The Pianigiani-Yorke Measure for Topological Markov Chains.

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Abstract

We prove the existence of a Pianigiani-Yorke measure for a Markovian factor of a topological Markov chain. This measure induces a Gibbs measure in the limit set. The proof uses the contraction properties of the Ruelle - Perron - Frobenius operator.

1. Introduction and Main Result.

Let S be a finite set, $X = S^{\mathbb{N}}$ and $\sigma : X \to X$ be the shift transformation. For a transition matrix $L = (\ell_{ij} \in \{0,1\} : i,j \in S)$ we denote by $X_L = \{x \in X : \ell_{x_n,x_{n+1}} = 1$ $\forall n \in \mathbb{N}\}$, the topological Markov chain defined by L, and by $\sigma_L : X_L \to X_L$ the action of the shift on X_L.

For a compact metric space Y we denote by $\mathcal{C}(Y)$ the space of continuous real functions, by $\| \ \|_Y$ the supremum norm and by $\mathcal{M}(Y)$ the space of finite measures defined on Y.

For x and y in X, we define $N_{xy} = \max\{n : x_k = y_k \quad \forall k \le n\}$. Let $\phi : X \to \mathbb{R}$ be a potential which satisfies the usual Hölder hypotheses, that is there exists $c_0 > 0$ and $\gamma \in (0,1)$ such that

$$|\phi(x) - \phi(y)| \le c_0 \gamma^{N_{xy}} \quad \text{for any} \quad x, y \in X. \tag{1}$$

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By $\mathcal{L}_L$ we denote the Ruelle - Perron - Frobenius operator acting on $\mathcal{C}(X_L)$, namely

$$\mathcal{L}_L f(x) = \sum_{y \in \sigma_L^{-1}\{x\}} e^{\phi(y)} f(y) \quad \text{for} \quad x \in X_L, f \in \mathcal{C}(X_L). \tag{2}$$

We will denote by $\mathcal{L}_L^*$ the adjoint of the operator $\mathcal{L}_L$. We assume L is irreducible and aperiodic, namely there exists a positive number q such that all the entries of the matrix L^q are strictly positive. Under these hypothesis there exists (see Bowen [1]) a unique number $\alpha_L > 0$, a unique function $h_L \in \mathcal{C}(X_L)$, $h_L > 0$, and a unique probability measure $\nu_L \in \mathcal{M}(X_L)$ satisfying

$$\mathcal{L}_L h_L = \alpha_L h_L, \quad \mathcal{L}_L^* \nu_L = \alpha_L \nu_L, \quad \nu_L(h_L) = 1, \tag{3}$$

and

$$\forall f \in \mathcal{C}(X_L) : \|\alpha_L^{-n} \mathcal{L}_L^n f - h_L \nu_L(f)\|_{X_L} \to 0 \text{ when } n \to \infty. \tag{4}$$

Now let $L' = (\ell'_{ij} \in \{0,1\} : i,j \in S)$ be a different (irreducible and aperiodic) transition matrix imposing less constraints than L, that is $\ell'_{ij} = 0$ implies $\ell_{ij} = 0$, hence $X_L \subset X_{L'}$. As for L there are unique $\alpha_{L'} > 0$, $h_{L'} \in \mathcal{C}(X_{L'})$ with $h_{L'} > 0$, and $\nu_{L'} \in \mathcal{M}(X_{L'})$ verifying the analogous properties (3) and (4) for the operator $\mathcal{L}_{L'}$ defined as in (2). Observe that it is enough to assume that ϕ is defined in $X_{L'}$ to define $\mathcal{L}_L$ and $\mathcal{L}_{L'}$.

