\beta)$ on Λ_∞, namely

1. $$\int_{\Lambda_\infty} \varphi \circ \Pi_{m_1}\, d\mu_G(\beta) = \lim_{m\to\infty} \int_{\Lambda_m} \varphi \circ \Pi_{m_1}(x,\xi) e^{-\beta q_m(x,\xi)} \frac{dx\, d\xi}{Z_m}, \quad \forall \varphi \in C^0(\mathbb{R}^{2m_1+1}) \tag{2.8}$$

   where

   $$Z_m(\beta) = \int_{\Lambda_m} e^{-\beta q_m(x,\xi)}\, dx\, d\xi \tag{2.9}$$

   is the m-particle partition function;

2. The Gibbs measure is invariant and ergodic with respect to the flow $\phi(t; x, \xi)$, namely the continuous dynamical system $(\Lambda_\infty, \phi_t, d\mu_G(\beta))$ is ergodic.

An exemple of an infinite matrix satisfying (H1)-(H2) is given by $V = W$ where

$$W_{ij} = 0, \qquad |i-j| \geq 2, \qquad W_{ii} = 1, \qquad W_{i,i+1} = W_{i,i-1} = \alpha \tag{2.10}$$

with $|\alpha| < 1$, $\alpha \in \mathbb{R}$. The properties (H1), (H2) are proved e.g. in [Sj].

To state our result we need to establish some further notation. For $f \in L^2(\mathbb{R}^{2m+1})$ and $(x, \xi) \in \Lambda_m$, we introduce the Wigner function of f

$$w_f(x, \xi) = \int_{\mathbb{R}^{2m+1}} e^{iu\xi} f(x - \frac{u}{2}) \overline{f(x + \frac{u}{2})}\, du \tag{2.11}$$

and we restrict our attention to a random set of states f in the following sense : for all $m \in \mathbb{N}$, we consider a measure space (X_m, ν_m) with positive measure ν_m, and a family $(f_\lambda)_{\lambda \in X_m}$ of functions in $L^2(\mathbb{R}^{2m+1})$ such that :

(H3) For $dxd\xi$-almost all $(x, \xi) \in \Lambda_m$, the application $X_m \ni \lambda \mapsto w_{f_\lambda}(x, \xi)$ is in $L^1(X_m, d\nu_m)$ with non negative values, and the quantity $\int_{X_m} w_{f_\lambda}(x, \xi) d\nu_m(\lambda)$ is ($dxd\xi$-almost everywhere) constant with respect to (x, ξ).

In the last section we develop an example (the so-called coherent states) where (H3) is satisfied. Note that in any case, $w_f(x, \xi)$ is real and satisfies : $\int w_f(x, \xi) dx d\xi = (2\pi)^{2m+1} \|f\|^2$. As we shall see, (H3) implies among other things that

$$\int_{X_m} \|e^{-\beta H_m} f_\lambda\|^2 d\nu_m(\lambda) < +\infty \tag{2.12}$$

and we set :

$$d\nu'_m(\lambda) = \frac{\|e^{-\beta H_m} f_\lambda\|^2 d\nu_m(\lambda)}{\int_{X_m} \|e^{-\beta H_m} f_\lambda\|^2 d\nu_m(\lambda)} \tag{2.13}$$

Now let

$$W_\beta = \sqrt{2} V^{-\frac{1}{2}} \tanh \frac{\beta V^{\frac{1}{2}}}{\sqrt{2}} \tag{2.14}$$

(which is well defined on $\ell^2(\mathbb{Z})$), and for $\beta > 0$, denote $\hat{\mu}_\beta$ the Gaussian probability measure on Λ_∞ with mean zero and covariance given by :

$$\begin{aligned} E[x_i x_j] &= \langle (V W_\beta)^{-1} e_i, e_j \rangle_{\ell^2(\mathbb{Z})} \\ E[\xi_i \xi_j] &= \langle W_\beta^{-1} e_i, e_j \rangle_{\ell^2(\mathbb{Z})} \\ E[x_i \xi_j] &= 0 \end{aligned} \tag{2.15}$$

where $e_i = (\delta_{ij})_{j \in \mathbb{Z}}$. Then our main result is :

Theorem 2.1 *Assume* (H1)-(H3). *Then :*

(i) For any $\beta > 0$, *the dynamical system* $(\Lambda_\infty, \phi_t, \hat{\mu}_\beta)$ *is ergodic ;*

(ii) For $m_1 \in \mathbb{N}$ *fixed and* $a \in C_0^\infty(\Lambda_{m_1})$, *denote*

$$g_{m,\beta,\lambda} = \frac{e^{-\beta H_m} f_\lambda}{\|e^{-\beta H_m} f_\lambda\|}$$

and

$$A(m,n,T,\lambda) = \frac{1}{T}\int_0^T \left\langle e^{itH_n} Op^W(a \circ \Pi_{m_1}) e^{-itH_n} g_{m,\beta,\lambda}\, ,\, g_{m,\beta,\lambda} \right\rangle_{L^2(\mathbb{R}^{2m+1})} dt.$$

Then one has

$$\lim_{T\to\infty} \limsup_{n\to\infty} \limsup_{m\to\infty} \int_{X_m} \left| A(m,n,T,\lambda) - \int_{\Lambda_\infty} a \circ \Pi_{m_1}\, d\hat{\mu}_{2\beta} \right| d\nu'_m(\lambda) = 0.$$

Remarks.

