The Geometrical Approach to Multidimensional Inverse Scattering *

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Abstract

We prove that in multidimensional short–range potential scattering the high velocity limit of the scattering operator of an N–body system determines uniquely the potential. For a given long–range potential the short–range potential of the N–body system is uniquely determined by the high velocity limit of the modified Dollard scattering operator. Moreover, we prove that any one of the Dollard scattering operators determines uniquely the total potential. We obtain as well a reconstruction formula with error term. Our simple proof uses a geometrical time–dependent method.

I Introduction

Let us consider first an N–body quantum mechanical system in $n \geq 2$ space dimensions with interactions given by local short–range pair potentials (multiplication operators). By

* AMS Classification 35P, 35Q, 81U; PACS number: 03.65.Nk
† Fellow Sistema Nacional de Investigadores. Research Partially supported by Deutscher Akademischer Austauschdienst

S we denote the scattering operator between the free channels, where there are no bounded subsystems asymptotically. In Theorem 1.1 we prove that S determines the pair potentials uniquely. In fact, it is enough to know S on states where the relative velocity between any two particles tends to infinity. Moreover, we give a formula with error term for the reconstruction of the pair potentials from the high velocity limit of $(S - I)$ (see equation (4.25)).

If, in addition, the particles interact with long–range forces it is necessary to know (the far out "tail" of) the long–range potential in order to define a scattering operator, for example the Dollard scattering operator S^D. In this case we prove (Theorem 1.2) that S^D determines uniquely the short–range part of the potential. We prove as well that any one of the S^D's determines uniquely the total potential. As in the short–range case we give in Theorem 4.1 a formula with error term for the reconstruction of the potential from the high velocity limit of $(S^D - I)$.

We also study the channel where one particle scatters off a target consisting of a bound state of $N-1$ particles. In this case we prove in Section V that the high velocity limit of the appropriate scattering operator determines uniquely the effective multiparticle potential that gives the interaction between the incoming particle and the bound state, and we give a formula with error term for its reconstruction (equation (5.14)). Our proof, which is novel and simpler even in the two–body case, uses geometrical time-dependent methods.

We denote by m_j and $\tilde{\mathbf{x}}_j \in I\!\!R^n, j = 1, 2, \ldots, N$, respectively, the masses and the positions of the particles. The free Hamiltonian

$$\tilde{H}_0 = \sum_{j=1}^{N} (2m_j)^{-1}\, \tilde{\mathbf{p}}_j^2\,, \quad \tilde{\mathbf{p}}_j = -i\nabla_{\tilde{\mathbf{x}}_j},$$

generates the free time evolution. We study the system in the center of mass frame and we separate off the motion of the center of mass

$$H_{CM} = \left(2\sum_{j=1}^{N} m_j\right)^{-1} \left(\sum_{j=1}^{N} \tilde{\mathbf{p}}_j\right)^2$$

in order to obtain the free Hamiltonian $H_0 := \tilde{H}_0 - H_{CM}$.

As a general reference for multiparticle scattering see e.g. [1]. In the center of mass frame the space of states is the Hilbert space, $\mathcal{H}$, represented in configuration space by wave functions ϕ in

$$L^2(\mathbf{X}), \quad \mathbf{X} = \left\{ (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N) \middle| \sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j = 0 \right\} \cong I\!\!R^{n(N-1)}$$

with the measure induced on $\mathbf{X}$ by the following norm on $I\!\!R^{nN}$: $|||\mathbf{x}||| = \left[\sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j^2 \right]^{1/2}$. The space

$$L^2(\hat{\mathbf{X}}), \quad \hat{\mathbf{X}} = \left\{ (\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_N) \middle| \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0 \right\} \cong I\!\!R^{n(N-1)},$$

where $\hat{\mathbf{X}}$ is equipped with the dual metric induced by $\left[\sum_{j=1}^{N} (m_j)^{-1} \, \tilde{\mathbf{p}}_j^2 \right]^{1/2}$ on $I\!\!R^{nN}$ is the set of momentum space wave functions $\hat{\phi}$. Fourier transform maps unitarily $L^2(\mathbf{X})$ onto $L^2(\hat{\mathbf{X}})$. The measures on $\mathbf{X}$ and $\hat{\mathbf{X}}$ are equivalent to Lebesgue measure. Given an (abstract) state $\Phi \in \mathcal{H}$ we use both its configuration or momentum space wave functions where appropriate. H_0 is self–adjoint on its domain $D(H_0) = W^{2,2}(\mathbf{X})$. We assume that the potential is a sum of pair potentials that are multiplication operators

$$V = \sum_{j<k} V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j). \tag{1.1}$$

Each pair potential is split into parts of short range or long range depending on their differentiability and decay rate towards infinity

$$V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) = V_{jk}^{s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^{\ell}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j). \tag{1.2}$$

We say that an operator is Kato–small in $L^2(I\!\!R^n)$ if it is bounded with respect to the Laplacian with relative bound zero (see [2] for definitions). Then we define

$$\mathcal{V}_{SR} = \Big\{ V^s = \sum_{j<k} V_{jk}^{s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \;\Big|\; V_{jk}^{s} \text{ is Kato–small in } L^2(I\!\!R^n),$$
$$\int_0^\infty dR \left\| V_{jk}^{s}(\mathbf{y}) \, (-\Delta + I)^{-1} F(|\mathbf{y}| \geq R) \right\| < \infty \Big\} \tag{1.3}$$

where for any set $O \subset I\!\!R^n$ we denote by $F(\mathbf{x} \in O)$ the multiplication operator with the characteristic function of O. The decay property in (1.3) is for multiplication operators equivalent to the more intuitive condition that

$$\left\| F(|\mathbf{y}| \geq R)\, V^s_{jk}(\mathbf{y})\, (-\Delta + I)^{-1} \right\|$$

decays integrably. The set of admissible long–range potentials is

$$\mathcal{V}_{LR} = \left\{ V^\ell = \sum_{j<k} V^\ell_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \,\middle|\, V^\ell_{jk} \in C^1_\infty(I\!\!R^n),\ |\nabla V^\ell_{jk}(\mathbf{y})| \leq C\,(1+|\mathbf{y}|)^{-\gamma},\ \gamma > 3/2 \right\}, \tag{1.4}$$

to simplify notation later we will use only $\gamma < 2$. C^1_∞ is the space of all continuously differentiable functions that tend to zero towards infinity. The splitting into short– and long–range parts is not unique. Without loss of generality (see [3]) we can make it e.g. in such a way that in addition

$$V^\ell_{jk} \in C^4(I\!\!R^n),\ |D^\alpha V^\ell_{jk}(\mathbf{y})| \leq C\,(1+|\mathbf{y}|)^{-1-|\alpha|(\varepsilon+1/2)} \tag{1.5}$$

for $1 \leq |\alpha| \leq 4,\ 0 < \varepsilon < 1/2$, where D^α denotes the derivatives with the usual multi–index notation. We define the interacting Hamiltonian as

$$H = H_0 + V = H_0 + V^s + V^\ell.$$

It is a self–adjoint operator on $D(H) = D(H_0)$. The time evolution of the interacting system is given by e^{-itH}.

We first consider the case when there are no long–range potentials: $V^\ell_{ij} = 0$ for all $i < j$. The wave operators for the free channel (i.e. no bounded subsystems asymptotically) are

$$\Omega_\pm = s - \lim_{t\to\pm\infty} e^{itH}\ e^{-itH_0}. \tag{1.6}$$

It is well known that the strong limits exist. It also follows as a consequence of our estimates (see in particular (4.18)). Moreover, $(\Omega_\pm)^*\,\Omega_\pm = I,\ \ e^{-itH}\,\Omega_\pm = \Omega_\pm\, e^{-itH_0}$, and

$$(\Omega_+ - \Omega_-)\,\Phi = i\int_{-\infty}^{+\infty} dt\ e^{itH}\ (H - H_0)\ e^{-itH_0}\,\Phi$$

for any state $\Phi \in D(H_0)$ for which the integral is well defined. The scattering operator S from the free channel to the free channel is defined as

$$S = (\Omega_+)^*\,\Omega_-, \tag{1.7}$$

and we have that $S - I = (\Omega_+ - \Omega_-)^* \, \Omega_-$. We call S as a mapping from $\mathcal{V}_{SR}$ into the set of bounded operators $\mathcal{L}(\mathcal{H})$, $S = S(V^S)$, the scattering map.

THEOREM 1.1. *The scattering map $S(\,\cdot\,) : \mathcal{V}_{SR} \to \mathcal{L}(\mathcal{H})$ is injective.*

∎

In the general case when the long–range potential is present we assume that a splitting according to (1.2) – (1.5) has been made and is kept fixed. In the free channel the Dollard modified free time evolution is generated by the time–dependent Hamiltonian

$$H^D(t) := H_0 + \sum_{j<k} V^\ell_{jk}(t\,\mathbf{p}_{jk}/\mu_{jk}),$$

where $(\mathbf{p}_{jk}/\mu_{jk}) = (\tilde{\mathbf{p}}_k/m_k) - (\tilde{\mathbf{p}}_j/m_j)$ is the relative velocity of particles j and k, and $\mu_{jk} = m_j m_k/(m_j + m_k)$ is the reduced mass. The Dollard propagator is a unitary multiplication operator in momentum space that for initial time 0 is given by

$$U^D(t) := e^{-itH_0} \; \exp\Big[- i \sum_{j<k} \int_0^t ds \; V^\ell_{jk}(s\,\mathbf{p}_{jk}/\mu_{jk}) \Big].$$

We omit the fixed initial time in our notation. The modified Dollard wave operators for the free channel are defined as

$$\Omega^D_\pm = s - \lim_{t \to \pm\infty} e^{itH} \; U^D(t). \tag{1.8}$$

It is known that the strong limits exist, and in fact it also follows from our estimates (see (4.18), (4.19)). Moreover, $(\Omega^D_\pm)^* \, \Omega^D_\pm = I$. We define the Dollard scattering operator between the free channels as follows

$$S^D = (\Omega^D_+)^* \, \Omega^D_- = S^D(V^\ell; V^S), \tag{1.9}$$

and as before $S^D - I = (\Omega^D_+ - \Omega^D_-)^* \, \Omega^D_-$.