Now set,

$$\underline{X} = \{x \in X_{L'} : \ell_{x_0,x_1} = 1\}$$

and observe that $\underline{\sigma} : \underline{X} \to X_{L'}$ is a map from $\underline{X}$ onto $X_{L'}$. We have $\underline{\sigma}^{-1}\{x\} = \{y \in \sigma_{L'}^{-1}\{x\} : \ell_{y_0,y_1} = 1\}$, and similarly $\underline{\sigma}^{-n}\{x\} = \{y \in \sigma_{L'}^{-n}\{x\} : \ell_{y_k,y_{k+1}} = 1, \quad \forall k < n\}$. Hence $X_L = \bigcap_{n \geq 0} \underline{\sigma}^{-n} X_{L'}$ and if $x \in X_L$ then $\underline{\sigma}^{-n}\{x\} = \sigma_L^{-n}\{x\}$ for any $n \in \mathbb{N}$. We define the operator $\underline{\mathcal{L}}$ acting on $\mathcal{C}(X_{L'})$ by

$$\underline{\mathcal{L}} f(x) = \sum_{y \in \underline{\sigma}^{-1}\{x\}} e^{\phi(y)} f(y) \text{ for } x \in X_{L'}, f \in \mathcal{C}(\underline{X}). \tag{5}$$

From the definition of $\underline{\mathcal{L}}$ and $\mathcal{L}_{L'}$, it follows easily that $\mathcal{L}_{L'}(f 1_{\underline{X}}) = \underline{\mathcal{L}} f$ for $f \in \mathcal{C}(X_{L'})$. On the other hand for $x \in X_L, f \in \mathcal{C}(\underline{X})$: $\mathcal{L}_L(f|_{X_L})(x) = (\underline{\mathcal{L}} f)(x)$. Therefore for any $f \in \mathcal{C}(X_{L'})$ we have

$$\mathcal{L}_{L'}^n (f 1_{\underline{\sigma}^{-n} X_{L'}}) = \underline{\mathcal{L}}^n f \text{ and } \mathcal{L}_L^n (f|_{X_L}) = (\underline{\mathcal{L}}^n f)|_{X_L}. \tag{6}$$

We recall that if $T : A \to TA$ is an expanding smooth transformation such that TA contains A strictly, a Pianigiani-Yorke measure μ constructed in [5] is a measure satisfying

$$\mu \circ T^{-1} = \alpha \mu$$

for a number $\alpha > 0$. In [2] we proved that this measure was related to the Gibbs measure on the Cantor set $\bigcap_{n\geq 0} T^{-n}A$ by using quasi-compactness properties of an operator related to g-measures (for definitions see Walters [6]). This result is close to the work of Keller [2].

We are going to prove below the existence of the Pianigiani-Yorke measure for mixing topological Markov chains.

We note that the method of proof of the next theorem also gives a new proof for the original result established in [5]. Our main result is the following one.

Theorem. Let L, L' be irreducible and aperiodic and such that $\ell'_{ij} = 0$ implies $\ell_{ij} = 0$. Then there exists $\underline{h} \in \mathcal{C}(X_{L'}), \underline{h} > 0$, such that

$$\underline{\mathcal{L}}\underline{h} = \alpha_L \underline{h}, \quad \underline{h}|_{X_L} = h_L \tag{7}$$

and

$$\forall f \in \mathcal{C}(X_{L'}) : \quad \|\alpha_L^{-n}\underline{\mathcal{L}}^n f - \underline{h}\nu_L(f|_{X_L})\|_{X_{L'}} \to 0 \text{ when } n \to \infty. \tag{8}$$

Moreover $\alpha_L \leq \alpha_{L'}$ and the following properties hold, for any Borel set $D \subset X_{L'}$

$$\int_{\underline{\sigma}^{-n}D} \underline{h} d\nu_{L'} = (\alpha_L \alpha_{L'}^{-1})^n \int_D \underline{h} d\nu_{L'} \quad \text{ for } n \in \mathbb{N}; \tag{9}$$

$$\nu_{L'}(\underline{\sigma}^{-n}D|\underline{\sigma}^{-n}X_{L'}) \underset{n\to\infty}{\longrightarrow} \left(\int_{X_{L'}} \underline{h} d\nu_{L'}\right)^{-1} \int_D \underline{h} d\nu_{L'}. \tag{10}$$

$$\nu_{L'}(D|\underline{\sigma}^{-n}X_{L'}) \underset{n\to\infty}{\longrightarrow} (\nu_L(X_L))^{-1}\nu_L(1_{D\cap X_L}). \tag{11}$$

■

From (9) the Pianigiani-Yorke measure of $\underline{\sigma} : \underline{X} \to X_{L'}$ is given by

$$\mu_{PY}(D) = \int_D \underline{h} d\nu_{L'} \text{ for } D \subset X_{L'} \text{ any Borel set }. \tag{12}$$

In fact, it follows from (9) that $\mu_{PY} \circ \underline{\sigma}^{-1} = \alpha^{-1}\mu_{PY}$, in our case $\alpha = \alpha_L \alpha_{L'}^{-1}$ (which is ≤ 1).