1. In particular, $A(m,n,T,\lambda)$ can be made arbitrarily close to $\int a \circ \Pi_{m_1}\, d\hat{\mu}_{2\beta}$ in $L^1(X_m, d\nu'_m(\lambda))$ by first choosing T, then $n = n(T)$, then $m = m(n,T)$ all of them sufficiently large. The *pointwise* convergence of $A(m,n,T,\lambda)$ will be proved in Proposition 5.1 below, choosing for f_λ a particular set of coherent states.

2. Note that $A(m,n,T,\lambda)$ is well defined since the action of $e^{itH_n} Op^W(a\circ\Pi_{m_1}) e^{-itH_n}$ on $g_{m,\beta,\lambda}$ which is a C^∞ function on $\mathbb{R}^{2m+1}$ is well defined for $n \leq m$.

3. For β small, we have $W_\beta = \beta I + \mathcal{O}(\beta^3)$ and therefore the covariance of $\hat{\mu}_\beta$ coincides with the one of the usual Gibbs measure $\mu_G(\beta)$ up to a $\mathcal{O}(\beta^3)$-error term. In this sense, we can say that $\hat{\mu}_\beta$ and $\mu_G(\beta)$ are asymptotically equal for small β's (that is for large temperatures).

4. The measure $\hat{\mu}_\beta$ can be seen as the limit when $m \to +\infty$ of the probability measure on Λ_m obtained by normalizing $e^{-q_{\beta,m}}(x,\xi)dxd\xi$, where

$$q_{\beta,m}(x,\xi) = q_m(W_{\beta,m}^{1/2}x, W_{\beta,m}^{1/2}\xi) \tag{2.16}$$

and

$$W_{\beta,m} = \sqrt{2}V_m^{-\frac{1}{2}}\tanh\frac{\beta V_m^{\frac{1}{2}}}{\sqrt{2}}. \tag{2.17}$$

This relation between $\hat{\mu}_\beta$ and the usual Gibbs measure reflects the relation between the usual quantum Gibbs measure $e^{-\beta H_m}$ and the Weyl quantization of the classical Gibbs measure $e^{-\beta q_m}$, namely (see Lemma 3.1 below) :

$$e^{-\beta H_m} = C_{\beta,m} Op^W(e^{-q_{\beta,m}})$$

where $C_{\beta,m}$ is a constant. In particular, if we denote # the Weyl composition of symbols, we get (with some other constant $C'_{\beta,m}$):

$$e^{-q_{\beta,m}} \# e^{-q_{\beta,m}} = C'_{\beta,m} e^{-q_{2\beta,m}} \tag{2.18}$$

which also explains the fact that $\hat{\mu}_{2\beta}$ appears in the result.

Under an additional assumption on V the result of Theorem 2.1 admits a less cumbersome formulation which eliminates the necessity of the double limit with respect to m and n. The further assumption is :

(H4) For all $m \geq 0$ there exists a $(2m+1) \times (2m+1)$ real-symmetric matrix $\tilde{V}_m$ satisfying the same assumption (H1) as V_m, and such that :
(i) $\forall i, j \in \mathbb{Z}$, $\langle \tilde{V}_m^{-1} e_i, e_j \rangle$ tends to $\langle V^{-1} e_i, e_j \rangle_{\ell^2}$ as $m \to +\infty$;
(ii) $\forall m, n \in \mathbb{Z}$, the operator $\Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n$ becomes independent of n for n sufficiently large ;
(iii) The operator $\tilde{V}_m^{-\frac{1}{2}} \Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n$ tends strongly to the identity on each $\mathcal{H}_k$ ($k \in \mathbb{N}$) as $m \to +\infty$.

It is not very difficult to verify that an example of such V satisfying (H4) (in addition to (H1)-(H2)) is given by $V = W^2$ where W is as in the example (2.10) : in this case one can take $\tilde{V}_m = (W_m)^2$ where W_m is extracted from W as in (1.4).

Under assumption (H4), we define for $0 < \beta < (2M)^{-1/2}$:

$$\tilde{q}_{\beta,m}(x, \xi) = \langle F_{\beta,m} \xi, \xi \rangle + \langle G_{\beta,m} x, x \rangle \tag{2.19}$$

where (denoting I_m the identity on $\mathbb{R}^{2m+1}$) :

$$F_{\beta,m} = \frac{1}{2\beta} \tilde{V}_m^{-1} \left(I_m - (I_m - 2\beta^2 \tilde{V}_m)^{\frac{1}{2}} \right) \tag{2.20}$$

$$G_{\beta,m} = 2\tilde{V}_m F_{\beta,m}. \tag{2.21}$$

We also consider on X_m the probability measure :

$$d\tilde{\nu}'_m(\lambda) = \frac{\|Op^W(e^{-\tilde{q}_{\beta,m}}) f_\lambda\|^2 d\nu_m(\lambda)}{\int_{X_m} \|Op^W(e^{-\tilde{q}_{\beta,m}}) f_\lambda\|^2 d\nu_m(\lambda)}. \tag{2.22}$$

Then the result is :

