Ground State Structure in a Highly Disordered Spin Glass Model

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Abstract: We propose a new Ising spin glass model on $\mathbf{Z}^d$ of Edwards-Anderson type, but with highly disordered coupling magnitudes, in which a greedy algorithm for producing ground states is exact. We find that the procedure for determining (infinite volume) ground states for this model can be related to invasion percolation with the number of ground states identified as $2^{\mathcal{N}}$, where $\mathcal{N} = \mathcal{N}(d)$ is the number of distinct global components in the "invasion forest". We prove that $\mathcal{N}(d) = \infty$ if the invasion connectivity function is square summable. We argue that the critical dimension separating $\mathcal{N} = 1$ and $\mathcal{N} = \infty$ is $d_c = 8$. When $\mathcal{N}(d) = \infty$, we consider free or periodic boundary conditions on cubes of side length L and show that frustration leads to chaotic L dependence with *all* pairs of ground states occuring as subsequence limits. We briefly discuss applications of our results to random walk problems on rugged landscapes.

1 Introduction

Theoretical work on spin glasses has focused overwhelmingly on the short-ranged Edwards-Anderson (EA) model [10] and its infinite-ranged counterpart, the Sherrington-Kirkpatrick (SK) model [27]. The EA Hamiltonian is:

$$\mathcal{H} = -\sum_{<xy>} J_{xy}\sigma_x\sigma_y \quad , \tag{1}$$

where the sum runs only over nearest-neighbor pairs of sites on a regular d-dimensional lattice. We will confine ourselves in this paper to Ising models, *i.e.,* the spins σ_x take on the values ± 1. The couplings J_{xy} are i.i.d. random variables chosen from a distribution

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symmetric about zero. One standard choice is a mean zero Gaussian, though one needn't be confined to that case.

There are a number of open questions [4] associated with the EA model. These include the question of whether it displays a thermodynamic phase transition, and if so, in which dimensions; the nature of its large-scale dynamical behavior for standard local spin-flip dynamics; the relationship between the behavior of this model and others, such as the SK model or long-ranged models; and others. The question we are concerned with here is the number of ground state pairs the model displays in d dimensions. (Because of the spin-flip symmetry in the Hamiltonian, ground states always come in pairs.) To avoid local degeneracies, we will assume that as in the Gaussian case, the common distribution of the J_{xy}'s is continuous.

Parisi's analysis [24] of the SK model displays the striking feature of an infinite number of low-temperature phases [25] organized in an ultrametric fashion [18]. A number of workers in the field have assumed that a similar result applies also to short-ranged spin glasses, and to other models with frustration and quenched disorder.

However, an alternative point of view arose in the mid-1980's following a scaling *ansatz* due to MacMillan [17], Bray and Moore [5], and Fisher and Huse [12]. Their work led to the opposite conclusion [14], namely, that the EA model has only a *single* pair of ground states (or of low-temperature pure states) in all finite dimensions. Thus, over the past decade two opposing viewpoints have dominated the literature. The first, following the work of the Parisi school, asserts that short-ranged spin glasses have an infinite number of ground states (in infinite volume) for all "nontrivial" dimensions, *i.e.,* $d \geq 2$. The second, proceeding from scaling arguments, argues that such spin glass models possess only a single pair of ground states in all finite dimensions. (Both of the above statements are normally interpreted to hold for almost every coupling realization.)

In a recent paper [20], hereafter referred to as I, we invented a model (see also [9]) simpler than (but related to) the EA spin glass in which we could explore these issues more fully. The model is analytically tractable, yet complex enough so that it displays a surprisingly rich ground state behavior. We will argue that our model undergoes a *transition in ground state multiplicity* at eight dimensions: below it has a single pair of ground states, while above it has infinitely (in fact, uncountably) many.

We are able to demonstrate explicitly the mechanism by which multiple ground state pairs arise. This is the first such demonstration of which we are aware for a short-ranged spin glass model in finite dimension. The nature of the mechanism is related to a mapping of the ground state structure of our model to invasion percolation [16, 7, 29]. We will show that solving the problem of ground state multiplicity in our model requires the solution of an interesting problem in invasion percolation.

To summarize, we propose and analyze a tractable short-ranged finite dimensional spin glass model (although, like the SK model, it is not physically realistic). Our techniques enable us to study the relationship within the model among quenched disorder, frustration, dimensionality, and ground state multiplicity. We will find in particular that the role of frustration is subtle but important, and our model is instructive in illustrating the larger role frustration may play in more realistic spin glass models.

The basic features of our model and its analysis have appeared in I. In this paper, we supply a number of arguments omitted for brevity in I, extend our work in several directions, and provide proofs for various conclusions drawn in I. Among these is a proof of a theorem (see Section 5) describing conditions under which certain random growth processes avoid intersection in d dimensions; as such, its utility extends beyond its particular use in this paper.

The outline of the paper is as follows: In Section 2, we define our model. In Section 3 we provide a simple algorithm for finding the ground state configuration for any volume with a specified boundary condition, and discuss some of its properties. In Section 4 we demonstrate a second algorithm for finding the ground state, thereby mapping our problem onto invasion percolation and showing how the ground state multiplicity problem in our model is equivalent to the question of nonintersection of invasion regions in invasion percolation. In Section 5 we provide a heuristic argument for the dimension-dependence of the above questions, make part of that argument rigorous by relating it to the square-summability of a connectivity function in invasion percolation, and discuss the dimensionality of the invasion region as a function of space dimensionality. This yields the previously mentioned transition at eight dimensions. In Section 6 we discuss the role of frustration, and point out an important difference between the nature of multiple ground states in spin glasses and that in random ferromagnets. Finally, in Section 7 we summarize our results and briefly discuss their application to some dynamical problems of random walks on rugged landscapes [21, 22].