Now we can write (9) in terms of conditional measures.

$$\mu_{PY}(\underline{\sigma}^{-n}D|\underline{\sigma}^{-n}X_{L'}) = \mu_{PY}(D|X_{L'}) \text{ for } D \subset X_{L'} \text{ a Borel set }. \tag{13}$$

This property is known in Markov chains theory, as conditional invariance and property (10) means that (the normalization of) μ_{PY} is the limit conditional measure of $\nu_{L'}$. For these properties in the context of Markov chains see [3].

2. Proof of the Main Result.

The proof of the theorem will follow from several lemmas. Let us first introduce some notations. For $x \in X_{L'}$ define $N(x) = \inf\{n : \ell_{x_n, x_{n+1}} = 0\}$. Note that $N(x) = \infty$ if and only if $x \in X_L$.

We first fix $z \in X_L$. Since L is aperiodic there exists $q > 0$ such that $\ell_{ij}^{(q)} > 0$ for any i, j, one can chose a map ψ from S to S^{q-1} (depending on z) such that for any $s \in S$

$$\begin{aligned} \ell_{s,\psi_1(s)} &= 1 \\ \ell_{\psi_j(s),\psi_{j+1}(s)} &= 1 \quad \text{for} \quad j = 1, \cdots, q-2 \\ \ell_{\psi_{q-1}(s), z_0} &= 1 \end{aligned}$$

We now define $\pi : X_{L'} \to X_L$ by $\pi(x) = x$ when $x \in X_L$ and for $x \in X_{L'} \backslash X_L$

$$\pi(x) = (x_0, \ldots, x_{N(x)}; \psi(x_{N(x)}); z) = (x_0, ..., x_{N(x)}, \psi_1(x_{N(x)}), ..., \psi_{q-1}(x_{N(x)}), z_0, z_1, ...)$$

Observe that x and $\pi(x)$ start with the same symbol x_0.

We denote by $\mathcal{C}_p(X_{L'})$ the set of real functions defined on $X_{L'}$ which are p-cylindrical, that is depending only on the first p coordinates of the point. By $\mathcal{C}_p^{b,+}(X_{L'})$ we denote the subset of strictly positive functions in $\mathcal{C}_p(X_{L'})$. Note that for any function $f \in \mathcal{C}_p^{b,+}(X_{L'})$ there exists $B(f) \in (0,1)$ satisfying $B(f) < f < B(f)^{-1}$. We recall that $\gamma \in (0,1)$ is the Hölder constant associated to ϕ in (1).

Lemma 1. There exists $c > 0$ such that for any $p > 0$, for any $k \geq p$, and for any $f \in \mathcal{C}_p^{b,+}(X_{L'})$

$$e^{-c\gamma^{N(x)}} \leq \frac{\underline{\mathcal{L}}^k f(x)}{\underline{\mathcal{L}}^k f(\pi(x))} \leq e^{c\gamma^{N(x)}} \tag{14}$$

Proof. We shall use the fact that for finite sets of strictly positive numbers $(c_i)_{i\in I}, (d_i)_{i \in I}$ we have

$$\frac{\sum\limits_{i\in I} c_i}{\sum\limits_{i \in I} d_i} \in \left(\min_{i\in I} \frac{c_i}{d_i}, \max_{i \in I} \frac{c_i}{d_i} \right).$$

For $x_0 \in S$ we define $I(x_0; k) = \{(x_{-k}, \ldots, x_{-1}) \in S^k : \ell_{x_i, x_{i+1}} = 1 \text{ for } i = -k, \ldots, -1\}$. Note that since L is irreducible, $I(x_0;k)$ is non empty. Below we denote by $x(-k,-1) = (x_{-k}, \ldots, x_{-1})$ and if $y \in S^{\mathbb{N}}$ we denote by $(x(-k,-1); y)$ the point of $S^{\mathbb{N}}$ with coordinates $(x_{-k}, \ldots, x_{-1}, y_0, y_1, \ldots)$. We have

$$\underline{\mathcal{L}}^k f(x) = \sum_{x(-k,-1) \in I(x_0;k)} e^{\sum\limits_{i=1}^{k} \phi(x(-i,-1);x)} f(x(-k,-1);x). \tag{15}$$