Theorem 2.2 *Assume* (H1)-(H4), *and, for $m_1 \in \mathbb{N}$ fixed, $a \in C_0^\infty(\Lambda_{m_1})$ and $0 < \beta < (2M)^{-1/2}$, denote*

$$\tilde{g}_{m,\beta,\lambda} = \frac{Op^W(e^{-\tilde{q}_{\beta,m}}) f_\lambda}{\|Op^W(e^{-\tilde{q}_{\beta,m}}) f_\lambda\|}$$

and

$$\tilde{A}(m,T,\lambda)=\frac{1}{T}\int_0^T\left\langle e^{itH_m}Op^W(a\circ\Pi_{m_1})e^{-itH_m}\tilde{g}_{m,\beta,\lambda}\,,\,\tilde{g}_{m,\beta,\lambda}\right\rangle_{L^2(\mathbb{R}^{2m+1})}dt.$$

Then one has

$$\lim_{T\to\infty}\limsup_{m\to\infty}\int_{X_m}\left|\tilde{A}(m,T,\lambda)-\int_{\Lambda_\infty}a\circ\Pi_{m_1}\,d\mu_G(2\beta)\right|d\tilde{\nu}'_m(\lambda)=0.$$

Remarks.

1. Here, the choice of the quadratic form $\tilde{q}_{\beta,m}$ is dictated from the fact that we have (see Lemma 4.3 below) :

$$e^{-\tilde{q}_{\beta,m}}\#e^{-\tilde{q}_{\beta,m}}=C''_{\beta,m}e^{-2\beta\tilde{q}_m} \tag{2.23}$$

where

$$\tilde{q}_m(x,\xi)=\frac{1}{2}|\xi|^2+\langle\tilde{V}_m x,x\rangle.$$

In view of assumption (H4)(i), this explains why we get the usual Gibbs measure in the limit $m\to+\infty$. Note that we also have $F_{\beta,m}=\frac{\beta}{2}I_m+\mathcal{O}(\beta^3)$ et $G_{\beta,m}=\beta V_m+\mathcal{O}(\beta^3)$, so that $\tilde{q}_{\beta,m}$ is asymptotically equal to $\beta\tilde{q}_m$ as $\beta\to 0_+$.

2. For β small it is possible to compare $Op^W(e^{-\tilde{q}_{\beta,m}})$ with $e^{-\beta\tilde{H}_m}$, where $\tilde{H}_m=Op^W(\tilde{q}_m)=\frac{1}{2}(\sum_{j=1}^{2m+1}D^2_{x_j})+\langle\tilde{V}_m x,x\rangle$. Actually, denoting $\tilde{\lambda}_1,...,\lambda_{2m+1}$ the eigenvalues of $\tilde{V}_m$, we get by standard formulas (see (3.29) below) that $Op^W(e^{-\tilde{q}_{\beta,m}})$ is unitarily equivalent to

$$\bigotimes_{j=1}^{2m+1}\frac{1}{\sqrt{1-\beta^2\tilde{\lambda}_j/2}}\exp\left[-\frac{1}{2}\left(\ln\frac{1+\beta\sqrt{\tilde{\lambda}_j/2}}{1-\beta\sqrt{\tilde{\lambda}_j/2}}\right)(D^2_{x_j}+x_j^2)\right] \tag{2.24}$$

while under the same unitary transformation $e^{-\beta\tilde{H}_m}$ becomes

$$\bigotimes_{j=1}^{2m+1}e^{-\beta\sqrt{\tilde{\lambda}_j/2}(D^2_{x_j}+x_j^2)}. \tag{2.25}$$

Since for small β we have :

$$\frac{1}{2}\ln\frac{1+\beta\sqrt{\tilde{\lambda}_j/2}}{1-\beta\sqrt{\tilde{\lambda}_j/2}}=\beta\sqrt{\frac{\tilde{\lambda}_j}{2}}\big(1+\mathcal{O}(\beta)\big)$$

we get from (2.24)-(2.25) :

$$Op^W(e^{-\tilde{q}_{\beta,m}}) = e^{-\beta\tilde{H}_m(1+\beta R_{\beta,m})} \tag{2.26}$$

where $R_{\beta,m}$ commutes with $\tilde{H}_m$ and $\frac{1}{2m+1}R_{\beta,m}$ is uniformly bounded with respect to β and m.

3. The previous remark proves that for β small, $Op^W(e^{-\tilde{q}_{\beta,m}})$ commutes with $\tilde{H}_m$. This is actually true for all positive β, as it can be deduced from formula (3.30) below, and the fact that

$$e^{-\tilde{q}_{\beta,m}} \circ \exp tH_{\tilde{q}_m}(x,\xi) = e^{-\tilde{q}_{\beta,m}} \circ \exp tH_{\tilde{q}_{\beta,m}}((2F_{\beta,m})^{-\frac{1}{2}}x, (2F_{\beta,m})^{-\frac{1}{2}}\xi)$$

is constant with respect to $t \in \mathbb{R}$.

4. Since the choice of the lattice $\mathbb{Z}$ does not play any role at all in our proofs, it can be replaced without modification by any lattice $\Gamma \subset \mathbb{R}^d$ $(d \geq 1)$ of the type considered in [LL], in which case the model describes an infinite harmonic crystal in $\mathbb{R}^d$. Also the choice of Λ_m is unessential, in the sense that any other choice $\Lambda'_m \to \Lambda_\infty$ in a reasonable way leads to the same results.