2 Model

We work on a cubic lattice in d dimensions, *i.e.,* of sites $x \in \mathbf{Z}^d$ and edges connecting nearest-neighbor sites only. The Hamiltonian is of the standard Edwards-Anderson Ising form given in Eq. (1). The difference with the usual EA Ising model is in the coupling distribution, which now depends on the system size. That is, we apply a nonlinear scaling to the couplings which "spreads them out" so much that each coupling magnitude exists on its own scale — more precisely, for large enough system size each coupling will have at least twice the magnitude of the next smaller coupling.

We therefore consider a region Λ_L, which is an L^d cube centered at the origin. We achieve the required condition on the couplings by separating their signs and magnitudes in the following manner. Let ϵ_{xy} be a set of independent identically distributed (i.i.d.) symmetric ± 1 valued random variables, and let K_{xy} be a set of i.i.d. continuous random variables (*e.g.,* uniform on $[0,1]$). The ϵ_{xy}'s and K_{xy}'s are defined on a common probability space $(\Omega, \mathcal{F}, P)$ and are independent of each other. A point ω in Ω may be thought of as a particular realization of all ϵ_{xy}'s and K_{xy}'s. Then we set

$$J_{xy} = J_{xy}^{(L)} = c_L \epsilon_{xy} \ e^{-\lambda^{(L)} K_{xy}} \quad , \tag{2}$$

where c_L is a linear scaling factor which plays no role in ground state selection and where the nonlinear scaling factor $\lambda^{(L)}$ is chosen to diverge fast enough as $L \to \infty$ to ensure that (with probability one) for all large L, each $|J_{xy}^{(L)}|$ in Λ_L is larger than at least twice the next smaller one.

To see that such a choice of $\lambda^{(L)}$ is possible, note that for any distinct pair of edges, the function,

$$\begin{aligned} g(\lambda) &= P(\frac{1}{2} \leq e^{-\lambda K_{xy}}/e^{-\lambda K_{x'y'}} \leq 2) \\ &= P(\quad |K_{xy} - K_{x'y'}| \leq \tfrac{ln2}{\lambda}), \end{aligned} \tag{3}$$

tends to zero as $\lambda \to \infty$ because K_{xy} and $K_{x'y'}$ are independent *continuous* random variables. The probability that our desired condition on the $|J_{xy}^{(L)}|$'s in Λ_L is *not* satisfied is bounded by

$$\sum_{<xy>} \sum_{<x'y'>} g(\lambda^{(L)}) = O(L^{2d} g(\lambda^{(L)})), \tag{4}$$

where the sums are over edges in Λ_L. If we choose $\lambda^{(L)}$ so that $g(\lambda^{(L)}) = O(L^{-(2d+1+\epsilon)})$ for some $\epsilon > 0$, then the sum of (4) over L is finite, so by the Borel-Cantelli lemma it follows that (with probability one) our desired condition will be valid for $L \geq$ some finite $L^*(\omega)$. For example, if the K_{xy}'s are uniform on $[0,1]$, then $g(\lambda) = O(\frac{1}{\lambda})$ and so $\lambda^{(L)} \geq L^{(2d+1+\epsilon)}$ is a sufficiently fast divergence.

It may be helpful to note that the couplings (2) can be expressed in a simpler form if we choose each $\lambda^{(L)}$ to be an *odd* integer. Letting $\hat{J}_{xy}$ denote the *L-independent* coupling $\epsilon_{xy}\exp(-K_{xy})$, Eq. (2) then becomes

$$J_{xy}^{(L)} = c_L (\hat{J}_{xy})^{\lambda^{(L)}}. \tag{5}$$

Any continuous symmetric distribution for the $\hat{J}_{xy}$'s (such as Gaussian) is possible.

We will show in the next section that we have constructed a model whose ground state for large L can be found with a type of greedy algorithm. While this may seem to suggest that interesting ground state behavior cannot occur, we shall see that, surprisingly, this is not the case. We point out here, however, that the model has no interesting behavior at nonzero temperature. Its use is only as a means of studying ground state structure.

3 Greedy Ground State Algorithm

We begin with a formal definition of infinite-volume ground states for a specific coupling realization ω. We first consider the finite volume Λ_L and for the moment assume some fixed, boundary condition, $\hat{\sigma}$, on $\partial\Lambda_L$, the boundary of Λ_L. We define σ_L^* to be a ground state on Λ_L (with boundary condition $\hat{\sigma}$) if it minimizes $\mathcal{H}_L$, where $\mathcal{H}_L$ is the Hamiltonian of Eq. (1) with the summation confined to couplings in Λ_L (including those between Λ_L and $\partial\Lambda_L$). The set of *all* infinite volume ground states given ω is then the set of all subsequence limits as $L \to \infty$ of σ_L^* with some $\hat{\sigma}_L$ (*i.e.,* the set of all possible limits with all possible boundary conditions).

For the volume Λ_L with a fixed boundary condition on $\partial\Lambda_L$, we now describe an algorithm for finding the ground state σ_L^*. (Non-fixed boundary conditions, such as free and periodic, will be discussed in Section 6.) To do this we rank order the couplings in the following manner: the coupling with largest magnitude (corresponding to the smallest value of K_{xy}) in Λ_L will be said to have rank one; the next larger coupling will have rank two, and so on. Now select the coupling with rank one; *i.e.,* that with the smallest value of K_{xy}, and choose the spins σ_x and σ_y on its endpoints such that $\sigma_x \epsilon_{xy} \sigma_y > 0$, *i.e.,* the coupling is satisfied. Then find the coupling of rank two, and choose the spins on its endpoints to satisfy it. Repeat this procedure, *unless* a resulting closed loop (or path connecting two boundary sites) with previously satisfied couplings forbids it. When that happens, simply proceed to the coupling next in order, and continue until every coupling has been tested.