Since x and $\pi(x)$ begin with the some symbol x_0 we get

$$\frac{\underline{\mathcal{L}}^k f(x)}{\underline{\mathcal{L}}^k f(\pi(x))} \leq \max_{x(-k,-1)\in I(x_0;k)} \left\{ e^{\sum_{i=1}^{k} \phi(x(-k,-1);x)-\phi(x(-i,-1);\pi(x))} \frac{f(x(-k,-1);x)}{f(x(-k,-1);\pi(x))} \right\}$$

Since f is p-cylindric, and $k \geq p$, we have $\frac{f(x(-k,-1);x)}{f(x(-k,-1);\pi(x))} = 1$. On the other hand $(x(-i,-1);x)$ and $(x(-i,-1);\pi(x))$ coincide in the first $i+N(x)$ coordinates, therefore $|\phi(x(-i,-1);x) - \phi(x(-i,-1);\pi(x))| \leq c_0\gamma^{i+N(x)}$.

This implies

$$\sum_{i=1}^{k} |\phi(x(-i,-1);x) - \phi(x(-i,-1);\pi(x))| \leq c\gamma^{N(x)}$$

with $c = c_0 \sum_{i=1}^{\infty} \gamma^i$. It follows that $\frac{\underline{\mathcal{L}}^k f(x)}{\underline{\mathcal{L}}^k f(\pi(x))} \leq e^{c\gamma^{N(x)}}$. In an analogous way we prove the inequality $\geq e^{-c\gamma^{N(x)}}$. ■

Lemma 2. There exists $r \in (0,1)$ and $c(f) > 0$ such that for any $n > 2p$ and for any $f \in \mathcal{C}_p^{b,+}(X_L)$ we have

$$e^{-c(f)r^n} \leq \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{n+1} f(x)}{\underline{\mathcal{L}}^n f(x)} \leq e^{c(f)r^n} \text{ for all } x \in X_{L'}.$$

Proof. From the results of Bowen [1] proving the exponential convergence in (4) and since $h_L > 0$, we conclude that there exists $\rho \in (0,1)$ such that for any $g \in \mathcal{C}_p^{b,+}(X_L)$ there exists $c'(g) > 0$ such that

$$e^{-c'(g)\rho^k} \leq \frac{\alpha_L^{-k} \mathcal{L}_L^k g(y)}{h_L(y)\nu_L(g)} \leq e^{c'(g)\rho^k} \quad \forall y \in X_L, \forall k > 0.$$

We apply this result to the restriction $f|_{X_L}$. Since $(\underline{\mathcal{L}}^k f)|_{X_L} = \mathcal{L}_L^k f|_{X_L}$ and $\pi(x) \in X_L$ we get the existence of $c'(f) > 0$ such that

$$e^{-c'(f)\rho^k} \leq \frac{\alpha_L^{-k} \underline{\mathcal{L}}^k f(\pi(x))}{h_L(\pi(x))\nu_L(f|_{X_L})} \leq e^{c'(f)\rho^k} \quad \forall x \in X_{L'}, \forall k > 0. \tag{16}$$

We now have for any $x \in X_{L'}$

$$\alpha_L^{-1} \frac{\underline{\mathcal{L}}^{k+1} f(x)}{\underline{\mathcal{L}}^k f(x)} = \frac{\underline{\mathcal{L}}^{k+1} f(x)}{\underline{\mathcal{L}}^{k+1} f(\pi(x))} \frac{\underline{\mathcal{L}}^k f(\pi(x))}{\underline{\mathcal{L}}^k f(x)} \frac{\alpha_L^{-(k+1)} \underline{\mathcal{L}}^{k+1} f(\pi(x))}{h_L(\pi(x))\nu_L(f|_{X_L})} \frac{h_L(\pi(x))\nu_L(f|_{X_L})}{\alpha_L^{-k} \underline{\mathcal{L}}^k f(\pi(x))}.$$

If $k \geq p$, then by using (14) and (16) we obtain

$$e^{-c''(f)(\gamma^{N(x)}+\rho^k)} \leq \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{k+1} f(x)}{\underline{\mathcal{L}}^k f(x)} \leq e^{c''(f)(\gamma^{N(x)}+\rho^k)} \quad \text{for any} \quad x \in X_{L'} \tag{17}$$

with $c''(f) = 2(c + c'(f)(1+\rho))$.