3 Proof of Theorem 2.1

Now we turn to the proof of Theorem 2.1. The fact that $(\Lambda_\infty, \phi_t, \hat{\mu}_\beta)$ is ergodic essentially follows from the arguments of [LL], but for the sake of completeness, we give a sketch of the proof. First, the invariance of $\hat{\mu}_\beta$ under ϕ_t is a consequence of the commutativity between the operator B defined in (2.5) and the operator $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \ni (x,\xi) \mapsto (W_\beta^{\frac{1}{2}}x, W_\beta^{\frac{1}{2}}\xi)$. Now, denote $\hat{h}_1$ the closure in $L^2(\Lambda_\infty, d\hat{\mu}_\beta)$ of the set of all finite sums of $(a_jx_j + b_j\xi_j)$'s, $(a_j, b_j \in \mathbb{C}\ ;\ j \in \mathbb{Z})$. Then, denoting $d(\mathbb{Z})$ the elements of $\ell^2(\mathbb{Z})$ with finite support, the application

$$\begin{array}{rcl} \theta\ :\ d(\mathbb{Z}) \oplus d(\mathbb{Z}) & \to & \hat{h}_1 \\ a \oplus b & \mapsto & \sum_j (a_jx_j + b_j\xi_j) \end{array}$$

can be extended into an isomorphism from $D((VW_\beta)^{-1/2}) \oplus D(W_\beta^{-1/2})$ to $\hat{h}_1$ (where $D(A)$ denotes the domain of the operator A). Moreover, identifying $D((VW_\beta)^{-1/2}) \oplus D(W_\beta^{-1/2})$ with $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ in an obvious way, we see that the action of ϕ_t on $\hat{h}_1$

is represented on $\ell^2(\mathbb{Z})\oplus\ell^2(\mathbb{Z})$ (via the two previous identifications) by its infinitesimal generator $U = \begin{pmatrix} 0 & -V^{-1/2} \\ V^{1/2} & 0 \end{pmatrix}$. Since, by assumption (H2), U has no point spectrum the result (i) follows by an abstract argument (see [LL]Prop.4.2).

To prove (ii) we first show :

Lemma 3.1 *There exists a constant $C_{\beta,m}$ such that :*

$$e^{-\beta H_m} = C_{\beta,m} Op^W(e^{-q_{\beta,m}})$$

Proof -Let $\lambda_1, ..., \lambda_{2m+1}$ be the eigenvalues of V_m, and denote $y = (y_1, ..., y_{2m+1})$ the coordinates in $\mathbb{R}^{2m+1}$ corresponding to an orthonormal basis of eigenvectors of V_m. Then H_m becomes:

$$H'_m = -\frac{1}{2}\Delta_y + \sum_j \lambda_j y_j^2$$

while the operator $K_m = Op^W(e^{-q_{\beta,m}})$ is transformed into :

$$K'_m = \bigotimes_{j=1}^{2m+1} Op^W\left(e^{-\left(\sqrt{\frac{2}{\lambda_j}}\tanh\left(\beta\sqrt{\frac{\lambda_j}{2}}\right)\eta_j^2 + \sqrt{2\lambda_j}\tanh\left(\beta\sqrt{\frac{\lambda_j}{2}}\right)y_j^2\right)}\right).$$

(Here, we have denoted η the dual variable of y, and we have used the fact that $x = My$ with M orthogonal, and therefore $\xi = ({}^tM)^{-1}\eta = M\eta$.) Then, the change of variables

$$y_j \mapsto z_j = (2\lambda_j)^{1/4} y_j$$

transforms H'_m into :

$$H''_m = \sum_j \sqrt{\frac{\lambda_j}{2}}(D_{z_j}^2 + z_j^2) \tag{3.27}$$

and K'_m into :

$$K''_m = \bigotimes_{j=1}^{2m+1} Op^W\left(e^{-\tanh\left(\beta\sqrt{\frac{\lambda_j}{2}}\right)(\zeta_j^2+z_j^2)}\right). \tag{3.28}$$

Next, consider the well known one-dimensional identity valid for $0 < a < 1$

$$Op^W(e^{-a(x^2+\xi^2)}) = \frac{1}{\sqrt{1-a^2}} e^{-\frac{a}{2}x^2} e^{-\frac{a}{1-a^2}D_x^2} e^{-\frac{a}{2}x^2}$$

which can be for instance verified by explicit computation of the Weyl symbol of the r.h.s.. Then, using the formula (see e.g.[He])

$$e^{-\frac{x^2}{2}} e^{-tD_x^2} e^{-\frac{x^2}{2}} = \exp\left[-\frac{\ln(z+\sqrt{z^2-1})}{4k\sqrt{z^2-1}}(D_x^2 + 4k^2(z^2-1)x^2)\right]$$

with $k = \dfrac{1}{4t}$, $z = 2t+1$ $(t>0)$, we get in particular for $0 < a < 1$:

$$Op^W(e^{-a(x^2+\xi^2)}) = \frac{1}{\sqrt{1-a^2}} \exp\left[-\frac{1}{2}\left(\ln\frac{1+a}{1-a}\right)(D_x^2+x^2)\right] \tag{3.29}$$

Taking $a = \tanh(\beta\sqrt{\frac{\lambda_j}{2}})$, the Lemma follows from (3.27)-(3.29). $\square$

Now, since the flow generated by q_n defines a linear canonical transformation on Λ_n, we have :

$$e^{itH_n} Op^W(a) e^{-itH_n} = Op^W(a \circ \exp tH_{q_n}), \qquad \forall a \in S_n(1) \tag{3.30}$$

Note that the relation (3.30) (which can be considered as an "exact Egorov theorem", and goes back at least to Van Hove (see e.g.[Fo])) is valid only for the Weyl quantization [BS]§5.2.