The only known long–range potential in physics is the Coulomb (or gravitational) potential. If we know the charges q_j of the particles we can describe the long–range part by $V^\ell_{jk}(\mathbf{y}) = q_j q_k (1 + \mathbf{y}^2)^{-1/2}$ and all other effects are of short range. In the general case let $V = \bar{V}^S + \bar{V}^\ell$ be given. If necessary we adapt the splitting in such a way that $V = V^S + V^\ell$

where V^ℓ satisfies in addition (1.5). We then keep this V^ℓ fixed and we use it in the definition of H^D, U^D, $\Omega^D_\pm$, and S^D. In the next Theorem we prove in particular that for a given long–range tail of the potential the short–range part is uniquely determined by the scattering map.

THEOREM 1.2. *Let a long–range potential $V^\ell \in \mathcal{V}_{LR}$ be given (satisfying without loss of generality also (1.5)). Then the scattering map $S^D(V^\ell; \cdot) : \mathcal{V}_{SR} \to \mathcal{L}(\mathcal{H})$ is injective. Moreover, any one of the Dollard scattering operators S^D determines uniquely the total potential $V(\mathbf{x})$.*

■

We prove below Theorems 1.1 and 1.2 with a simple time–dependent geometrical method and we give reconstruction formulae with error terms as well, see Theorems 2.4, 3.5, 4.1, and equations (4.25), (5.14).

Theorem 1.1 was previously proven in the two–body case with stationary methods by Faddeev (1956) [4], Berezanskii (1958) [5], and Saito (1984) [6]. The last author proved it for continuous potentials that satisfy $|V^s(\mathbf{y})| \le C(1+|\mathbf{y}|)^{-1-\varepsilon}$. Also a stationary proof of uniqueness at fixed energy was given by Ramm (1987) [7] and Novikov (1988) [8] for potentials of compact support. It is known that uniqueness at fixed energy fails for our class of short–range potentials (see Chadan and Sabatier (1989) [9], Section XII.7). However, it was proven in Weder (1991) [10], also with stationary methods, that uniqueness is restored if the far out "tail" at infinity of the short–range potential is a priori known. This is similar to the situation in Theorem 1.2 when long–range potentials are present. Moreover, a stationary proof of uniqueness at fixed energy for exponentially decreasing potentials was given in Novikov (1994) [11]. In Wang (1994) [12] a proof of uniqueness is given with stationary methods in the N–body case with regular short–range potentials. They are covered by our method as well. Isozaki and Kitada (1986) [13] studied the two-body case with C^∞ long–range potential. They proved that the potential is uniquely determined by a particular scattering operator with time–independent modifier.

Furthermore, one has studied with stationary methods the question of existence of a potential - the characterization problem - that is to give necessary and sufficient conditions on an operator, S, in order that it is the scattering operator of an unique potential in a given class of functions satisfying appropriate conditions on regularity and decay at infinity. This problem was solved in the two–body, short–range case by means of the Faddeev solutions to the Schrödinger equation (Faddeev (1966) [14], (1976) [15]) the non linear $\bar{\partial}$–equation (Nachman and Ablowitz (1984) [16], Beals and Coiffman (1985) [17]) and the general limiting absorption principle (Weder (1991) [18]). See Novikov and Henkin (1987) [19] where potentials in the Schwartz class are studied and Weder (1991) [20] where potentials in $C_0^\infty(\mathbb{R}^n)$ and in weighted Sobolev spaces are considered and a different proof is given. We do not address the characterization problem here. For complementary information about multidimensional inverse scattering see Weder (1990) [21], Eskin and Ralston (1991) [22], Newton (1989) [23], and Chadan and Sabatier (1989) [9].

The paper is organized as follows. In Section II we obtain our reconstruction formula (Theorem 2.4) and we prove Theorem 1.1 in the two–body, short–range case. These results are generalized to the two–body, long–range case in Section III. Moreover, we obtain our reconstruction formula in the N–body case (Theorem 4.1) and we prove Theorems 1.1 and 1.2 in Section IV. Finally in Section V we obtain our reconstruction formula for the effective multibody potential for two cluster scattering (equation (5.14)). A preliminary version of Theorems 1.1 and 1.2 in the case where the short–range pair potentials are continuous was given in Enss and Weder (1993) [24] in the two–body case and in Enss and Weder (1994) [25] in the N–body case.

II The Two–Body Case with Short–Range Potentials

In this section we consider the two–body case when the long–range part of the potential, $V^\ell(\mathbf{x})$, is zero. The basic kinematical input of our proof is the following propagation estimate of Enss (1983) [26] that expresses in a convenient way the fact that solutions to the free Schrödinger equation have rapid decay away from the classically allowed region. We denote by B_η the open ball of center zero and radius η in $\mathbb{R}^n$ and by m the reduced mass of the two particles.

PROPOSITION 2.1. *For any $f \in C_0^\infty(\mathbb{R}^n)$ with* $\mathrm{supp} f \subset B_{m\eta}$*, for some $\eta > 0$, and any $\ell = 1, 2, 3, \ldots$, there is a constant C_ℓ such that the following estimate is true*

$$\left\| F(\mathbf{x} \in \mathcal{M}')\, e^{-itH_0}\, f(\mathbf{p} - m\mathbf{v})\, F(\mathbf{x} \in \mathcal{M}) \right\| \leq C_\ell\, (1 + r + |t|)^{-\ell} \tag{2.1}$$

for every $\mathbf{v} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and any measurable sets $\mathcal{M}, \mathcal{M}'$ with the property that $r := \mathrm{dist}(\mathcal{M}', \mathcal{M} + \mathbf{v}t) - \eta|t| \geq 0$.

Proof: See Proposition 2.10 of [26]. ■

Under translation in momentum or configuration space generated by $\mathbf{x}$ or $\mathbf{p}$, respectively, we get

$$e^{-im\mathbf{v}\cdot\mathbf{x}}\, f(\mathbf{p})\, e^{im\mathbf{v}\cdot\mathbf{x}} = f(\mathbf{p} + m\mathbf{v}), \tag{2.2}$$

in particular

$$e^{-im\mathbf{v}\cdot\mathbf{x}}\, e^{-iH_0 t}\, e^{im\mathbf{v}\cdot\mathbf{x}} = e^{-i\mathbf{p}\cdot\mathbf{v}t}\, e^{-iH_0 t}\, e^{-imv^2 t/2}, \tag{2.3}$$

where $v := |\mathbf{v}|$, and

$$e^{i\mathbf{p}\cdot\mathbf{v}t}\, f(\mathbf{x})\, e^{-i\mathbf{p}\cdot\mathbf{v}t} = f(\mathbf{x} + \mathbf{v}t). \tag{2.4}$$

The decay condition (1.3) implies integrable decay of

$$\|V^s(\mathbf{x})\, g(\mathbf{p})\, F(|\mathbf{x}| \geq R)\|$$

for all $g \in C_0^\infty(\mathbb{R}^n)$, see e.g. Corollary 2.4 in [26]. Thus condition (2.5) below is satisfied with $\rho = 0$ for any potential in $\mathcal{V}_{SR}$. Larger ρ means faster decay.

LEMMA 2.2. *Suppose that $V^s \in \mathcal{V}_{SR}$ and that for some $0 \leq \rho \leq 1$ and all $g \in C_0^\infty(\mathbb{R}^n)$*

$$(1 + R)^\rho \left\| V^s(\mathbf{x})\, g(\mathbf{p})\, F(|\mathbf{x}| \geq R) \right\| \in L^1\big((0, \infty),\, dR\big). \tag{2.5}$$

Then for any function $f \in C_0^\infty(B_{m\eta})$ there is a function h with $(1 + \tau)^\rho h(\tau) \in L^1\big((0, \infty)\big)$ such that for every $\mathbf{v} \in \mathbb{R}^n$ satisfying $v \geq 4\eta$ we have

$$\left\| V^s(\mathbf{x})\, e^{-itH_0} f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-3/2} \right\| = \left\| V^s(\mathbf{x} + t\mathbf{v})\, e^{-itH_0} f(\mathbf{p})\, (1 + \mathbf{x}^2)^{-3/2} \right\| \leq h(|vt|). \tag{2.6}$$

Proof: Equality of the norms is evident from (2.2) - (2.4). Take $g \in C_0^\infty(B_{m\eta})$ with $g \equiv 1$ on the support of f. Then by (2.2)

$$\left\| V^s(\mathbf{x})\, g(\mathbf{p} - m\mathbf{v}) \right\| = \left\| V^s(\mathbf{x})\, g(\mathbf{p}) \right\|$$

is independent of $\mathbf{v}$. Proposition 2.1 with $\mathcal{M} = \{|\mathbf{x}| \leq |vt|/8\}, \mathcal{M}' = \{|\mathbf{x} - \mathbf{v}t| > |vt|\, 5/8\}$, and $r \geq |vt|/4$ yields

$$\left\| V^s\, g(\mathbf{p} - m\mathbf{v}) \right\| \left\| F(|\mathbf{x} - \mathbf{v}t| > |vt|\, 5/8)\, e^{-itH_0}\, f(\mathbf{p} - m\mathbf{v})\, F(|\mathbf{x}| < |vt|/8)\, (1 + \mathbf{x}^2)^{-3/2} \right\|$$
$$\leq C\, (1 + |vt|/4)^{-3}\,. \tag{2.7}$$

Moreover,

$$\left\| V^s\, g(\mathbf{p} - m\mathbf{v})\, F\, (|\mathbf{x} - \mathbf{v}t| > |vt|\, 5/8)\ e^{-itH_0}\, f(\mathbf{p} - m\mathbf{v})\, F(|\mathbf{x}| \geq |vt|/8)\, (1 + \mathbf{x}^2)^{-3/2} \right\|$$
$$\leq C \left\| F(|\mathbf{x}| \geq |vt|/8)\, (1 + \mathbf{x}^2)^{-3/2} \right\| \leq C\, (1 + |vt|)^{-3}. \tag{2.8}$$

Furthermore,

$$\left\| V^s\, g(\mathbf{p} - m\mathbf{v})\, F(|\mathbf{x} - \mathbf{v}t| \leq |vt|\, 5/8)\, e^{-itH_0}\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-3/2} \right\|$$
$$\leq C \left\| V^s\, g(\mathbf{p})\, F(|\mathbf{x}| \geq |vt|\, 3/8) \right\| =: h_1(|vt|) \tag{2.9}$$

where $(1 + \tau)^\rho\, h_1(\tau) \in L^1\big((0, \infty)\big)$ by (2.5). The Lemma follows from (2.7), (2.8), and (2.9).