Note that until a cluster of spins (connected by tested edges) reaches the boundary, only the relative orientations of the spins in the cluster are known. With a fixed boundary condition, the sign of each spin in a cluster will be determined as soon as it connects to the boundary.

It is not hard to see that this algorithm always provides the ground state for any Λ_L in which every coupling magnitude is greater than the sum of all those of lower order in Λ_L. This will be the case for all large L (with probability one) because each coupling magnitude is greater than at least twice that of the next smaller one. To see then that in the ground state a given coupling *must* be satisfied providing this doesn't violate the previously satisfied couplings of higher rank (or the boundary conditions), consider the clusters formed by the previously satisfied couplings. Under the proviso, the two endpoints of the given coupling must belong to distinct clusters, at least one of which does not touch $\partial\Lambda_L$. If the given coupling were not satisfied in a spin configuration, then flipping all the spins in that cluster (the one not touching $\partial\Lambda_L$) would lower the energy and so the spin configuration would not be a ground state.

The algorithm outlined here is easily recognizable as simply a version of the greedy algorithm — in effect, we have invented a spin glass model whose exact ground states can be found via the greedy algorithm (which for most models is generally a relatively poor algorithm for finding ground state configurations or energies). It is not at all clear at this point, however, that the procedure that we outline, when repeated for ever-increasing volumes, will have a natural infinite-volume limit — but we will show in the next section that this is in fact the case. Before we do that, however, we explore some of the properties of our model in light of the ground state algorithm just described.

3.1 Statement of the Problem

The question of whether this model has multiple infinite volume ground state pairs is equivalent to whether, as $L \to \infty$, a change in boundary conditions can change a fixed coupling deep in the interior from being satisfied to unsatisfied, or vice-versa.

We therefore ask whether any bond $< x_1 x_2 >$ exists with the following property: for all large L, before any path of satisfied couplings joining x_1 and x_2 within Λ_L (i.e., not touching $\partial\Lambda_L$) is formed according to the greedy algorithm described above, there already exist two disjoint paths, one joining x_1 to the boundary, and the other joining x_2 to the boundary. If such a bond exists, then whether its coupling is satisfied or unsatisfied will be determined by the boundary conditions (for all large L).

When the boundary of Λ_L is sufficiently far from some fixed interior region R, it may be that no such coupling exists within R — each coupling in R is either itself tested before two such disjoint paths can be found, or else its endpoints are first connected via some path (not touching $\partial\Lambda_L$) of previously tested couplings. If that is the case for every finite R, then only a single pair of spin-flip-related ground states exists in the thermodynamic limit. Otherwise, the system possesses multiple pairs of ground states.

3.2 "Always Satisfied" Bonds

One can distinguish between two kinds of satisfied bonds in our model, for a given coupling realization ω: there is the kind which is satisfied in ω but which would become *unsatisfied* in ω', which is simply ω with the *sign* (*i.e.,* the ϵ_{xy}) of that particular bond reversed. That is, whether this bond is satisfied depends on the specific sign of its corresponding ϵ_{xy}. The second set of satisfied bonds are those which are satisfied *regardless* of their sign; we will call them $S1$ bonds (a precise definition is given below). It is easy to see, for example, that the coupling of highest rank is $S1$. In fact, each site in Λ_L has at least one $S1$ bond attached to it, namely that whose rank is the highest of all bonds which connect to that site. We note that $S1$ bonds appear in the ordinary EA model also (*e.g.,* any bond whose magnitude is greater than the sum of the magnitudes of the adjoining bonds at either of its ends) but they don't appear to play the crucial role that they play in our model.

Because $S1$ bonds play an important role in what follows, we will devote some space to study their properties here. In particular, it is they which determine the ground state structure in our model. We begin with a precise definition of $S1$ bonds for a given Λ_L.

Definition. A bond will be denoted $S1$ if the following is true: its rank must be greater (*i.e.,* its coupling must be of larger magnitude) than at least one coupling in any path (not using that bond) connecting its endpoints. In this definition, we treat all boundary points as automatically connected so that the union of a path from x to $\partial\Lambda_L$ and a path from y to $\partial\Lambda_L$ is considered to connect x and y. We remark that this notion of connection within Λ_L is a consequence of our dealing with a fixed boundary condition $\hat{\sigma}$; when we treat boundary conditions such as free or periodic (in Section 6), the notion of connection will be modified accordingly.

According to the definition, an $S1$ coupling is chosen to be satisfied by the greedy algorithm before any other path of similarly chosen bonds connects its endpoints. It is apparent that it is the $S1$ couplings which determine the ground state configuration. Satisfied bonds which are not $S1$ (call them $S2$) play no role in determining any part of the ground state spin configuration.

We now present some properties of $S1$ bonds which will be useful later.

1) The set of all $S1$ bonds forms a union of trees. This claim is obvious from the definition of $S1$ bonds.

2) The set of all $S1$ bonds spans the set of all sites in Λ_L, *i.e.,* every site belongs to at least one $S1$ tree; and furthermore every $S1$ tree touches the boundary of Λ_L.

This second claim comes in two parts: the first has already been shown above by explicit construction. The second part is easily shown by contradiction: suppose a given $S1$ tree "dies" before reaching the boundary of Λ_L. Consider all edges which connect a point in this tree to a point not in it. The coupling of highest rank within this set must be $S1$. Therefore, all trees formed of $S1$ couplings reach the boundary.