For $n \le m$ and $(x,\xi) \in \Lambda_m$, denote

$$\rho(m,\lambda) = \|e^{-\beta H_m} f_\lambda\|^2 \tag{3.31}$$

$$a_{n,T}(x,\xi) = \frac{1}{T}\int_0^T (a \circ \Pi_{m_1} \circ \exp tH_{q_n} \circ \Pi_n)(x,\xi))dt. \tag{3.32}$$

Using Lemma 3.1, (3.30) and (2.11), we get :

$$A(m,n,T,\lambda) = \frac{C_{\beta,m}^2}{\rho(m,\lambda)} \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi) w_{f_\lambda}(x,\xi)dxd\xi \tag{3.33}$$

where $\#$ is the Weyl composition of symbols on Λ_m :

$$(a\#b)(x,\xi) = \pi^{-2(2m+1)} \int_{\Lambda_m^2} a(x+y,\xi+\eta)b(x+z,\xi+\zeta)e^{2i(\zeta y - z\eta)}dyd\eta dzd\zeta. \tag{3.34}$$

Using the assumption (H3), we get from (3.33)

$$\int_{X_m} A(m,n,T,\lambda)\rho(m,\lambda)d\nu_m(\lambda) = C_0 \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)dxd\xi \tag{3.35}$$

$$\int_{X_m} |A(m,n,T,\lambda)|\rho(m,\lambda)d\nu_m(\lambda) \le C_0 \int_{\Lambda_m} |\left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)|dxd\xi \tag{3.36}$$

where $C_0 = C_0(\beta,m)$ is a constant, which can be computed by taking $a \equiv 1$ in (3.35):

$$C_0 = \left(\int e^{-q_{\beta,m}} \# e^{-q_{\beta,m}} dxd\xi\right)^{-1} \int_{X_m} \rho(m,\lambda)d\nu_m(\lambda). \tag{3.37}$$

This also proves (2.12) so that, using the notation (2.13) we can rewrite (3.35)-(3.36) as :

$$\int_{X_m} A(m,n,T,\lambda)d\nu'_m(\lambda) = C_1 \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)dxd\xi \tag{3.38}$$

$$\int_{X_m} |A(m,n,T,\lambda)|d\nu'_m(\lambda) \le C_1 \int_{\Lambda_m} |\left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)|dxd\xi \tag{3.39}$$

with

$$C_1 = \left(\int e^{-q_{\beta,m}} \# e^{-q_{\beta,m}} dx d\xi \right)^{-1}.$$

Now we use the two following properties of the operation # (valid e.g. for any a, b, c in $\mathcal{S}(\Lambda_m)$) :

$$\int_{\Lambda_m} (a\#b\#c)(x,\xi) dx d\xi = \int_{\Lambda_m} (b\#c\#a)(x,\xi) dx d\xi \tag{3.40}$$

$$\int_{\Lambda_m} (a\#b)(x,\xi) dx d\xi = \int_{\Lambda_m} a(x,\xi) b(x,\xi) dx d\xi. \tag{3.41}$$

The property (3.40) is just a consequence of the cyclicity of the trace of operators, and (3.41) comes from a direct computation using (3.34).

Here our symbol a is not supposed to be in $\mathcal{S}(\Lambda_m)$, but an easy argument of density allows us to deduce from (3.38) and (3.40)-(3.41) (using also (2.18)):

$$\int_{X_m} A(m,n,T,\lambda) d\nu'_m(\lambda) = \int_{\Lambda_m} a_{n,T}(x,\xi) \left[e^{-q_{2\beta,m}(x,\xi)} dx d\xi \right]_N \tag{3.42}$$

where we have used the notation :

$$\int_E f \, [d\mu]_N = \frac{1}{\mu(E)} \int_E f d\mu \tag{3.43}$$

for any finite positive measure μ on a set E.

Now the problem is to rewrite also (3.39) in this way, despite the appearance of the modulus. To do this, we use :

Lemma 3.2 *There exists a positive definite quadratic form $Q_{\beta,m}(x,\xi,y,\eta)$ on Λ_m^2 such that for all $a \in S_m(1)$:*

$$e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}} = C'_{\beta,m} \tilde{a} e^{-q_{2\beta,m}}$$

where $C'_{\beta,m}$ is the constant appearing in (2.18), and

$$\tilde{a}(x,\xi) = \int_{\Lambda_m} a(y,\eta) \left[e^{-Q_{\beta,m}(x,\xi,y,\eta)} dy d\eta \right]_N.$$

Proof - See Appendix.

We deduce in particular from Lemma 3.2 the existence of a positive C^∞ function $\gamma(x,\xi)$ on Λ_m such that for all $a \in S_m(1)$:

$$\int_{\Lambda_m} (e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}}) dx d\xi = \int_{\Lambda_m} a(x,\xi) \gamma(x,\xi) dx d\xi \tag{3.44}$$

$$\int_{\Lambda_m} |e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}}| dx d\xi \leq \int_{\Lambda_m} |a(x,\xi)| \gamma(x,\xi) dx d\xi. \tag{3.45}$$

By (3.38), (3.42) and (3.44) we get that γ equals a constant times $e^{-q_{2\beta,m}}$, which by (3.39) and (3.45) allows us to conclude that :

$$\int_{X_m} |A(m,n,T,\lambda)|\, d\nu'_m(\lambda) \leq \int_{\Lambda_m} |a_{n,T}(x,\xi)| \left[e^{-q_{2\beta,m}(x,\xi)} dx d\xi\right]_N. \tag{3.46}$$