■

Let $\Phi_0 \in \mathcal{H}$ be a normalized asymptotic configuration with momentum support in $B_{m\eta}$ and momentum space wave function

$$\hat{\phi}_0 \in C_0^\infty(B_{m\eta}).$$

The boosted configuration

$$\Phi_\mathbf{v} = e^{im\mathbf{v}\cdot\mathbf{x}}\, \Phi_0 \ \Leftrightarrow\ \hat{\phi}_\mathbf{v}(\mathbf{p}) = \hat{\phi}_0(\mathbf{p} - m\mathbf{v}) \tag{2.10}$$

has compact velocity support contained in the open ball around $\mathbf{v}$ with radius η and it satisfies $\|(1 + \mathbf{x}^2)^{3/2}\, \Phi_\mathbf{v}\| \leq$ const uniformly in $\mathbf{v}$. We will use the high velocity limit in an arbitrary fixed direction $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|,\ v = |\mathbf{v}| \to \infty$.

COROLLARY 2.3. *Assume that $V^S \in \mathcal{V}_{SR}$ then uniformly for $t \in I\!\!R$*

$$\left\| (\Omega_\pm - I)\, e^{-itH_0}\, \Phi_{\mathbf{v}} \right\| = O(v^{-1}). \tag{2.11}$$

Proof: It follows from Duhamel's formula that

$$(\Omega_+ - I)\, e^{-itH_0}\, \Phi_{\mathbf{v}} = i \int_0^\infty dt'\, e^{it'H}\, V^S\, e^{-it'H_0}\, e^{-itH_0}\, \Phi_{\mathbf{v}}.$$

Take $f \in C_0^\infty(B_{m\eta})$ such that $f(\mathbf{p} - m\mathbf{v})\, \Phi_{\mathbf{v}} = \Phi_{\mathbf{v}}$. Then

$$\begin{aligned} \left\| (\Omega_+ - I)\, e^{-itH_0}\, \Phi_{\mathbf{v}} \right\| &\leq C \int_{-\infty}^{\infty} dt'' \left\| V^S\, e^{-it''H_0}\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-3/2} \right\| \\ &\leq \frac{C}{v} \int_{-\infty}^{\infty} d(vt'')\; h(|vt''|), \end{aligned} \tag{2.12}$$

and the Corollary follows for Ω_+. The proof for Ω_- is similar. ■

THEOREM 2.4. (Reconstruction Formula)
Suppose that $V^S \in \mathcal{V}_{SR}$ and that (2.5) is satisfied. Then for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$, as in (2.10)

$$v\left(i(S - I)\, \Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}} \right) = \left(\int_{-\infty}^{+\infty} d\tau\, V^S(\mathbf{x} + \tau \hat{\mathbf{v}})\, \Phi_0,\ \Psi_0 \right) + \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1. \end{cases} \tag{2.13}$$

Proof: It follows from the definition of the wave and scattering operators that

$$S - I = (\Omega_+ - \Omega_-)^*\, \Omega_-.$$

Then by Duhamel's formula

$$i(S - I)\, \Phi_{\mathbf{v}} = \int_{-\infty}^{+\infty} dt\, e^{itH_0}\, V^S\, \Omega_-\, e^{-itH_0}\, \Phi_{\mathbf{v}},$$

where we used the intertwining relations $e^{-itH}\, \Omega_\pm = \Omega_\pm\, e^{-itH_0}$. Note that $\Omega_-\, D(H_0) \subset D(H)$, and from this it follows that $\Omega_-\, e^{-itH_0}\, \Phi_{\mathbf{v}} \in D(V^S)$. Then

$$v\left(i(S - I)\, \Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}} \right) = \int_{-\infty}^{\infty} \ell_{\mathbf{v}}(\tau)\, d\tau + R(\mathbf{v}), \tag{2.14}$$

where $\tau = vt$ and the leading and remainder terms, respectively, are

$$\ell_{\mathbf{v}}(\tau) := \left(V^s(\mathbf{x})\, e^{-iH_0\,\tau/v}\, \Phi_{\mathbf{v}},\; e^{-iH_0\,\tau/v}\, \Psi_{\mathbf{v}} \right), \tag{2.15}$$

$$R(\mathbf{v}) := v \int_{-\infty}^{+\infty} dt \left((\Omega_- - I)\, e^{-itH_0}\, \Phi_{\mathbf{v}},\; V^s\, e^{-itH_0}\, \Psi_{\mathbf{v}} \right). \tag{2.16}$$

By Lemma 2.2 and Corollary 2.3 we obtain that

$$|R(\mathbf{v})| \leq C \int_{-\infty}^{+\infty} \left\| V^s\, e^{-itH_0}\, \Phi_{\mathbf{v}} \right\| dt \leq \frac{C}{v} \int_{-\infty}^{+\infty} h(|\tau|)\, d\tau. \tag{2.17}$$

Thus $R(\mathbf{v})$ satisfies the remainder estimate in (2.13) for all $0 \leq \rho \leq 1$. Recalling that $\Phi_{\mathbf{v}} = e^{im\mathbf{v}\cdot\mathbf{x}}\, \Phi_0$ and (2.3), (2.4) a simple calculation gives

$$\ell_{\mathbf{v}}(\tau) = \left(V^s(\mathbf{x} + \tau\hat{\mathbf{v}})\, e^{-iH_0\,\tau/v}\, \Phi_0,\; e^{-iH_0\,\tau/v}\, \Psi_0 \right). \tag{2.18}$$

Note that pointwise in τ

$$\lim_{v\to\infty} \ell_{\mathbf{v}}(\tau) = (V^s(\mathbf{x} + \tau\hat{\mathbf{v}})\, \Phi_0,\; \Psi_0). \tag{2.19}$$

Moreover, Lemma 2.2 gives uniformly in $\mathbf{v}$ the integrable bound

$$|\ell_{\mathbf{v}}(\tau)| \leq C \left\| V^s(\mathbf{x})\, e^{-iH_0\,\tau/v}\, \Phi_{\mathbf{v}} \right\| \leq C \left\| V^s(\mathbf{x})\, e^{-iH_0\,\tau/v}\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-3/2} \right\| \leq C\, h(|\tau|). \tag{2.20}$$

Then (2.13) in the case $\rho = 0$ follows from the dominated convergence theorem. Moreover,

$$\ell_{\mathbf{v}}(\tau) - (V^s(\mathbf{x} + \tau\hat{\mathbf{v}})\, \Phi_0,\; \Psi_0) = \ell^{(1)}_{\mathbf{v}}(\tau) + \ell^{(2)}_{\mathbf{v}}(\tau) \tag{2.21}$$

where

$$\ell^{(1)}_{\mathbf{v}}(\tau) := \left(V^s(\mathbf{x} + \tau\hat{\mathbf{v}})\, e^{-iH_0\,\tau/v}\, \Phi_0,\; (e^{-iH_0\,\tau/v} - I)\, \Psi_0 \right), \tag{2.22}$$

$$\ell^{(2)}_{\mathbf{v}}(\tau) := \left((e^{-iH_0\,\tau/v} - I)\, \Phi_0,\; V^s(\mathbf{x} + \tau\hat{\mathbf{v}})\, \Psi_0 \right). \tag{2.23}$$

Since $\hat{\psi}_0$ has compact support, $\left\| (e^{-iH_0\,\tau/v} - I)\, \Psi_0 \right\| \leq \|H_0\, \Psi_0\|\, |\tau|/v$ and, moreover, $\left\| (e^{-iH_0\,\tau/v} - I)\, \Psi_0 \right\| \leq 2$. It follows that

$$\left\| (e^{-iH_0\,\tau/v} - I)\, \Psi_0 \right\| \leq C \left(\frac{|\tau|}{v} \right)^{\rho}, \tag{2.24}$$

for all $0 \leq \rho \leq 1$. Then by Lemma 2.2

$$|\ell_{\mathbf{v}}^{(1)}(\tau)| \leq \frac{C}{v^\rho} |\tau|^\rho \, h(|\tau|), \tag{2.25}$$

and since for $0 \leq \rho < 1$, $\lim\limits_{v \to \infty} v^\rho \ell_{\mathbf{v}}^{(1)}(\tau) = 0$, it follows from dominated convergence that

$$\int_{-\infty}^{+\infty} \ell_{\mathbf{v}}^{(1)}(\tau) \, d\tau = \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1. \end{cases} \tag{2.26}$$

Moreover, denoting $\hat{x} = \mathbf{x} \cdot \hat{\mathbf{v}}$

$$\begin{aligned} &\int_{-\infty}^{+\infty} (1 + |\tau|)^\rho \left\| V^s(\mathbf{x} + \tau \hat{\mathbf{v}}) \, \Psi_0 \right\| d\tau \\ &\leq \int_{-\infty}^{+\infty} (1 + |\tau|)^\rho \Big[\left\| V^s(\mathbf{x} + \tau \hat{\mathbf{v}}) \, g(\mathbf{p}) \, F(|\hat{x} + \tau| \geq |\tau|/2) \right\| \\ &\qquad + \left\| V^s(\mathbf{x} + \tau \hat{\mathbf{v}}) \, g(\mathbf{p}) \right\| \left\| F(|\hat{x} + \tau| < |\tau|/2) \, \Psi_0 \right\| \Big] d\tau < \infty, \end{aligned} \tag{2.27}$$

because of (2.5) and

$$(1 + \tau)^\rho \left\| F(|\hat{x}| > |\tau|/2) \, \Psi_0 \right\| \in L^1\big((0, \infty)\big). \tag{2.28}$$

Then we show as in the proof of (2.26) that

$$\int_{-\infty}^{+\infty} \ell_{\mathbf{v}}^{(2)}(\tau) d\tau = \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1, \end{cases} \tag{2.29}$$

and this completes the proof of Theorem 2.4.