It may happen, for a given ω, d, and Λ_L, that the $S1$ couplings form either a single tree or a union of disjoint trees. Note that this tree partition of Λ_L is the same for all boundary conditions $\hat{\sigma}$. Within each tree the relative sign of the spins is fixed by the $S1$ couplings; the overall sign for each tree is determined by $\hat{\sigma}$. We can now address the question posed earlier — different boundary conditions can give rise to different infinite volume ground states if and only if there exist fixed neighboring sites which belong to *disjoint* trees of Λ_L for arbitrary large L's.

We are left, however, with several important questions — how is the tree partition for a particular L related to that for some $L' > L$? More specifically, how can one be sure that the procedure we have proposed has a natural infinite-volume limit? In order to answer these questions, we present an alternative algorithm in the next section, which will provide a mapping to invasion percolation.

4 Invasion Percolation Algorithm

Before describing our alternative algorithm for obtaining the ground states, let us note that although the absolute ranks of the K_{xy}'s depend on L, the K_{xy}'s themselves, and hence their relative ordering by rank, does not change with L. This will allow us to analyze the $L \to \infty$ limit of our algorithm.

We begin by defining (in all of $\mathbf{Z}^d$) and for a given ω (and hence a fixed relative ordering of all the $K_{xy}(\omega)$'s), a growing sequence of trees $T_0(u), T_1(u), T_2(u), \ldots$, starting from $u \in \mathbf{Z}^d$, with $T_n(u)$ containing n edges and $n+1$ sites (including u). $T_0(u)$ consists of u alone, and in general $T_{n+1}(u)$ is obtained from $T_n(u)$ by considering all edges from sites in $T_n(u)$ to new sites and adjoining the edge $e_{n+1}(u) =< x_n, x_{n+1} >$ (and new site x_{n+1}) with the smallest value of K_{xy}. This procedure is identical to that employed in invasion percolation on $\mathbf{Z}^d$ (except that we include only edges which connect to new sites).

For a given L and $u \in \Lambda_L$, let $N_L(u)$ denote the smallest n such that $T_n(u)$ touches $\partial\Lambda_L$. The crucial point is that when every coupling magnitude in Λ_L is greater than the sum of all those of lower order, then for any boundary condition $\hat{\sigma}$ on $\partial\Lambda_L$ (and any choice of ϵ_{xy}'s), in the ground state $\sigma^*(\hat{\sigma})$, every coupling in $T^{(L)}(u) \equiv T_{N_L(u)}(u)$ must be satisfied. To see this, note that if $J_{x_n x_{n+1}}$ were not satisfied (here $n+1 \le N_L(u)$) in a given spin configuration, then flipping all the spins in $T_n(u)$ would lower the energy because $J_{x_n x_{n+1}}$ is the coupling on the boundary of $T_n(u)$ of largest magnitude. σ_u^* is then determined by the tree $T^{(L)}(u)$, the coupling signs ϵ_{xy} on that tree and the boundary condition $\hat{\sigma}_x$ at the boundary site x touched by that tree.

It is clear from the preceding discussion (and that of Section 3) that every bond in $T^{(L)}(u)$ (for every u) is an $S1$ bond. Since every $T^{(L)}(u)$ touches $\partial\Lambda_L$ and the union of all these edges (for all u's in Λ_L) clearly touches every site in Λ_L, it must be that this union is the same tree partition of Λ_L as obtained in Section 3 from the union of all $S1$ bonds. On the other hand, it is clear from the last paragraph, that (for $u, v \in \Lambda_L$) the relative sign $\sigma_u^*(\hat{\sigma})\sigma_v^*(\hat{\sigma})$ is the same for all choices of $\hat{\sigma}$ if and only if $T^{(L)}(u)$ and $T^{(L)}(v)$ are nondisjoint. Furthermore, one has the following dichotomy concerning the infinite volume invasion trees, $T_\infty(u) = \lim_{n\to\infty} T_n(u) = \lim_{L\to\infty} T^{(L)}(u)$: If $T_\infty(u)$ and $T_\infty(v)$ are nondisjoint, then $T^{(L)}(u)$ and $T^{(L)}(v)$ are nondisjoint for all large L; if $T_\infty(u)$ and $T_\infty(v)$ are disjoint, then $T^{(L)}(u)$ and $T^{(L)}(v)$ are disjoint for all L (such that $u, v \in \Lambda_L$). We are thus led to the following conclusions, concerning the trees $T_\infty(u)$ and their union,

$$F_\infty = \cup_{u\in\mathbf{Z}^d} T_\infty(u),$$

which we call the invasion forest (note that both the trees and the forest depend only on the K_{xy}'s and not the ϵ_{xy}'s):

(1) In every infinite volume ground state, every coupling in F_∞ is satisfied.

(2) F_∞ is either a single (infinite) tree or else a union of $\mathcal{N} \geq 2$ distinct (infinite) trees; in either case it spans all of $\mathbf{Z}^d$.

(3) The former case happens if for every u, v, the trees $T_\infty(u)$ and $T_\infty(v)$ intersect. In this case there is a single infinite volume ground state pair.

(4) The latter case happens if $T_\infty(u)$ and $T_\infty(v)$ are disjoint for some u, v. In this case the number of ground state pairs is $2^{\mathcal{N}-1}$ and so is uncountable if $\mathcal{N}$ is infinite.

In the next section we will discuss the dependence of the value of $\mathcal{N}$, and hence of the ground state multiplicity, on the spatial dimension d.