Without loss of generality, we can assume from now on that $\int_{\Lambda_\infty} (a \circ \Pi_{m_1}) d\hat{\mu}_{2\beta} = 0$, and then it remains to estimate the r.h.s. of (3.46). Since $a_{n,T}(x,\xi)$ depends only on $\Pi_n(x,\xi)$, we can let m go to $+\infty$ in (3.46) and we get :

$$\limsup_{m\to\infty} \int_{X_m} |A(m,n,T,\lambda)|\, d\nu'_m(\lambda) \leq \int_{\Lambda_\infty} |a_{n,T}(x,\xi)| d\hat{\mu}_{2\beta}. \tag{3.47}$$

Then we use (2.7) to let n go to $+\infty$ in (3.48). By the dominated convergence theorem, we then obtain :

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \int_{X_m} |A(m,n,T,\lambda)|\, d\nu'_m(\lambda) \leq \int_{\Lambda_\infty} |\frac{1}{T}\int_0^T (a \circ \Pi_{m_1} \circ \phi_t)(x,\xi) dt| d\hat{\mu}_{2\beta}. \tag{3.48}$$

Finally, we let T go to $+\infty$. By the ergodicity property, we have that

$$\frac{1}{T}\int_0^T (a \circ \Pi_{m_1} \circ \phi_t)(x,\xi) dt \to \int_{\Lambda_\infty} (a \circ \Pi_{m_1}) d\hat{\mu}_{2\beta} = 0$$

for $\hat{\mu}_{2\beta}$-almost all (x,ξ) in Λ_∞. Therefore, applying again the dominated convergence theorem, we get from (3.48) :

$$\lim_{T\to+\infty} \limsup_{n\to\infty} \limsup_{m\to\infty} \int_{X_m} |A(m,n,T,\lambda)|\, d\nu'_m(\lambda) \leq 0 \tag{3.49}$$

which completes the proof of Theorem 2.1. □

4 Proof of Theorem 2.2

We first notice :

Lemma 4.3 *For any pair of positive definite real-symmetric matrices F and G on Λ_m, there exists a constant $C = C(F,G,m)$ such that :*

$$e^{-(\langle F\xi,\xi\rangle + \langle Gx,x\rangle)} \# e^{-(\langle F\xi,\xi\rangle + \langle Gx,x\rangle)} = C e^{-2\left(\langle F(F+G^{-1})^{-1}G^{-1}\xi,\xi\rangle + \langle (F+G^{-1})^{-1}x,x\rangle\right)}.$$

Proof - See Appendix.

In particular, taking $F = F_{\beta,m}$ and $G = G_{\beta,m}$ defined in (2.20)-(2.21) we get easily (2.23) from Lemma 4.3. Then computations analogous to those of the previous section ((3.33) through (3.46)) lead to :

$$\int_{X_m} \left|\tilde{A}(m,T,\lambda)\right| d\tilde{\nu}'_m(\lambda) \leq \int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-2\beta\tilde{q}_m(x,\xi)} dx d\xi\right]_N. \tag{4.50}$$

Now in the r.h.s. (4.50) we make the change of variables :

$$x = \tilde{V}_m^{-\frac{1}{2}} y \qquad \xi = \eta$$

which gives :

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-2\beta \tilde{q}_m(x,\xi)} dx d\xi \right]_N = \int_{\Lambda_m} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}} y, \eta)| \left[e^{-2\beta(\eta^2+y^2)} dy d\eta \right]_N . \quad (4.51)$$

Since the quadratic form in the exponent is now diagonal, we can integrate over $n \geq m$ variables so that

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-2\beta \tilde{q}_m(x,\xi)} dx d\xi \right]_N = \int_{\Lambda_n} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}} \Pi_m y, \Pi_m \eta)| \left[e^{-2\beta(\eta^2+y^2)} dy d\eta \right]_N \quad (4.52)$$

for any $n \geq m$. Coming back to the old variables on Λ_n, this gives :

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-2\beta \tilde{q}_m(x,\xi)} dx d\xi \right]_N = \int_{\Lambda_n} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}} \Pi_m \tilde{V}_n^{\frac{1}{2}} x, \Pi_m \xi)| \left[e^{-2\beta \tilde{q}_n(x,\xi)} dx d\xi \right]_N . \quad (4.53)$$

Now, by assumption (H4)(ii), for n large enough, the function $a_{m,T}(\tilde{V}_m^{-\frac{1}{2}} \Pi_m V_n^{\frac{1}{2}} x, \Pi_m \xi)$ depends only on a fixed number of variables independent of n. Then, by assumption (H4)(i) and standard results on the Gaussian measures (see e.g. [LL]), letting n tend to $+\infty$ in (4.53), we get :

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-2\beta \tilde{q}_m(x,\xi)} dx d\xi \right]_N = \int_{\Lambda_\infty} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}} \Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n x, \Pi_m \xi)| d\mu_G(2\beta). \quad (4.54)$$

Finally, using assumption (H4)(iii), (2.7), and the uniform (with respect to $m \geq 0$) continuity of $\exp tH_{q_m} \circ \Pi_m$ on each $\mathcal{H}_k$, we see that for all $(x,\xi) \in \Lambda_\infty$ and $t \in \mathbb{R}$:

$$(a \circ \Pi_{m_1} \circ \exp tH_{q_m})(\tilde{V}_m^{-\frac{1}{2}} \Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n x, \Pi_m \xi) \to (a \circ \Pi_{m_1} \circ \phi_t)(x,\xi) \quad \text{as} \quad m \to +\infty. \quad (4.55)$$