On the other hand,

$$\alpha_L^{-1} \frac{\underline{\mathcal{L}}^{n+1} f(x)}{\underline{\mathcal{L}}^n f(x)} = \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{n-k}(\underline{\mathcal{L}}^{k+1} f)(x)}{\underline{\mathcal{L}}^{n-k}(\underline{\mathcal{L}}^k f)(x)} =$$

$$\frac{\sum_{x(-n-k,-1)\in I(x_0;n-k)} \alpha_L^{-(k+1)} (\underline{\mathcal{L}}^{k+1} f)(x(-n-k,-1);x) e^{\sum_{i=1}^{n-k} \phi(x(-i,-1);x)}}{\sum_{x(-n-k,-1)\in I(x_0;n-k)} \alpha_L^{-k} (\underline{\mathcal{L}}^{k} f)(x(-n-k,-1);x) e^{\sum_{i=1}^{n-k} \phi(x(-i,-1);x)}}$$

$$\leq \max_{x(-n-k,-1)\in I(x_0;n-k)} \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{k+1} f(x(-n-k,-1);x)}{\underline{\mathcal{L}}^k f(x(-n-k,-1);x)} .$$

A lower bound can be proved similarly. If $n-k \geq p$ we can use (17) and the fact that $N(x_{-(n-m)}, \ldots, x_{-1}; x) \geq (n-k)$ to get

$$e^{-c''(f)(\gamma^{n-k}+\rho^k)} \leq \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{n+1} f(x)}{\underline{\mathcal{L}}^n f(x)} \leq e^{c''(f)(\gamma^{n-k}+\rho^k)} \quad \text{for all} \quad x \in X_{L'} .$$

Let $k = \left[\frac{n}{2}\right]$, since $n > 2p$ we have $k \geq p$, and also $n-k \geq p$. Taking $c(f) = 2c''(f)\max(\gamma^{-\frac{1}{2}}, \rho^{-\frac{1}{2}})$ and $r = \max(\gamma^{\frac{1}{2}}, \rho^{\frac{1}{2}}) < 1$, since $c''(f)(\gamma^{n-[\frac{n}{2}]} + \rho^{[\frac{n}{2}]}) \leq c(f) r^n$ we get the result. ■

Lemma 3. For any $f \in \bigcup_{p\geq 1} \mathcal{C}_p^{b,+}(X_{L'})$

(i) $(\alpha_L^{-n} \underline{\mathcal{L}}^n f)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{C}(X_{L'})$.

(ii) $\underline{h} = \lim_{n\to\infty} \frac{\alpha_L^{-n} \underline{\mathcal{L}}^n f}{\nu_L(f|X_L)}$ does not depend on the function $f \in \bigcup_{p\geq 1} \mathcal{C}_p^{b,+}(X_{L'})$ and satisfies

$$\underline{\mathcal{L}} \underline{h} = \alpha_L \underline{h} .$$

Proof. (i) By using lemma 2 we get for $n > 2p$ and $x \in X_{L'}$:

$$\alpha_L^{-n} \underline{\mathcal{L}}^n f(x) = \left(\prod_{k=2p+1}^{n-1} \alpha_L^{-1} \frac{\underline{\mathcal{L}}^{k+1} f(x)}{\underline{\mathcal{L}}^k f(x)} \right) \alpha_L^{-(2p+1)} \underline{\mathcal{L}}^{2p+1} f(x)$$

$$\in (K(f)^{-1}\alpha_L^{-(2p+1)}\underline{\mathcal{L}}^{(2p+1)}f(x), K(f)\alpha_L^{-(2p+1)}\underline{\mathcal{L}}^{(2p+1)}f(x)),$$

with $K(f) = e^{c(f)\sum\limits_{k=2p+1}^{\infty} r^k}$.