5 Nonintersection in invasion percolation

We have mapped the problem of multiplicity of ground states in our model to that of whether invasion percolation has nonintersecting invasion regions. We can already answer the question of multiplicity of states of our model in two dimensions. Because it is known that for $d = 2$ invasion percolation the trees $T_\infty(u)$ and $T_\infty(v)$ always intersect (in fact are the same modulo finitely many sites) [8], it follows that our model has only a single pair of ground states in two dimensions. Whether any two such trees in higher dimensions must intersect is an interesting problem in invasion percolation, which we now consider.

To proceed (mostly nonrigorously), we use a well-known feature of invasion percolation: that the invaded region asymptotically approaches the so-called incipient infinite cluster (i.e., at the critical percolation probability p_c) in the independent bond percolation problem on the same lattice [28]. The fractal dimension D of the incipient cluster in the independent bond problem on the d-dimensional cubic lattice is known from both numerical studies and scaling arguments; in particular, D is dimension-dependent (increasing with d) below six dimensions, but $D = 4$ for $d \geq 6$ [28].

The following heuristic argument might then provide an intuitive picture of our model's behavior. Consider the infinite-volume invasion trees $T_\infty(u)$ and $T_\infty(v)$ introduced in the last section. If each has a fractal dimension less than $d/2$, the probability that they will "miss" each other is greater than zero; if it is greater than $d/2$, they will intersect with probability one.

This suggests that if the fractal dimension D_i of $T_\infty(u)$ is equal to D, then the critical dimension of our model is eight. Below eight dimensions invasion regions should always intersect, and hence there would be only one pair of ground states in our spin glass model; above that there should be an infinite number.

To make this line of reasoning a bit more precise, let us define a *pair connectedness function* $G(y - x)$ as the probability that the site $y \in T_\infty(x)$. We can then define D_i by the relation [11, 28]

$$G(y - x) \sim 1/\|y - x\|^{d-D_i} \tag{6}$$

as $\|y - x\| \to \infty$. Note that summing Eq. (6) over all y in a box of side length L centered at x yields L^{D_i} as the order of the (mean) number of sites in that box which belong to $T_\infty(x)$.

Our main task is then to determine D_i as a function of space dimension d. First, however, we provide a precise statement of a condition for nonintersection of invasion trees, which also provides rigorous justification for part of the heuristic argument presented above.

Theorem. If $\sum_{x \in Z^d} G(x)^2 < \infty$, then (with probability one) there are infinitely many (random) sites $x_1, x_2, \ldots$ such that $T_\infty(x_i) \cap T_\infty(x_j) = \emptyset$.

Remark. Although we state this theorem in the context of invasion percolation, the proof we will now present shows that it remains valid for a fairly general class of random growth processes in place of $T_n(x)$. The ingredients of the proof are (statistical) translation and reflection invariance and the fact that the events $\{T_n(x) = A\}$ and $\{T_n(y) = B\}$ are independent as long as A and B are separated in $\mathbf{Z}^d$ by some fixed distance. This fact is in turn a consequence of the "local dependence" of events like $\{T_n(x) = A\}$ on the underlying K_{xy} variables (and the mutual independence of those variables).

Proof. Let A_n denote the event that there exist some n sites, $x_1, \ldots, x_n$ with $T_\infty(x_i) \cap T_\infty(x_j) = \emptyset$ for $1 \leq i < j \leq n$. If we can show that for every $\epsilon > 0$ and every n, $P(A_n) \geq 1 - \epsilon$, then $P(A_n) = 1$ for each n and the desired result follows by letting $n \to \infty$. The desired lower bound on $P(A_n)$ would itself be a consequence of showing that $P(T_\infty(x) \cap T_\infty(y) = \emptyset) \to 1$ as $\|x - y\| \to \infty$. To see this, pick deterministic $y_1, \ldots, y_n$ with $\|y_i - y_j\|$ large enough, for $i \neq j$, so that $P(T_\infty(y_i) \cap T_\infty(y_j) \neq \emptyset) \leq \epsilon/\binom{n}{2}$; then

$$\begin{aligned} P(A_n) &\geq P(T_\infty(y_i) \cap T_\infty(y_j) = \emptyset \text{ for } 1 \leq i < j \leq n) \\ &\geq 1 - \sum_{1 \leq i < j \leq n} P(T_\infty(y_i) \cap T_\infty(y_j) \neq \emptyset) \end{aligned}$$

$$\geq \ 1-\binom{n}{2}\cdot\epsilon/\binom{n}{2}=1-\epsilon \quad . \tag{7}$$

It remains to obtain a suitable lower bound for $P(T_\infty(x)\cap T_\infty(y)=\emptyset)$ when $\|x-y\|$ is large.

Denote by $\rho\left(T_\infty(x),T_\infty(y)\right)$ the minimum Euclidean distance between some site in $T_\infty(x)$ and some site in $T_\infty(y)$. Furthermore, let $T'_\infty(y)$ denote an invasion region constructed using a completely independent duplicate set of variables $\{K'_{xy}\}$ (so that $T_\infty(x)$ and $T'_\infty(y)$ are independent random trees). An elementary but crucial observation is that

$$P\left(T_\infty(x)\cap T_\infty(y)=\emptyset\right)\geq P\left(\rho(T_\infty(x),T_\infty(y))>1\right)=P\left(\rho(T_\infty(x),T'_\infty(y))>1\right) \quad , \tag{8}$$

where the inequality is trivial and the equality follows from a fairly standard type of argument (see, e.g., Ref. [2]) given in the next paragraph which, roughly speaking, leads to the conclusion that $T_\infty(x)$ and $T_\infty(y)$ are independent, *conditional* on $\rho\left(T_\infty(x),T_\infty(y)\right)>1$.