It follows from (4.55), (4.54), (4.50) and the dominated convergence theorem that :

$$\limsup_{m \to +\infty} \int_{X_m} \left| \tilde{A}(m,T,\lambda) \right| d\tilde{\nu}'_m(\lambda) \leq \int_{\Lambda_\infty} \left| \frac{1}{T} \int_0^T (a \circ \Pi_{m_1} \circ \phi_t)(x,\xi) dt \right| d\mu_G(2\beta). \quad (4.56)$$

Letting T tend to infinity in (4.56), the ergodicity of the system $(\Lambda_\infty, \phi_t, \mu_G(2\beta))$ and the fact that we can restrict to the case $\int a \circ \Pi_{m_1} d\mu_G(2\beta) = 0$ yield the assertion.

□

5 Sharpening the Result in the Case of the Coherent States

Now we take $X_m = \Lambda_m$, $d\nu_m(\lambda) = d\lambda$, and for $\lambda = (\lambda_x, \lambda_\xi) \in \Lambda_m$:

$$f_\lambda(x,\xi) = e^{ix\lambda_\xi - (x-\lambda_x)^2/2}. \quad (5.57)$$

Then a direct computation gives :

$$w_{f_\lambda}(x,\xi) = 2^{2m+1}\pi^{m+\frac{1}{2}}e^{-(\xi-\lambda_\xi)^2-(x-\lambda_x)^2} \tag{5.58}$$

so that (H3) is obviously satisfied. Therefore the results of Theorems 2.1 and 2.2 hold for V satisfying (H1)-(H2) (respectively (H1)-(H3)). The new fact which appears in this situation is :

Lemma 5.4 *Under (5.57), the two measures $d\nu'_m(\lambda)$ and $d\tilde{\nu}'_m(\lambda)$, defined respectively by (2.13) and (2.22), are Gaussian probability measures on $X_m = \Lambda_m$.*

Proof - In each case, the measure is of the form $C\|Op^W(e^{-q})f_\lambda\|^2 d\lambda$ where C is a constant and q is a positive definite quadratic form on Λ_m. Moreover, by computations analogous e.g. to those for (3.33) we have :

$$\|Op^W(e^{-q})f_\lambda\|^2 = C'\int_{\Lambda_m}(e^{-q}\#e^{-q})(x,\xi)w_{f_\lambda}(x,\xi)dxd\xi \tag{5.59}$$

where C' is another constant. Then the result follows immediately by (5.58) and the fact that $e^{-q}\#e^{-q} = C''e^{-q'}$ where q' is a positive definite quadratic form on Λ_m and C'' is a constant. □

Now denote L_m and $\tilde{L}_m$ the two real-symmetric positive definite $(4m+2)\times(4m+2)$-matrices defined by :

$$d\nu'_m(\lambda) = \left[e^{-\langle L_m\lambda,\lambda\rangle}d\lambda\right]_N \tag{5.60}$$

$$d\tilde{\nu}'_m(\lambda) = \left[e^{-\langle \tilde{L}_m\lambda,\lambda\rangle}d\lambda\right]_N. \tag{5.61}$$

Then for any bounded function $A(\lambda)$ we have :

$$\int_{\Lambda_m} A(\lambda)d\nu'_m(\lambda) = \int_{\Lambda_m} A(L_m^{-\frac{1}{2}}\lambda)\left[e^{-|\lambda|^2}d\lambda\right]_N \tag{5.62}$$

and therefore, by an argument similar to the one leading to (4.54) :

$$\int_{\Lambda_m} A(\lambda)d\nu'_m(\lambda) = \int_{\Lambda_\infty} A(L_m^{-\frac{1}{2}}\Pi_m\lambda)d\nu_G(\lambda) \tag{5.63}$$

where $d\nu_G(\lambda)$ is the infinite dimensional Gibbs measure obtained by taking the limit of $\left[e^{-|\lambda|^2}d\lambda\right]_N$ on Λ_n as $n\to+\infty$. A formula analogous to (5.63) is also true for $d\tilde{\nu}'_m(\lambda)$, and therefore it follows from Theorems 2.1 and 2.2 that we have in this situation :

$$\lim_{T\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\int_{\Lambda_\infty}\left|A(m,n,T,L_m^{-\frac{1}{2}}\Pi_m\lambda) - \int_{\Lambda_\infty} a\circ\Pi_{m_1}\,d\hat{\mu}_{2\beta}\right|d\nu_G(\lambda) = 0 \tag{5.64}$$

$$\lim_{T\to\infty}\limsup_{m\to\infty}\int_{\Lambda_\infty}\left|\tilde{A}(m,T,\tilde{L}_m^{-\frac{1}{2}}\Pi_m\lambda) - \int_{\Lambda_\infty} a\circ\Pi_{m_1}\,d\mu_G(2\beta)\right|d\nu_G(\lambda) = 0. \tag{5.65}$$

Finally, using a very standard argument of measure theory, we easily deduce from (5.64) and (5.65) the following :

Proposition 5.1 *Assume (H1)-(H2) and choose the set of coherent states (5.57). Then there exist sequences* $(T_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$, $(n_k)_{k\in\mathbb{N}}$ *simultaneously tending to* $+\infty$ *such that for* ν_G*-almost all* $\lambda \in \Lambda_\infty$ *:*

$$\lim_{k\to+\infty} A(m_k, n_k, T_k, L_{m_k}^{-\frac{1}{2}}\Pi_{m_k}\lambda) = \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\hat{\mu}_{2\beta}.$$

If moreover (H3) is satisfied, there exist sequences $(T_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ *both tending to* $+\infty$ *such that for* ν_G*-almost all* $\lambda \in \Lambda_\infty$ *:*

$$\lim_{k\to+\infty} \tilde{A}(m_k, T_k, \tilde{L}_{m_k}^{-\frac{1}{2}}\Pi_{m_k}\lambda) = \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\mu_G(2\beta).$$

Remarks.