Since there exists $B'(f) \in (0,1)$ such that $\alpha_L^{-(2p+1)}\underline{\mathcal{L}}^{2p+1}f(x) \in (B'(f), B'(f)^{-1})$ we conclude that there exists $B''(f) \in (0,1)$ such that $B''(f) \leq \alpha_L^{-n}\underline{\mathcal{L}}^n f(x) \leq B''(f)^{-1}$ for any $n > 2p$ and $x \in X_{L'}$.

Therefore $\|\alpha_L^{-(n+1)}\underline{\mathcal{L}}^{n+1}f - \alpha_L^{-n}\underline{\mathcal{L}}^n f\|_{X_{L'}} = \|\alpha_L^{-n}\underline{\mathcal{L}}^n f(\alpha_L^{-1}\frac{\underline{\mathcal{L}}^{n+1}f}{\underline{\mathcal{L}}^n f} - 1)\|_{X_{L'}} \leq B''(f)^{-1}(e^{c(f)r^n} - 1)$. Since the series $\sum\limits_{n\geq 1}(e^{c(f)r^n} - 1)$ is summable the Cauchy property follows.

(ii) Let f_1, f_2 be in $\bigcup_{p\geq 1}\mathcal{C}_p^{b,+}(X_{L'})$. We can assume that for some p, f_1 and f_2 belong to $\mathcal{C}_p^{b,+}(X_{L'})$. Take k such that $k > p$, and n such that $n-k > p$. As in the proof of lemma 2 we have

$$\frac{\alpha_L^{-n}\underline{\mathcal{L}}^n f_1(x)}{\alpha_L^{-n}\underline{\mathcal{L}}^n f_2(x)} \leq \max_{x(-n-k,-1)\in I(x_0;n-k)} \frac{\underline{\mathcal{L}}^k f_1(x(-n-k,-1);x)}{\underline{\mathcal{L}}^k f_2(x(-n-k,-1);x)}$$

$$\leq \frac{\nu_L(f_1|X_L)}{\nu_L(f_2|X_L)} \max_{(x_{-(n-k)},\ldots,x_{-1})\in I(x_0;n-k)} \left\{ \frac{\underline{\mathcal{L}}^k f_1(x(-n-k,-1);x)}{\underline{\mathcal{L}}^k f_1(x(-n-k,-1);\pi(x))} \frac{\nu_L(f_2|X_L)}{\nu_L(f_1|X_L)} \right.$$

$$\left. \frac{\underline{\mathcal{L}}^k f_1(x(-n-k,-1);\pi(x))}{h_L(x(-n-k,-1);\pi(x))} \frac{h_L(x(-n-k,-1);\pi(x))}{\underline{\mathcal{L}}^k f_2(x(-n-k,-1);\pi(x))} \frac{\underline{\mathcal{L}}^k f_2(x(-n-k,-1);\pi(x))}{\underline{\mathcal{L}}^k f_2(x(-n-k,-1);(x))} \right\}.$$

Then, by using (l6) and (14) we get,

$$e^{-2c\gamma^{n-k}-(c'(f_1)+c'(f_2))\rho^k} \leq \frac{\alpha_L^{-n}\underline{\mathcal{L}}^n f_1(x)}{\alpha_L^{-n}\underline{\mathcal{L}}^n f_2(x)} \frac{\nu_L(f_2|X_L)}{\nu_L(f_1|X_L)} \leq e^{2e\gamma^{n-k}+(c'(f_1)+c'(f_2))\rho^k}$$

for any $k > p, n-k > p$. By taking as before $k = [\frac{n}{2}]$ we conclude that the above expression converges to 1. Therefore $\lim\limits_{n\to\infty} \frac{\alpha_L^{-n}\underline{\mathcal{L}}^n f_i(x)}{\nu_L(f_i|X_L)}$ is the same for $i = 1, 2$. The equality $\underline{\mathcal{L}}\underline{h} = \alpha_L\underline{h}$ follows immediately. ■

Proof of the Theorem. Consider the $\underline{h}$ constructed in lemma 3 (ii). From the proof of lemma 3 (i) we have $\underline{h} \geq \frac{B''(f)}{\nu_L(f|X_L)}$ for any $f \in \mathcal{C}_p^{b,+}(X_{L'})$, therefore $\underline{h} > 0$. From lemma 3 (ii) it verifies $\underline{\mathcal{L}}\underline{h} = \alpha_L\underline{h}$. On the other hand the equality $\alpha_L^{-n}\underline{\mathcal{L}}^n f(x) = \alpha_L^{-n}\mathcal{L}_L^n f(x)$ for $x \in X_L$ implies $\underline{h}|_{X_L} = h_L$. Hence (7) holds.