To explain more concretely the equality in Eq. (8), let us note that for any possible configuration A of $T_n(x)$, the event that $T_n(x)=A$ depends only on the K_{xy}'s with $<xy>\in \overline{\mathcal{E}}(A)$, where $\overline{\mathcal{E}}(A)$ denotes the set of nearest neighbor edges which touch either one or two vertices of A. If B is a possible configuration for $T_n(y)$ with $\rho(A,B)>1$ (or equivalently with $\overline{\mathcal{E}}(A)\cap\overline{\mathcal{E}}(B)=\emptyset$) then the events $\{T_n(x)=A\}$ and $\{T_n(y)=B\}$ are independent since they depend on disjoint sets of the independent K_{xy}'s. Hence

$$\begin{aligned}
P\left(\rho\Big(T_n(x),T_n(y)\Big)>1\right) &= \sum_{A,B:\rho(A,B)>1} P\Big(T_n(x)=A,T_n(y)=B\Big)\\
&= \sum_{A,B:\rho(A,B)>1} P\Big(T_n(x)=A\Big)\Big(T_n(y)=B\Big)\\
&= \sum_{A,B:\rho(A,B)>1} P\Big(T_n(x)=A,T'_n(y)=B\Big)\\
&= P\left(\rho\Big(T_n(x),T'_n(y)\Big)>1\right) \ .
\end{aligned} \tag{9}$$

Letting $n\to\infty$ gives the desired equality.

Let $\hat{I}_z$ denote the event that $z\in T_\infty(x)$ and $\rho(z,T'_\infty(y))\leq 1$. Then $\rho(T_\infty(x),T'_\infty(y))\leq 1$ if and only if $\hat{I}_z$ occurs for some z and hence

$$\begin{aligned}
P\left(\rho\Big(T_\infty(x),T'_\infty(y)\Big)>1\right) &= 1-P(\cup_{z\in\mathbf{Z}^d}\hat{I}_z)\\
&\geq 1-\sum_{z\in\mathbf{Z}^d}P(\hat{I}_z)\\
&= 1-\sum_{z\in\mathbf{Z}^d}P(z\in T_\infty(x))P\left(\rho(z,T'_\infty(y))\leq 1\right)\\
&\geq 1-\sum_{z\in\mathbf{Z}^d}G(z-x)\sum_{\|z'-z\|\leq 1}G(z'-y) \ .
\end{aligned} \tag{10}$$

By using the reflection invariance of $G(x)$, the last expression can be rewritten as

$$1-\sum_{\|w\|\leq 1}\left(\sum_{z\in\mathbf{Z}^d}G(x-y+w-z)G(z)\right) \quad . \tag{11}$$

To complete the proof, it clearly suffices to show that

$$\Big(G \star G\Big)(x) \equiv \sum_{z \in \mathbf{Z}^d} G(x-z)G(z) \to 0 \qquad \text{as } \|x\| \to \infty \ . \tag{12}$$

But

$$\Big(G \star G\Big)(x) = \int_{[-\pi,\pi]^d} \Big[\hat{G}(k)\Big]^2 e^{-ik\cdot x} dk \tag{13}$$

where

$$\hat{G}(k) = (2\pi)^{-d/2} \sum_{x \in \mathbf{Z}^d} G(x) e^{ik\cdot x} \quad . \tag{14}$$

Furthermore, $\hat{G}(k)$ is real since $G(x) = G(-x)$ and so

$$\int_{[-\pi,\pi]^d} \Big[\hat{G}(k)\Big]^2 dk = \int_{[-\pi,\pi]^d} \Big|\hat{G}(k)\Big|^2 dk = \sum_{x \in \mathbf{Z}^d} |G(x)|^2 < \infty \quad ; \tag{15}$$

thus $\Big[\hat{G}(k)\Big]^2 \in L^1\Big([-\pi,\pi]^d, dk\Big)$ and so by Eq. (13) and the Riemann-Lebesgue lemma, $(G \star G)(x) \to 0$ as $\|x\| \to \infty$, as desired. $\diamond$

Using this result and Eq. (6), it follows that the condition $D_i < d/2$ is sufficient for nonintersection of invasion regions in d dimensions, and correspondingly, for our model to have an uncountable number of ground states. We now consider this question.

Monte Carlo simulations of invasion percolation on square and simple cubic lattices [29] provide strong evidence that $D_i = D$ in dimensions two and three ($D = 91/48$ and $D = 2.53$ [28]). In higher dimensions, less is known. There does exist, however, an exact solution for invasion percolation on a Cayley tree [23].

One can deduce the fractal dimension D_i of the invasion region using two different measures. One of these is to compute the radius of gyration (i.e., the root of the mean square cluster radius) R after the invasion process has completed n steps. For the Cayley tree, it was found using Monte Carlo simultations [23] that, as $n \to \infty$, $R \sim n^{1/4}$, consistent with $D_i = 4$.

The second measure uses the exact solution mentioned above and is much closer to Eq. (6). The shape function, S_m^n, which is the mean number of invaded sites on level m of the Cayley tree for an invasion of n steps, is computed. By analyzing Eq. (9) in Ref. [23] (valid for the simplest Cayley tree, i.e., with coordination number 3) in the limits $\beta \to 1$ and $\alpha \to 1$ (in that order), we find that S_m^∞ is proportional to m (with logarithmic corrections) as $m \to \infty$. The *total* number of sites invaded up to level m thus scales as m^2. The usual measure of distance on a Cayley tree places level m at distance $\sqrt{m}$ from the origin, leading again to $D_i = 4$.

We therefore conclude (nonrigorously) that $D_i = 4$ is an *upper bound* for the fractal dimension of an invasion tree on a lattice in finite dimension. Given that we expect $D_i = D$ in any dimension, and given the known values of D, we conclude that the critical dimension in our problem is eight.