1. A similar result also holds if instead of (5.57) we consider the more general case $f_\lambda(x,\xi) = e^{ix\lambda_\xi - \langle F_m(x-\lambda_x), x-\lambda_x\rangle}$ where F_m is any positive definite symmetric matrix.

2. Actually, in the result of the Proposition one can replace L_m by any other symmetric matrix L'_m such that the matrix $K_m = (L'_m)^{-1/2}L_m(L'_m)^{-1/2}$ is diagonal and the measure $\left[e^{-\langle K_m\lambda,\lambda\rangle}d\lambda\right]_N$ on Λ_m admits a limit $d\nu_\infty$ as $m\to+\infty$. In this case the "ν_G-almost all λ" of the Proposition must be replaced by "ν_∞-almost all λ".

6 Appendix

1. Proof of Lemma 3.2

Using (3.34), we see that $e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}$ can be put under the form :

$$(e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}})(x,\xi) = C_1\int_{\Lambda_m^4} a(Y_1)e^{-q_1(x,\xi,Y_1,Y_2,Y_3,Y_4)}dY_1dY_2dY_3dY_4 \qquad (6.66)$$

where all along this proof C_j $(j = 1,2...)$ will denote complex constants, q_1 is a complex quadratic form on Λ_m^5, and the integral (6.66) is oscillatory. Moreover, a direct computation gives :

$$\int_{\Lambda_m^3} e^{-q_1(x,\xi,Y_1,Y_2,Y_3,Y_4)}dY_2dY_3dY_4 = C_2e^{-Q(x,\xi,Y_1)} \qquad (6.67)$$

where $C_2\in\mathbb{R}$ and Q is a positive definite quadratic form. Actually, this can also be seen without computation in the following way : the existence of the complex constant C_2 and the complex quadratic form Q such that (6.67) holds is clear, and if a is real then $Op^W(e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}) = Op^W(e^{-q_{\beta,m}})Op^W(a)Op^W(e^{-q_{\beta,m}})$ is a symmetric operator. As a consequence $e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}$ must be real for a real, which implies that Ce^{-Q} is real (and hence both C and Q are). Moreover, one can show easily that the application $S_m(1) \ni a \mapsto e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}$ maps continuously $S_m(1)$ into $\mathcal{S}(\Lambda_m)$, so that Q is necessarily positive definite.

When $a \equiv 1$, we get from (2.18), (6.66), (6.67) :

$$C_1C_2\int_{\Lambda_m} e^{-Q(x,\xi,Y_1)}dY_1 = C'_{\beta,m}e^{-\tilde{q}_{2\beta,m}}. \tag{6.68}$$

Then the result follows from (6.66), (6.67), (6.68) by writing :

$$(e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}})(x,\xi) = \left(C_1C_2\int_{\Lambda_m} e^{-Q(x,\xi,Y_1)}dY_1\right)\int_{\Lambda_m} a(Y_1)\left[e^{-Q(x,\xi,Y_1)}dY_1\right]_N.$$

□

2. Proof of Lemma 4.3

From (3.34) we get easily :

$$\begin{aligned} e^{-(\langle F\xi,\xi\rangle+\langle Gx,x\rangle)}\#e^{-(\langle F\xi,\xi\rangle+\langle Gx,x\rangle)} &= \pi^{-2(2m+1)}\left|\int_{\Lambda_m} e^{2i\zeta y-\langle G(x+y),x+y\rangle-\langle F(\xi+\zeta),\xi+\zeta\rangle}dyd\zeta\right|^2 \\ &= \pi^{-2(2m+1)}|I|^2 \end{aligned} \tag{6.69}$$

where

$$I = e^{-\langle Gx,x\rangle-\langle F\xi,\xi\rangle}\int e^{iy(2\zeta+2iGx)-\langle Gy,y\rangle-2\langle F\xi,\zeta\rangle-\langle F\zeta,\zeta\rangle}dyd\zeta. \tag{6.70}$$

Making the change of variables $y' = \sqrt{2}G^{\frac{1}{2}}y$ and integrating first with respect to y' we get :

$$I = Ce^{-\langle Gx,x\rangle-\langle F\xi,\xi\rangle}\int e^{-\langle G^{-1}(\zeta+iGx),\zeta+iGx\rangle-2\langle F\xi,\zeta\rangle-\langle F\zeta,\zeta\rangle}d\zeta \tag{6.71}$$

where C is a constant, and therefore, setting $\zeta' = \sqrt{2}(F+G^{-1})^{\frac{1}{2}}\zeta$, this gives :

$$I = C'e^{-\langle F\xi,\xi\rangle-\langle (F+G^{-1})^{-1}(x-iF\xi),x-iF\xi\rangle} \tag{6.72}$$

where C' is a constant. Then (6.69) and (6.72) yield the result. □

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