We now prove (8). From lemma 3 (ii) we must extend the convergence from $\mathcal{C}_p^{b,+}(X_{L'})$ to $\mathcal{C}(X_{L'})$. First if $f \in \mathcal{C}_p(X_{L'})$ we can decompose $f = f^+ - f^-$ with $f^+ > 0$ and $f^- > 0$ and the result holds for f because $\underline{\mathcal{L}}^n f = \underline{\mathcal{L}}^n f^+ - \underline{\mathcal{L}}^n f^-$ and $\nu_L(f|_{X_L}) = \nu_L(f^+|_{X_L}) - \nu_L(f^-|_{X_L})$. Now if $g \in \mathcal{C}(X_{L'})$ there exist $f_1, f_2 \in \mathcal{C}_p(X_{L'})$ such that $f_1 \le g \le f_2$ and $\|f_1 - f_2\|_{X_{L'}} < \varepsilon$. The result follows because $\underline{\mathcal{L}}^n$ preserves the order and $|\nu_L(f_1|_{X_L}) - \nu_L(f_2|_{X_L})| < \varepsilon$.

From $\underline{h} > 0$ on $X_{L'}$ we get $\nu_{L'}(\underline{h}) > 0$. Using $\mathcal{L}_{L'}\underline{h} \ge \underline{\mathcal{L}}\underline{h}$, the convergence property (4) for matrix L' and relation (8) we derive the inequality $\alpha_L \le \alpha_{L'}$.

We now prove (9) and (10). Let $f, g \in \mathcal{C}(X_{L'})$, we first observe that

$$\underline{\mathcal{L}}^n(f 1_{\underline{\sigma}^{-n}X_{L'}} g \circ \underline{\sigma}^n) = g\underline{\mathcal{L}}^n(f 1_{\underline{\sigma}^{-n}X_{L'}}) = g\underline{\mathcal{L}}^n f.$$

On the other hand for $n \ge 1$ we have $\underline{\mathcal{L}}^n(f \cdot 1_{\underline{\sigma}^{-n}X_{L'}} \cdot g \circ \underline{\sigma}^n) = \mathcal{L}_{L'}^n(f \cdot 1_{\underline{\sigma}^{-n}X_{L'}} \cdot g \circ \underline{\sigma}^n)$. Therefore

$$\begin{aligned} \int_{\underline{\sigma}^{-n}X_{L'}} f g \circ \underline{\sigma}^n \, d\nu_{L'} &= \int_{X_{L'}} f 1_{\underline{\sigma}^{-n}X_{L'}} g \circ \underline{\sigma}^n \alpha_{L'}^{-n} \, d(\mathcal{L}_{L'}^{*n}\nu_{L'}) \\ &= \alpha_{L'}^{-n} \int_{X_{L'}} \mathcal{L}_{L'}^n(f 1_{\underline{\sigma}^{-n}X_{L'}} g \circ \underline{\sigma}^n) \, d\nu_{L'} = \alpha_{L'}^{-n} \int_{X_{L'}} \underline{\mathcal{L}}^n(f 1_{\underline{\sigma}^{-n}X_{L'}} g \circ \underline{\sigma}^n) \, d\nu_{L'} \\ &= \alpha_{L'}^{-n} \int_{X_{L'}} g\underline{\mathcal{L}}^n f \, d\nu_{L'}. \end{aligned} \tag{18}$$

Now let $D \subset X_{L'}$ be a Borel set.

Let $g = 1_D$, $f = \underline{h}$, and using $\underline{\mathcal{L}}^n \underline{h} = \alpha_L^n \underline{h}$ in the last equalities (18) we obtain $\int 1_{\underline{\sigma}^{-n}D} \underline{h} d\nu_{L'} = (\alpha_L \alpha_{L'}^{-1})^n \int 1_D \underline{h} d\nu_{L'}$, which proves (9).