6 The Role of Frustration

We have so far argued that our model has a transition in ground state multiplicity at eight dimensions. However, the nature of the "spin-glassiness" of our model, and in particular the role played by frustration, has not been clarified. In fact, suppose that one were to construct a model of a random *ferromagnet* using the Hamiltonian (1) and couplings (2) but with all $\epsilon_{xy} = +1$. For this system, the ground state in any finite volume with specified boundary conditions can be found using the same greedy algorithm described in Section 3. It is clear that once again many (infinite volume) ground states (in $d > 8$) can be generated with appropriate choices of fixed boundary conditions. (We recall that this is also the case for uniform ferromagnets in any dimension or ordinary random ferromagnets [15] in $d > 5$).

So how is the ground state structure of the above random ferromagnet different from that of our spin glass? The answer can be found through analyzing the behavior of each in the presence of *spin-symmetric*, coupling-independent boundary conditions, such as free or periodic.

In an earlier paper, [19] the authors argued that multiplicity of ground states in the EA Ising spin glass should be associated with nonexistence of a single limiting Gibbs distribution, in the thermodynamic limit, for *any* coupling-independent boundary conditions. We find that the same association holds here. While the conclusion is fairly clear for any sequence of *fixed* boundary conditions, in which each boundary spin is assigned a definite value, the mechanism for spin-symmetric (e.g., free or periodic) boundary conditions is more subtle. Its investigation provides deeper insight into the nature of our spin glass model, and in particular the role played by frustration.

We first discuss fixed boundary conditions. Let us now denote the always satisfied bonds in Λ_L, as defined in Sec. 3, as $S1^w$ bonds, where the w superscript (for "wired") is to remind us of the convention that all boundary points are treated as automatically connected. (This is analogous to wired boundary conditions in the Fortuin-Kasteleyn random cluster representation of Ising and Potts models [1].) Let us denote by F_L^w the union of all the $S1^w$ bonds, or equivalently the union over $u \in \Lambda_L$ of the invasion trees $T^{(L)}(u)$ (stopped when they touch $\partial\Lambda_L$). F_L^w depends on the K_{xy}'s while the overall "sign" of any individual tree in F_L^w (*i.e.,* the sign of a single spin in that tree) is determined finally by the boundary spin (at the particular site where the tree touches $\partial\Lambda_L$) and the coupling sign ϵ_{xy} of the $S1^w$ bond touching that boundary site. For $d > 8$, as L increases with a given sequence $\hat{\sigma}^{(L)}$ of fixed, coupling-independent boundary conditions, the sign of each tree will randomly flip, and no single limiting ground state is obtained in the thermodynamic limit.

Indeed the size dependence is such that *all* of the uncountably many ground states will appear as limits along coupling-dependent subsequences of volumes. To see that, we first condition on all the K_{xy}'s and let $T_1, T_2, ...$ be a list (in some order) of all the distinct trees from F_∞. Let $\eta_i^{(L)}$ denote the sign of (some fixed σ_{x_i} in) T_i, as determined by the ϵ_{xy}'s and the boundary condition $\hat{\sigma}^{(L)}$. From the previous discussion it should be clear that (i) for each L, $\eta^{(L)} = \{\eta_i^{(L)} : x_i \in \Lambda_L\}$ consists of independent random variables, equally likely to be $+1$ or -1, and (ii) for varying L, the $\eta^{(L)}$'s are independent. It follows (for almost every coupling realization) that for every m and for every assignment $\overline{\eta}^{[m]}$ of signs $\overline{\eta}_i^{[m]} = \pm 1$ for $i \in \{1, ..., m\}$, the collection $\{L : \eta_i^{(L)} = \overline{\eta}_i^{[m]}$ for $i = 1, ..., m\}$ is an infinite subsequence of L's. By a standard subsequence diagonalization argument, it follows that for every one of the uncountably many assignments $\overline{\eta}$ of signs $\overline{\eta}_i = \pm 1$ for all $i \geq 1$, there exists a subsequence $\{L_j(\overline{\eta}) : j \geq 1\}$ so that for all i, $\eta_i^{(L_j(\overline{\eta}))} \to \overline{\eta}_i$ as $j \to \infty$.

Consider now the case of the finite cube $\Lambda_L \subset Z^d$, with $d > 8$, and with, say, free boundary conditions. (We briefly discuss periodic and antiperiodic b.c.'s later.) At first it might seem that the connection between ground states and invasion percolation is no longer valid, because *for every finite volume there is only a single cluster of always satisfied couplings.* This is because, unlike when the boundary spins are fixed, the procedure of satisfying couplings of successively smaller magnitude will be continued until all sites are connected. To explain this more fully, we define a bond $< xy >$ with *both* sites x and y in Λ_L to be an $S1^f$ bond exactly as in the definition of an $S1$ bond in Sec. 3 except that a path is said to connect the points x and y only if the path stays entirely within Λ_L, never touching the boundary. In this case, it is not hard to see that the set F_L^f of all $S1^f$ bonds forms a *single* tree spanning all of Λ_L.

To clarify the relation between free b.c. ground states and the invasion forest F_∞, we begin with a simple but important observation, that every $S1^w$ bond with both endpoints in Λ_L (*i.e.,* leaving out any $S1^w$ bond touching $\partial\Lambda_L$) is also an $S1^f$ bond. Let us denote by E_L^f the set of all $S1^f$ bonds which are *not* also $S1^w$ bonds. These are the bonds which connect together the distinct trees of F_L^w to create the single spanning tree F_L^f. Note that every such bond $< xy >$ must have its two endpoints in distinct trees of F_L^w. We now make a crucial claim: as $L \to \infty$, the edges of E_L^f will move out to infinity. More precisely stated, the intersection of E_L^f with any fixed set of bonds (say, all bonds in Λ_{L_0} with L_0 fixed) will, for all large L, be the empty set. To see this, first observe that any such edge $< xy >$ not moving out to infinity would have two properties: (a) There would be no path in Z^d between x and y with all K values along the path strictly below K_{xy}; (b) $T_\infty(x) \cap T_\infty(y) = \emptyset$. It was proved by Alexander [3] that such edges do not exist in any dimension.