■

Proof of Theorem 1.1 in the Two–Body Case:

Let $V^{S,1}(\mathbf{x})$ and $V^{S,2}(\mathbf{x})$ be any two potentials in $\mathcal{V}_{SR}$ which have the same scattering operator. Denote $V(\mathbf{x}) = V^{S,1}(\mathbf{x}) - V^{S,2}(\mathbf{x})$. We identify any $\mathbf{z} = (z_1, z_2) \in I\!R^2$ with the vector $(z_1, z_2, 0, \ldots, 0) \in I\!R^n$ (or any other two–dimensional plane). Then for any normalized Φ, Ψ with $\hat{\phi}$, $\hat{\psi} \in C_0^\infty(I\!R^n)$ we define by

$$\Phi(\mathbf{z}) = e^{-i\mathbf{p} \cdot \mathbf{z}} \, \Phi, \;\; \Psi(\mathbf{z}) = e^{-i\mathbf{p} \cdot \mathbf{z}} \, \Psi,$$

the states translated by $\mathbf{z}$, and

$$f(\mathbf{z}) := (V\,\Phi(\mathbf{z}),\ \Psi(\mathbf{z})). \tag{2.30}$$

Note that $f(\mathbf{z})$ is a continuous and bounded function. Moreover, with $g(\mathbf{p})\,\Phi = \Phi$

$$\begin{aligned} &|f(\mathbf{z})| \leq \|V\ g(\mathbf{p})\ \Phi(\mathbf{z})\| \\ &\leq \left\|V\ g(\mathbf{p})\ F\left(|\mathbf{x}| > |\mathbf{z}|/2\right)\right\| + \left\|V\ g(\mathbf{p})\right\|\ \left\|F\left(|\mathbf{x}| < |\mathbf{z}|/2\right) e^{-i\mathbf{p}\cdot\mathbf{z}}\Phi\right\| \in L^2(I\!\!R^2, d\mathbf{z}), \end{aligned} \tag{2.31}$$

where we used the rapid decay of Φ, the short–range condition (1.3), and boundedness. The Radon transform of $f(\mathbf{z})$ is given for $\hat{\mathbf{v}}$ in the $\mathbf{z}$-plane by

$$\tilde{f}(\hat{\mathbf{v}};\,\mathbf{z}) := \int_{-\infty}^{+\infty} f(\mathbf{z}+\tau\hat{\mathbf{v}})\,d\tau = \int_{-\infty}^{+\infty} \Big(V(\mathbf{x}+\tau\hat{\mathbf{v}})\ \Phi(\mathbf{z}),\ \Psi(\mathbf{z})\Big)\,d\tau \equiv 0 \tag{2.32}$$

by Theorem 2.4. But since $f \in L^2(I\!\!R^2, d\mathbf{z})$ it follows from a classical result on the Radon transform (see Theorem 2.17 in Chapter I of Helgason [27]) that $f(\mathbf{z}) \equiv 0$. This implies that $f(0) = (V(\mathbf{x})\,\Phi,\ \Psi) = 0$ for all $\Phi,\ \Psi$ with $\hat{\phi},\ \hat{\psi} \in C_0^\infty(I\!\!R^n)$, i.e. $V(\mathbf{x}) = 0$ as an operator and also $V(\mathbf{x}) = 0$ *a.e.* as a function. ■

The proof of Theorem 1.1 given above shows that for the reconstruction of the potential it is enough to consider in the left hand side of (2.13) only $\mathbf{v}$ in a two dimensional subspace. The problem is reduced to the inversion of the Radon transform of a bounded, continuous, square integrable function on $I\!\!R^2$.

III The Two–Body Case with Long–Range Potentials

We first prove some results that we will need later. The correction in the Dollard propagator $U^D(t)$ is

$$\tilde{U}(t) := \exp\left[-i\int_0^t V^\ell(s\,\mathbf{p}/m)\,ds\right]. \tag{3.1}$$

PROPOSITION 3.1. *For every $f \in C_0^\infty(B_{m\eta})$ and $\mathbf{v} \in I\!\!R^n$ with $v \geq 2\eta$*

$$\left\|\mathbf{x}\,\tilde{U}(t)\,f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-1/2}\right\| \leq C\,(1+v^{-2}\,|vt|^{2-\gamma}) \leq C\,(1+|vt|^{2-\gamma}) \tag{3.2}$$

and for any $\kappa > 0$

$$\left\| F(|\mathbf{x}| \geq \kappa |vt|)\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-2} \right\| \leq C\, (1 + |vt|)^{-2-\varepsilon}, \tag{3.3}$$

for some $\varepsilon > 0$.

Proof: Since $\mathbf{x} = i\, \partial/\partial \mathbf{p}$

$$\begin{aligned} &\left\| \mathbf{x}\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-1/2} \right\| \\ \leq &\left\| \int_0^t (\nabla V^\ell)(s\, \mathbf{p}/m)\, \frac{s}{m}\, ds\, f(\mathbf{p} - m\mathbf{v}) \right\| + \left\| \left[i\, (\nabla f)(\mathbf{p} - m\mathbf{v}) + f(\mathbf{p} - m\mathbf{v})\, \mathbf{x} \right] (1 + \mathbf{x}^2)^{-1/2} \right\|. \end{aligned}$$

Then (3.2) follows from (1.4) noting that $|\mathbf{p}| > m\, v/2$ in the support of $f(\mathbf{p} - m\mathbf{v})$. In order to prove (3.3) we observe that

$$\begin{aligned} &\kappa^4\, |vt|^4 \left\| F(|\mathbf{x}| > \kappa |vt|)\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-2} \right\| \\ &\leq \left\| |\mathbf{x}|^4\, F(|\mathbf{x}| > \kappa |vt|)\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-2} \right\| \\ &\leq \left\| |\mathbf{x}|^4\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-2} \right\|. \end{aligned} \tag{3.4}$$

The norm of $|\mathbf{x}|^4\, \tilde{U}(t)\, f(\mathbf{p} - m\mathbf{v})\, (1 + \mathbf{x}^2)^{-2}$ is bounded by a finite sum of terms of the form

$$C \prod_{j=1}^{4} I_{\alpha_j},$$

where $I_{\alpha_j} = 1$ if the multiindex $\alpha_j = 0$ and if $\alpha_j \neq 0$ we have

$$I_{\alpha_j} = \left\| \int_0^t (D^{\alpha_j}\, V)(s\, \mathbf{p}/m)\, |s|^{|\alpha_j|}\, ds\, g(\mathbf{p} - m\mathbf{v}) \right\|,$$

where $g \in C_0^\infty(B_{m\eta})$ and $g \equiv 1$ on the support of f and, moreover, $\sum_{j=1}^4 |\alpha_j| \leq 4$. But since $|\mathbf{p}| > m\, v/2$ for $\mathbf{p}$ in the support of $g(\mathbf{p} - m\mathbf{v})$ it follows from (1.5) that for some $\varepsilon > 0$

$$I_{\alpha_j} \leq C\, (1 + |vt|)^{|\alpha_j|(-\varepsilon + 1/2)},$$

and then $C \prod_{j=1}^4 I_{\alpha_j} \leq C\, (1 + |vt|)^{2 - 4\varepsilon}$. Finally it follows from this estimate of (3.4) and the boundedness of the norm that (3.3) is satisfied. ■

LEMMA 3.2. *Suppose that $V^s \in \mathcal{V}_{SR}$, $V^\ell \in \mathcal{V}_{LR}$, and that (2.5) is satisfied. Then for $f \in C_0^\infty(B_{m\eta})$ there is a function h with $(1+\tau)^\rho h(\tau) \in L^1\big((0,\infty)\big)$ such that for any $\mathbf{v} \in I\!R^n$ satisfying $v \geq 4\eta$ we have that*

$$\left\| V^s(\mathbf{x})\, U^D(t)\, f(\mathbf{p} - m\mathbf{v})\, (1+\mathbf{x}^2)^{-2} \right\| \leq h(|vt|). \tag{3.5}$$

Proof: Take $g \in C_0^\infty(B_{m\eta})$ with $g \equiv 1$ on the support of f. Then we prove as in (2.7) that

$$\begin{aligned} &\Big\| V^s(\mathbf{x})\, g(\mathbf{p}-m\mathbf{v})\, F(|\mathbf{x}-\mathbf{v}t| > |vt|\, 5/8)\, e^{-itH_0}\, g(\mathbf{p}-m\mathbf{v})\, F(|\mathbf{x}| < |vt|/8) \\ &\qquad \times \tilde{U}(t)\, f(\mathbf{p}-m\mathbf{v})\, (1+\mathbf{x}^2)^{-2} \Big\| \leq C\, (1+|vt|/4)^{-3}. \end{aligned} \tag{3.6}$$

Moreover, by (3.3)

$$\begin{aligned} &\Big\| V^s(\mathbf{x})\, g(\mathbf{p}-m\mathbf{v})\, F(|\mathbf{x}-\mathbf{v}t| > |vt|\, 5/8)\, e^{-itH_0}\, g(\mathbf{p}-m\mathbf{v}) \\ &\qquad \times F(|x| \geq |vt|/8)\, \tilde{U}(t)\, f(\mathbf{p}-m\mathbf{v})\, (1+\mathbf{x}^2)^{-2} \Big\| \\ &\leq C \left\| F(|\mathbf{x}| \geq |vt|/8)\, \tilde{U}(t)\, f(\mathbf{p}-m\mathbf{v})\, (1+\mathbf{x}^2)^{-2} \right\| \leq C\, (1+|vt|)^{-2-\varepsilon}. \end{aligned} \tag{3.7}$$

Finally, as in (2.9) we prove that

$$\left\| V^s(\mathbf{x})\, g(\mathbf{p}-m\mathbf{v})\, F(|\mathbf{x}-\mathbf{v}t| \leq |vt|\, 5/8)\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\, (1+\mathbf{x}^2)^{-2} \right\| \leq h_1(|vt|), \tag{3.8}$$

with $(1+\tau)^\rho\, h_1(\tau) \in L^1\big((0,\infty)\big)$. The Lemma follows from (3.6)–(3.8)

■

For the proof of Lemma 3.3 we need the following formula given in [28].

$$\begin{aligned} &V(\mathbf{x} + t\,\mathbf{p}/m) - V(t\,\mathbf{p}/m) \\ &= \int_0^1 \Big[(\nabla V)(s\,\mathbf{x} + t\,\mathbf{p}/m) \cdot \mathbf{x} + \frac{it}{2m}\, (\Delta V)(s\,\mathbf{x} + t\,\mathbf{p}/m) \Big]\, ds \end{aligned} \tag{3.9}$$

where $V(\mathbf{x})$ and its first and second order derivatives are continuous and decay towards infinity. The simple proof follows by using the Fourier representation and the Baker–Campbell–Hausdorff formula $\exp[i\,q \cdot (\mathbf{x}\,s + \mathbf{p}\,t/m)] = \exp[i\,q \cdot \mathbf{x}\,s]\ \exp[i\,q \cdot \mathbf{p}\,t/m]\ \exp[i\,st\,q^2/2m]$ for all $q \in I\!R^n$.