Relation (18) also implies (taking g as before and $f = 1$

$$\nu_{L'}(\underline{\sigma}^{-n}D) = \int_{\underline{\sigma}^{-n}(X_{L'})} 1_D \circ \underline{\sigma}^n d\nu_{L'} = \alpha_{L'}^{-n} \int_{X_{L'}} 1_D \cdot (\underline{\mathcal{L}}^n 1_{X'_L}) d\nu_{L'}.$$

Therefore

$$\nu_{L'}(\underline{\sigma}^{-n}D | \underline{\sigma}^{-n}X_{L'}) = \frac{\int 1_D \underline{\mathcal{L}}^n 1_{X_{L'}} d\nu_{L'}}{\int 1_{X_{L'}} \underline{\mathcal{L}}^n 1_{X_{L'}} d\nu_{L'}} = \frac{\int 1_D \alpha_L^{-n} \underline{\mathcal{L}}^n 1_{X_{L'}} d\nu_{L'}}{\int 1_{X_{L'}} \alpha_L^{-n} \underline{\mathcal{L}}^n 1_{X_{L'}} d\nu_{L'}}$$

which, by (8), converges to $\left(\int_{X_{L'}} \underline{h} d\nu_{L'} \right)^{-1} \int_D \underline{h} d\nu_{L'}$. This proves (10).

Finally, let us prove (11). We have from (18) (with $g = 1$ and $f = 1_D$)

$$\nu_{L'}(D \cap \underline{\sigma}^{-n} X_{L'}) = \int\limits_{\underline{\sigma}^{-n} X_{L'}} 1_D d\nu_{L'} = \alpha_{L'}^{-n} \int\limits_{X_{L'}} \underline{\mathcal{L}}^n(1_D) d\nu_{L'}$$

$$= (\alpha_L \alpha_{L'}^{-1})^n \int\limits_{X_{L'}} \alpha_L^{-n} \underline{\mathcal{L}}^n(1_D) d\nu_{L'}.$$

Moreover, from (8) we have

$$\int\limits_{X_{L'}} \alpha_L^{-n} \underline{\mathcal{L}}^n(1_D) d\nu_{L'} \underset{n\to\infty}{\longrightarrow} \nu_L(1_{D\cap X_L}) \int\limits_{X_{L'}} \underline{h} d\nu_{L'}.$$

Hence the result follows. ■

Example. Consider the trivial case $\phi = 0$ and L' such that $\ell'_{ij} = 1$ for all i, j. Then $X_{L'}$ is the full shift, $\alpha_{L'} = |S|, h_{L'} \equiv 1$ and $\nu_{L'}$ is the uniform Bernoulli measure. α_L the Perron-Frobenius eigenvalue of L and $\underline{h}(x_0 x_1 ...) = v_{x_0}$ where $v = (v_i : i \in S)$ is the left Perron-Frobenius eigenvector of L. Now apply (9) to $D = [i]_0 = \{x \in X_{L'} : x_0 = i\}$. We have $\int\limits_{[i]_0} \underline{h} d\nu_{L'} = v_i \cdot |S|^{-1}$. On the other hand $\underline{\sigma}^{-n}[i]_0 = \{y \in X : \ell_{y_k, y_{k+1}} = 1$ for $k < n$ and $y_n = i\}$. Hence if we denote by M_i^n the number $\#\{L$-words of length n finishing at $i\}$, we obtain $\nu_{L'}(\underline{\sigma}^{-n}[i]_0) = |S|^{-n} \cdot M_i^n$. Therefore, it follows from the theorem that

$$\nu_{L'}(\underline{\sigma}^{-n}[i]_0 | \underline{\sigma}^{-n} X_{L'}) = \frac{M_i^n}{\sum\limits_{j \in S} M_j^n} \underset{n\to\infty}{\longrightarrow} \frac{v_i}{\sum\limits_{j\in S} v_j},$$

or equivalently if we normalize the vector $(M_i^n : i \in S)$, it converges to the normalized vector $(v_i : i \in S)$. An analogous result holds for the right Perron-Frobenius eigenvector of L.

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