We will next argue that the *relative* sign between trees flips randomly, and so for our spin glass model there is no single limiting pair of ground states for a sequence of volumes (chosen independently of the couplings) with free boundary conditions. In fact, we will see that in our model, all of the uncountably many ground state pairs arise via *coupling-dependent* subsequences. This *chaotic size dependence*, i.e., absence of a limiting ground state for spin-symmetric, coupling-independent boundary conditions, is similar to that which would be found in the EA model were it to possess many ground state pairs [19]. We argued in Ref. [19] that this is, in fact, the signature of multiple ground state pairs in spin glasses, and sets them apart from other systems which may also possess multiple ground state pairs, such as random ferromagnets.

The argument for random flipping of relative signs between trees in the free b.c. case is like the one used above for flipping of absolute signs of trees in the fixed b.c. case, with a few modifications. Again, we condition on all K_{xy}'s and let $T_1, T_2, ...$ be the distinct trees of F_∞. Now let $\eta_{ij}^{(L)}$ be the relative sign $\sigma_{x_i}\sigma_{x_j}$ in the free b.c. ground state. This is determined (finally) by the signs ϵ_{xy} of the bonds $< xy >$ in E^f_L. Since these bonds move off to infinity, we can pick a (sufficiently rapidly growing) subsequence $L_k \to \infty$ so that the sets $E^f_{L_k}$ are disjoint from each other for different k's. It follows that the relative signs $\eta_{1i}^{(L_k)}$ between the first and i'th trees will be independent as both i and k vary. Consequently, the same diagonalization procedure as in the fixed b.c. case can be applied to obtain the requisite further subsequences of L_k.

Finally, we note that for periodic (or antiperiodic) b.c.'s, $S1^p$ bonds are defined by regarding every boundary point as automatically connected to its periodic image point, but with no other automatic connections. Thus, like in the free b.c. case, every $S1^w$ bond is also an $S1^p$ bond, while the other $S1^p$ bonds must move to infinity as $L \to \infty$. This leads to the same conclusions as in the free b.c. case.

Let us now contrast this picture with that of the random ferromagnetic version of our model. There, a sequence of coupling-independent spin-symmetric boundary conditions *would* yield a single pair of ground states as $L \to \infty$ — namely, all spins up and all spins down. This will be the case in any dimension. As with the usual models of ferromagnets and spin glasses, either of which might have many ground states, the difference from spin glasses is revealed most sharply through the use of coupling-independent, spin-symmetric boundary conditions.

7 Conclusions

In this paper we have expanded our original discussion in I, supplied additional proofs and arguments, and looked at our model in greater depth. In this section we briefly discuss some consequences of our studies, particularly what (if any) conclusions can be drawn about more realistic spin glass models, and how our approach can be used to study other topics, particularly certain dynamical problems.

We first recap our main conclusions, based on both rigorous and nonrigorous arguments:

- In invasion percolation on $\mathbf{Z}^d$, the invasion forest F_∞ has only one tree in dimensions less than eight, and infinitely many above eight.
- In the spin-glass model introduced in I and discussed here, there exist only a single pair of ground states below eight dimensions, and infinitely many above eight.
- Elucidation of the ground state structure exactly *at* eight dimensions cannot be addressed by the techniques provided in this paper, and requires further study.
- The random ferromagnetic verison of our model (all $\epsilon_{xy} = 1$ in Eq. (2)) also has a transition in ground state multiplicity at eight dimensions. However, exactly as in the Edwards-Anderson model, the special feature of spin-glass ground state multiplicity manifests itself in chaotic size dependence for spin-symmetric, coupling-independent boundary conditions, such as periodic or free.
- The special properties of this model render it unsuitable for drawing firm conclusions about more realistic spin glass models. However, it suggests that a third conjecture (besides those of the Parisi picture and the scaling picture) should be entertained — that of

a transition in ground state multiplicity at some finite (and, obviously, greater than one) dimension.

• Our study can resolve one perennially cloudy issue in the literature; namely, that of whether the joint presence of quenched disorder and frustration has any universal implications for ground state structure or multiplicity. The results presented here make it clear that no such *a priori* implications can be drawn.

We close with a brief discussion of applications of our results to other problems. The form of the exponential expression in Eq. (2) suggests that our techniques might apply in some way to the problem of a random walk in a strongly inhomogeneous environment. If K_{xy} is thought of as an energy barrier between sites x and y, and λ now represents inverse temperature, then $\exp(-\lambda K_{xy})$ may be interpreted as an Arrhenius factor proportional to the *rate* of making a transition from x to y at fixed temperature λ^{-1}. Random K_{xy}'s, like those used throughout this paper, correspond to a rugged energy landscape. We can then prove that, in the limit as temperature gets small, the order in which sites are visited for the first time corresponds exactly to the invasion order in invasion percolation [21]. We further develop conclusions drawn from the theorems proved in Ref. [21], and study the nature of broken ergodicity on a system whose state space resembles a rugged landscape, i.e, has many metastable states with random fixed barriers separating them [22]. This leads to a surprising degree of emergent structure from what appears initially as a rather featureless landscape. The multiplicity of trees in the invasion forest for high dimensions plays an important role in that work. We are presently studying how our analysis here and in Refs. [21, 22] can be applied in other contexts.

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