LEMMA 3.3. *Suppose that $V^\ell \in \mathcal{V}_{LR}$. Then for $f \in C_0^\infty(B_{m\eta})$*

$$\left\|\Big(V^\ell(\mathbf{x}) - V^\ell(t\,\mathbf{p}/m)\Big)\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \le C\,(1+|vt|)^{-1-\varepsilon} \tag{3.10}$$

for every $\mathbf{v} \in I\!\!R^n$ with $v \ge 4\eta$ and some $\varepsilon > 0$.

Proof: We first split the long–range potential V^ℓ into an inner part and an outer part with easily controllable decay properties. Let $\chi \in C^\infty$ satisfy, $0 \le \chi \le 1$, $\chi(\lambda) = 1$ for $\lambda \ge 1/2$, and $\chi(\lambda) = 0$ for $\lambda \le 1/4$; and define $V^\ell_{vt}(\mathbf{x}) = V^\ell(\mathbf{x})\,\chi(|\mathbf{x}/vt|)$.
Then $\operatorname{supp}(V^\ell - V^\ell_{vt}) \subset B_{|vt|/2}$, $\|V^\ell - V^\ell_{vt}\| \le \|V^\ell\|$, and it follows from (1.4) and (1.5) that

$$\|\nabla V^\ell_{vt}\| = \sup_{\mathbf{x}} |(\nabla V^\ell_{vt})(\mathbf{x})| \le C\,|vt|^{-\gamma}, \gamma > 3/2, \tag{3.11}$$

$$\|\Delta V^\ell_{vt}\| \le C\,|vt|^{-(2+\varepsilon)},\ \varepsilon > 0. \tag{3.12}$$

We estimate the norm in (3.10) as follows

$$\begin{aligned} &\left\|\Big(V^\ell(\mathbf{x}) - V^\ell(t\,\mathbf{p}/m)\Big)\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ \le\ &\left\|\Big(V^\ell - V^\ell_{vt}\Big)(\mathbf{x})\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ &+\left\|\Big(V^\ell_{vt}(\mathbf{x}) - V^\ell_{vt}(t\,\mathbf{p}/m)\Big)\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\|, \end{aligned} \tag{3.13}$$

where we used that $V^\ell(t\,\mathbf{p}/m) = V^\ell_{vt}(t\,\mathbf{p}/m)$ on the support of $f(\mathbf{p}-m\mathbf{v})$. But with $g \in C_0^\infty(B_{m\eta})$ and $g \equiv 1$ on the support of f

$$\begin{aligned} &\left\|\Big(V^\ell - V^\ell_{vt}\Big)(\mathbf{x})\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ \le\ & C\left\|F(|\mathbf{x}-\mathbf{v}t| \ge |vt|/2)\, e^{-itH_0}\, g(\mathbf{p}-m\mathbf{v})\, F(|\mathbf{x}| < |vt|/8)\right\| \\ &+C\left\|F(|\mathbf{x}| \ge |vt|/8)\, \tilde U(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \le C\,(1+|vt|)^{-2-\varepsilon}, \end{aligned} \tag{3.14}$$

where in the last inequality we used Proposition 2.1 with $r = ((3v/8)-\eta)\,t \ge |vt|/8$ and (3.3). Moreover, since $e^{itH_0}V(\mathbf{x})e^{-itH_0} = V(\mathbf{x}+t\,\mathbf{p}/m)$ and using (3.9), (3.11), (3.12), and (3.2) we obtain that

$$\begin{aligned} &\left\|\Big(V^\ell_{vt}(\mathbf{x}) - V^\ell_{vt}(t\,\mathbf{p}/m)\Big)\, U^D(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ =\ &\left\|\Big(V^\ell_{vt}(\mathbf{x}+t\,\mathbf{p}/m) - V^\ell_{vt}(t\,\mathbf{p}/m)\Big)\, \tilde U(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ \le\ &\int_0^1 ds\left\|\Big[(\nabla V^\ell_{vt})(s\,\mathbf{x}+t\,\mathbf{p}/m)\cdot\mathbf{x} + \frac{it}{2m}\,(\Delta V^\ell_{vt})(s\,\mathbf{x}+t\,\mathbf{p}/m)\Big]\tilde U(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| \\ \le\ & C\Big[|vt|^{-\gamma}\left\|\mathbf{x}\,\tilde U(t)\, f(\mathbf{p}-m\mathbf{v})\,(1+\mathbf{x}^2)^{-2}\right\| + |vt|^{-1-\varepsilon}\Big] \le C\,|vt|^{-1-\varepsilon}. \end{aligned} \tag{3.15}$$

Actually, we have used only the weaker form of (3.2). Then noting that the left–hand side of (3.10) is uniformly bounded in $|vt|$ the Lemma follows from (3.13)–(3.15). ∎

COROLLARY 3.4. *Suppose that $V^S \in \mathcal{V}_{SR}$, $V^\ell \in \mathcal{V}_{LR}$, then for $\Phi_{\mathbf{v}}$ as in (2.10)*

$$\left\| \left(e^{-itH}\, \Omega_\pm^D - U^D(t)\right) \Phi_{\mathbf{v}} \right\| = O\left(v^{-1}\right), \tag{3.16}$$

uniformly for $t \in I\!\!R$.

Proof:

$$\begin{aligned} e^{-itH}\, \Omega_\pm^D - U^D(t) &= \lim_{t' \to \pm\infty} e^{-itH}\, e^{it'H}\, U^D(t') - U^D(t) \\ &= \lim_{T \to \pm\infty} e^{iTH}\, U^D(T+t) - U^D(t) = i \int_0^{\pm\infty} ds\, e^{isH}\, [H - H^D(s+t)]\, U^D(s+t). \end{aligned}$$

Then by Lemmas 3.2 and 3.3 and since $\Phi_{\mathbf{v}}$ has rapid decay in configuration space

$$\begin{aligned} \sup_{t \in I\!\!R} v \left\| \left(e^{-itH}\, \Omega_\pm^D - U^D(t)\right) \Phi_{\mathbf{v}} \right\| &\leq v \int_{-\infty}^{+\infty} dt \left\| \left(V^S(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(t\,\mathbf{p}/m)\right) U^D(t)\, \Phi_{\mathbf{v}} \right\| \\ &\leq \int_{-\infty}^{+\infty} \left[h(|vt|) + C\,(1+|vt|)^{-1-\varepsilon}\right] d(vt) < \infty, \end{aligned}$$

and the Corrollary follows. ∎

THEOREM 3.5. (Reconstruction Formula)
Suppose that $V^S \in \mathcal{V}_{SR}$, $V^\ell \in \mathcal{V}_{LR}$, and (2.5) is satisfied. Then for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as in (2.10)

$$\begin{aligned} & v\left[\left(i(S^D - I)\, \Phi_{\mathbf{v}},\, \Psi_{\mathbf{v}}\right) - \left(\int_{-\infty}^{+\infty} dt\, \left(V^\ell(\mathbf{x}) - V^\ell(t\,\mathbf{p}/m)\right) U^D(t)\, \Phi_{\mathbf{v}},\, U^D(t)\, \Psi_{\mathbf{v}}\right)\right] \\ &= \left(\int_{-\infty}^{+\infty} d\tau\, V^S(\mathbf{x} + \tau\hat{\mathbf{v}})\, \Phi_0,\, \Psi_0\right) + \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1, \end{cases} \end{aligned} \tag{3.17}$$

and in particular

$$\lim_{v \to \infty} v \left(i(S^D - I)\, \Phi_{\mathbf{v}},\, \Psi_{\mathbf{v}}\right) = \left(\int_{-\infty}^{+\infty} d\tau\, \left[V^S(\mathbf{x} + \tau\hat{\mathbf{v}}) + V^\ell(\mathbf{x} + \tau\hat{\mathbf{v}}) - V^\ell(\tau\hat{\mathbf{v}})\right] \Phi_0,\, \Psi_0\right). \tag{3.18}$$

Proof: We prove as in the short–range case that

$$v\left[\left(i(S^D-I)\,\Phi_{\mathbf v},\ \Psi_{\mathbf v}\right) \ - \ \int_{-\infty}^{+\infty} dt\,\left(\left(V^\ell(\mathbf x)-V^\ell(t\,\mathbf p/m)\right)\,U^D(t)\,\Phi_{\mathbf v},\ U^D(t)\,\Psi_{\mathbf v}\right)\right]$$
$$= \int_{-\infty}^{+\infty} \ell_{\mathbf v}(\tau)\,d\tau + R(\mathbf v),$$

where $\tau = vt$ and

$$\ell_{\mathbf v}(\tau) := (V^S(\mathbf x)\,U^D(\tau/v)\,\Phi_{\mathbf v},\ U^D(\tau/v)\,\Psi_{\mathbf v})\,, \tag{3.19}$$

$$R(\mathbf v) := v\int_{-\infty}^{+\infty}\left(\left(e^{-itH}\,\Omega_-^D - U^D(t)\right)\,\Phi_{\mathbf v},\ \left(V^S(\mathbf x)+V^\ell(\mathbf x)-V^\ell(t\,\mathbf p/m)\right)\,U^D(t)\,\Psi_{\mathbf v}\right)\,dt.$$

By Lemmas 3.2, 3.3, Corollary 3.4 and since $\Phi_{\mathbf v},\ \Psi_{\mathbf v}$ have rapid decay in configuration space

$$R(\mathbf v) = O\left(v^{-1}\right).$$

Moreover, by (2.3), (2.4) we obtain analogous to (2.18) that

$$l_{\mathbf v}(\tau) = \left(V^S(\mathbf x+\tau\hat{\mathbf v})\,e^{-iH_0\,\tau/v}\,\tilde U(\mathbf v,\tau/v)\,\Phi_0,\ e^{-iH_0\,\tau/v}\,\tilde U(\mathbf v,\tau/v)\,\Psi_0\right),$$

where

$$\tilde U(\mathbf v,t) := \exp\left[-i\int_0^t ds\,V^\ell(s\,\mathbf v + s\,\mathbf p/m)\right]. \tag{3.20}$$

Note that $s-\lim_{v\to\infty}\tilde U(\mathbf v,\tau/v) = I$ for fixed τ. Then pointwise in τ

$$\lim_{v\to\infty} l_{\mathbf v}(\tau) = (V^S(\mathbf x+\tau\hat{\mathbf v})\,\Phi_0,\ \Psi_0),$$

and Lemma 3.2 gives an integrable bound uniformly in $\mathbf v$. We complete the proof of (3.17) as in the short–range case (Theorem 2.4). Moreover, by (2.2)–(2.4),

$$v\int_{-\infty}^{+\infty} dt\,\left(\left(V^\ell(\mathbf x)-V^\ell(t\,\mathbf p/m)\right)\,U^D(t)\,\Phi_{\mathbf v},\ U^D(t)\,\Psi_{\mathbf v}\right)$$

$$= \int_{-\infty}^{+\infty} d\tau\left(\left(V^\ell(\mathbf x+\tau\hat{\mathbf v}) - V^\ell(\tau(\mathbf p/mv)+\tau\hat{\mathbf v})\right)e^{-iH_0\tau/v}\,\tilde U(\mathbf v,\tau/v)\,\Phi_0,\ e^{-iH_0\tau/v}\,\tilde U(\mathbf v,\tau/v)\,\Psi_0\right).$$

But $s-\lim_{v\to\infty} V^\ell(\tau\,(\mathbf p/mv)+\tau\hat{\mathbf v}) = V^\ell(\tau\hat{\mathbf v})$ for fixed τ. Then by Lemma 3.3

$$\lim_{v\to\infty} v\int_{-\infty}^{+\infty} dt\,\left(\left(V^\ell(\mathbf x)-V^\ell(t\,\mathbf p/m)\right)\,U^D(t)\,\Phi_{\mathbf v},\ U^D(t)\,\Psi_{\mathbf v}\right)$$
$$= \int_{-\infty}^{+\infty} d\tau\,\left(\left(V^\ell(\mathbf x+\tau\hat{\mathbf v}) - V^\ell(\tau\hat{\mathbf v})\right)\,\Phi_0,\ \Psi_0\right),$$

and it follows from (3.17) that (3.18) is satisfied.

■

Proof of Theorem 1.2 in the Two–Body Case: The fact that the scattering map is injective follows from (3.18) as in the two–body, short–range case.

We now prove that (3.18) allows us to uniquely reconstruct the total potential $V(\mathbf{x})$ from any one of the Dollard scattering operators S^D. We consider as in Weder (1995) [29] the high velocity limit of the commutator of S^D with a component of $\mathbf{p}$, say p_1.

$$\lim_{v\to\infty} iv\,([S^D,\, p_1]\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}}) = \lim_{v\to\infty} iv\,([S^D,\, p_1 - mv_1]\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}})$$

$$= \int_{-\infty}^{+\infty} d\tau\ \left\{(V^S(\mathbf{x}+\tau\hat{\mathbf{v}})\,p_1\,\Phi_0,\ \Psi_0)\ -\ (V^S(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_0,\ p_1\,\Psi_0)\right\}$$

$$+i\int_{-\infty}^{+\infty} d\tau\ \Big(\,((\partial_1 V^\ell)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_0,\ \Psi_0\Big)\,, \tag{3.21}$$

where we used (3.18) and that $(p_1 - mv_1)\Phi_{\mathbf{v}} = (p_1\Phi)_{\mathbf{v}}$. For any pair of states $\Phi(\mathbf{z}),\ \Psi(\mathbf{z})$, as defined in the proof for the two–body case of Theorem 1.1 we denote

$$f(\mathbf{z}) := (V^S\,p_1\,\Phi(\mathbf{z}),\ \Psi(\mathbf{z})) - (V^S\,\Phi(\mathbf{z}),\ p_1\,\Psi(\mathbf{z})) + i\,\Big((\partial_1 V^\ell)\,\Phi(\mathbf{z}),\ \Psi(\mathbf{z})\Big)\,. \tag{3.22}$$

$f(\mathbf{z})$ is a bounded continuous function in $L^2(\mathbb{R}^2)$ and by (3.21) its Radon transform for $\hat{\mathbf{v}}$ in the $\mathbf{z}$-plane (which contains the 1-direction if p_1 is chosen) is given by

$$\tilde{f}(\hat{\mathbf{v}};\mathbf{z}) = \int_{-\infty}^{+\infty} d\tau\ f(\mathbf{z}+\hat{\mathbf{v}}\tau) = \lim_{v\to\infty} iv\,([S^D,\, p_1]\,\Phi_{\mathbf{v}}(\mathbf{z}),\ \Psi_{\mathbf{v}}(\mathbf{z}))\,. \tag{3.23}$$

Then from the high speed limit of $[S^D, p_1]$ we obtain $\tilde{f}(\hat{\mathbf{v}};\mathbf{z})$ and inverting the Radon transform we compute $f(\mathbf{z})$. But

$$\frac{\partial}{\partial z_1}(V\,\Phi(\mathbf{z}),\ \Psi(\mathbf{z})) = -if(\mathbf{z}),$$

and since by (1.3), (1.4) the function $f(\mathbf{z})$ is integrable along any line and $\lim_{\mathbf{z}\to\infty}(V\,\Phi(\mathbf{z}),\ \Psi(\mathbf{z})\,) = 0$, we have that

$$(V\,\Phi,\ \Psi) = i\int_0^\infty f(z_1,0)\,dz_1.$$

■

IV The N–Body Case

For any given pair out of the N particles we construct a sequence of states where all particles have high velocity relative to each other. The properly rescaled deviation of the scattering operator from the identity gives the Radon or X–ray transform of the corresponding pair potential or of its short–range part. This implies the uniqueness theorems given in the introduction. We choose the family of states here by mathematical convenience. Other choices are also possible. We have to introduce some kinematical notation before we define the states. In order to determine the potential for a given pair we use a numbering of the particles such that the pair of interest consists of particles 1 and 2. Once we have separated out the motion of the center of mass we use various coordinates that describe the relative positions and momenta of the particles. As it is usual for two–body systems one of the n–dimensional variables is the relative position $\mathbf{x}$ of the distinguished pair with its conjugate momentum $\mathbf{p}$.

$$\mathbf{x} := \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \quad \mathbf{p} = -i\nabla_{\mathbf{x}} = m[(-i\nabla_{\tilde{\mathbf{x}}_2}/m_2) - (-i\nabla_{\tilde{\mathbf{x}}_1}/m_1)],$$

where $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the pair (1 2). In addition, we use the position $\mathbf{x}_j$ and the momentum $\mathbf{p}_j$ of the $j - th$ particle $j = 1, \ldots, N$, relative to the center of mass of the distinguished pair

$$\mathbf{x}_j := \tilde{\mathbf{x}}_j - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1 + m_2), \quad j = 1, \ldots, N, \tag{4.1}$$

$$\mathbf{p}_j = \mu_j(\tilde{\mathbf{p}}_j/m_j - (\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2)/(m_1 + m_2)), \quad j = 1, \ldots, N, \tag{4.2}$$

where μ_j is the reduced mass of the $j - th$ particle with respect to the center of mass of the distinguished pair,

$$\mu_j = \frac{m_j(m_1 + m_2)}{(m_j + m_1 + m_2)}, \quad j = 1, \ldots, N, \tag{4.3}$$

and $\tilde{\mathbf{p}}_j = -i\nabla_{\tilde{\mathbf{x}}_j}$, $j = 1, \ldots, N$ is the momentum relative to some origin. $\mathbf{p}_j/\mu_j$, $j = 1, \ldots, N$ are the relative velocities with respect to the center of mass of the pair. $\{\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N\}$ and $\{\mathbf{p}, \mathbf{p}_3, \ldots, \mathbf{p}_N\}$ are sets of $N - 1$ independent n–dimensional configuration and momentum coordinates in the total center of mass frame. The relative momentum of particle j and k is

$$\mathbf{p}_{jk} = -i\nabla_{(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)}, \quad j, k = 1, 2, \ldots, N, \tag{4.4}$$

where in the derivative the positions of all other particles as well as of the center of mass are kept fixed. Their relative velocity is

$$\frac{\mathbf{p}_{jk}}{\mu_{jk}} = \frac{\tilde{\mathbf{p}}_k}{m_k} - \frac{\tilde{\mathbf{p}}_j}{m_j} = \frac{\mathbf{p}_k}{\mu_k} - \frac{\mathbf{p}_j}{\mu_j}, \quad j,k = 1,\ldots,N, \tag{4.5}$$

where μ_{jk} is the reduced mass of the pair j,k

$$\mu_{jk} = \frac{m_j m_k}{m_j + m_k}, \quad j,k = 1,2,\ldots,N. \tag{4.6}$$

Note that the coordinates $(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)$ and $\mathbf{p}_{jk}$ are conjugate to each other.

In order to construct our high velocity states we choose asymptotic configurations $\Phi_0 \in \mathcal{H}$ with product wave functions of the following form in momentum space

$$\Phi_0 \sim \hat{\phi}_{12}(\mathbf{p})\, \hat{\phi}_3(\mathbf{p}_3,\ldots,\mathbf{p}_{N}) \tag{4.7}$$

where $\hat{\phi}_{12} \in C_0^\infty(I\!R^n)$ varies while $\hat{\phi}_3 \in C_0^\infty(I\!R^{n(N-2)})$ is a fixed normalized function with support in $\{ (\mathbf{p}_3,\ldots,\mathbf{p}_{N}) : |\mathbf{p}_j| < \mu_j \}$ i.e. particles 3 to N have speed smaller than one relative to the pair (1 2). Take η such that the velocity support of $\hat{\phi}_{12}$ is contained in $B_{m\eta}$, i.e. $\hat{\phi} \in C_0^\infty(B_{m\eta})$. We define our high velocity states as

$$\Phi_{\mathbf{v}} \sim \hat{\phi}_{12}(\mathbf{p} - m\mathbf{v})\, \hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_{N} - \mu_{N}\mathbf{v}_{N}), \tag{4.8}$$

with $\mathbf{v} = v\hat{\mathbf{v}},\ |\hat{\mathbf{v}}| = 1,\ \mathbf{v}_j = v^2\,\mathbf{e}_j$, with $\mathbf{e}_j \neq 0$ for $j = 3,\ldots,N$, and where we assume that $\mathbf{e}_j - \mathbf{e}_k \neq 0$ for $j,k = 3,\ldots,N$. We also define $\mathbf{v}_1 = -\mathbf{v}\, m/m_1$, and $\mathbf{v}_2 = \mathbf{v}\, m/m_2$. Then denoting by

$$\mathbf{v}_{jk} = \mathbf{v}_k - \mathbf{v}_j, \quad v_{jk} = |\mathbf{v}_{jk}|, \quad j,k = 1,\ldots,N \tag{4.9}$$

the approximate relative velocities of particles j and k we have that

$$\begin{aligned}
\mathbf{v}_{1j} &= v^2\left(\mathbf{e}_j + \frac{m}{m_1}\frac{\hat{\mathbf{v}}}{v}\right) \neq 0 \text{ if } v > \frac{m}{m_1}(e_j)^{-1} \\
\mathbf{v}_{2j} &= v^2\left(\mathbf{e}_j - \frac{m}{m_2}\frac{\hat{\mathbf{v}}}{v}\right) \neq 0 \text{ if } v > \frac{m}{m_2}(e_j)^{-1} \\
\mathbf{v}_{jk} &= v^2(\mathbf{e}_k - \mathbf{e}_j) \neq 0, \quad j,k = 3,\ldots,N.
\end{aligned}$$

So in our high velocity states the relative average velocity of the distinguished pair is v while all other particles travel with minimal velocity proportional to v^2 with respect to the particles in the distinguished pair as well as relative to each other. Note that $\Phi_0 \in \mathcal{S}(I\!R^{n(N-1)})$ and that

$$\Phi_{\mathbf{v}} = e^{im\mathbf{v}\cdot\mathbf{x}} \prod_{j=3}^{N} e^{i\mu_j \mathbf{v}_j\cdot\mathbf{x}_j} \Phi_0. \tag{4.10}$$

Moreover, by (4.1)

$$|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j| = |\mathbf{x}_k - \mathbf{x}_j| \le |\mathbf{x}_k| + |\mathbf{x}_j|, \quad j,k = 1,\ldots,N$$

and $|\mathbf{x}_1|$, $|\mathbf{x}_2| \le |\mathbf{x}|$, thus we have good initial localization uniformly in $\mathbf{v}$

$$\left\|(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^2\, \Psi_{\mathbf{v}}\right\| \le C, \quad j,k = 1\ldots,N. \tag{4.11}$$

It follows from (4.5) that there are functions $f_{jk} \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ such that

$$\Phi_{\mathbf{v}} = f_{jk}(\mathbf{p}_{jk} - \mu_{jk}\mathbf{v}_{jk})\,\Phi_{\mathbf{v}}, \tag{4.12}$$

where $\eta_{12} = \eta$, $\eta_{1j} = 2(1+\eta\, m/m_1)$, $\eta_{2j} = 2(1+\eta\, m/m_2)$, $j = 3,\ldots,N$, and $\eta_{jk} = 4$ for $j,k = 3,\ldots,N$. (4.11) and (4.12) allow us to reduce in an effective way the proofs in the N–body case to the ones for two particles. Let again $\tilde{U}$ be the Dollard correction

$$\tilde{U}(t) := \exp\Big[-i\int_0^t ds \sum_{j<k} V_{jk}^{\ell}(s\,\mathbf{p}_{jk}/\mu_{jk})\Big]. \tag{4.13}$$

Note that by (4.1)

$$\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial\mathbf{p}}, \quad \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j = i\frac{\partial}{\partial\mathbf{p}_k} - i\frac{\partial}{\partial\mathbf{p}_j}, \quad j,k = 3,\ldots,N, \tag{4.14}$$

$$\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial\mathbf{p}_j} + \frac{m}{m_1} i\frac{\partial}{\partial\mathbf{p}}, \quad \tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_2 = i\frac{\partial}{\partial\mathbf{p}_j} - \frac{m}{m_2} i\frac{\partial}{\partial\mathbf{p}}, \quad j = 3,\ldots,N. \tag{4.15}$$

Take v_0 so large that $|\mathbf{p}_{jk}| \ge v_{jk}\,\mu_{jk}/2$ in the support of $f_{jk}(\mathbf{p}_{jk} - \mu_{jk}\,\mathbf{v}_{jk})$ for $v \ge v_0$. Then as in Proposition 3.1 we prove that for $v \ge v_0, j<k$

$$\begin{aligned}
&\left\|(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\,\tilde{U}(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\,\mathbf{v}_{j'k'})\,(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-1/2}\right\| \\
&\le C\begin{cases}(1+v^{-2}\,|vt|^{2-\gamma}), & \text{if } \ j = 1,2, \\[2ex] (1+v_{jk}^{-2}\,|v_{jk}\,t|^{2-\gamma}), & j = 3,4,\ldots,N,\end{cases} \\
&\le C\,(1+|v_{jk}\,t|^{2-\gamma}),
\end{aligned} \tag{4.16}$$

and for $\kappa > 0$

$$\left\| F(|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j| \geq \kappa |v_{jk}\, t|)\, \tilde{U}(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\, \mathbf{v}_{j'k'})\, (1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2} \right\|$$
$$\leq C\, (1 + |v_{jk} t|)^{-2-\varepsilon}. \tag{4.17}$$

Then as in Lemma 3.2 we prove that if V^{S}_{ij} satisfies (2.5) as an operator in $L^2(I\!\!R^n)$

$$\left\| V^{s}_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\, U^{D}(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\, \mathbf{v}_{j'k'})\, (1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2} \right\|$$
$$\leq h_{jk}(|v_{jk} t|) \tag{4.18}$$

where $(1+\tau)^{\rho}\, h_{jk}(\tau) \in L^1\big((0,\infty)\big)$. For this purpose note that for any pair (jk), $H_0 = \mathbf{p}^2_{jk}/2\mu_{jk} + \tilde{H}_{jk}$, where $\tilde{H}_{jk}$ commutes with $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j$. This follows for example taking Jacobi coordinates with $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j$ as first coordinate. Moreover, as in Lemma 3.3 we prove that

$$\left\| \Big(V^{\ell}_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V^{\ell}_{jk}(t\, \mathbf{p}_{jk}/\mu_{jk}) \Big)\, U^{D}(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\, \mathbf{v}_{j'k'})\, (1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2} \right\|$$
$$\leq C\, (1 + |v_{jk}\, t|)^{-1-\varepsilon}, \tag{4.19}$$

Then using (4.11) and (4.12) we prove as in Corollary 3.4 that

$$\left\| \left(e^{-itH}\, \Omega^{D}_{\pm} - U^{D}(t) \right) \Phi_{\mathbf{v}} \right\| = O\left(v^{-1} \right), \tag{4.20}$$

uniformly in t. Note that $v_{12} = v$ and $v_{jk} = O(v^2)$ for $j < k = 3, \ldots, N$.

THEOREM 4.1. *Suppose that $V^{S} \in \mathcal{V}_{SR}$, $V^{\ell} \in \mathcal{V}_{LR}$ and that V^{S}_{12} satisfies (2.5). Then for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as in (4.8) with some fixed normalized $\hat{\phi}_3$*

$$v\left[\Big(i(S^{D} - I)\, \Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}} \Big) - \Big(\int_{-\infty}^{+\infty} dt\ \Big(V^{\ell}_{12}(\mathbf{x}) - V^{\ell}_{12}(t\, \mathbf{p}/m) \Big)\, U^{D}(t)\, \Phi_{\mathbf{v}},\ U^{D}(t)\, \Psi_{\mathbf{v}} \Big) \right]$$
$$= \Big(\int_{-\infty}^{+\infty} d\tau\, V^{S}_{12}(\mathbf{x} + \tau \hat{\mathbf{v}})\, \Phi_{12},\ \Psi_{12} \Big) + \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1, \end{cases} \tag{4.21}$$

and in particular for all $0 \leq \rho \leq 1$

$$\lim_{v\to\infty} v \Big(i(S^{D} - I)\, \Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}} \Big) = \Big(\int_{-\infty}^{+\infty} d\tau\ \Big(V^{S}_{12}(\mathbf{x} + \tau \hat{\mathbf{v}}) + V^{\ell}_{12}(\mathbf{x} + \tau \hat{\mathbf{v}}) - V^{\ell}_{12}(\tau \hat{\mathbf{v}}) \Big)\ \Phi_{12},\ \Psi_{12} \Big). \tag{4.22}$$

Proof: As in the proof of Theorem 3.5 for the two–body case we obtain using Duhamel's formula that

$$v\ \left[\Big(i(S^D - I)\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}}\Big) - \int_{-\infty}^{+\infty} dt\,\Big(\Big(V^\ell_{12}(\mathbf{x}) - V^\ell_{12}(t\,\mathbf{p}/m)\Big)\ U^D(t)\,\Phi_{\mathbf{v}},\ U^D(t)\,\Psi_{\mathbf{v}}\Big)\right]$$
$$= \int_{-\infty}^{+\infty} d\tau\ \ell_{\mathbf{v}}(\tau) + R(\mathbf{v}),$$

where $\tau = vt$ and

$$\ell_{\mathbf{v}}(\tau) := \left(V^S_{12}(\mathbf{x})\,U^D(\tau/v)\,\Phi_{\mathbf{v}},\ U^D(\tau/v)\,\Psi_{\mathbf{v}}\right), \tag{4.23}$$

$$\begin{aligned} R(\mathbf{v}) \ :=\ & v \sum_{j<k=3}^{N} \int\limits_{-\infty}^{+\infty} dt\Big(\Big[V^S_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V^\ell_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V^\ell_{jk}(t\,\mathbf{p}_{jk}/\mu_{jk})\Big]\,U^D(t)\,\Phi_{\mathbf{v}},\ U^D(t)\,\Psi_{\mathbf{v}}\Big) \\ & + v \sum_{j<k} \int\limits_{-\infty}^{+\infty} dt\,\Big(\Big(e^{-itH}\,\Omega^D_- - U^D(t)\Big)\,\Phi_{\mathbf{v}}, \\ & \qquad \times \Big[V^S_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V^\ell_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V^\ell_{jk}(t\,\mathbf{p}_{jk}/\mu_{jk})\Big]\ U^D(t)\,\Psi_{\mathbf{v}}\Big). \end{aligned}$$

But by (4.11), (4.12), (4.18) – (4.20) and since $v_{jk} = O(v^2)$ for $j < k = 3, \ldots, N$,

$$R(\mathbf{v}) = O\left(v^{-1}\right). \tag{4.24}$$

At this point we complete the proof of the Theorem as in the two–body case.

■

Proof of Theorems 1.1 and 1.2: Theorem 1.2 follows from (4.22) applied to each pair of particles as in the two–body case. Theorem 1.1 is a particular case of Theorem 1.2.

■

Note that in the special case when there are no long–range forces, *i.e.* $V^\ell_{ij} \equiv 0$, the reconstruction formula (4.21) simplifies to

$$v\,\Big(i(S - I)\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}}\Big) = \Big(\int_{-\infty}^{+\infty} d\tau\ V^S_{12}(\mathbf{x} + \tau\hat{\mathbf{v}})\,\Phi_{12},\ \Psi_{12}\Big) + \begin{cases} o\left(v^{-\rho}\right), & 0 \le \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1. \end{cases} \tag{4.25}$$

If in addition to the pair potentials one has three–body or multiparticle forces then we determine only the two–body potentials with the approximating sequences used so far, the analogous approach using states with three particles having relative velocities of order v and all the others of order v^2 determines then all three–body potentials, and so on. See Wang(1994) [12] for a related approach.

V Two-Cluster Scattering

In this section we illustrate with one example how our method applies to other scattering channels. We consider a two-cluster channel where the first particle is scattered by a target that is a cluster of $N-1$ particles in a bound state. This is a typical situation in scattering experiments. Since in the high speed limit the interaction between the incoming particle and the cluster of $N-1$ particles goes to zero it is natural to consider the channel where the cluster is in the same bound state before and after the scattering takes place. Let us denote by $\mathbf{x}$ the Jacobi coordinate of the first particle relative to the center of mass of the particles $2, \ldots, N$ and by $\mathbf{p}$ the corresponding momentum

$$\mathbf{x} = -(\tilde{\mathbf{x}}_1 - \mathbf{x}_{CM}), \quad \mathbf{p} = -i\nabla_{\mathbf{x}}, \tag{5.1}$$

$$\mathbf{x}_{CM} = \left(\sum_{j=2}^{N} m_j \mathbf{x}_j \right) \left(\sum_{j=2}^{N} m_j \right)^{-1}. \tag{5.2}$$

In these coordinates the Hilbert space of states $\mathcal{H}$ decomposes as

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_{(N-1)}, \tag{5.3}$$

where $\mathcal{H}_0 \cong L^2(I\!R^n)$ and $\mathcal{H}_{(N-1)}$ is the space of the states of the cluster that in the configuration representation is isomorphic to a L^2 space on the relative coordinates of particles 2 to N. For simplicity we assume that there are no long–range forces *i.e.* $V^{\ell}_{jk} = 0,\ j,k = 1, \ldots, N$. Then we define

$$H^{(2)} := H - \sum_{j=2}^{N} V_{1j}(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_1) = T_0 + H^{(N-1)}$$

with

$$T_0 = \mathbf{p}^2/2m, \quad m = m_1 \left(\sum_{j=2}^{N} m_j \right) \left(\sum_{j=1}^{N} m_j \right)^{-1},$$

the kinetic energy of the incoming particle relative to the center of mass of particles $2, \ldots, N$, and $H^{(N-1)}$ the internal Hamiltonian of the cluster of $N-1$ particles. We assume that the $N-1$ particles are in a bound state $\Phi_b \in \mathcal{H}_{(N-1)}$ of energy E and we denote by P_{N-1} the projector onto the corresponding eigenspace $\mathcal{H}_0 \otimes \Phi_b \subset \mathcal{H}$. Then

$$H^{(N-1)}\, P_{N-1} = E\, P_{N-1}. \tag{5.4}$$

The channel wave operators are defined as

$$\Omega_{\pm}^{(2)} := s - \lim_{t\to\pm\infty} e^{itH}\, e^{-itH^{(2)}}\, P_{N-1} = s - \lim_{t\to\pm\infty} e^{itH}\, e^{-it(T_0+E)}\, P_{N-1}. \tag{5.5}$$

As is well known the strong limits exist, and this also follows from the estimate (5.10) below. The channel scattering operator is given by

$$S^{(2)} = (\Omega_{+}^{(2)})^{*}\, \Omega_{-}^{(2)}. \tag{5.6}$$

By the Duhamel formula

$$\left(\Omega_{\pm}^{(2)} - I\right) P_{N-1} = \pm i \int_0^{\pm\infty} dt\, e^{itH} \sum_{j=2}^{N} V_{1j}^{S}(\mathbf{x}+\mathbf{y}_j)\, e^{-itT_0}\, e^{-iEt}\, P_{N-1} \tag{5.7}$$

where we define for $j = 2, \ldots, N$

$$\mathbf{y}_j = \tilde{\mathbf{x}}_j - \mathbf{x}_{CM}$$

the coordinate of particle j relative to the center of mass of the $N-1$ particles, such that

$$\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_1 = \mathbf{x} + \mathbf{y}_j.$$

Our high velocity states are defined as follows. Take

$$\Phi_0 = \Phi^{(1)} \otimes \Phi_b$$

with $\hat{\phi}^{(1)} \in C_0^{\infty}(I\!R^n)$ and Φ_b the normalized bound state for the $N-1$ particles with energy E

$$\begin{aligned}
\Phi_{\mathbf{v}}^{(1)} &= e^{im\mathbf{v}\cdot\mathbf{x}}\, \Phi^{(1)} \quad &\Leftrightarrow\quad \hat{\phi}_{\mathbf{v}}^{(1)} = \hat{\phi}^{(1)}(\mathbf{p} - m\mathbf{v}),\\
\Phi_{\mathbf{v}} &= e^{im\mathbf{v}\cdot\mathbf{x}}\, \Phi_0 \quad &\Leftrightarrow\quad \hat{\phi}_{\mathbf{v}} = \hat{\phi}^{(1)}(\mathbf{p} - m\mathbf{v})\, \phi_b,\\
P_{N-1}\Phi_{\mathbf{v}} &= \Phi_{\mathbf{v}} \quad \text{for all } \mathbf{v}
\end{aligned}$$

where now by $\hat{}$ we denote the momentum representation only for the first particle. We define as a bound on the "effective potential" V_{eff} introduced below

$$q(\mathbf{x}) := \left\| \sum_{j=2}^{N} V_{1j}^{S}(\mathbf{x}+\mathbf{y}_j)\, \Phi_b \right\|_{\mathcal{H}_{(N-1)}} \tag{5.8}$$

to be considered as a function or as a multiplication operator in $\mathcal{H}_0$ and we assume that

$$\int_0^\infty dR\,(1+R)^\rho \sup_{|\mathbf{x}|\geq R} q(\mathbf{x}) < \infty \tag{5.9}$$

for some $0 \leq \rho \leq 1$. This will be true if the pair potentials decay as in (2.5) and the eigenfunction has enough decay. Typically eigenfunctions decay exponentially which is more than enough. Then by Lemma 2.2

$$\left\| \sum_{j=2}^N V_{1j}^S(\mathbf{x}+\mathbf{y}_j)\, e^{-itH^{(2)}}\, \Phi_{\mathbf{v}} \right\| \leq \left\| q(\mathbf{x})\, e^{-itT_0}\, \Phi_{\mathbf{v}}^{(1)} \right\| \leq h(|vt|) \tag{5.10}$$

with $(1+\tau)^\rho\, h(\tau) \in L^1\big((0,\infty)\big)$. It follows from (5.7) as in Corollary 2.3 that

$$\left\| (\Omega_\pm^{(2)} - I)\, e^{-itH^{(2)}}\, \Phi_{\mathbf{v}} \right\| = O\left(v^{-1}\right) \tag{5.11}$$

uniformly in $t \in I\!R$. Moreover, by Duhamel's formula

$$v\left(i(S^{(2)} - I)\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}} \right) = \int_{-\infty}^{+\infty} l_{\mathbf{v}}(\tau)\, d\tau + R(\mathbf{v}),$$

with $\tau = vt$ and

$$\begin{aligned} \ell_{\mathbf{v}}(\tau) &:= \left(V_{\text{eff}}(\mathbf{x})\, e^{-iT_0\tau/v}\, \Phi_{\mathbf{v}}^{(1)},\ e^{-iT_0\tau/v}\, \Psi_{\mathbf{v}}^{(1)} \right)_{\mathcal{H}_0} \\ R(\mathbf{v}) &:= v \int_{-\infty}^{+\infty} dt \left((\Omega_-^{(2)} - I)\, e^{-itH^{(2)}}\, \Phi_{\mathbf{v}},\ \sum_{j=2}^N V_{1j}^S(\mathbf{x}+\mathbf{y}_j)\, e^{-itH^{(2)}}\, \Psi_{\mathbf{v}} \right)_{\mathcal{H}} \end{aligned}$$

and where

$$V_{\text{eff}}(\mathbf{x}) := \sum_{j=2}^N \left(V_{1,j}^S(\mathbf{x}+\mathbf{y}_j)\, \Phi_b,\ \Phi_b \right)_{\mathcal{H}(N-1)}$$

is the effective potential for the interaction between particle 1 and the cluster. Note that since we integrate over all relative coordinates $\mathbf{y}_j$ the effective potential depends only on $\mathbf{x}$. But, by (5.10), (5.11)

$$R(\mathbf{v}) = O\left(v^{-1}\right). \tag{5.12}$$

Moreover, by (2.3), (2.4)

$$\ell_{\mathbf{v}}(\tau) = \left(V_{\text{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}})\, e^{-iT_0\tau/v}\, \Phi^{(1)},\ e^{-iT_0\tau/v}\, \Psi^{(1)} \right)_{\mathcal{H}_0}, \tag{5.13}$$

and as in the proof of Theorem 2.4 we obtain the reconstruction formula with error term

$$
\begin{aligned}
& v\left(i(S^{(2)}-I)\,\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}}\right) \\
&= \left(\int_{-\infty}^{+\infty} d\tau\, V_{\mathrm{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi^{(1)},\ \Psi^{(1)}\right)_{\mathcal{H}_0} + \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1. \end{cases}
\end{aligned} \tag{5.14}
$$

Finally, as in the proof of Theorem 1.1 in the two–body case we prove that the high velocity limit of $(S^{(2)}-I)$ determines $V_{\mathrm{eff}}(\mathbf{x})$ uniquely.

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